Enhanced Precedence Theorems for $1||\sum T_j$

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1. Introduction

Scheduling jobs onto a single machine to minimize the total tardiness ($1||\sum T_j$) is a well-known problem of practical importance in virtually all industrial operations. Moreover, it is known to be strongly NP-hard (Lawler, 1977). As the problem is NP-hard and therefore requires enumeration procedures like dynamic programming or branch & bound for its solution, precedence (dominance) theorems can help in determining pair-wise relations between jobs, e.g., that a job *j* precedes a job *k* in an optimum solution. We shall use the notation "$j \prec k$" to mean "j precedes k in some optimum sequence." Each time we discover such a relation, the search space is reduced by as much as one half. Precedence relations are transitive, e.g., $i \prec j$ and $j \prec k \Rightarrow i \prec k$, so the effect of their discovery can accumulate. Such theorems were first introduced by Emmons (1969) and later extended by Rinnooy Kan, Lageweg and Lenstra (1975) and Kanet (2007). Schedules adhering to such precedences are called *dominant* schedules.

The proof tactics employed by Emmons to test if job $j \prec k$ is to start with an assumed optimal sequence S where k precedes j. Now a schedule S' is constructed in which j precedes k with minimal disturbance to the other jobs in the schedule. The two tactics are:

- Tactic 1: Swapping positions of jobs k and j
- Tactic 2: Inserting job k immediately after job j

If it can be shown that the maneuver provides tardiness for S' not higher than that for S, then it is possible to say that schedules in which k precedes j are not uniquely optimum, thus $j \prec k$. The new theorems follow exactly those tactics but consider one or more additional jobs in the maneuver. This report describes the current progress of this research effort. In the section to follow, we first introduce notation (2.1) and then describe in more detail two of the seven new theorems (2.2). Section 2.3 summarizes the seven new theorems. Section 3 provides initial findings of the computational results illustrating the marginal advantage (beyond Emmons's theorems) that the two theorems provide. Section 4 projects the remaining research underway.

2. The New Theorems

2.1. Notation

We consider the sequencing of a set N of n jobs available for processing at time t = 0 by a continuously available machine. Each job *i* is characterized by a processing time $p_i > 0$ and a due date $d_i > 0$. The goal is to determine the scheduled completion time for job *i*, C_i $\forall$ i, such that total tardiness is minimum. We let B_i represent the set of jobs already known to precede i in some optimum sequence; A_i represents the set of jobs known to follow i in some optimum sequence. We use the notations $B_i' = N\text{-}B_i$, and $A_i' = N\text{-}A_i$. We use the notation P(X) to mean the total processing time of jobs in set X. In determining if job j ≺ k, the new theorems make use of $P(B_k)$, $P(A_j)$, as well as the information about one or more other jobs $w \in B_k$ or $z \in A_j$. In proving the new theorems we use the concept of the tardiness improvement, $TI_i(d_i)$, tardiness decrement, $TD_i(d_i)$, for a job i that results from a maneuver (be it swapping or inserting after), where TI, TD are functions of the job's due date. Finally, the notation UB(.), LB(.) denotes upper bound, lower bound, respectively.

2.2. Using the swap and insert-after tactic with job $w \in B_k$

Theorem SW1: Given jobs j, k, $w \in B_k$ and $p_j \leq p_w$,

if $d_j \leq max\{d_w, P(B_w)+p_w\}+min\{max\{0, P(B_k)+p_k-d_k\}, p_w-p_j\}$ *then* $j \prec k$.

The proof involves beginning with a dominate sequence in which w and k precede j, then swapping w with j and observing if $UB(TD_w) \leq LB(TI_j)+LB(TI_k)$. Figure 1 illustrates.

