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CHARACTERISTIC CLASSES OF THE HILBERT SCHEMES OF POINTS ON NON-COMPACT SIMPLY-CONNECTED SURFACES

MARC A. NIEPER-WISSKIRCHEN

Institut für Mathematik
Johannes-Gutenberg-Universität
55099 Mainz, Germany
e-mail: marc@nieper-wisskirchen.de

Abstract

We prove a closed formula expressing any multiplicative characteristic class evaluated on the tangent bundle of the Hilbert schemes of points on a non-compact simply-connected surface.

As a corollary, we deduce a closed formula for the Chern character of the tangent bundles of these Hilbert schemes.

We also give a closed formula for the multiplicative characteristic classes of the tautological bundles associated to a line bundle on the surface.

We finally remark which implications the results here have for the Hilbert schemes of points of an arbitrary surface.

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The theorems in this paper supersede the results of the author's earlier paper "Equivariant cohomology, symmetric functions and the Hilbert scheme of points on the total space of the invertible sheaf $\mathcal{O}_{\mathbf{P}^1}(-2)$ over the projective line".

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1. Introduction

Let X be a quasi-projective connected smooth surface over the complex numbers (*surface* for short). We denote by $X^{[n]}$ the Hilbert scheme of n points on X, parametrising zero-dimensional subschemes of X of length n. It is a quasi-projective variety [7] and smooth of dimension $2n$ [6].

The direct sum $\bigoplus_{n\geq 0} H^*(X^{[n]}, \mathbf{Q}[2n])$ of the (shifted) cohomology spaces of all Hilbert schemes carries a natural grading given by the cohomological degree and a weighting given by the number of points n. Likewise, the symmetric algebra $S^*(t^{-1}H^*(X, \mathbf{Q}[2])[t^{-1}])$ carries a grading by cohomological degree and a weighting. The weighting is defined such that $t^{-n}H^*(X, \mathbf{Q}[2])$ is of pure weight n.

There is a natural vector space isomorphism

$$S^*(t^{-1}H^*(X, \mathbf{Q}[2])[t^{-1}]) \to \bigoplus_{n\geq 0} H^*(X^{[n]}, \mathbf{Q}[2n]),$$

respecting the grading and weighting [8, 14]. The image of 1 under the isomorphism is denoted by $|\,0\,\rangle$ and called the *vacuum*. For each $l > 0$ and $\alpha \in H^*(X, \mathbf{Q}[2])$, the linear operator on $\bigoplus_{n\geq 0} H^*(X^{[n]}, \mathbf{Q})$ that corresponds via this isomorphism to multiplying by $t^{-l}\alpha$ is denoted by $q_l(\alpha)$. These operators are called *creation operators*.

One open problem in the study of Hilbert schemes is to express any characteristic class of $X^{[n]}$ by a closed formula in terms of the creation operators applied on the vacuum.

The following general result holds [3, 4]: For each multiplicative characteristic class ϕ with values in a $\mathbf{Q}$-algebra [9], there are unique series $(a_\phi^\lambda)_\lambda, (b_\phi^\lambda)_\lambda, (c_\phi^\lambda)_\lambda, (d_\phi^\lambda)_\lambda$ of elements in A, indexed by the set $\{\lambda\}$ of all partitions, such that

$$\sum_{n\geq 0} \phi(\mathcal{T}_{X^{[n]}}) = \exp \sum_\lambda (a_\phi^\lambda q_\lambda(1) + b_\phi^\lambda q_\lambda(K_X) + c_\phi^\lambda q_\lambda(e_X) + d_\phi^\lambda q_\lambda(K_X^2))|\,0\,\rangle \tag{1}$$

holds for all smooth surfaces X (where we have to view the equation in the completion $\prod_{n\geq 0} H^*(X^{[n]}, \mathbf{Q}[2])$ with respect to the weighting).

