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ASYMPTOTIC PROPERTIES OF OPTIMAL SOLUTIONS IN PLANAR DISCOUNTED CONTROL PROBLEMS*

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Abstract. The classical Poincaré-Bendixson Theorem on limit sets of solutions of planar differential equations is generalized to solutions of planar optimal control problems maximizing a discounted present value that does not depend explicitly on the control function.

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1. Introduction. The main result of this paper is a generalization of the classical Poincaré-Bendixson Theorem for the following class of optimal control problems (P) (with $n=2$):

$$\text{(1.1)} \qquad \text{Maximize} \quad \int_0^\infty e^{-\delta t} R(x(t))\, dt,$$

subject to

$$\text{(1.2)} \qquad \dot{x}_j(t) = x_j(t)\left[f_0^j(x(t)) + \sum_{i=1}^m u_i(t) f_i^j(x(t))\right] \quad \text{a.a. } t \in \mathbb{R}_+, \quad j=1,\cdots,n,$$

$$\text{(1.3)} \qquad u(t) = (u_i(t)) \in \Omega \subset \mathbb{R}^m \quad \text{a.a. } t \in \mathbb{R}_+,$$

$$\text{(1.4)} \qquad x(0) = x \in \mathbb{R}_+^n;$$

here $\delta > 0$ and $R:\mathbb{R}_+^n \to \mathbb{R}$, $f_i = (f_i^j):\mathbb{R}_+^n \to \mathbb{R}$ are locally Lipschitz continuous, the set Ω of control values is compact and convex, and the control functions are chosen in

$$\text{(1.5)} \qquad U_{\text{ad}} = U_{\text{ad}}(\mathbb{R}_+) := \{u:\mathbb{R}_+ \to \Omega, \text{ measurable}\}.$$

Thus we consider discounted optimal control problems where the integral of the performance index does not depend explicitly on the control, and the system equation has the "ecological form" (1.2) with control appearing linearly. Our original motivation for considering asymptotic properties of optimal solutions comes from bioeconomics. Here the study of such problems is often decomposed into two parts.

First an optimal equilibrium point e is searched for and then a determination optimal approach path from the initial point x_0 to e is tried (compare, e.g., Clark [5, p. 317]). This approach is justified in the case of a single-state variable ($n=1$), since here, in general, bounded solution $x(\cdot)$ of (P) converge monotonically to an optimal equilibrium as t tends to $+\infty$ (see Theorem 2.7 below). For two state variables ($n=2$) the classical Poincaré-Bendixson Theorem describes the asymptotic behavior of the special class of (uncontrolled) differentiable dynamical systems; here the limit set $\omega(x)$ of a trajectory either is a periodic trajectory or consists of trajectories connecting

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equilibria. The Poincaré–Bendixson theorem generalizes within the framework of local (nondifferentiable) dynamical systems (see Hajek [10]). However, optimal solutions of (P) are not, in general, unique (cf. the examples given in § 6 below). Hence they need not form a local dynamical system. Nevertheless, the present paper shows that the Poincaré–Bendixson Theorem can be generalized to optimal control problems of the type above. The problem of nonuniqueness must be met at the following crucial steps in the reconstruction of the classical argument for proving the Poincaré–Bendixson Theorem.

(1) We want to form Jordan arcs from parts of solutions; however, solutions may be self-intersecting. This problem is solved via the optimality principle: if an optimal solution x returns at time $t_2 > t_1$ into the same state as at time t_1, then it is also optimal to run through the same piece of trajectory $x|[t_1, t_2]$ again and again. The solution obtained this way is optimal and periodic after time t_1. This is our justification for studying only the limit sets of nonself-intersecting solutions.

(2) To construct "flow boxes" at nonequilibria we cannot rely on a local parallelization theorem such as in dynamical systems. To prove existence of transversal sections and appropriately defined "flow boxes" we use that the integrand in (1.1) does not depend explicitly on the control u. Sometimes more general problems can be reduced to this form (cf. Remarks 2.12, 4.8). The possibility of defining optimality by means of such a functional has been exploited by Clark in a number of resource management problems.

The literature on the asymptotic behavior of optimal solutions for (P) concentrates mainly on establishing sufficient conditions for convergence to equilibrium. We only mention Arrow [2], Rockafellar [15], [16], Feinstein and Luenberger [8], and Feinstein and Oren [9]. The convexity assumptions made here are quite restrictive and are usually not satisfied in resource management. Haurie [12], [13] relaxes the convexity condition, such that they are, e.g., applicable to Volterra–Lotka equations. However, he must assume existence of optimal equilibria (with additional properties).

Oscillatory behavior of optimal solutions is often attributed to nonlinear cost effects and to age structure (Clark [5, pp. 166, 293], Deklerk and Gatto [7]). In § 5 we present an example that possesses neither of these attributes.

For a problem arising in economics, Benhabib and Nishimura [4] analyze the optimality system resulting from the Pontryagin maximum principle. Taking the discount rate δ as a bifurcation parameter, they show that Hopf bifurcations occur. The corresponding periodic solutions are optimal due to convexity assumptions.

The paper is organized as follows. Section 2 contains the basic assumptions and what is needed later about convergent subsequences of solutions and their limit sets. Furthermore the key lemma about transversal segments is proved as well as the existence of "flow boxes." Section 3 is a study of optimal equilibria. As a consequence of Pontryagin's maximum principle it is shown that in "general" there are only finitely many optimal equilibria, and a sufficient condition for attractivity of optimal equilibria is established. The main result is Theorem 3.5, which settles a case in the Poincaré–Bendixson Theorem. Section 4 contains the proof of the Poincaré–Bendixson Theorem for nonself-intersecting solutions. Section 5 discusses resource management problems. A predator-prey system where the predator is subject to harvesting is analyzed. As a consequence of the Poincaré–Bendixson Theorem, there are optimal solutions having as limit set an optimal periodic solution. Section 6 discusses nonuniqueness arising when an optimal periodic trajectory does not contain an optimal equilibrium in its interior as well as nonuniqueness in an example of a symmetric system of two harvested competing species. Here nonuniqueness stems from the bifurcation of behavior in the nonharvested system.

