ASYMPTOTIC BEHAVIOUR OF OPTIMAL CONTROL SYSTEMS WITH LOW DISCOUNT RATES*†

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Let x be the path and u the control that maximizes the cumulative return discounted at rate δ. The purpose is to analyze the dependence of x and u on the rate δ. Under certain conditions, as δ tends to 0, the corresponding solution (x, u) tends to the solution for rate 0. We use results from geometric control theory and the theory of semiflows.

1. Introduction. This paper deals with discounted optimal control problems $\mathscr{F}(\delta)$ of the following form:

$$\text{(1.1)} \quad \text{Maximize } V(x, u, \delta) := \int_0^\infty e^{-\delta t}\left\{ g_0(x(t)) + \sum_{i=1}^m u_i(t) g_i(x(t))\right\} dt \quad \text{s.t.}$$

$$\text{(1.2)} \quad \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \quad \text{a.a. } t \in R_+ := [0, \infty),$$

$$\text{(1.3)} \quad x(0) = x \in R^n$$

$$\text{(1.4)} \quad u \in \mathscr{U}_{ad} := \{u\colon R_+ \to \Omega, \text{ measurable}\},$$

$$\text{(1.5)} \quad x(t) \in K \quad \text{for all } t \in R_+,$$

where $g_i\colon R^n \to R$, $f_i\colon R^n \to R^n$, $i = 0, 1, \ldots, m$, are Lipschitz continuous on every bounded set, $\Omega \subset R^m$ is convex and compact, and $K \subset R^n$ is closed; δ is a positive constant.

Optimal control problems of this type frequently occur in economics. Sometimes the family of problems $\mathscr{F}(\delta)$, $0 < \delta < \infty$, is supplemented by the following problem referred to as a "problem with zero discount rate",

$\mathscr{F}(0)$

$$\text{Maximize } V(x, u, 0) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\{ g_0(x(t)) + \sum_{i=1}^m u_i(t) g_i(x(t))\right\} dt$$

s.t. (1.2)–(1.5).

The problem $\mathscr{F}(0)$ can also be interpreted as maximizing the average yield.

We call $(x, u) \in R^n \times \mathscr{U}_{ad}$ optimal for $\mathscr{F}(\delta)$ if it satisfies (1.2)–(1.5) and for all other pairs (x, v) with these properties $V(x, u, \delta) \geqslant V(x, v, \delta)$.

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It is the purpose of the present paper to analyse continuity properties of optimal solutions with respect to δ and, in particular, the relation between $\mathscr{F}(\delta)$ for low δ and $\mathscr{F}(0)$.

§2 shows that under a controllability assumption optimal solutions (x, u^k) of $\mathscr{F}(\delta_k)$ with $\delta_k \to 0$ contain a subsequence converging on finite intervals to an optimal solution of $\mathscr{F}(0)$. Using some results from geometric control theory, it is shown that this controllability assumption holds in the interior of "invariant control sets".

§3 goes one step further: It is not only interesting to analyse convergence properties on finite intervals, but also to see if the omega limit sets of optimal solutions depend continuously on δ. We cannot study, directly, omega limit sets. Instead we show—under a dissipativity assumption—weak upper semicontinuity of the set-valued function $A(\delta)$, where $A(\delta)$ in the set of all bounded optimal R-solutions for $\mathscr{F}(\delta)$. These sets contain all omega limit sets.

This is motivated by an analogous theory of semiflows given in Hale, Magalhaes and Oliva [6]. Note, however, that $\mathscr{F}(\delta)$ does not define a semiflow on R^n, since optimal solutions need not be unique. Nonuniqueness cannot, in general, be eliminated by a slight change in the system parameters, as shown in [5, §6].

2. Convergence on finite intervals. This section gives convergence results on finite intervals for optimal solutions of $\mathscr{F}(\delta_k)$ as $\delta_k \to 0$.