Here, the operator $q_\lambda(\alpha)$ for an unordered r-tuple $\lambda = (\lambda^1, \ldots, \lambda^r)$ (e.g., a partition of length r) and a class $\alpha \in H^*(X, \mathbf{Q}[2])$ is defined as follows: Let $\delta^r : X \to X^r$ be the diagonal embedding and $\delta_!^r : H^*(X, \mathbf{Q}[2]) \to H^*(X, \mathbf{Q}[2])^{\otimes r}$ the induced proper push-forward map (which is of degree $2r - 2$). We then set $q_\lambda(\alpha) := \sum q_{\lambda_1}(\alpha_{(1)}) \cdots q_{\lambda_r}(\alpha_{(r)})$, where $\delta_!^r \alpha = \sum \alpha_{(1)} \otimes \cdots \otimes \alpha_{(r)}$ (in Sweedler notation).

It remains to express the coefficients $(a_\phi^\lambda), \ldots, (d_\phi^\lambda)$ by a closed formula depending on ϕ. In this paper, we completely solve the problem for all non-compact simply-connected surfaces.

We also give a similar result for the *tautological bundles* $\mathcal{F}^{[n]}$ on the Hilbert schemes of X associated to a line bundle $\mathcal{F}$ on X [11]. The tautological bundle $\mathcal{F}^{[n]}$ is a vector bundle of rank n on $X^{[n]}$, whose fiber over a point ξ in $X^{[n]}$ (i.e., a zero-dimensional subscheme of length n on X) is given by $H^0(X, \mathcal{F} \otimes \mathcal{O}_\xi)$.

By universality of the formulas for the characteristic classes [2, 3], the results found here also apply to compact or non-simply-connected surfaces. For an arbitrary surface, however, this gives only partial results.

Let us finally remark that we have not made any assumptions on the canonical divisor of X. This is noteworthy insofar as many results for Hilbert schemes have been obtained in closed form only for $K_X = 0$.

2. The Results

From now on, let ϕ be a fixed multiplicative characteristic class with values in the $\mathbf{Q}$-algebra A. It is given by a power series $f \in 1 + xA[[x]]$ such that $\phi = \prod_i f(x_i)$, where x_i are the Chern roots [9].

Let $g \in xA[[x]]$ be the compositional inverse of the power series $G := \frac{x}{f(x)f(-x)}$.

The first main result of this paper is the following:

Theorem 2.1. *Let X be non-compact simply-connected surface. The multiplicative class ϕ evaluated on the tangent bundle of the Hilbert schemes of points on X is given by*

$$\sum_{n\geq 0} \phi(\mathcal{T}_{X^{[n]}}) = \exp\left(\sum_{k\geq 1}(a_k q_k(1) + b_k q_k(K_X)) + \sum_{k,l\geq 1} a_{k,l} q_{(k,l)}(1)\right) |\,0\,\rangle,$$

where the A-valued sequences $(a_k)_k$, $(b_k)_k$, and $(a_{k,l})_{k,l}$ are defined by

$$\sum_{k\geq 1} k a_k x^k = g(x), \quad \sum_{k\geq 1} b_k = \frac{1}{2}\log\frac{f^2(-g(x))}{g'(x)},$$

and

$$\sum_{k,l\geq 1} a_{k,l} x^k y^l = \frac{1}{2}\log\frac{(x-y)f(g(x)-g(y))f(g(y)-g(x))}{g(x)-g(y)}.$$

The theorem is proven in the following section.

From this theorem, we can deduce a formula for the Chern character.

