Recent advances in composite finite elements

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Recent advances in Composite Finite Elements

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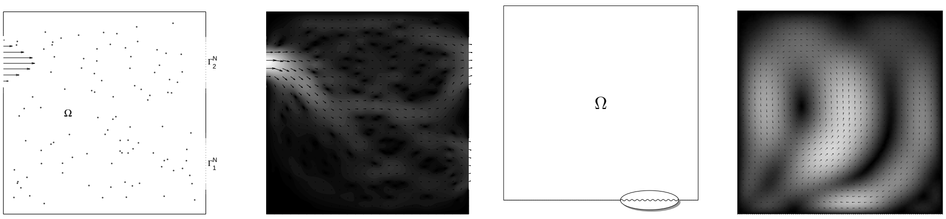
(joint work with Stefan A. Sauter)

Many physical processes that can be modeled by partial differential equations such as groundwater or ocean flows take place in complex environments (shore lines are rarely smooth). Finite element methods are known to be very powerful tools in the numerical investigation of such processes. In principle, the concept of finite elements is sufficient to handle problems on complicated domains, but the standard requirement saying that the underlying finite element mesh has to resolve the boundary of the physical domain is too restrictive if the domain contains small geometric details such as rough boundaries or holes. The resolution condition links the number of elements to the number (and size) of geometric details. Therefore, the minimal dimension of the approximation space reaches a size which is not feasible to solve with a standard computer. Neither can spaces based on resolving grids serve as coarse grid spaces in multilevel solvers. In practice, one is often interested in a moderate accuracy that cannot be achieved at a moderate effort if the mesh has very fine parts used to resolve the geometry. Furthermore, the mesh density of coarse shape regular triangulations of complicated domains is determined by the geometry and *not* by the smoothness properties of the solution.

To be more precise, consider the simple setting of the Poisson equation $-\Delta u = f$ on a polyhedral domain $\Omega \subset \mathbb{R}^2$ having N_Ω sides. In case of Dirichlet boundary condition the discrete weak variational problem reads

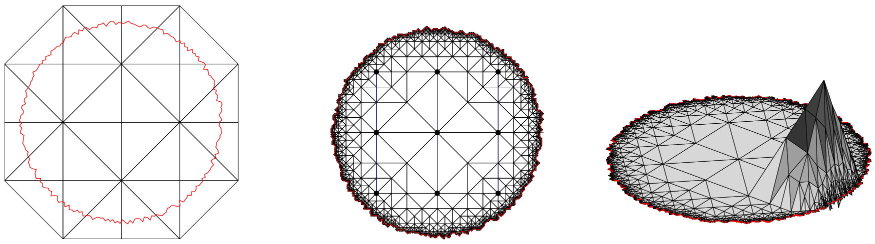
$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in V, \tag{1}$$

where $V = V_\mathcal{T} \subset H_0^1(\Omega)$ contains typically continuous piecewise polynomials with respect to some regular triangulation $\mathcal{T}$ of Ω. The a priori error for a piecewise



(a) Model domain Ω with tiny holes. (b) Solution velocity (black= 0, white= 1). (c) Model domain Ω with oscillating bottom boundary. (d) Solution velocity (black= 0, white= 0.5).

Figure 1. Model problems: (a-b) Stokes flow on the unit square with 100 tiny holes, a Dirichlet inflow boundary and two Neumann outflow boundaries. (c-d) Force driven Stokes flow in a domain with rough slip bottom boundary.



(a) Initial coarse triangulation.

(b) Structured refinement of triangulation in 2a, degrees of freedom (•).

(c) Composite basis function (fulfills the Dirichlet boundary condition in an approximative way).

FIGURE 2. Structured overlapping triangulation of a two dimensional domain with a rough boundary and composite finite element basis function.