2. Limit sets and flow boxes. We will denote solutions of (1.2), (1.4) by $\varphi(t, x, u)$, $t \geqq 0$, and always assume global existence of $\varphi(\cdot, x, u)$ on $\mathbb{R}_+$ (uniqueness follows from local Lipschitz continuity). Let $\varphi(x, u) := \{\varphi(t, x, u): t \in \mathbb{R}_+\}$ and denote the value of (1.1) corresponding to (x, u) by $V(x, u)$. If not specified otherwise, convergence in U_{ad} means weak convergence in the L_2-sense on compact intervals. We will also use

$$U_{\mathrm{ad}}(\mathbb{R}) := \{u : \mathbb{R} \to \Omega, \text{ measurable}\}.$$

Throughout this paper, we assume that the following hypothesis is satisfied:

(2.1) For every compact subset $K \subset \mathbb{R}^n_+$, the set $\{\varphi(t, x, u): t \in \mathbb{R}_+, x \in K, u \in U_{\mathrm{ad}}\}$ is bounded.

DEFINITION 2.1. A pair $(x, u) \in \mathbb{R}^n_+ \times U_{\mathrm{ad}}$ is called optimal if for all $v \in U_{\mathrm{ad}}$ we have

$$V(x, u) \geqq V(x, v).$$

A pair $(e, u^e) \in \mathbb{R}^n_+ \times \Omega$ is called an optimal equilibrium, if $e = \varphi(t, e, u^e)$ for all $t \in \mathbb{R}_+$ and the pair (e, u^e) is optimal (here u^e is identified with the constant control in U_{ad} with value u^e). For an optimal pair (x, u), we let $V(x) := V(x, u)$.

Remark 2.2. The notion of optimality above keeps the initial point $x(0) = x$ fixed and considers only the effect of different control actions v.

Remark 2.3. Frequently it will—instead of (2.1)—be sufficient that for a fixed (optimal) pair (x, u), we have that $\varphi(x, u) \subset \mathbb{R}^n_+$ is bounded.

Remark 2.4. Let $(x, u) \in \mathbb{R}^n_+ \times U_{\mathrm{ad}}$ be given and suppose that for some $t > 0$, we have $\varphi_j(t, x, u) = 0$ for all $j \in J \subset \{1, 2, \cdots, n\}$. Define

$$\psi_i(s, x, u) = \begin{cases} 0, & i \in J, \\ \varphi_i(s, x, u), & i \notin J, \end{cases}$$

for s in a neighborhood of t. Then ψ also solves (1.2) and $\psi(t, x, u) = \varphi(t, x, u)$. Hence by the uniqueness of solutions of ordinary differential equations $\varphi = \psi$ in a neighborhood of t. Hence either $\varphi_j(s, x, u) = 0$ for all $s \geqq 0$ or $\varphi_j(s, x, u) > 0$ for all $s \geqq 0$. Therefore none of the species can become extinct in finite time and for any $J \subset \{1, \cdots, n\}$ $\{y \mid y_j = 0, j \in J\} \cap \mathbb{R}^n_+ = \mathbb{R}^{(J)}_+$ is invariant and the restriction of the system to $\mathbb{R}^{(J)}_+$ is a system of the same form.

LEMMA 2.5. *Suppose $x^k \to x^0$ in $\mathbb{R}^n_+$ and $u^k \to u^0$ in U_{ad}. Then $\varphi(\cdot, x^k, u^k) \to \varphi(\cdot, x^0, u^0)$ uniformly on bounded intervals and $V(x^k, u^k) \to V(x^0, u^0)$.*

Proof. The first assertion follows in a standard way from Gronwall's inequality. For the second one, take $\varepsilon > 0$. Then for T and k large enough and $x^k(t) := \varphi(t, x^k, u^k)$, $t \in \mathbb{R}_+$, $k = 0, 1, 2, \cdots$,

$$\begin{aligned} \left| V(x^0, u^0) - \int_0^\infty e^{-\delta t} R(x^k(t))\, dt \right| &\leqq \left| V(x^0, u^0) - \int_0^T e^{-\delta t} R(x^0(t))\, dt \right| \\ &\quad + \left| \int_0^T e^{-\delta t} [R(x^0(t)) - R(x^k(t))]\, dt \right| \\ &\quad + \left| \int_T^\infty e^{-\delta t} R(x^k(t))\, dt \right| \\ &\leqq 3\varepsilon, \end{aligned}$$

using the first assertion and (2.1).

COROLLARY 2.6. *Let $(x^k, u^k) \in \mathbb{R}^n_+ \times U_{\mathrm{ad}}$ $(k \in \mathbb{N})$ be optimal and $(x^k)_k$ bounded. Then there are a subsequence (x^{k_i}, u^{k_i}) $(i \in \mathbb{N})$ and an optimal (x, u) such that $\lim_i \varphi(\cdot, x^{k_i}, u^{k_i}) = \varphi(\cdot, x, u)$ locally uniformly and $\lim_i u^{k_i} = u$ in U_{ad}.*

Proof. Existence of a subsequence (x^{k_i}, u^{k_i}) converging to (x, u) follows from boundedness. Now let $v \in U_{\mathrm{ad}}$. Then, by optimality of (x^{k_i}, u^{k_i}),

$$V(x, u) = \lim V(x^{k_i}, u^{k_i}) \geqq \lim V(x^{k_i}, v) = V(x, v).$$

This proves optimality of (x, u).

Let $(x, u) \in \mathbb{R}^n_+ \times U_{\mathrm{ad}}$ be optimal and define $x(\cdot) := \varphi(\cdot, x, u)$.

THEOREM 2.7. (1) *Suppose that $n = 1$ and $x(t)$ is neither increasing nor decreasing; then $\alpha < \beta$ exist such that each $e \in (\alpha, \beta)$ is an optimal equilibrium.*

(2) *If $e = \lim_{t \to \infty} x(t)$ then e is an optimal equilibrium.*

Proof. (1) If $x(t)$ is neither increasing nor decreasing, there exist $r < s < t$ such that $x(r) = x(t)$, $x(s) \neq x(r)$. We may choose s such that either $x(s) = \min x([r, t])$ or $x(s) = \max x([r, t])$, say $x = \max x([r, t])$. Choose any $b \in (\alpha, \beta) = (x(r), x(s))$. There is a first instant $r_\varepsilon > r$ such that $x(r_\varepsilon) = b$, a first instant $s_\varepsilon > r_\varepsilon$ such that $x(s_\varepsilon) = b + \varepsilon$, and a first instant $t_\varepsilon > s_\varepsilon$ such that $x(t_\varepsilon) = b$. Let s'_ε be the last instant $< t_\varepsilon$ such that $x(s'_\varepsilon) = b + \varepsilon$. Then define

$$\begin{aligned} u_\varepsilon(\sigma) &= u(r_\varepsilon + \sigma) && \text{for } 0 \leqq \sigma \leqq s_\varepsilon - r_\varepsilon, \\ u_\varepsilon(s_\varepsilon - r_\varepsilon + \sigma) &= u(s'_\varepsilon + \sigma) && \text{for } 0 \leqq \sigma \leqq t_\varepsilon - s_\varepsilon. \end{aligned}$$