Corollary 2.2. *The Chern character of the tangent bundle of the Hilbert schemes of points on X is given by*

$$\sum_{n\geq 0} \mathrm{ch}(\mathcal{T}_{X^{[n]}}) = \left(\sum_{k\geq 1}(a_k q_k(1) + b_k q_k(K_X)) + \sum_{k,l\geq 1} a_{k,l} q_{(k,l)}(1)\right) \exp(q_1(1))|\,0\,\rangle,$$

where the rational sequences $(a_k)_k$, $(b_k)_k$, and $(a_{k,l})_{k,l}$ are defined by

$$\sum_{k\geq 1} a_k x^k = \sum_{m\geq 0} \frac{2}{(2m+1)!}\frac{x^{2m+1}}{2m+1},$$

$$\sum_{k\geq 1} b_k x^k = \sum_{m\geq 0} \frac{-1}{(2m+1)!}(x^{2m+1} + x^{2m+2})$$

and

$$\sum_{k,\,l\geq 1} a_{k,\,l}x^k x^l = \sum_{m\geq 1}\frac{1}{(2m)!}\sum_{k+l=2m}\left((-1)^k\binom{2m}{k}-1\right)x^k y^l.$$

Proof. Set $\widetilde{\mathrm{ch}} := \mathrm{ch} - \mathrm{ch}_0$. Let ϕ be the multiplicative class with values in the ring $\mathbf{Q}[\varepsilon]$ of dual numbers that corresponds to the power series $f(x) := 1 + \varepsilon(\exp x - 1)$. Then $\widetilde{\mathrm{ch}} = [\varepsilon]\phi$. Here, we use the notation $[t^n]F(t)$ to denote the t^n-coefficient of a polynomial $F(t)$.

For the choice of ϕ at hand, the power series G is given by $G(x) = x - 2\varepsilon x(\cosh x - 1)$. Its compositional inverse is $g(x) = x + 2\varepsilon x(\cosh x - 1)$. Write

$$\sum_{n\geq 0}\phi(\mathcal{T}_{X^{[n]}}) = \exp\left(\sum_{k\geq 1}(\tilde{a}_k q_k(1) + \tilde{b}_k q_k(K_X)) + \sum_{k,\,l\geq 1}\tilde{a}_{k,\,l}q_{(k,\,l)}(1)\right)|\ 0\ \rangle.$$

By Theorem 2.1, we have

$$\sum_{k\geq 1}k\tilde{a}_k x^k = x + 2\varepsilon x(\cosh x - 1),\ \sum_{k\geq 1}\tilde{b}_k x^k = -\varepsilon(1+x)\sin x \tag{2}$$

and

$$\sum_{k,\,l\geq 1}\tilde{a}_{k,\,l}x^k y^l = \varepsilon\left(\cosh(x-y) - \frac{x\cosh x - y\cosh y}{x-y}\right).$$

Because of $\widetilde{\mathrm{ch}} = [\varepsilon]\phi$ and the fact that $(\mathrm{ch} - \widetilde{\mathrm{ch}})(\mathcal{T}_{X^{[n]}}) = 2n\frac{q_1^n(1)}{n!}|\ 0\rangle$, the corollary follows.

By the same methods we will use in the proof of the preceding theorem, one can also prove our second theorem, which is about the tautological bundles.

Let $h \in xA[[x]]$ be the compositional inverse of the power series $H := \dfrac{x}{f(-x)}$.

Theorem 2.3. *Let X be a non-compact simply-connected surface and $\mathcal{F}$ be a line bundle on X with first Chern class F. The multiplicative class ϕ evaluated on the tautological bundles $\mathcal{F}^{[n]}$ of the Hilbert schemes of points on X is given by*

$$\sum_{n\geq 0}\phi(\mathcal{F}^{[n]}) = \exp\left(\sum_{k\geq 1}(a_k q_k(1) + b_k q_k(K_X) + c_k q_k(F)) + \sum_{k,l\geq 1} a_{k,l} q_{(k,l)}(1)\right)|\, 0\,\rangle,$$

where the A-valued sequences $(a_k)_k$, $(b_k)_k$, $(c_k)_k$, and $(a_{k,l})_{k,l}$ are defined by

$$\sum_{k\geq 1} k a_k x^k = h(x), \quad \sum_{k\geq 1} b_k = \frac{1}{2}\log\frac{h^2(x)}{x^2 h'(x)}, \quad c_k = \log\frac{x}{h(x)}$$

and

$$\sum_{k,l\geq 1} a_{k,l} x^k y^l = \frac{1}{2}\log\frac{(x-y)h(x)h(y)}{xy(h(x)-h(y))}.$$

The proof of this Theorem is omitted as it is very similar to the one of Theorem 2.1.

