in this case. The main result of the present article is that the conjecture does indeed hold true and that the solutions to (1) do again converge to those of (2) as $\varepsilon \to 0$. This time, the limiting constant $\bar{V}$ is given by

$$\bar{V} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{\overline{\Phi}(t)}{\sqrt{t}}\, dt\ ,$$

where we have set $\overline{\Phi}(s) := \int_{\mathbb{R}} \Phi(x,s)dx$.

The techniques employed in the present article are very different from [7]: instead of relying on probabilistic techniques, we adapt the analytical techniques from [5].

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Spectrum-preserving two-scale decompositions with applications to numerical homogenization and eigensolvers

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(joint work with A. Målqvist)

This note summarizes some recent results on linear elliptic problems with rough multiscale coefficients in the absence of strong assumptions such as periodicity or scale separation. Given a polyhedral domain $\Omega \subset \mathbb{R}^d$ and functions $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ with uniform spectral bounds $\sigma(A(\cdot)) \subset [\alpha,\beta] \subset]0,\infty[$ and $g \in L^2(\Omega)$, consider the second order linear elliptic problem

(LP) find $u \in V := H^1_0(\Omega)$ such that

$$a(u,v) := \int_\Omega (A\nabla u)\cdot\nabla v = \int_\Omega gv =: (g,v) \quad \text{for all } v \in V,$$

and the associated eigenproblem

(EP) find pairs $(\lambda^{(\ell)}, u^{(\ell)}) \in \mathbb{R}_{>0} \times V \setminus \{0\}$, $\ell \in \mathbb{N}$ such that

$$a(u^{(\ell)}, v) = \lambda^{(\ell)} \int_\Omega u^{(\ell)} v =: (\lambda^{(\ell)} u^{(\ell)}, v) \quad \text{for all } v \in V.$$

If the diffusion coefficient A is highly variable then the numerical approximation of either problem is challenging. The underlying mesh width has to be sufficiently

small to resolve the oscillations and heterogeneities of A. For problems in geophysics or material sciences with characteristic geometric features on microscopic length scales, this so-called resolution condition is often so restrictive that the initial mesh must be chosen very fine and further refinement exceeds computer capacity.

In this note, a new numerical homogenization method is presented. It is related to some low-dimensional macroscopic generalized finite element space. The assembling of the corresponding linear system of equations requires microscopic computations only in local vertex patches of diameter $H\log(1/H)$; H being the macroscopic mesh size. The energy error of the method converges linearly with respect to H without any pre-asymptotic effects. Moreover, small (leading) eigenvalues of the operator are preserved on the macroscopic level in a superconvergent way (at least quartic with respect to H).

The framework presented here was mainly developed in [4, 5] and further investigated in [2, 3, 1]. References to alternative methods for numerical homogenization or upscaling may be found therein and in several contributions to this Oberwolfach workshop, e.g., by A. Abdulle, L. Berlyand, Y. Efendiev, R. Lipton, A. Lozinski, or L. Zhang. The question of how the different approaches compare in certain scenarios remains open. As major advantages of our approach we consider its generality, rigorousness and spectral properties.

1. Two-scale Decompositions

Our method is based on novel multiscale decompositions of H^1 into some macroscopic/coarse part V_{cs} plus some microscopic/fine part V_{fs}. Let $V_H \subset V$ denote the classical $P1$ finite element space with respect to some coarse mesh of width H. It is spanned by nodal basis functions ϕ_z for interior vertices $z \in \mathcal{N}$. The key tool is the Clément-type (quasi-)interpolation operator $\mathcal{I}_H : V \to V_H$ with $\mathcal{I}_H v := \sum_{z\in\mathcal{N}} \frac{(v,\phi_z)}{(1,\phi_z)}\phi_z$. Throughout this note, its kernel $V_{\mathrm{fs}} := \operatorname{kernel}\mathcal{I}_H$ defines microscopic functions.