This way $u_\varepsilon(\sigma)$ is defined for $0 \leqq \sigma \leqq s_\varepsilon - r_\varepsilon + t_\varepsilon - s'_\varepsilon = \pi_\varepsilon$. Now extend u_ε to obtain a periodic function on $\mathbb{R}_+$ with period π_ε. Define x_ε in the same way as u_ε using $x(t)$ instead of $u(t)$. Then x_ε satisfies $\dot{x}_\varepsilon(t) = f(x_\varepsilon(t), u_\varepsilon(t))$ almost everywhere on $\mathbb{R}_+$ and $(x_\varepsilon, u_\varepsilon)$ is a solution of (1.1)-(1.4) with $x_\varepsilon(0) = b$. Note that for all $t \geqq 0$ we have $|x_\varepsilon(t) - b| \leqq \varepsilon$. Let $\varepsilon_n > 0$ tend to zero. Then Corollary 2.6 implies that b is an optimal equilibrium.

(2) Suppose $e = \lim_{t \to \infty} x(t)$. For $n \in \mathbb{N}$ put $x_n(t) = x(t + n)$, $u_n(t) = u(t + n)$. Then (x_n, u_n) solves (1.1)-(1.4) with $x_n(0) = x(n)$. Hence e is an optimal equilibrium by Corollary 2.6.

Next we introduce the central notions of this paper.

DEFINITION 2.8. For $(x, u) \in \mathbb{R}^n_+ \times U_{\mathrm{ad}}$ define the omega limit set $\omega(x, u)$ by

$$\omega(x, u) := \{y \in \mathbb{R}^n \colon \text{there exist } t_k \in \mathbb{R}_+ \text{ such that } t_k \to \infty \text{ and } \varphi(t_k, x, u) \to y\} = \bigcap_{n \in \mathbb{N}} \mathrm{cl}\, \{\varphi(t, x, u) \colon t \geqq n\}. \tag{2.2}$$

For $I = \mathbb{R}_+$ or $I = \mathbb{R}$, we call $(x, u) \in \mathbb{R}^n_+ \times U_{\mathrm{ad}}(I)$ an optimal I-solution if the corresponding solution $\varphi(\cdot, x, u)$ of (1.2) exists on I and for all $t \in I$

$$V(\varphi(t, x, u), u(t + \cdot)) = V(\varphi(t, x, u)).$$

Frequently, we call optimal $\mathbb{R}_+$-solutions simply optimal. If (x, u) is an optimal $\mathbb{R}$-solution, define the alpha limit set $\alpha(x, u)$ by

$$\alpha(x, u) := \bigcap_{n \in \mathbb{N}} \mathrm{cl}\, \{\varphi(t, x, u) \colon t \leqq -n\}. \tag{2.3}$$

Finally define for optimal (x, u)

(2.4) $\hat{\omega}(x, u) := \{(y, v) \colon (y, v)$ is an optimal $\mathbb{R}$-solution and there are $t_k \in \mathbb{R}_+$ such that $t_k \to \infty$ and $\varphi(t_k + \cdot, x, u) \to \varphi(\cdot, y, v)$ locally uniformly on $\mathbb{R}$ and $u(t_k + \cdot) \to v$ in $U_{\mathrm{ad}}\}$.

DEFINITION 2.9. A subset L of $\mathbb{R}^n_+$ is called (*positively*) *invariant* if for all $y \in L$ there is an optimal ($\mathbb{R}_+$-) $\mathbb{R}$-solution (y, v) with $\varphi(\mathbb{R}, y, v) \subset L$ (respectively, $\varphi(\mathbb{R}_+, y, v) \subset L$).

PROPOSITION 2.10. *Let (x, u) be optimal. Then $\omega(x, u)$ is nonvoid, compact, and connected. For every $y \in \omega(x, u)$ there is $v \in U_{\mathrm{ad}}$ such that $(y, v) \in \hat{\omega}(x, u)$ and $\varphi(\mathbb{R}, y, v) \subset \omega(x, u)$. In particular, $\omega(x, u)$ is invariant.*

Proof. Using (2.1), we see that $\omega(x, u)$ is nonvoid, connected, and compact. Let $y \in \omega(x, u)$. Then there are $t_k \to \infty$ with $\varphi(t_k, x, u) \to y$. By Corollary 2.6, we can, without loss of generality, assume that $s \to u(t_k + s)$, $s \in \mathbb{R}_+$, converges in U_{ad} to some $u_0 \in U_{\mathrm{ad}}$ and $s \to \varphi(t_k + s, x, u)$, $s \in \mathbb{R}_+$, converges uniformly on bounded intervals to $\varphi(\cdot, y, u_0)$. Taking again, if necessary, subsequences, $s \to u(t_k - 1 + s)$, $s \in \mathbb{R}_+$, converges weakly on bounded intervals to $u_{-1}: [-1, \infty) \to \Omega$ and $s \to \varphi(t_k - 1 + s, x, u)$ converges uniformly on bounded intervals to $\varphi(\cdot, y, u_{-1}): [-1, \infty) \to \mathbb{R}^n$ with $u_0 = u_{-1}$ and $\varphi(\cdot, y, u_0) = \varphi(\cdot, y, u_{-1})$ on $[0, \infty)$. By successively taking subsequences of (t_k) we obtain sequences $u_{-n}: [-n, \infty) \to \Omega$, $\varphi(\cdot, y, u_{-n}): [-n, \infty) \to \mathbb{R}^n$ with $u_{-n} = u_{-n+1}$ on $[-n+1, \infty)$ and

$$\frac{d}{dt} \varphi(t, y, u_{-n}) = f(\varphi(t, y, u_{-n}), u_{-n}(t)),$$

$$V(\varphi(-n, y, u_{-n}), u_{-n}(-n + \cdot)) = V(\varphi(-n, y, u_{-n})).$$

Defining

$$v(t) = u_{-n}(t) \quad \text{on } [-n, \infty),$$

we obtain an optimal $\mathbb{R}$-solution (y, v).

The following lemma is our key for the construction of local transversal sections.