Remark 2.4. In fact, our theorems hold for all surfaces for which $H^4(X, \mathbf{Q}) = 0$ and for which $H^1(X, \mathbf{Q}) = 0$ or $H^3(X, \mathbf{Q}) = 0$. In particular, this condition is fulfilled for X being non-compact and simply-connected.

3. The Proof

By our assumptions on X, it is $e_X = 0 = K_X^2$. Furthermore, it is $\delta_!^r 1 = 0$ for $r \geq 3$ and $\delta_!^r K_X = 0$ for $r \geq 2$. The general formula (1) thus specialises to the following: There are unique A-valued sequences

$(a_k)_k$, $(b_k)_k$, and $(a_{k,l})_{k,l}$ with $a_{k,l} = a_{l,k}$ such that

$$\sum_{n\geq 0} \phi(\mathcal{T}_{X^{[n]}}) = \exp\left(\sum_{k\geq 1}(a_k q_k(1) + b_k q_k(K_X)) + \sum_{k,l\geq 1} a_{k,l} q_{(k,l)}(1)\right)|\ 0\ \rangle$$

for all non-compact simply-connected surfaces X. (The uniqueness can be proven as in the general case; see [3].)

By degree reasons, it follows that

$$\sum_{n\geq 0} \phi_{n-1}(\mathcal{T}_{X^{[n]}}) = \left(\sum_{k\geq 1} a_k q_k(1)\right)|\ 0\ \rangle$$

and

$$\sum_{n\geq 0} \phi_n(\mathcal{T}_{X^{[n]}}) = \exp\left(\sum_{k\geq 1} b_k q_k(K_X) + \sum_{k,l\geq 1} a_{k,l} q_{(k,l)}(1)\right)|\ 0\ \rangle, \tag{3}$$

where ϕ_n denotes the component of ϕ of cohomological degree $2n$. The results in [3] already give the claimed values of the a_k in Theorem 2.1.

It remains to prove the claims about the values of the b_k and the $a_{k,l}$.

The idea of the proof is to explicitly calculate the left hand side of (3) when X is the total space of a line bundle $\mathcal{O}_{\mathbf{P}^1}(-\gamma)$ over $\mathbf{P}^1$ by means of equivariant cohomology. Here, $\gamma > 1$. By comparing coefficients, we will be able to deduce the generating series for the a_k, b_k, and $a_{k,l}$.

We note the following cohomological facts about X: The cohomological fundamental class h of a fibre of the line bundle $\mathcal{O}_{\mathbf{P}^1}(-\gamma)$ spans $H^2(X, \mathbf{Q})$. The canonical class is given by $K_X = (\gamma - 2)h$ and it is $\delta_!^2 1 = -\gamma(h \otimes h)$.

On X, we fix the following $\mathbf{C}^\times$-action, which is described in [12]: First of all, X can be regarded as the quotient space of $(\mathbf{C}^2 \setminus \{0\}) \times \mathbf{C}$ by the

$\mathbf{C}^\times$-action defined by

$$s \cdot (x,\ y,\ t) := (sx,\ sy,\ s^{-\gamma}t)$$

for $s \in \mathbf{C}^\times$. Denote the equivalence class of $(x,\ y,\ z)$ in X by $[x,\ y,\ z]$. Then we let $\mathbf{C}^\times$ act on X by

$$s \cdot [x,\ y,\ t] := [s^{-1}x,\ y,\ st]$$

for $s \in \mathbf{C}^\times$.

This $\mathbf{C}^\times$-action has the two isolated fix points $[1, 0, 0]$ and $[0, 1, 0]$. It induces a $\mathbf{C}^\times$-action on the Hilbert scheme $X^{[n]}$, also with isolated fix points. To each pair (λ^0, λ^1) of partitions with $|\lambda|^0 + |\lambda|^1 = n$, i.e., to each bipartition of n, corresponds exactly one fix point $\xi_{\lambda^0, \lambda^1}$ [12].