Throughout this paper we assume that for every initial value $x \in K$ and every control function $u \in \mathscr{U}_{ad}$, the corresponding solution $\varphi(t, x, u)$ of (1.2), (1.3) exists an R_+; it is unique by the assumed local Lipschitz continuity of f_i. Furthermore we assume the control system $\mathscr{F}$ given by (1.2)–(1.5) satisfies for all $x \in K$ the following condition

$$\{\varphi(t, x, u) \colon t \in R_+,\ u \in \mathscr{U}_{ad}\} \quad \text{is bounded.} \tag{2.1}$$

REMARK 2.1. We have, in particular, bioeconomic problems [2, 3] in mind, where usually a priori bounds on the trajectories $\varphi(t, x, u)$ can be given due to limited resources.

The following lemma is a slight modification of [5, Lemma 2.4, Corollary 2.5]. For the sake of completeness, the proof is sketched.

LEMMA 2.2. *Consider the control system $\mathscr{F}$, let $\delta_k \to \delta_0 \in (0, \infty)$ and assume that $(x^k, u^k) \in K \times \mathscr{U}_{ad}$, $k = 0, 1, 2, \ldots$ satisfy $x^k \to x^0$ and $u^k \to u^0$ weakly in L^2 on bounded intervals, and $\varphi(t, x^k, u^k)$ is uniformly bounded. Then $\varphi(\cdot, x^k, u^k) \to \varphi(\cdot, x^0, u^0)$ uniformly on bounded intervals and $V(x^k, u^k, \delta_k) \to V(x^0, u^0, \delta_0)$.*

PROOF. With $x^k(t) := \varphi(t, x^k, u^k)$, $t \in R_+$, $k = 0, 1, 2, \ldots$ one obtains

$$\begin{aligned}|x^k(t) - x^0(t)| \leqslant |x^k - x^0| + \int_0^t \Big\{ &|f_0(x^k(t)) - f_0(x^0(t))| \\ &+ \sum_{i=1}^m |u_i^k(t)[f_i(x^k(t)) - f_i(x^0(t))]| \Big\}\, dt \\ &+ \left| \int_0^t \sum_{i=1}^m [u_i^k(t) - u_i^0(t)]\, f_i(x^0(t))\, dt \right|.\end{aligned}$$

The third summand converges to zero by weak convergence of (u^k) to u^0; by uniform boundedness of $\varphi(t, x^k, u^k)$ and local Lipschitz continuity the second one is bounded by $L\int_0^t |x^k(t) - x^0(t)|\,dt$ where $L > 0$. Hence Gronwall's inequality implies the asserted convergence. Convergence of $V(x^k, u^k, \delta^k)$ to $V(x^0, u^0, \delta^0)$ follows using the assertion just proved. □

We will show that for $\delta_k \to 0$ optimal solutions of $\mathscr{F}(\delta_k)$ tend to optimal solutions of $\mathscr{F}(0)$ provided that a certain finite time controllability condition is satisfied. Define the "first hitting time map" by

$$h\colon R^n \times R^n \to R_+ \cup \{\infty\},$$

$$h(x, y) := \inf\{t > 0\colon \text{there is } u \in \mathscr{U}_{ad} \text{ with } \varphi(t, x, u) = y\}.$$

Observe that the inf in this definition actually is a minimum. Next we formulate the required controllability condition. Let $x \in R^n$, and $(u^k) \subset \mathscr{U}_{ad}$ be fixed.

$$\text{(2.2)} \qquad \begin{array}{l} \text{For all } v \in \mathscr{U}_{ad} \text{ there are } T_1, \bar{h} > 0 \text{ such that} \\ \text{for all } k \in N \text{ and all } T \geqslant T_1 \\ h\big(\varphi(T, x, v), \varphi(T, x, u^k)\big) \leqslant \bar{h}. \end{array}$$

Thus we require that every point $\varphi(T, x, v)$ on a trajectory starting in x can be steered —in uniformly bounded time—to $\varphi(T, x, u^k)$ provided that T is large enough.

The following theorem gives the fundamental relation between $\mathscr{F}(\delta)$ and $\mathscr{F}(0)$.