linear finite element approximation $u_{\mathcal{T}} \in V$ can be estimated by

$$\|u - u_{\mathcal{T}}\| \lesssim \inf_{v \in V} \|u - v\|_{H^1(\Omega)} \lesssim h^r |u|_{H^{1+r}(\Omega)}, \tag{2}$$

where h denotes the maximal meshwidth of $\mathcal{T}$ and $r \in (\frac{1}{2}, 1]$. The crucial condition for this estimate in case of N_Ω large is the so called conformity condition $V \subset H_0^1(\Omega)$ since it demands $\mathcal{T}$ to be exact. No matter which accuracy one is interested in, the dimension of V is always bounded from below by N_Ω. The resulting linear system might be too large to be solved efficiently. The situation can be even worse in three space dimensions where mesh generation is still a bottle neck in many cases. Overlapping triangulations $\mathcal{T}$ (cf. Figure 2a) allow the definition of low dimensional approximation spaces, but the resulting approximation error will be reflected truly by the sum

$$\inf_{v \in V} \|u - v\|_{H^1(\Omega)} + \sup_{v \in V \setminus \{0\}} \frac{\|v\|_{L^2(\partial\Omega)}}{\|v\|_{H^1(\Omega)}}.$$

While the infimum still can be estimated in terms of the maximal mesh width h, the supremum has a negative effect (pollution) on the overall approximation. To overcome this problem we define coarse finite element spaces (cf. [4]) that preserve the a priori bound given in (2) without the crucial coupling between domain geometry and space dimension. This work bases on the concept of composite finite elements introduced by Hackbusch and Sauter (cf. [2], [3]). Starting from a possibly coarse, overlapping triangulation (cf. Figure 2a) all triangles that intersect the boundary are refined successively (cf. Figure 2b). Note, that the number of refinement steps does not depend on the complicated geometry but only on the meshwidth h of the initial grid. Additionally, the degrees of freedom are the same as in the initial coarse triangulation. New nodes do not enlarge the space dimension, since they become slave nodes. Composite shape functions u^{cfe} are defined by mapping shape functions u according to the initial triangulation to the finite element space with respect to the refined triangulation. The (linear) mapping is explicitly given by the simple formula

$$u^{\mathrm{cfe}}(x) = \begin{cases} u(x), & x \text{ interior node} \\ u(x) - u(x^{\partial\Omega}), & \text{else,} \end{cases} \tag{3}$$

where $x^{\partial\Omega}$ denotes an (approximative) projection of x to the boundary of Ω. A typical basis function of the resulting space V^{cfe} is depicted in Figure 2c. The dimension of the composite space V^{cfe} does *not* depend on N_Ω. Apart from this result, the composite finite element approximation fulfills an a priori error bound which is optimal in the meshwidth parameter h (cf. [5]):

$$\|u - u^{\mathrm{cfe}}\| \lesssim h^r |u|_{H^{1+r}(\Omega)}.$$

The estimate remains true for Lipschitz domains in two as well as three space dimensions (cf. [4] and [1]). Recently, the concept of composite finite elements has further been extended to Stokes problem with mixed Dirichlet, Neumann, slip and leak boundary conditions (cf. [4], [1]). It allows to compute Stokes flows with reasonable accuracy with only a few degrees of freedom. In many cases (see for instance Figure 1) their number can be chosen much smaller than the number of geometric details (N_Ω).

References

[1] Peterseim, D. and Sauter, S. A., *The Composite Mini Element: Coarse mesh computation of Stokes flows on complicated domains*, to appear in Siam J. Num. Anal.

[2] Hackbusch, W. and Sauter, S. A., *Composite finite elements for the approximation of PDEs on domains with complicated micro-structures*, Numer. Math., **75(4)** (1997), 447–472.

[3] Hackbusch, W. and Sauter, S. A., *Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results*, Computing and Visualization in Science **1** (1997), 15–25.

[4] Peterseim, D. *The Composite Mini Element: A mixed FEM for the Stokes equations on complicated domains*, Dissertation, Universität Zürich (2007).

[5] Rech, M., Sauter, S. and Smolianski, A., *Two-scale composite finite element method for Dirichlet problems on complicated domains*, Numer. Math. **102 (4)** (2006), 681–708.

Numerical methods for two-phase incompressible flows

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(joint work with Maxim Olshanskii)

Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain containing two different immiscible incompressible phases. The time dependent subdomains containing the two phases are denoted by $\Omega_1(t)$ and $\Omega_2(t)$ with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. We assume that Ω_1 and Ω_2 are connected and $\partial\Omega_1 \cap \partial\Omega = \emptyset$ (i. e., Ω_1 is completely contained in Ω). The interface is denoted by $\Gamma(t) = \bar{\Omega}_1(t) \cap \bar{\Omega}_2(t)$. A typical example is a rising air bubble or liquid droplet in a surrounding fluid. The standard model for