The torus $\mathbf{C}^\times$ acts on the tangent space at the fix point $\xi_{\lambda^0, \lambda^1}$ with weights $W_{\lambda^0}(-1, -1)$ and $W_{\lambda^1}(\gamma - 1, 1)$, where $W_\lambda(\alpha, \beta)$, is a multiset defined by

$$W_\lambda(\alpha, \beta) := \{\alpha l(w) + \beta(a(w) + 1),\ -\alpha(l(w) + 1) - \beta a(w) \mid w \in D_\lambda\}$$

for a partition λ. In the definition, D_λ denotes the Young diagram of λ, and $l(w)\,(a(w))$ denotes the leg (arm) length of a cell w in the Young diagram. For the notion of leg and arm length, see [13]; the description of the weights is from [12].

As $X^{[n]}$ has no cohomology in odd degrees and is thus equivariantly formal, we can apply the localisation formula in equivariant cohomology [1, 5]. For the equivariant characteristic class $\phi^{\mathbf{C}^\times}$, the localisation formula gives

$$u^n \phi_n^{\mathbf{C}^\times}(\mathcal{T}_{X^{[n]}}) = \sum_{\substack{\lambda^0, \lambda^1 \\ |\lambda^0| + |\lambda^1| = n}} [\xi_{\lambda^0, \lambda^1}]_{\mathbf{C}^\times} [u^n] \prod_{w \in W_{\lambda^0} \coprod W_{\lambda^1}} \frac{f(wu)}{w} \tag{4}$$

in $H^*_{\mathbf{C}^\times}(X^{[n]}, \mathbf{Q})$, which is a $H^*(B\mathbf{C}^\times, \mathbf{Q}) = \mathbf{Q}[u]$-module. Here $[\xi_{\lambda^0, \lambda^1}]_{\mathbf{C}^\times}$ is the equivariant cohomological fundamental class of the fix point $\xi_{\lambda^0, \lambda^1}$.

In [12], the authors introduce equivariant cohomology classes $[\lambda^0, \lambda^1] \in H^{2n}_{\mathbf{C}}(X^{[n]}, \mathbf{Q})$ for which

$$\frac{u^n[\lambda^0, \lambda^1]}{c'_{\lambda^0}(-1, -1)c'_{\lambda^1}(\gamma - 1, 1)} = \frac{[\xi_{\lambda^0, \lambda^1}]_{\mathbf{C}^\times}}{\prod_{w \in W_{\lambda^0} \coprod W_{\lambda^1}} w}$$

holds. Here $c'_\lambda(\alpha, \beta) := \prod_{w \in D_\lambda} (\alpha l(w) + \beta(a(w) + 1))$ for a partition λ.

Let Λ be the ring of symmetric functions over the rationals in the variables $x_1, x_2, \ldots$. The power symmetric functions are denoted by $p_n := \sum_i x_i^n$. Let $\psi : \Lambda \otimes \Lambda \to \bigotimes_{n \geq 0} H^*(X^{[n]}, \mathbf{Q})$ be the linear map that maps $(p_{\lambda_1} \cdots p_{\lambda_r}) \otimes (p_{\mu_1} \cdots p_{\mu_s})$ to $q_{\lambda_1}(h) \cdots q_{\lambda_r}(h) q_{\mu_1}(h) \cdots q_{\mu_s}(h) | \, 0 \rangle$.

Let us denote by $j^* : H^*_{\mathbf{C}^\times}(X^{[n]}, \mathbf{Q}) \to H^*(X^{[n]}, \mathbf{Q})$ the natural map from equivariant to ordinary cohomology. In [12] it is proven that $j^*([\lambda^0, \lambda^1]) = \psi(P^1_{\lambda^0} \otimes P^{(\gamma-1)^{-1}}_{\lambda^1})$. Here $P^\alpha_\lambda \in \Lambda$ is the Jack polynomial to the parameter α [13].