THEOREM 2.3. *Assume that the control system* ($\mathscr{F}$) *satisfies* (2.1), *let* $\delta_k \to 0$ *and let* (x, u^k) *be optimal for* $\mathscr{F}(\delta_k)$.

Then there are $u \in \mathscr{U}_{ad}$ *and a subsequence* (k_i) *such that for all* $T > 0$
$u^{k_i}|[0, T]$ *converges weakly to* $u|[0, T]$ *in* $L^2(0, T; R^m)$,
$\varphi(t, x, u^{k_i})$ *converges uniformly to* $\varphi(t, x, u)$ *for* $t \in [0, T]$.
If (2.2) *holds then* (x, u) *is optimal for* $\mathscr{F}(0)$.

PROOF. We abbreviate

$$g(x, u) = g_0(x) + \sum_{i=1}^{m} u_i g_i(x).$$

Existence of the subsequence (k_i) and of $u \in \mathscr{U}_{ad}$ with the asserted convergence properties follow from Lemma 2.2. It remains to prove optimality of (x, u) for $\mathscr{F}(0)$, i.e. for all $v \in \mathscr{U}_{ad}$

$$\text{(2.3)} \quad \limsup_{T\to\infty} \frac{1}{T}\int_0^T g(\varphi(t, x, u), u(t))\,dt \geqslant \limsup_{T\to\infty} \frac{1}{T}\int_0^T g(\varphi(t, x, v), v(t))\,dt.$$

In order to prove (2.3) it suffices to show that for every $\epsilon > 0$ there is $T(\epsilon) > 0$ such that for all $T > T(\epsilon)$ there is $N = N(\epsilon, T)$ s.t. for all $k \geqslant N$

$$\text{(2.4)} \qquad \frac{1}{T}\int_0^T \big\{g(\varphi(t, x, v), v(t)) - g\big(\varphi(t, x, u^k), u^k(t)\big)\big\}\,dt \leqslant \epsilon c_0$$

where c_0 is a positive constant. If (2.4) holds, one obtains that for all $\epsilon > 0$ there is

$T(\epsilon) > 0$ such that for all $T > T(\epsilon)$

$$\frac{1}{T}\int_0^T g(\varphi(t,x,u),u(t))\,dt = \lim_{k\to\infty}\frac{1}{T}\int_0^T g(\varphi(t,u^k),u^k(t))\,dt$$

$$\geqslant \frac{1}{T}\int_0^T g(\varphi(t,x,v),v(t))\,dt - \epsilon c_0.$$

Hence (2.3) follows by taking the lim sup for $T\to\infty$. In the following proof of (2.4), c_0 denotes a generic constant. Fix $v\in\mathscr{U}_{ad}$, $\epsilon>0$, $T>0$. Then for all $k\in N$

$$(2.5)\quad \int_0^T\{g(\varphi(t,x,v),v(t)) - g(\varphi(t,x,u^k),u^k(t))\,dt$$

$$= \int_0^T e^{-\delta_k t}\{g(\varphi(t,x,v),v(t)) - g(\varphi(t,x,u^k),u^k(t))\}\,dt$$

$$+ \int_0^T[1-e^{-\delta_k t}]\{g(\varphi(t,x,v),v(t)) - g(\varphi(t,x,u^k),u^k(t))\}\,dt.$$

For $k\geqslant N(\epsilon,T)$ the integrand of the second summand is bounded by

$$[1-e^{-\delta_k t}]c_0 \leqslant [1-e^{-\delta_k T}]c_0 \leqslant \epsilon c_0.$$

Hence the second summand is bounded by $\epsilon T c_0$. It remains to estimate the first summand. Observe that it only involves the restriction of v to $[0,T]$.