Lemma 1 (L^2-orthogonal two-scale decomposition)**.**

$$V = V_H \oplus V_{\mathrm{fs}} \quad \textit{and} \quad (V_{\mathrm{cs}}, V_{\mathrm{fs}}) = 0.$$

The decomposition is orthogonalized with respect to the scalar product a induced by the problems (LP) and (EP). Let $\mathfrak{F} : V \to V_{\mathrm{fs}}$ denote the a-orthogonal projection onto the finescale space V_{fs}, that is $a(\mathfrak{F}v, w) = a(v, w)$ for all $w \in V_{\mathrm{fs}}$. A modified (operator-dependent) coarse space is then given by

$$V_{\mathrm{cs}} := (1-\mathfrak{F})V_H = \operatorname{span}\{(1-\mathfrak{F})\phi_z \mid z \in \mathcal{N}\}.$$

Lemma 2 (a-orthogonal two-scale decomposition)**.**

$$V = V_{\mathrm{cs}} \oplus V_{\mathrm{fs}} \quad \textit{and} \quad a(V_{\mathrm{cs}}, V_{\mathrm{fs}}) = 0.$$

The lemma immediately yields the desired error estimates for the Galerkin method with respect to V_{cs} (see Section 2). A further main observation is that the a-orthogonal decomposition remains almost orthogonal in L^2.

Lemma 3 (L^2-quasi-orthogonality of the a-orthogonal decomposition)**.**

$$\forall v_{\mathrm{cs}} \in V_{\mathrm{cs}} \forall v_{\mathrm{fs}} \in V_{\mathrm{fs}} : \ (v_{\mathrm{cs}}, v_{\mathrm{fs}}) \lesssim \alpha^{-1} H^2 \left|\left|A^{1/2}\nabla v_{\mathrm{cs}}\right|\right| \left|\left|A^{1/2}\nabla v_{\mathrm{fs}}\right|\right|.$$

Here and throughout the paper, the constant hidden in the notation $\lesssim$ only depends on Ω and interior angles of the finite element mesh. Lemma 3 indicates that V_{cs} is suitable for the discretization of the eigenproblem (see Section 3).

2. Galerkin approximation and sparse representation of V_{cs}

Let $u_{\mathrm{cs}} \in V_{\mathrm{cs}}$ denote the Galerkin approximation of u with respect to V_{cs}, i.e.,

$$a(u_{\mathrm{cs}}, v) = (f, v) \quad \text{for all } v \in V_{\mathrm{cs}}.$$

The error of this discretization is small (of order H) because $g \in L^2(\Omega)$ and u_{cs} is exactly the coarse part of u in the decomposition of Lemma 2.

Lemma 4 (Discretization error)**.**

$$\left|\left|A^{1/2}\nabla(u - u_{\mathrm{cs}})\right|\right| \lesssim \alpha^{-1/2} \left|\left|Hg\right|\right|.$$

Observe that V may be replaced with any subspace $V_h \supset V_H$. In practical computations, V_h is some high resolution FE space that is sufficiently rich to capture the characteristic scales of the problem. In addition to this finescale discretization, we need to find approximations of the corrector function $\mathfrak{F}\phi_z$ with local support in order to turn (2) into a feasible method. We introduce a new parameter, the localization parameter $k \in \mathbb{N}$ and define nodal patches $\omega_{z,1} := \operatorname{supp} \phi_z$ and $\omega_{z,k} := \cup\{T \in \mathcal{T}_H \mid T \cap \omega_{z,k-1} \neq \emptyset\}$ for $k \geq 2$.