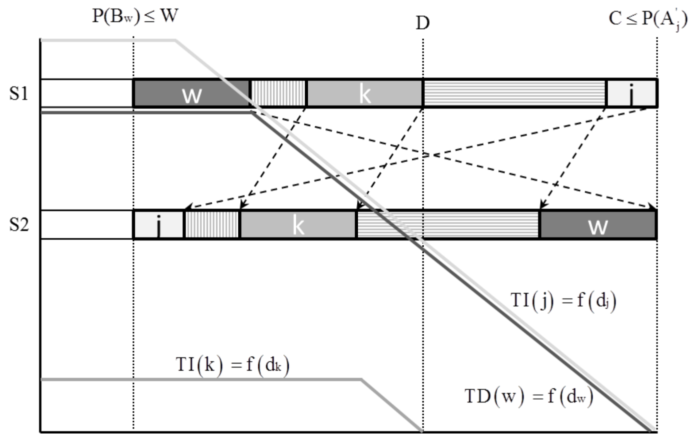

Figure 1. New swap theorem considering three jobs

Theorem IA1: Given jobs j, k, $w \in B_k$,

if $max\{d_w, P(B_w)+p_w\} \geq max\{d_j, P(A_j')-p_w\}-min\{max\{0, P(B_k)+p_k-d_k\}, p_w\}$ *then* $j \prec k$.

As above, the proof involves beginning with a dominate sequence in which w and k precede j, then inserting w after j and observing if $UB(TD_w) \leq LB(TI_j)+LB(TI_k)$. Figure 2 illustrates.

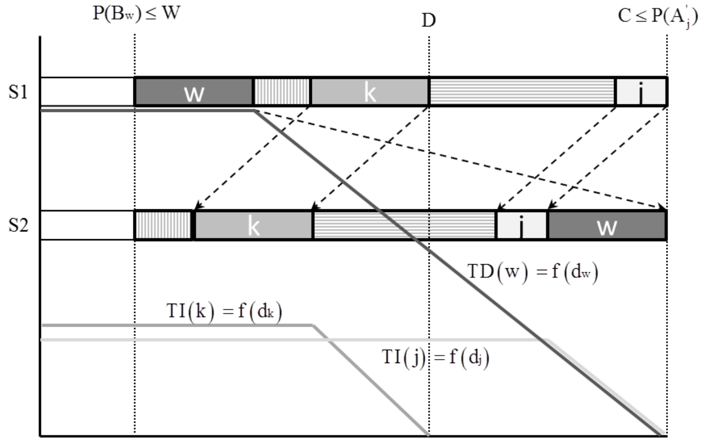

Figure 2. New insert-after-theorem considering three jobs

2.3. The set of seven possible theorems

We currently envision seven new theorems which are under investigation. Three of the possible new theorems follow the swap tactic while four new theorems follow the insert-after tactic. Figure 3 summarizes those concepts. SW1, SW3, IA1, and IA 3 consider the case when one or more jobs $w \in B_k$. Theorems SW2, IA2, and IA4 have one or more jobs $z \in A_j$. There is no theorem SW4 because its condition would be too restrictive to discover further precedences ($p_j + \sum z_{wi} <= p_k$).

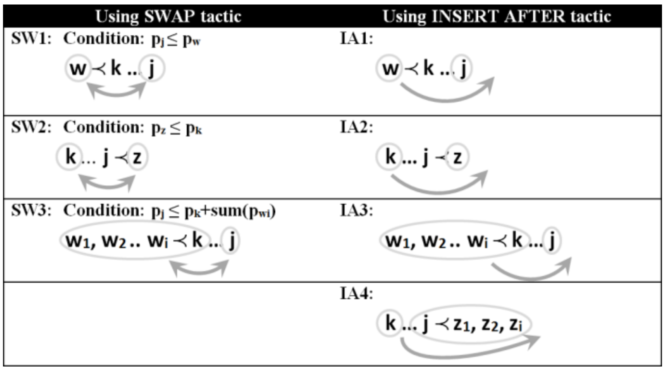

Figure 3. Current and possible new theorems

3. Computational Results

To gain an insight into the benefit of the new theorems a computational study was conducted to observe the number of precedence discoveries by the original theorems of Emmons (1969) and the additional discoveries of the new theorems over a specific set of known problem instances. For this analysis, we use a set of twelve 20-job problem instances from Baker and Trietsch (2009) and 40-job and 50-job problems from Beasley (1990) with 125 instances each. The study was performed in several steps. At first, we calculated the optimum objective function value for each instance with the CPLEX solver to create a validation reference. We then started to activate the theorems one after the other and let the program calculate until no further precedences were discovered. In this state, we captured the number of "hits" produced by the activated theorems.