As j^* maps the equivariant characteristic class $\phi^{\mathbf{C}^\times}$ to the ordinary characteristic class ϕ, equation (4) thus yields

$$\phi_n(\mathcal{T}_{X^{[n]}}) = \sum_{\substack{\lambda^0, \lambda^1 \\ |\lambda^0| + |\lambda^1| = n}} \psi(P^1_{\lambda^0} \otimes P^{(\gamma-1)^{-1}}_{\lambda^1}) \frac{[u^n] \prod_{w \in W_{\lambda^0} \coprod W_{\lambda^1}} f(wu)}{c'_{\lambda^0}(-1, -1) c'_{\lambda^1}(\gamma - 1, 1)}. \quad (5)$$

Let us define a linear map $\rho : H^{2n}(X^{[n]}, \mathbf{Q}) \to A[x, y]$ that maps $q_{\lambda_1}(h) \cdots q_{\lambda_r}(h) | \, 0 \rangle$ to $(x^{\lambda_1} + y^{\lambda_1}) \cdots (x^{\lambda_r} + y^{\lambda_r})$ for each partition $(\lambda_1, \ldots, \lambda_r)$ of n.

For an element $f_0 \otimes f_1 \in \Lambda \otimes \Lambda$ of total degree n, we have $\rho(\psi(f_0 \otimes f_1)) = f_0(x, y) f_1(x, y)$. Here, for a symmetric function $f \in \Lambda$, the expression $f(x, y)$ means to substitute x_1 by x, x_2 by y and x_i for $i \geq 3$ by 0.

We apply the map ρ on both sides of (5). In view of (3), this yields

$$\begin{aligned}\sum_{n\geq 0}\rho(\phi_n(\mathcal{T}_{X^{[n]}})) &= \exp\left(\sum_{k\geq 1}(\gamma-2)b_k(x^k+y^k)\right.\\ &\qquad\left. -\gamma\sum_{k,l\geq 1}a_{k,l}(x^k+y^k)(x^l+y^l)\right)\\ &= \sum_{\lambda^0,\lambda^1} P^1_{\lambda^0}(x, y)\otimes P^{(\gamma-1)-1}_{\lambda^1}(x, y)\\ &\qquad\cdot\frac{[u^{|\lambda^0|+|\lambda^1|}]\prod_{w\in W_{\lambda^0}\coprod W_{\lambda^1}} f(wu)}{c'_{\lambda^0}(-1,-1)c'_{\lambda^1}(\gamma-1, 1)}\\ &=: Z_\gamma(x, y). \end{aligned} \tag{6}$$

From this, we can read off

$$\sum_{k,l\geq 1} a_{k,l}x^ky^l = -\frac{1}{4}(\log Z_2(x, y) - \log Z_2(x, 0) - \log Z_2(0, y)),$$

and

$$\sum_{k\geq 1} b_kx^k = \log Z_3(x, 0) - \frac{3}{2}\log Z_2(x, 0).$$

By Lemma 4.1 and Lemma 4.2, Theorem 2.1 is proven.

4. Two Combinatorial Equalities

The following two Lemmas give the values for the power series $Z_2(x, y)$ and $Z_3(x, 0)$. Both were needed in the proof of Theorem 2.1 in the previous section.