LEMMA 2.11. *Let L be a compact positively invariant set and $R(e) = \sup \{R(x) \mid x \in L\}$ for some $e \in L$. Then one of the following conditions is satisfied:*

(i) *e is an optimal equilibrium;*

(ii) *L contains a point x^0 with $0 \notin f(x^0, \Omega)$.*

Proof. If (ii) is violated there is $v^e \in \Omega$ such that $f(e, v^e) = 0$. By invariance of L we find $v \in U_{\mathrm{ad}}$ such that (e, v) is optimal with $\varphi(\mathbb{R}_+, e, v) \subset L$. Hence

$$\begin{aligned} V(e) &= \int_0^\infty e^{-\delta t} R(\varphi(t, e, v))\, dt \leqq \int_0^\infty e^{-\delta t} R(e)\, dt \\ &= \int_0^\infty e^{-\delta t} R(\varphi(t, e, v^e))\, dt = V(e, v^e). \end{aligned}$$

Thus (e, v^e) is an optimal equilibrium, i.e., (i) holds.

Remark 2.12. In § 5, we will consider a two-dimensional problem from resource management ($n = 2$), where the integrand of the performance criterion depends also on u. However, the problem can be transformed into one in which in the *interior* of $\mathbb{R}^2_+$ we obtain a criterion of the form (1.1) ($R(x)$ becomes unbounded for $x \to \partial \mathbb{R}^2_+$). In fact, Lemma 2.11 remains true here, since it holds in the following general situation.

Suppose (1.1) is replaced by

$$\int_0^\infty e^{-\delta t} \left[g_0(x(t)) + \sum_{i=1}^m u_i(t) g_i(x(t)) \right] dt, \tag{2.5}$$

with $g_i: \mathbb{R}^2_+ \to \mathbb{R}$, $i = 0, 1, \cdots, m$, locally Lipschitz continuous, and the following condition holds:

(2.6) There is a continuous function $R: \operatorname{int} \mathbb{R}^2_+ \to \mathbb{R}$ such that

(α) $\int_0^\infty e^{-\delta t} R(\varphi(t, a, u))\, dt$ converges for every $a \in \operatorname{int} \mathbb{R}^2_+$, $u \in U_{\mathrm{ad}}$

to a real number $V_R(a, u)$.

(β) A pair $(a, u) \in \operatorname{int} \mathbb{R}^2_+ \times U_{\mathrm{ad}}$ is optimal if and only if $V_R(a, u) \geqq V_R(a, v)$ for all $v \in U_{\mathrm{ad}}$.

First observe that Corollary 2.6, Proposition 2.10, and Theorem 2.7 remain true for the criterion (2.5). If $L \subset \operatorname{int} \mathbb{R}^2_+$ this follows from Lemma 2.11. Otherwise, $L \cap \partial \mathbb{R}^2_+ \neq \emptyset$; suppose, for example, $L_1 = L \cap \pi_1^{-1}(0) \neq \emptyset$ where $\pi(x_1, x_2) = x_1$. By Remark 2.4, the restriction of the system to $\pi_1^{-1}(0)$ is well defined and L_1 is compact and invariant for this system. Since the restricted system again is of the form (2.5), (1.2)-(1.4), Theorem 2.7 yields the assertion.

We first consider case (ii) of Lemma 2.11, and show that it translates into a geometric condition.

DEFINITION 2.13. Let $x^0 \in \mathbb{R}^n$, $l: \mathbb{R}^n \to \mathbb{R}$ linear and $\alpha > 0$. If

$$lf(y, u) > \alpha$$

for all y in a neighborhood W of x^0 and all $u \in \Omega$, then

$$S := W \cap l^{-1}(x^0)$$

is called a local transversal section through x^0.

PROPOSITION 2.14. *Suppose that $0 \notin f(x^0, \Omega)$. Then x^0 possesses a local transversal section. Hence a compact positively invariant set L either contains an optimal equilibrium or a point possessing a local transversal segment.*

Proof. In view of Lemma 2.11, we only have to show the first assertion. If $0 \notin f(x^0, \Omega)$, then by the Hahn-Banach Theorem this assertion follows, since $f(x^0, \Omega)$ is compact and convex.

Obviously, trajectories "can cross a local transversal section only from one side." The next result presents an important consequence from the existence of a local transversal section.

We need the following definition.

DEFINITION 2.15. Let S be a local transversal section through x^0, and let $V_1 \subset V_0$ be neighborhoods of x^0. Then the triple (V_0, V_1, S) is called a flow box around x^0, if it has the following property:

If $\varphi(\cdot, x^0, u)$ satisfies

$$\varphi(t_0, x^0, u) \notin V_0, \quad \varphi(t_1, x^0, u) \in V_1, \quad \varphi(t_2, x^0, u) \notin V_0$$

for some $0 \leqq t_0 < t_1 < t_2$, then there exists $t \in (t_0, t_2)$ such that $\varphi(t, x^0, u) \in S$ and $\varphi(s, x^0, u) \in V_0$ for all s between t and t_1.

THEOREM 2.16. *Let S be a local transversal section through x^0. Then there are neighborhoods V_0 and V_1 of x^0 such that (V_0, V_1, S) is a flow box around x^0.*

Proof. There exist a linear map $l: \mathbb{R}^n \to \mathbb{R}$, a constant $\alpha > 0$, and a neighborhood W of x^0 such that $S \supset W \cap l^{-1}(x^0)$ and

$$l(f(y, v)) > \alpha \quad \text{for all } y \in W, \quad v \in \Omega.$$

Choose a ball $V_0 = B(r_0, x^0)$ around x^0 with radius $r_0 > 0$ such that $V_0 \subset W$ and put $c := \sup \{|f(y, u)| \mid y \in V_0, v \in \Omega\}$. Then choose $r_1 \in (0, r_0)$ so small that

$$lz - \alpha/2c(r_0 - r_1) \leqq ly \leqq lz + \alpha/2c(r_0 - r_1) \tag{2.7}$$

for all $z, y \in V_1 = B(r_1, x^0)$. We have for $t > t' \geqq 0$:

$$\varphi(t, x, u) = \varphi(t', x, u) + \int_{t'}^{t} f(\varphi(s, x, u), u(s))\, ds$$

and hence

$$l(\varphi(t, x, u)) = l(\varphi(t', x, u)) + \int_{t'}^{t} lf(\varphi(s, x, u), u(s))\, ds$$
$$\geqq l(\varphi(t', x, u)) + \alpha(t - t')$$

provided that $\varphi(s, x, u) \in W$, $t' \leqq s \leqq t$. Without loss of generality, we may assume

$$\varphi(s, x, u) \in V_0 \quad \text{for all } t_0 \leqq s \leqq t_2$$

replacing, if necessary, t_0 by the last time before t_1 at which $\varphi(t, x, u)$ is in the complement of V_0 and t_2 by the first time after t_1 at which $\varphi(t, x, u)$ leaves V_0. We have

$$r_0 - r_1 \leqq |\varphi(t_1, x, u) - \varphi(t_0, x, u)| \leqq c(t_1 - t_0),$$
$$r_0 - r_1 \leqq |\varphi(t_2, x, u) - \varphi(t_1, x, u)| \leqq c(t_2 - t_1).$$

If $l\varphi(t_0, x, u) \leqq lx^0 \leqq l\varphi(t_2, x, u)$, or $l\varphi(t_2, x, u) \leqq lx^0 \leqq l\varphi(t_0, x, u)$, the assertion follows by continuity of $t \to lf(t, x, u)$. Hence we only have to consider the following two cases.