Table 1. Emmons 1,2,3 & new theorems applied on twelve Baker-/Trietsch' 20-job instances

Experiment(s)	E123	E123+SW1			E123+IA1			E123+SW1+IA1		
	HITS	TOTAL HITS	INCR. [%]	+ HITS (SOLOS)	TOTAL HITS	INCR. [%]	+ HITS (SOLOS)	TOTAL HITS	INCR. [%]	+ HITS (SOLOS)
Baker 1	148	149	0,67	1 (0)	148	0,00	0 (0)	149	0,67	1 (0)
Baker 2	195	195	0,00	0 (0)	195	0,00	0 (0)	195	0,00	0 (0)
Baker 3	234	234	0,00	0 (0)	235	0,43	1 (0)	235	0,43	1 (0)
Baker 4	182	182	0,00	0 (0)	183	0,55	1 (0)	183	0,55	1 (0)
Baker 5	128	128	0,00	0 (0)	128	0,00	0 (0)	128	0,00	0 (0)
Baker 6	179	182	1,65	3 (0)	181	1,10	2 (0)	184	2,72	5 (0)
Baker 7	122	122	0,00	0 (0)	123	0,81	1 (0)	123	0,81	1 (0)
Baker 8	204	204	0,00	0 (0)	205	0,49	1 (0)	205	0,49	1 (0)
Baker 9	189	194	2,58	5 (0)	190	0,53	1 (0)	195	3,08	6 (0)
Baker 10	216	216	0,00	0 (0)	216	0,00	0 (0)	216	0,00	0 (0)
Baker 11	115	115	0,00	0 (0)	115	0,00	0 (0)	115	0,00	0 (0)
Baker 12	176	176	0,00	0 (0)	177	0,56	1 (0)	177	0,56	1 (0)

We compared the number of total theorem hits when applying the new theorems to the number of hits we received with Emmons' theorems solely. The result is provided in the "Increase"-column. Furthermore, the number of hits provided by SW1, IA1, or both is provided in the last column of each test. The number in brackets specifies the quantity of solo hits, which means that only SW1 or only IA1 discovered a precedence relation.

The results for the small Baker instances (cf. Table 1) reveal first hits of the new theorems, though providing no new information (no solo hits). This can be explained due to the fact that the new theorems are more likely to hit if there are many already known precedence relations ("Given jobs (…), $w \in B_k$").

In contrast to this, table 2 depicts results for the bigger Beasley instances showing a lot of solo hits, especially by SW1. As there are 125 instances in each of the Beasley sets, the results are condensed. They provide the average number of hits throughout all the instances. Furthermore, the average and maximum increase of discovered precedences is stated. The last column in each test shows the amount of additional (solo) hits provided by the new theorem(s).

Table 2. Emmons 1,2,3 and new theorems applied on 125 Beasley's 40- and 50-job instances

Experiment(s)	E123	E123+SW1				E123+IA1				E123+SW1+IA1			
	Ø HITS	Ø TOTAL HITS	Ø INCR. [%]	MAX INCR. [%]	+ HITS (SOLOS)	Ø TOTAL HITS	Ø INCR. [%]	MAX INCR. [%]	+ HITS (SOLOS)	Ø TOTAL HITS	Ø INCR. [%]	MAX INCR. [%]	+ HITS (SOLOS)
Beasley's 125 40-job instances	575	616	**10,11**	**31,70**	**5781 (596)**	581	**1,05**	**5,41**	**714 (8)**	620	**11,05**	**32,45**	**6415 (599)**
Beasley's 125 50-job instances	862	919	**9,49**	**30,83**	**7802 (672)**	870	**1,01**	**5,17**	**958 (2)**	925	**10,42**	**31,39**	**8692 (674)**

4. Outlook

The results and first insights into the benefit of all the new theorems necessitate further research. At first, the full set of new theorems should be specified in detail. In regard of computational studies we expect a performance boost for the runtime of the solver which has not been analysed in detail yet. Those effects will arise especially when solving larger instances. After proving a benefit from those new theorems virtually all existing approaches for solving $1||\sum T_j$ and its variants might be well served by application of those.

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