Lemma 4.1. *The value of* $Z_2(x, y)$ *in* $A[[x, y]]$ *is given by*

$$\begin{aligned} Z_2(x, y) &= g'(x)g'(y)\left(\frac{G(g(x)-g(y))}{x-y}\right)^2\\ &= g'(x)g'(y)\left(\frac{g(x)-g(y)}{(x-y)f(g(x)-g(y))f(g(y)-g(x))}\right)^2. \end{aligned}$$

Proof. For $\gamma = 2$, the Jack polynomials $P_\lambda^{(\gamma-1)^{-1}} = P_\lambda^1$ are exactly the Schur polynomials s_λ [13]. It is $s_\lambda(x, y) = 0$ for any partition λ of length greater than two and $s_{(a,b)}(x, y) = \frac{x^{a+1}y^b - y^{a+1}x^b}{x - y}$ for $a \geq b \geq 0$. In particular only the summands corresponding to partitions of length two or less contribute to the sum $Z_2(x, y)$. For these partitions, we have

$$W_{(a,b)}(1,1) = W_{(a,b)}(-1,-1) = \{\pm 1, \dots, \pm b, \pm 1, \dots, \pm \widehat{(a-b+1)}, \dots, \pm (a+1)\},$$

$c'_{(a,b)}(1, 1) = \frac{(a+1)!\,b!}{a-b+1}$, and $c_{(a,b)}(-1, -1) = (-1)^{a+b} c'_{(a,b)}(1, 1)$. Set $F(x) := f(x)f(-x)$. The defining equation (6) takes on the following form:

$$Z_2(x, y) = \frac{1}{(x-y)^2} \sum_{\substack{a \geq b \geq 0 \\ c \geq d \geq 0}}^{\infty} (-1)^{a+b} \frac{(a-b+1)(c-d+1)}{(a+1)!\,b!\,(c+1)!\,d!}$$

$$\cdot (x^{a+1}y^b - y^{a+1}x^b)(x^{c+1}y^d - y^{c+1}x^d)$$

$$\cdot [u^{a+b+c+d}] \frac{\prod_{w=-b}^{a+1} F(wu) \prod_{w=-d}^{c+1} F(wu)}{F((a-b+1)u)F(((c-d)+1)u)}$$

$$= -\frac{1}{(x-y)^2} \sum_{\substack{a > b \geq 0 \\ c > d \geq 0}}^{\infty} (-1)^{a+b} \frac{(a-b)(c-d)}{a!\,b!\,c!\,d!}$$

$$\cdot (x^a y^b - y^a x^b)(x^c y^d - y^c x^d)$$

$$[u^{a+b+c+d-2}] \frac{\prod_{w=-b}^{a} F(wu) \prod_{w=-d}^{c} F(wu)}{F((a-b)u)F((c-d)u)}$$

$$= -\frac{1}{(x-y)^2} \sum_{\substack{a, b \geq 0 \\ c, d \geq 0}}^{\infty} (-1)^{a+b} \frac{(a-b)(c-d)}{a!\,b!\,c!\,d!} \cdot x^{a+c} y^{b+d}$$

$$\cdot [u^{a+b+c+d-2}] \frac{\prod_{w=-b}^{a} F(wu) \prod_{w=-d}^{c} F(wu)}{F((a-b)u)F((c-d)u)}.$$

(The first equality is a simple index shift for the summation variables a and c. The second equality has been obtained by writing the term symmetrically in a and b, and c and d.) We introduce new summation variables $r = a + c$ and $s = b + d$. This and the fact that F is an even power series yields

$$Z_2(x, y) = -\frac{1}{(x-y)^2} \sum_{r,s\geq 0} x^r y^s [u^{r+s}] \sum_{a=0}^{r} \sum_{b=0}^{s} (-1)^{a+b} \cdot \frac{u^2(a-b)(r+s-(a-b))}{a!\, b!\, (r-a)!\, (s-b)!} \cdot \underbrace{\frac{\prod_{w=a-r}^{a} F(wu) \prod_{w=b-s}^{b} F(wu)}{F((a-b)u) F((r+s-(a+b))u)}}_{(*)}.$$

By Lemma 4.3, stated below, the term $(*)$ is equal to $\frac{F^{r+1}(au)F^{s+1}(bu)}{F((a-b)u)F((b-a)u)} + X(u, a, b)$, where $X(u, a, b) \in uA[[au, bu, u]]$. As

$$\sum_{a=0}^{r} \frac{(-1)^a a^k}{a!\,(r-a)!} = \begin{cases} 0 & \text{for } k < r \text{ and} \\ (-1)^r & \text{for } k = r \end{cases} \tag{7}$$