Case 1. $lx^0 < \min\{l\varphi(t_0, x, u), l\varphi(t_2, x, u)\}$. Here $l\varphi(t_1, x, u) \geqq l\varphi(t_0, x, u) + \alpha(t_1 - t_0) > lx^0 + \alpha/c(r_0 - r_1)$, contradicting (2.7) for $y = \varphi(t_1, x, u)$.

Case 2. $lx^0 > \max\{l\varphi(t_0, x, u), l\varphi(t_2, x, u)\}$. Here $l\varphi(t_2, x, u) \geqq l\varphi(t_1, x, u) + a(t_2 - t_1) > l\varphi(t_1, x, u) + \alpha/c(r_0 - r_1)$, again contradicting (2.7).

3. Optimal equilibria. In this section we first characterize optimal equilibria by necessary optimality conditions. It turns out that "in general" only finitely many optimal equilibria exist. Strong additional assumptions ensure that optimal equilibria in a limit set are already reached in finite time. Furthermore, limit sets $\omega(x, u)$ reduce to a single optimal equilibrium provided that $\omega(x, u)$ consists of equilibria only and contains at most finitely many optimal equilibria.

First we discuss the following problem:

$$\text{Maximize (2.5)} \quad \text{subject to (1.2)-(1.4)}$$

where Ω is a rectangle in $\mathbb{R}^2$ (in fact, the "ecological form" of (1.2) is not needed in this section, if not stated otherwise).

Abbreviate

$$g(x, u) = g_0(x) + \sum_{i=1}^{m} u_i g_i(x), \qquad f(x, u) = f_0(x) + \sum_{i=1}^{m} u_i g_i(x).$$

For any equilibrium $e = (x_1, x_2)$, there are the two equations (for x_1, x_2, u_1, u_2) defining an equilibrium, namely

$$o = f(x, u). \tag{3.1}$$

To derive a second set of equations we shall use Pontryagin's maximum principle (cf. Halkin [11]). Write

$$H = \lambda_0 e^{-\delta t} g + \lambda \cdot f,$$
$$\dot{\lambda}(t) = -\lambda_0 e^{-\delta t} g_x - f_x^t \lambda \quad \text{(adjoint equation)}.$$

Here

$$f = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} \quad \text{and} \quad f_x^t = \begin{pmatrix} f_{x_1}^1 & f_{x_1}^2 \\ f_{x_2}^1 & f_{x_2}^2 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Thus,

$$H = \lambda_0 e^{-\delta t} g_0 + \lambda_1 f_0^1 + \lambda_2 f_0^2 + u_1(\lambda_0 e^{-\delta t} g_1 + \lambda_1 f_1^1 + \lambda_2 f_1^2) + u_2(\lambda_0 e^{-\delta t} g_2 + \lambda_1 f_2^1 + \lambda_2 f_2^2).$$

Put $\mu = e^{\delta t}\lambda$. Then $\dot{\mu} = \delta\mu + e^{\delta t}\dot{\lambda}$. Hence the adjoint equation reads

$$\dot{\mu} = -\lambda_0 g_x + (\delta I - f_x^t)\mu,$$

and

$$H = h(t, x, \lambda) + u_1(\lambda_0 g_1 + \mu_1 f_1^1 + \mu_2 f_1^2) + u_2(\lambda_0 g_2 + \mu_1 f_2^1 + \mu_2 f_2^2).$$

Pontryagin's maximum principle implies that $(\lambda_0, \lambda(t) \neq 0$ for all $t \geqq 0$ and H attains its maximum over Ω in (u_1, u_2). We may assume that $\lambda_0 = 0$ or $\lambda_0 = 1$.

Now we discuss the possible numbers of optimal equilibria. There are three cases:

Case 1. $u = (u_1, u_2)$ is one of the corners of Ω.

Case 2. u lies in the relative interior of one of the edges of Ω.

Case 3. $u \in \operatorname{int} \Omega$.

Case 1. Recall that there are only four corners of Ω.

Case 2. One equation for u is given by the condition that u lies on one of the edges of Ω. Furthermore the derivative of H in direction, say $v = (v_1, v_2)$ (parallel to the edge of Ω containing u), vanishes, i.e.,

$$(\lambda_0 g_1 + \mu_1 f_1^1 + u_2 f_1^2)v_1 + (\lambda_0 g_2 + \mu_1 f_2^1 + \mu_2 f_2^2)v_2 = 0 \quad \text{for all } t.$$

Thus with $\varphi := (v_1 f_1^1 + v_2 f_2^1,\ v_1 f_1^2 + v_2 f_2^2)$

$$\varphi\mu = -\lambda_0(g_1 v_1 + g_2 v_2) \quad \text{for all } t. \tag{3.2}$$

Insertion into the adjoint equation yields

$$0 = \varphi\dot{\mu} = \varphi(-\lambda_0 g_x + (\delta I - f_x^t)\mu) \quad \text{or} \quad \varphi(\delta I - f_x^t)\mu = \lambda_0 \gamma_x \quad \text{for all } t. \tag{3.3}$$

If φ and $\varphi(\delta I - f_x^t)$ are linearly dependent we obtain with (3.1), the assumption that u lies on an edge of Ω and

$$\det(\varphi^t, (\delta I - f_x^t)\varphi^t) = 0, \tag{3.4}$$

four equations for the unknowns x_1, x_2, u_1, u_2. If φ and $\varphi(\delta I + f_x^t)$ are linearly independent, (3.2) and (3.3) imply that μ is constant and that $\lambda_0 = 1$. We assume $\det(\delta I - f_x^t) \neq 0$. Then by the adjoint equation and (3.2)

$$\varphi(\delta I - f_x^t)^{-1} g_x = -g_1 v_1 + -g_2 v_2, \tag{3.5}$$

and again we obtain four equations for x_1, x_2, u_1, u_2.

