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Comparison results for first-order FEMs

Mira Schedensack
(joint work with Carsten Carstensen, Daniel Peterseim)

Various first-order finite element methods are known for the Poisson Model Problem (1) and for linear elasticity (2). The recent publications [1] started the comparison between some of these methods for the Poisson Model Problem, which is completed in this presentation and its underlying paper [3].

Given a bounded polygonal Lipschitz domain Ω in the plane and data $f \in L^2(\Omega)$, the Poisson model problem seeks the weak solution $u \in H^1(\Omega)$ of

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

This presentation compares the error of three popular finite element methods (FEM) of Figure 1 for the numerical solution of (1), namely the conforming *Courant* FEM (CFEM) [4], the nonconforming *Crouzeix-Raviart* FEM (CR-NCFEM) [5], and the mixed *Raviart-Thomas* FEM (RT-MFEM) [7] with respective solutions u_{C}, u_{CR}, and $(p_{\mathrm{RT}}, u_{\mathrm{RT}})$ based on a shape-regular triangulation $\mathcal{T}$ of Ω into triangles. The finite element space of CFEM reads $P_1(\Omega) \cap C_0(\Omega)$ for

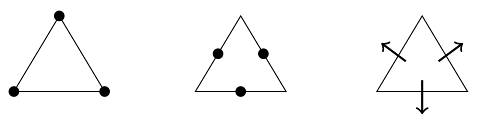

Figure 1. CFEM (left), CR-NCFEM (middle), RT-MFEM (right).

$C_0(\Omega)$ the continuous functions with zero boundary conditions. The finite element space of Crouzeix-Raviart $\mathrm{CR}_0^1(\mathcal{T})$ consists of all piecewise affines which are continuous at the midpoints of interior edges and vanish at the midpoints of exterior edges. The Raviart-Thomas finite element space for the flux approximation reads $\mathrm{RT}_0(\mathcal{T}) := \{p_{\mathrm{RT}} \in P_1(\mathcal{T}, \mathbb{R}^2) \cap H(\mathrm{div}, \Omega) \,|\, \forall T \in \mathcal{T}\, \exists\, a_T, b_T, c_T \in \mathbb{R} : p_{\mathrm{RT}}|_T(x) = (a_T, b_T) + c_T x\}$.

The comparison is stated in terms of $A \lesssim B$ which abbreviates the existence of some constant C which only depends on the minimal angle in $\mathcal{T}$, but not on the domain Ω and not on the mesh-size $h_{\mathcal{T}}$, such that $A \leq CB$. The comparison includes data oscillations, namely $\mathrm{osc}(f, \mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|$ where Π_0 denotes the L^2 orthogonal projection onto the piecewise constants.

The comparison result for CFEM, CR-NCFEM and RT-MFEM states that the errors of CFEM and CR-NCFEM are equivalent up to data oscillations, in the sense that

$$\|\nabla u - \nabla u_{\mathrm{C}}\| \lesssim \|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| \lesssim \|\nabla u - \nabla u_{\mathrm{C}}\| + \mathrm{osc}(f, \mathcal{T}).$$

The error of RT-MFEM is superior in the sense that

$$\|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| \lesssim \|h_{\mathcal{T}} f\| + \|\nabla u - p_{\mathrm{RT}}\| \lesssim \|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| + \mathrm{osc}(f, \mathcal{T}),$$

but the converse is false, i.e.,

$$\|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\| \not\lesssim \|\nabla u - p_{\mathrm{RT}}\| + \mathrm{osc}(f, \mathcal{T}).$$

The proof of the inequalities is an example of the medius analysis for it combines arguments of an *a priori* with those of an *a posteriori* error analysis. It is emphasised that no regularity assumption is made and the results hold for arbitrary coarse triangulations and not just in an asymptotic regime. The proof of the superiority of RT-MFEM considers a sequence of domains, on which the RT-MFEM has a steeper convergence rate than CR-NCFEM.