[3], the term $X(u, a, b)$ cannot contribute and we thus have

$$Z_2(x, y) = -\frac{1}{(x-y)^2} \sum_{r+s\geq 0} (-1)^{r+s} x^r y^s [a^r b^s] \frac{F^{r+1}(a)F^{s+1}(b)(a-b)(b-a)}{F(a-b)F(b-a)}.$$

Recall that we defined $G(z) = \frac{z}{F(z)}$. With this definition, we have

$$Z_2(x, y) = -\frac{1}{(x-y)^2} \sum_{r,s=0}^{\infty} (-x)^r (-y)^s \operatorname{res}_{(a,b)} \frac{G(a-b)G(b-a)}{G^{r+1}(a)G^{s+1}}.$$

By the Lagrange-Good formula [10], this is equivalent to

$$Z_2(-G(x), -G(y)) = \frac{1}{G'(x)G'(y)} \frac{G(x-y)}{G(x)-G(y)} \frac{G(y-x)}{G(y)-G(x)}.$$

As G is an odd power series, and g is defined as the inverse power series of G, the claim of the Lemma follows.

Lemma 4.2. *The value of* $Z_3(x, 0)$ *in* $A[[x]]$ *is given by*

$$Z_3(x, 0) = f(-g(x))g'(x).$$

Proof. The proof goes along the same lines as the proof of Lemma 4.1. By the definition of the Jack polynomials it is $P_\lambda^\alpha(x, 0) = 0$ for any partition of length greater than 1 and $P_{(a)}^\alpha(x) = x^a$ for $a \geq 0$ (independently of the parameter α). Thus only the summands corresponding to partitions of length one contribute to the sum $Z_3(x, 0)$. Note that $W_{(a)}(-1, -1) = \{\pm 1, \dots, \pm a\}$, $W_{(a)}(2, 1) = \{-(a+1), \dots, -2, 1, \dots, a\}$, $c'_{(a)}(-1, -1) = (-1)^a a!$, and $c'_{(a)}(2, 1) = a!$. As above, we set $F(x) := f(x)f(-x)$. The definition of $Z_3(x, 0)$ in (6) yields

$$Z_3(x, 0) = \sum_{n \geq 0}\sum_{a \geq 0}(-1)^a x^n [u^n] \frac{f(-(n-a+1)u)\prod_{w=a-n}^{a} F(wu)}{a!\,(n-a)!\,f(-u)}.$$

By Lemma 4.3, there exists a power series $X(a, u) \in uA[[au, u]]$ with

$$Z_3(x, 0) = \sum_{n \geq 0}\sum_{a \geq 0}(-1)^a x^n [u^n]\left(\frac{f(au)F^{n+1}(au)}{a!\,(n-a)!} + X(a, u)\right).$$

By (7), it follows that

$$Z_3(x, 0) = \sum_{n \geq 0}(-1)^n x^n [a^n] f(a)F^{n+1}(a) = \sum_{n \geq 0}(-x)^n \operatorname{res}_a \frac{f(a)}{G^{n+1}(a)}.$$

Again we can make use of the Lagrange-Good formula, which yields

$$Z_3(-G(x), 0) = \frac{f(x)}{G'(x)}.$$

As G is an odd power series, the claim of the Lemma follows.

Lemma 4.3. *Let* $f \in 1 + xA[[x]]$ *be any formal power series with constant coefficient* 1. *Let* n *and* a *be two integers with* $n \geq a \geq 0$. *Then there exists a power series* $X(a, u) \in uA[[ua, u]]$ *in each of which* *monomials the degree in a is strictly less than the degree in u such that*

$$\prod_{w=a-n}^{a} f(wu) = f^{n+1}(au) + X(a,u).$$

Proof. This statement has already appeared at the end of [3], where also a proof can be found.

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