Case 3. Put

$$F = \begin{pmatrix} f_1^1 & f_1^2 \\ f_2^1 & f_2^2 \end{pmatrix}.$$

Then

$$-\lambda_0 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = F\mu.$$

Suppose $\det F \neq 0$. Then

$$\mu = -\lambda_0 F^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and it follows that $\lambda_0 = 1$, and $\mu = 0$. The adjoint equation yields

$$0 = g_x + (\delta I - f_x^t)\mu \quad \text{and} \quad g_x = (\delta I - f_x^t)F^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{3.6}$$

Hence together with (3.1) we obtain four equations for (x_1, x_2, u_1, u_2).

Now suppose $\det F = 0$. By introducing new control variables we can eliminate one control variable in the system equation and proceed with the discussion as in Case 1 or Case 2 above.

We formulate the conclusion of this discussion in the following remark.

Remark 3.1. Consider problem (2.5), (1.2)-(1.4) with $n=m=2$ and $\Omega=[0, U_1]\times[0, U_2]$. Then every optimal pair (x, u) such that $u\in\Omega$, $0=f(x, u)$ must satisfy four equations in the unknowns x_1, x_2, u_1, u_2. In concrete examples, these equations may serve to compute all candidates for optimal equilibria (recall that the maximum principle is only a set of necessary conditions). On the other hand, these equations justify the statement that "in general" there exist at most finitely many optimal equilibria. We shall use this as a hypothesis further below in this section and in § 4.

We proceed to analyze finite-time reachability properties of optimal equilibria in limit sets.

DEFINITION 3.2. An equilibrium e is called strongly optimal if the constant function $x(t)=e$ is the unique optimal trajectory for start in e.

LEMMA 3.3. *Let (x, u) be optimal and suppose that e is a strongly optimal equilibrium in $\omega(x, u)$. Then for every $T>0$ and every neighborhood V of e there exists a neighborhood U of e such that $\varphi(t, x, u)\in U$ implies $\varphi([t, t+T], x, u)\subset V$.*

Proof. Assume, contrary to the assertion, that there exist a neighborhood V of e, $T>0$, and $t_n\to\infty$ with $\varphi(t_n, x, n)\to e$ and $\varphi(t_n+s_n, x, u)\notin V$ for some $s_n\in[0, T]$. Without loss of generality, $s_n\to s\in[0, T]$. We may assume that $\varphi(\cdot, \varphi(t_n, x, u), u(t_n+\cdot))$ converges uniformly on bounded intervals to an optimal $\varphi(\cdot, e, v)$. Since $\varphi(s, e, v)\notin V$ and $\varphi(0, e, v)=e$, this contradicts strong optimality of e.

THEOREM 3.4. *Let $(x, u)\in\mathbb{R}^n_+\times U_{\mathrm{ad}}$ be optimal for* (1.1)-(1.4). *Let $e\in\omega(x, u)$ with the following*:

(i) *e is a strongly optimal equilibrium in* $\operatorname{int}\mathbb{R}^n_+$;

(ii) *There is $u^e\in\operatorname{int}\Omega$ with $f(e, u^e)=0$;*

(iii) *$m\geqq n$ and $f_1(e), \cdots, f_n(e)$ are linearly independent*;

(iv) *R is a C^2-function in a neighborhood of e and $R'(e)=0$, $R''(e)$ is negative definite.*

Then for all $t>0$ sufficiently large $\varphi(t, x, u)=e$.

Proof. (a) Suppose $\psi: V\to\mathbb{R}^n$ is a coordinate change defined on an open neighborhood V of e. If $x(t)$ is in V and satisfies $x(t)=f_0(x(t))+\sum_{i=1}^n u_i(t)f_i(x(t))$, then $y(t)=\psi(x(t))$ satisfies $\dot y(t)=\dot\psi(x(t))(f_0\circ\psi^{-1})y(t)+\sum_{i=1}^n u_i(t)\dot\psi(x(t))(f_i\circ\psi^{-1})(y(t))$, which again is a system of equations of the type we are considering. Obviously our assumptions (i)-(iv) carry over. By (iv) and according to the Morse Lemma there is a coordinate change ψ such that $R\circ\psi^{-1}(x)=-\sum_{i=1}^n x_i^2$ for all x in a neighborhood W of $\psi(e)$.

Hence we may without loss of generality assume $R(e+x)=R(e)+\sum_{i=1}^n (x_i-e_i)^2$ in a ball $V(e, r)$ of center e and radius r.

(b) By (ii), (iii) and the implicit function theorem r may be chosen so small that a smooth function $u: V(0, r)\times V(e, r)\to\Omega$ exists such that for all $(y, x)\in V(0, r)\times V(e, r)$ we have $y=f(x, u(x, y))$. In fact, if $F(x)$ is the matrix with columns $f_1(x), \cdots, f_n(x)$ then

$$u(x, y)=F^{-1}(x)(y-f_0(x)).$$

In particular, if $x(t)=a+t(e-a)$, $\dot x(t)=e-a=f(x(t), u(x(t), e-a))$ provided $|e-a|<r$. Hence $x(t)=\varphi(t, a, u)$ is an admissible solution of our system at least up to $t=1$ $(u(t)=u(x(t), e-a))$.

(c) We now assume $e\notin\varphi(x, u)$ and try to reach a contradiction to (i). See Fig. 3.1. Choose $T>3$. According to Lemma 3.3 there is by (i) $t>0$ such that $\varphi((t, t+T), x, u)\subset V(e, r)$. By our assumption in (a) there is a first time $s_1\in(0, 1)$ such that for all $s\in(s_1, 1]$

$$R(x_0(s))>R(\varphi(t+s, x, u))$$

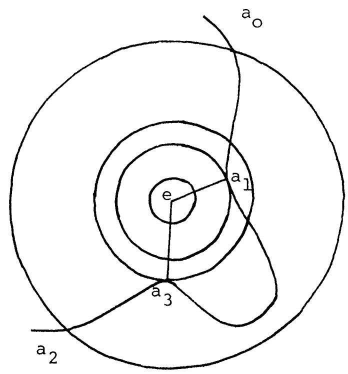

FIG. 3.1. $\hat{\varphi}$ *visits* a_0, a_1, e, a_3, a_2.

where

$$x_0(s)=a_0+s(e-a_0), \quad a_0=\varphi(t, x, u).$$

Put

$$a_1=\varphi(t+s_1, x, u),$$

$$x_1(s)=a_1+s\frac{|e-a_0|}{|e-a_1|}(e-a_1), \qquad 0\leqq s\leqq\frac{|e-a_1|}{|e-a_0|}=s_2;$$

then

$$R(x_1(t))>R(\varphi(t+s_1+s, x, u)), \qquad 0\leqq s\leqq s_2.$$

Put

$$a_2=\varphi(t+T, x, u),$$

$$x_2(s)=e+s(a_2-e), \qquad 0\leqq s\leqq 1=s_3.$$

Note that $s_1+s_2+s_3<3<T$.