By assumption (2.2), there are $T_1>0$, $\bar h>0$ such that for all $k\in N$ and all $T\geqslant T_1$

$$h(\varphi(T,x,v),\varphi(T,x,u^k)) \leqslant \bar h.$$

Thus for $T\geqslant T_1$ there are $w^k\in\mathscr{U}_{ad}$ and $0\leqslant t_k\leqslant \bar h$ (both depending on T) with

$$\varphi(t_k,\varphi(T,x,v),w^k) = \varphi(T,x,u^k).$$

Extend $v|[0,T]$ to an element $v^k\in\mathscr{U}_{ad}$ by

$$v^k(t) := \begin{cases} w^k(t-T), & t\in[T,T+t_k],\\ u^k(t-t_k), & t>T+t_k.\end{cases}$$

Thus

$$(2.6)\qquad v^k(t)=v(t),\quad \varphi(t,x,v^k)=\varphi(t,x,v)\quad \text{for } t\in[0,T]\quad\text{and}$$

$$(2.7)\qquad v^k(t+t_k)=u^k(t),\quad \varphi(t+t_k,x,v^k)=\varphi(t,x,u^k)\quad\text{for } t>T.$$

For the first summand in (2.5) we obtain

$$\int_0^T e^{-\delta_k t}\{g(\varphi(t,x,v),v(t)) - g(\varphi(t,x,u^k),u^k(t))\}\,dt$$

$$\leqslant \int_0^\infty e^{-\delta_k t}\{\cdots\}\,dt + \left|\int_T^\infty e^{-\delta_k t}\{\cdots\}\,dt\right|.$$

The first summand is nonpositive by optimality of (x, u^k) for $\mathscr{F}(\delta_k)$, the second one can be estimated above by

$$\left|\int_T^{T+t_k} e^{-\delta_k t} g(\varphi(t, x, v), v(t))\, dt\right|$$

$$+\left|\int_{T+t_k}^{\infty} e^{-\delta_k t} g(\varphi(t, x, v), v(t))\, dt\right.$$

$$\left. - \int_T^{\infty} e^{-\delta_k t} g(\varphi(t, x, u^k), u^k(t))\, dt\right|$$

$$\leqslant t_k c_0 + (e^{-\delta_k t_k} - 1)\left|\int_T^{\infty} e^{-\delta_k t} g(\varphi(t, x, u_k), u_k(t))\, dt\right|$$

using the transformation $t := t - t_k$ and (2.7)

$$\leqslant \bar{h} c_0 + (e^{-\delta_k t_k} - 1)/\delta_k c_0.$$

For $T > T(\epsilon)$ this is

$$\leqslant T\epsilon c_0 + (e^{-\delta_k \bar{h}} - 1)/\delta_k c_0.$$

For $k \geqslant N(\epsilon, T)$ this is

$$\leqslant 2T\epsilon c_0$$

since $\delta_k \to 0$.

This proves (2.4) and hence Theorem 2.3 is proved. □

Next we analyze the controllability assumption (2.2). We will use some notions and results from geometric control theory (cf. Sussmann [8], Isidori [7]). The positive orbit (or attainable set) of $x \in R^n$ at time t is

$$\mathscr{O}^+(x, t) := \{ y \in R^n\text{: there is } u \in \mathscr{U}_{ad} \text{ with } \varphi(t, x, u) = y\} \quad \text{and}$$

$$\mathscr{O}^+_{\leqslant T}(x) := \bigcup_{0 \leqslant t \leqslant T} \mathscr{O}^+(x, t), \quad \mathscr{O}^+(x) := \mathscr{O}^+_{<\infty}(x).$$

Similarly for the negative orbit $\mathscr{O}^-(x, t)$.

DEFINITION 2.4. A nonempty set $D \subset R^n$ is called a *control set* if $D \subset \operatorname{cl} \mathscr{O}^+(x)$ for all $x \in D$ and D is maximal with this property; a set containing a single point x is called a control set only if there is $\omega \in \Omega$ with $0 = f(x, \omega)$. An *invariant control set* C is a control set satisfying $\operatorname{cl} C = \operatorname{cl} \mathscr{O}^+(x)$ for all $x \in C$.

Existence of invariant control sets is guaranteed by the following result from Arnold and Kliemann [1].