The results for the Poisson Model Problem can be generalised for the Navier-Lamé equations from linear elasticity, which seek $u \in H_0^1(\Omega; \mathbb{R}^2)$ with

$$f + 2\mu\Delta u + (\mu + \lambda)\nabla(\mathrm{div}\, u) = 0 \quad \text{in } \Omega. \tag{2}$$

The compared FEMs are the conforming *Courant* FEM (CFEM) [2], the nonconforming *Kouhia-Stenberg* FEM (KS-NCFEM) [6], and the nonconforming *Crouzeix-Raviart* FEM (CR-NCFEM) [2] with respective solutions σ_{C}, σ_{KS} and σ_{CR}. The finite element space of KS-NCFEM reads $\mathrm{KS} := (P_1(\mathcal{T}) \cap C_0(\Omega)) \times \mathrm{CR}_0^1(\mathcal{T})$. The discretisation of CFEM and KS-NCFEM is based on the bilinear form

$$a(u_{\mathrm{KS}}, v_{\mathrm{KS}}) := \int_\Omega \varepsilon_{\mathrm{NC}}(u_{\mathrm{KS}}) : \mathbb{C}\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\, dx,$$

while the discretisation of CR-NCFEM involves the bilinear form

$$a(u_{\mathrm{CR}}, v_{\mathrm{CR}}) := \int_\Omega \big(\mu\, D_{\mathrm{NC}} u_{\mathrm{CR}} : D_{\mathrm{NC}} v_{\mathrm{CR}} + (\mu + \lambda)\, \mathrm{div}_{\mathrm{NC}}\, u_{\mathrm{CR}}\, \mathrm{div}_{\mathrm{NC}}\, v_{\mathrm{CR}}\big) dx.$$

The comparison result for linear elasticity involves the Lamé modulus λ, which effects the locking and the $\lesssim$ notation means, that, in addition, the underlying constants do not depend on the Lamé modulus λ. Then

$$\|\sigma - \sigma_{\mathrm{C}}\| \lesssim \lambda \|\sigma - \sigma_{\mathrm{KS}}\| \lesssim \lambda \left(\|\sigma - \sigma_{\mathrm{C}}\| + \operatorname{osc}(f, \mathcal{T})\right)$$

and

$$\|\sigma - \sigma_{\mathrm{KS}}\| + \operatorname{osc}(f, \mathcal{T}) \approx \|\sigma - \sigma_{\mathrm{CR}}\| + \operatorname{osc}(f, \mathcal{T}).$$

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Quasi optimal adaptive pseudostress approximation of the Stokes equations

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(joint work with Carsten Carstensen, Mira Schedensack)

The pseudostress-velocity formulation [3, 5] of the stationary Stokes equations

$$-\Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega \tag{1}$$

with Dirichlet boundary conditions along the polygonal boundary $\partial\Omega$ allows the stresses-like variables σ in Raviart-Thomas mixed finite element spaces [2] $\mathrm{RT}_k(\mathcal{T})$ with respect to a regular triangulation $\mathcal{T}$, and hence allows for higher flexibility in arbitrary polynomial degrees.

The weak form of problem (1) is formally equivalent and reads: Given $f \in L^2(\Omega;\mathbb{R}^2)$ and $g \in H^1(\Omega;\mathbb{R}^2) \cap H^1(\mathcal{E}(\partial\Omega);\mathbb{R}^2)$ with $\int_{\partial\Omega} g \cdot \nu \, ds = 0$ seek $\sigma \in H(\operatorname{div},\Omega;\mathbb{R}^{2\times 2})/\mathbb{R}$ and $u \in L^2(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned} \int_\Omega \operatorname{dev}\sigma : \tau \, dx + \int_\Omega u \cdot \operatorname{div}\tau \, dx &= \int_{\partial\Omega} g \cdot \tau \, \nu \, ds, \\ \int_\Omega v \cdot \operatorname{div}\sigma \, dx &= -\int_\Omega f \cdot v \, dx \end{aligned} \tag{2}$$