There is a last time $s_4\in(0, s_3)$ such that

$$R(x_2(s))>R(\varphi(t+T-s_3+s, x, u)), \qquad s\in(0, s_4).$$

Put

$$a_3=\varphi(t+T-s_3+s_4, x, u),$$

$$x_3(s)=e+s\frac{|a_2-e|}{|a_3-e|}(a_3-e), \qquad 0\leqq s\leqq\frac{|a_3-e|}{|a_2-e|}=s_4.$$

Then

$$R(x_3(s))>R(\varphi(t+T-s_3+s, x, u)), \qquad s\in(0, s_4).$$

Now we combine the x_j's to build a solution that performs better than $\varphi(\cdot, x, u)$. Put

$$\hat{\varphi}(s)=\begin{cases}\varphi(s, x, u), & 0\leqq s\leqq t+s_1,\\ x_1(s-t-s_2), & t+s_1\leqq s\leqq t+s_1+s_2,\\ e, & t+s_1+s_2\leqq s\leqq t+T-s_3,\\ x_3(s-t-T+s_3), & t+T-s_3\leqq s\leqq t+T-s_3+s_4,\\ \varphi(s, x, u), & t+T-s_3+s_4\leqq s.\end{cases}$$

The definition makes sense since $T>s_1+s_2+s_3$. Using (b) we may find $v\in U_{\text{ad}}$ such that $\hat{\varphi}(s)=\varphi(s, x, v)$. But

$$\sigma_1=t+s_1<t+T-s_3+s_4=\sigma_2$$

and

$$R(\varphi(s, x, v)) > R(\varphi(s, x, u)) \quad \text{for } s \in (\sigma_1, \sigma_2),$$

$$R(\varphi(s, x, v)) = R(\varphi(s, x, u)) \quad \text{for } s \notin (\sigma_1, \sigma_2).$$

Therefore $V(x, v) > V(x, u)$ in contradiction to the optimality of (x, u).

Next we give a sufficient condition for optimal solutions to converge to a single optimal equilibrium. This result will be used substantially in the next section.

THEOREM 3.5. *Let* $(x, u) \in \mathbb{R}_+^n \times U_{ad}$ *be optimal for* (1.1)–(1.4). *Suppose the following*:

(i) $\omega(x, u)$ *contains at most finitely many optimal equilibria.*

(ii) $\omega(x, u) \subset E = \{y \in \mathbb{R}_+^n \mid$ *there exists* $v \in \Omega$ *with* $o = f(y, v)\}$.

Then $\omega(x, u)$ *consists of a single optimal equilibrium.*

Proof. Suppose that

$$\# \omega(x, u) \geqq 2.$$

There is $e \in \omega(x, u)$ and a sequence of points $e_n \in \omega(x, u)$ such that $\lim_n e_n = e$ and

$$a = \sup \{R(y) \mid y \in \omega(x, u)\} = \lim_n R(e_n).$$

There is $\tilde{e} \in \omega(x, u)$ such that

$$b = R(\tilde{e}) < a.$$

Otherwise all points in $\omega(x, u)$ would be optimal equilibria and by connectedness $\omega(x, u)$ would contain infinitely many optimal equilibria contrary to (i). We choose numbers b_0, b_1, b_2 with

$$a > b_0 > b_1 > b_2 > b$$

such that there is at most one optimal equilibrium e_1 in $\omega(x, u)$ with $R(e_1) \geqq b_2$.

Choose $(t_k) \subset \mathbb{R}_+$ with $t_k \to \infty$ and $\varphi(t_k, x, u) \to e$. For every $k \in \mathbb{N}$ there are $s_i^k \geqq 0$, $i = 0, 1, 2$, with

$$s_2^k := \inf \{s \geqq 0 \colon R(\varphi(t_k + s, x, u)) < b_2\},$$

$$s_1^k := \sup \{s \leqq s_2^k \colon R(\varphi(t_k + s, x, u)) > b_1\},$$

$$s_0^k := \sup \{s \leqq s_1^k \colon R(\varphi(t_k + s, x, u)) > b_0\}.$$

We may assume that the functions

$$\varphi(t_k + s_0^k + \cdot, x, u)$$

converge uniformly on every bounded interval to an optimal trajectory $\varphi(\cdot, y, v) \subset \omega(x, u)$. Then $y \in E$ and $R(y) = b_0$. Next we show the following: The sequence $t_2^k = s_2^k - s_0^k$, $k \in \mathbb{N}$, is bounded. Otherwise we might assume that $t_2^k < t_2^{k+1}$ and $t_2^k \to \infty$. The functions

$$R(\varphi(t_k + s_0^k + t, x, u)), \qquad t \in [0, t_2^k]$$

have values in

$$V(b_0, b_2) := \{z \colon b_0 \geqq R(z) \geqq b_2\}.$$

Hence also $\varphi(\cdot, y, v)$, $\omega(y, v) \subset V(b_0, b_2)$.

Since $\omega(y, u) \subset E$, Lemma 2.11 implies that $\omega(y, v)$ contains an optimal equilibrium. This contradicts the choice of b_2.