PROPOSITION 2.5. *Assume that the following condition is satisfied:*

(2.8) *The Lie algebra L generated by the vector fields* $f_0(\cdot), f_1(\cdot), \ldots, f_n(\cdot)$ *is full at every point x* ($i.e.$ $\dim L(x) = n$).

Then there exists an invariant control set C and C is compact with nonempty interior.

The following result is proved in Colonius and Kliemann [4, Proposition 2.3]; here we only give a sketch of the proof.

Proposition 2.6. *Let $\mathscr{D}$ be a control set and assume that* (2.8) *holds. Then for all compact sets $K_1 \subset \mathscr{D}$, $K_2 \subset \operatorname{int} \mathscr{D}$ there exists $\bar{h} = \bar{h}(K_1, K_2)$ such that the first hitting time map h satisfies $h(x, y) \leqslant \bar{h}$ for all $x \in K_1$, $y \in K_2$.*

Sketch of proof. By a result of Sussmann [8], condition (2.8) implies local accessibility and hence $\mathcal{O}^+_{\leqslant T}(x)$ and $\mathcal{O}^-_{\leqslant T}(x)$ have nonempty interior for all $T > 0$. Using this and the approximate controllability required in the definition of control sets, one shows that $h(x, y)$ is bounded in neighborhoods of $x_1 \in K_1$, $y_1 \in K_2$. Then a compactness argument completes the proof. □

Using this result, we obtain the following corollary to Theorem 2.3:

Corollary 2.7. *In Theorem* (2.3) *replace* (2.2) *by* (2.8) *and the following assumption*:

(2.9) *There exist a compact invariant control set C and a compact set $K_2 \subset C$ such that*

$$\varphi(t, x, u^k) \in K_2 \quad \text{for all } t \geqslant 0,\ k \in N.$$

Then the limit (x, u) is optimal for $\mathscr{F}(0)$.

Proof. By Proposition 2.6, the assumptions (2.8) and (2.9) imply (2.2).

Remark 2.8. [4, §5] presents an example of a three-dimensional harvested predator-prey system satisfying the assumptions of the corollary above. This system possesses a (unique) invariant control set C, and hence for initial values x in the interior of C the above approximation result is valid.

3. Dissipativity and optimal *R*-solutions. In this section optimal R-solutions are defined and their behaviour in dependence of δ is studied. An additional boundedness condition—dissipativity—is assumed.

Let $\mathscr{U}_{ad}(R) := \{u\colon R \to \Omega, \text{ measurable}\}$.

Definition 3.1. A pair $(x, u) \in R^n \times \mathscr{U}_{ad}(R)$ is called an optimal R-solution of $\mathscr{F}(\delta)$, $0 \leqslant \delta < \infty$, if the corresponding trajectory $\varphi(t, x, u)$ exists on R and for all $t \in R$ one has $\varphi(t, x, u) \in K$ and

$$V(\varphi(t, x, u), u(t + \cdot), \delta) = \inf\{V(\varphi(t, x, u), v, \delta)\colon v \in \mathscr{U}_{ad}\}. \tag{3.1}$$

Remark 3.2. If (x, u) is an optimal pair for $\mathscr{F}(\delta)$, then (3.1) follows for all $t > 0$ by Bellman's principle.

Define the set $A(\delta)$, $0 \leqslant \delta < \infty$, by

$$A(\delta) := \{y \in K\colon \text{there exist } v \in \mathscr{U}_{ad}(R) \text{ s.t. } (y, v) \text{ is an optimal } R\text{-solution with } \{\varphi(t, y, v),\ t \in R\} \text{ bounded}\}. \tag{3.2}$$

Next we show that all limit points of optimal solutions are in $A(\delta)$. More precisely, let for $(x, u) \in R^n \times \mathscr{U}_{ad}$ the omega limit set be defined by

$$\omega(x, u) := \{y \in R^n\colon \text{there are } t_k \to \infty \text{ with } \varphi(t_k, x, u) \to y\}.$$