Lemma 5 (Decay of correctors)**.**

$$\forall z \in \mathcal{N} \forall k \in \mathbb{N} : \ \left|\left|A^{1/2}\nabla\mathfrak{F}\phi_z\right|\right|_{L^2(\Omega\setminus\omega_{z,k})} \lesssim e^{-\sqrt{\alpha/\beta}k} \left|\left|A^{1/2}\nabla\mathfrak{F}\phi_z\right|\right|.$$

The exponential decay motivates the approximation of $\psi_z = \mathfrak{F}\phi_z \in V_{\mathrm{fs}}$ by $\psi_{z,k} \in V_{\mathrm{fs}}(\omega_{z,k}) := \{v \in V_{\mathrm{fs}} \mid v|_{\Omega\setminus\omega_{x,k}} = 0\}$, where

$$a(\psi_{z,k}, v) = a(\phi_z, v) \quad \text{for all } v \in V_{\mathrm{fs}}(\omega_{z,k}).$$

Thus, the approximate modified coarse space V_{cs}^k has a local basis

$$V_{\mathrm{cs}}^k = \operatorname{span}\{\phi_z - \psi_{z,k} \mid z \in \mathcal{N}\}.$$

The corresponding Galerkin approximation of (LP) is denoted u_{cs}^k.

Theorem 6 (Discretization error after localization)**.**

$$\textit{If } k \gtrsim \sqrt{\beta/\alpha}|\log(H)| \textit{ then } \left|\left|A^{1/2}\nabla(u - u_{\mathrm{cs}}^k)\right|\right| \lesssim H \left|\left|g\right|\right|.$$

Estimates for fully discrete version and estimates in L^2-norm can be found in [4] (see [3] for improved results with regard to hidden constants).

3. Approximation of Eigenvalues and Eigenvectors

A third characterization of a macroscopic functions may be given via the eigenfunctions related to the ℓ smallest eigenvalues $E_\ell := \text{span}\{u^{(1)}, \dots, u^{(\ell)}\}$. By revisiting Lemma 3 we observe that those macroscopic functions are already represented very accurately by V_{cs} (or V_{cs}^k for k sufficiently large).

Corollary 7 (L^2-quasi-orthogonality of the a-orthogonal decomposition of macroscopic functions)**.** *Let $\ell \in \mathbb{N}$ and let $u = u_{\text{cs}} + u_{\text{fs}} \in E_\ell$ with $||u|| = 1$, where $u_{\text{cs}} \in V_{\text{cs}}$ (resp. $u_{\text{fs}} \in V_{\text{fs}}$) denotes the coarse scale part (resp. fine scale part) of u according to the a-orthogonal decomposition in Lemma 2. Then it holds*

$$(u_{\text{cs}}, u_{\text{fs}}) \lesssim \sqrt{\ell}\alpha^{-2}(\lambda^{(\ell)})^2 H^4.$$

We approximate eigenpairs by solutions of the discrete eigenvalue problem: find $\lambda_H^{(\ell)} \in \mathbb{R}$ and non-trivial $u_{\text{cs}}^{(\ell)} \in V_{\text{cs}}$ such that

$$a(u_{\text{cs}}^{(\ell)}, v) = \lambda_H^{(\ell)}(u_{\text{cs}}^{(\ell)}, v) \quad \text{for all } v \in V_{\text{cs}}.$$

Theorem 8 (Bound for the eigenvalue error)**.** *Let H be sufficiently small so that $H \lesssim \ell^{-1/4}\sqrt{\alpha/\lambda^{(l)}}$. Then it holds*

$$\lambda_H^{(\ell)} - \lambda^{(\ell)} \lesssim \sqrt{\ell}(\lambda^{(\ell)})^3\alpha^{-2}H^4 \quad \textit{for all } \ell = 1, 2, \dots, N_H.$$

See [5] for estimates of the error in the corresponding eigenfunctions. Again, discretization ($V \to V_h$) and localization ($V_{\text{cs}} \to V_{\text{cs}}^k$) as in Section 2 are applicable.

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Anomalous diffusion approximation for kinetic equations

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(joint work with N. B. Abdallah, A. Mellet)

We consider diffusion approximation for kinetic equations like linear Boltzmann or Fokker-Planck in the case where the equilibria are heavy tail functions, In the case, the classical diffusion scaling leads to an infinite diffusion coefficient. Then, we have to adapt the scaling and reorganize the different terms to obtain at the limit that the solution may be approximated by a equilibrium as a velocity profile multiplied by a density satisfying a fractional diffusion equation.