Proposition 3.3. *Assume that the control system $\mathscr{F}$ satisfies* (2.1). *Then for all $0 \leqslant \delta < \infty$ and all $(x, u) \in R^n \times \mathscr{U}_{ad}$ which are optimal for $\mathscr{F}(\delta)$*

$$\varnothing \neq \omega(x, u) \subset A(\delta).$$

PROOF. Consider first $0 < \delta < \infty$ and let $y \in \omega(x, u)$ for a pair (x, u) which is optimal for $\mathscr{F}(\delta)$. Let $t_k \to \infty$ with $x^k := \varphi(t_k, x, u) \to y$. Define $u^k := u(t_k + \cdot) \in \mathscr{U}_{ad}$. By Lemma 2.2 we may assume that u^k converges weakly on bounded intervals to an element $u_0 \in \mathscr{U}_{ad}$, $\varphi(t, x^k, u^k)$ converges uniformly on bounded intervals to $\varphi(t, y, u_0)$ and (y, u_0) is optimal.

Taking again if necessary a subsequence, $u_{-1}^k := u^k(-1 + \cdot)$ converges weakly on bounded intervals to $\tilde{v}_{-1}$: $R_+ \to \Omega$ and $\varphi(-1 + \cdot, x^k, u_{-1}^k)$ converges uniformly on bounded intervals to a function $\varphi_{-1}(\cdot, y, \tilde{v}_{-1})$. Define

$$v_{-1}(t) := \tilde{v}_{-1}(1 + t), \qquad t \in [-1, \infty),$$

$$\varphi_{-1}(t, y, v_{-1}) := \tilde{\varphi}(1 + t, y, \tilde{v}_{-1}), \qquad t \in [-1, \infty).$$

Then $v_0(t) = v_{-1}(t)$ and $\varphi(t, y, v_0) = \varphi_{-1}(t, y, v_{-1})$ for $t \in [0, \infty)$. Taking successively subsequences, one obtains

$$v_{-l}\colon [-l, \infty) \to \Omega \quad \text{and} \quad \varphi_{-l}(\cdot, y, v_{-l})\colon [-l, \infty) \to R^n$$

with $v_{-l}(t) = v_{-l+1}(t)$ for $t \in [-l + 1, \infty)$ and

$$\frac{d}{dt}\varphi_{-l}(t, y, v_{-l}) = f(\varphi_{-l}(t, y, v_{-l}), v_{-l}(t)) \quad \text{a.e. on } [-l, \infty),$$

$$V(\varphi_{-l}(-l, y, v_{-l}), v_{-l}(-l + \cdot)) = \inf\{V(\varphi_{-l}(-l, y, v_{-l}), w)\colon w \in \mathscr{U}_{ad}\}.$$

Defining $v(t) := v_{-l}(t)$ for $t \in [-l, \infty)$, $l \in N$ one obtains on optimal R-solution (y, v), since $\omega(x, u) \subset K$ implies $\varphi_{-l}(\cdot, y, v) \subset K$ for $y \in \omega(x, u)$.

The proof for $\delta = 0$ proceeds similarly.

The proposition above shows that $A(\delta)$ is a global attractor. Assuming the following additional boundedness condition, we will be able to show that $A(\delta)$ is compact and depends upper semicontinuously on δ.

DEFINITION 3.4. The family of optimal control systems $\mathscr{F}(\delta)$ given by (1.1)–(1.5) is dissipative if there exists a bounded set B such that for every compact set $C \subset K$ there is $t_0 = t_0(C)$ such that for all $(x, u) \in C \times \mathscr{U}_{ad}$ which are optimal for $\mathscr{F}(\delta)$, $0 < \delta < \infty$ and all $t \geqslant t_0$ one has $\varphi(t, x, u) \in B$.

REMARK 3.5. The definition given above is an adaptation of "compact dissipativity" employed in the theory of semiflows (cf. Hale, Magalhaes and Oliva [6, pp. 4, 46]).

PROPOSITION 3.6. *Assume that the optimal control system $\mathscr{F}(\delta)$ given by* (1.1)–(1.5) *is dissipative. Then for all $0 < \delta < \infty$ the set $A(\delta)$ is compact.*

PROOF. First we show that $A(\delta) \subset B$ and hence bounded. For $y \in A(\delta)$, there is $v \in \mathscr{U}_{ad}(R)$ such that (y, v) is an optimal R-solution with $\{\varphi(t, y, v),\ t \in R\}$ bounded. Hence $C := \mathrm{cl}\{\varphi(t, y, v),\ t \in R\}$ is compact. Using dissipativity (for fixed δ) one finds $t_0 = t_0(C)$ such that for all optimal $(x, u) \in C \times \mathscr{U}_{ad}$ and all $t \geqslant t_0$ one has $\varphi(t, x, u) \in B$.

Hence for every $s \in R$, $\varphi(s + t_0, y, v) \in B$ and hence $A(\delta) \subset B$. In order to prove closedness, let $x^k \in A(\delta)$, $x^k \to x$. There are $u^k \in \mathscr{U}_{ad}(R)$ such that (x^k, u^k) are optimal R-solutions, with $\varphi(t, x^k, u^k)$, $t \in R$, bounded. Proceeding as in the proof of Proposition 3.3 one obtains the assertion. □

Recall that a set-valued map A on R is upper semicontinuous at $\bar{\delta} \in R$ if for every $\epsilon > 0$ then exists $\eta > 0$ such that $A(\delta) \subset A(\bar{\delta}) + S^\epsilon$ provided $|\delta - \bar{\delta}| < \eta$; here S^ϵ is the closed ball with radius ϵ around the origin, $S^\epsilon := \{z \in R^n\colon |z| \leqslant \epsilon\}$.

THEOREM 3.7. *Assume that the optimal control system $\mathscr{F}$ is dissipative. Then the map $A(\delta)$, $0 < \delta < \infty$ is upper semicontinuous.*

PROOF. By Proposition 3.6, the value sets $A(\delta)$ are compact. Hence by Warga [9, Theorem I.7.2] it suffices to show that for $\delta_k \to \delta$, $x^k \in A(\delta_k)$ with $x^k \to x$ it follows that $x \in A(\delta)$. There are $u^k \in \mathscr{U}_{ad}(R)$ s.t. (x^k, u^k) is an optimal R-solution for $\mathscr{F}(\delta^k)$. We may assume that (u^k) tends weakly in L^2 on bounded intervals to $u \in \mathscr{U}_{ad}(R)$ and that for all $k \in N$, $t \in R$ and $v \in \mathscr{U}_{ad}$

$$V(\varphi(t, x^k, u^k), u^k(t + \cdot), \delta^k) \geqslant V(\varphi(t, x^k, u^k), v, \delta^k). \tag{3.3}$$

Let k tend to infinity. Then by Lemma 2.2 ($\varphi(t, x^k, u^k)$ is uniformly bounded by dissipativity)

$$V(\varphi(t, x^k, u^k), u^k(t + \cdot), \delta_k) \to V(\varphi(t, x, u), u(t + \cdot), \delta) \quad \text{and}$$

$$V(\varphi(t, x^k, u^k), v, \delta_k) \to V(\delta(t, x, u), v, \delta).$$

Hence

$$V(\varphi(t, x, u), u(t + \cdot), \delta) \geqslant V(\varphi(t, x, u), v, \delta)$$

and (x, u) is an optimal R-solution for $\mathscr{F}(\delta)$, i.e. $x \in A(\delta)$. □

REMARK 3.8. Hale, Magalhaes and Oliva [6, Theorem 5.5] give a result which shows that for semiflows defined by a functional differential equation $\dot{x} = f(x_t)$, $x_t(s) := x(t + s)$, $s \in [-r, 0]$, $r > 0$, the set $A(f)$ of all bounded solution defined on R depends in an upper-semicontinuous way on f. This result served as a motivation for the theorem above, which also studies the set of all bounded (optimal) R-solutions under perturbations of the data defining the system.

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