Thus (t_2^k) is bounded and, considering subsequences, we may even assume that $t_2^k \to t_2$ and $t_1^k = s_1^k - s_0^k \to t_1$ with $0 \leqq t_1,\ t_2 < \infty$. Hence,

$$\begin{aligned} &R(y) = R(\varphi(0, y, v)) = b_0, \\ &R(\varphi(t, y, v)) \leqq b_0 \quad \text{for } t \in [0, t_1], \\ &R(\varphi(t, y, v)) \leqq b_1 \quad \text{for } t \in [t_1, t_2], \\ &R(\varphi(t, y, v)) \leqq a \quad \text{for } t \in [t_2, \infty). \end{aligned}$$

We obtain

$$\int_0^\infty e^{-\delta t} R(\varphi(t, y, v))\, dt = \int_0^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^\infty \leqq \frac{b_0}{\delta}(1 - e^{t_1\delta}) + \frac{b_1}{\delta}(e^{-t_1\delta} - e^{-t_2\delta}) + \int_{t_2}^\infty .$$

Since $y \in E$ and (y, v) optimal

$$\frac{1}{\delta} R(y) = \frac{b_0}{\delta} \leqq \int_0^\infty e^{-\delta t} R(\varphi(t, y, v))\, dt;$$

hence

$$b_0 \leqq b_0 - b_0\, e^{-t_1\delta} + b_1\, e^{-t_1\delta} - b_1\, e^{-t_2\delta} + \delta \int_{t_2}^\infty ,$$

or

$$b_0 \leqq b_1(1 - e^{(t_1 - t_2)\delta}) + \delta\, e^{t_1\delta} \int_{t_2}^\infty R(\varphi(t, y, v))\, dt.$$

Note that the right-hand side is constructed independently of b_0. Hence if $a = +\infty$ we may let b_0 tend to $a = +\infty$, thus obtaining a contradiction since $\int_{t_2}^\infty$ converges to a finite value. Hence we may assume that a is finite and thus

$$b_0 \leqq b_1(1 - e^{(t_1 - t_2)\delta} + a\, e^{(t_1 - t_2)\delta}.$$

Letting b_0 tend to a, we find

$$a(1 - e^{(t_1 - t_2)\delta}) \leqq b_1(1 - e^{(t_1 - t_2)\delta}),$$

leading to the contradiction $a \leqq b_1$ since $\varphi(t_2, y, v) = b_2$, $\varphi(t_1, y, v) = b_1$, and hence $t_1 < t_2$.

Using similar arguments as in the proof above, we can show the analogous result for α-limit sets.

THEOREM 3.6. *Let the assumptions of Theorem* 3.5 *be satisfied for* $\alpha(x, u)$ *instead of* $\omega(x, u)$. *Then* $\alpha(x, u)$ *consists of a single optimal equilibrium.*

Remark 3.7. Let $n = 2$, replace (1.1) by (2.5), and suppose that (2.6) holds. Let $(x, u) \in \mathbb{R}_+^2 \times U_{\text{ad}}$ be optimal. Then when we assume (i), (ii), a slight change in the proof of Theorem 3.5 shows that either $\omega(x, u)$ consists of a single optimal equilibrium or is contained in the boundary of $\mathbb{R}_+^2$. Furthermore the following holds: Let $(y, u) \in \hat{\omega}(x, u)$ with $\omega(y, v) \subset \partial\mathbb{R}_+^2$ such that $\omega(y, v)$ contains only a finite number of optimal equilibria. Then $\omega(y, v)$ consists of a single optimal equilibrium.

Proof. Suppose the assertion is false. Then $\#\omega(y, v) \geqq 2$, and since $\omega(y, v)$ is connected, $\omega(y, v)$ contains infinitely many points. By assumption there is $z = (z_1, z_2) \in \omega(y, v)$, which is not an optimal equilibrium, say with $z_2 = 0$. There is $(z, w) \in \hat{\omega}(x, u)$. Since the first component $\varphi(\cdot, z, w)$ is not constant, it is by Theorem 2.7, say, increasing (if it is decreasing, analogous arguments will apply). By Remark 2.4, the second component $\varphi_2(t, z, w)$ vanishes for all t. Let $t>0$ such that the segment between z and $z'=\varphi(t, z, w)$ does not contain an optimal equilibrium. Since $z, z' \in \omega(x, u)$ there are $t_n, s_n \geqq 0$ such that

$$\lim_n t_n = +\infty, \quad \lim_n \varphi(t_n, x, u) = z', \quad \lim_n \varphi(t_n + s_n, x, u) = z$$

and for all $n \in \mathbb{N}$, $t \in [0, s_n]$,

$$z_1 \leqq \varphi_1(t_n + t, x, u) \leqq z_1', \qquad 0 \leqq \varphi_2(t_n + t, x, u) \leqq 1/n.$$

We may assume that the sequence of functions $t \to \varphi(t_n + t, x, u)$ converges locally uniformly to a function $\varphi(\cdot, z', w')$ such that $(z', w') \in \hat{\omega}(x, u)$. If $\lim s_n = +\infty$, then $\omega(z', w') \subset [z_1, z_1'] \times \{0\}$. By Theorem 2.7, $\omega(z', w')$ contains an optimal equilibrium, in contradiction to our assumption.

Therefore a subsequence of (s_n) converges to some $s \in [0, \infty)$. Obviously, $\varphi(s, z', w') = z$. Define

$$w''(\sigma) = w(\sigma) \qquad \text{for } 0 \leqq \sigma \leqq t,$$

$$w''(\sigma) = w'(\sigma - t) \qquad \text{for } t < \sigma \leqq s + t,$$

and extend w'' periodic on $\mathbb{R}_+$ with period $s+t$. Then (z, w'') is optimal and periodic. Now consider (z, w'') as an optimal pair with respect to the restriction of our system to $\mathbb{R}_+ \times \{0\} \simeq \mathbb{R}_+^1$ (Remark 2.4). Since $\varphi(\cdot, z, w'')$ is neither increasing nor decreasing, Theorem 2.7 implies the existence of infinitely many optimal equilibria contrary to the assumption.

4. Poincaré–Bendixson Theorem. The analysis of this section is restricted to two-dimensional systems (i.e., $n=2$). Our final result, Theorem 4.6, is a generalization of the classical Poincaré–Bendixson Theorem. If we drop the assumption of that theorem that $\varphi(\cdot, x, u)$ is nonself-intersecting, we obtain an optimal periodic solution in a trivial way according to the following proposition.

PROPOSITION 4.1. *Suppose that* $(x, u) \in \mathbb{R}_+^n \times U_{\text{ad}}$ *is optimal and* $\varphi(\cdot, x, u)$ *intersects itself, i.e., there are* $T_2 > T_1 \geqq 0$ *with* $\varphi(T_1, x, u) = \varphi(T_2, x, u)$. *Then there is* $\hat{u} \in U_{\text{ad}}$ *such that* $(x, \hat{u})$ *is optimal and* $\varphi(T_1 + s, x, u) = \varphi(T_1 + k(T_2 - T_1) + s, x, \hat{u})$ *for* $s \in [0, T_2 - T_1]$, $k \in \mathbb{N}$.

Proof. Define $\hat{u}(t) = u(t)$ for $t \in [0, T_1]$,

$$\hat{u}(T_1 + k(T_2 - T_1) + t) = u(T_1 + t) \quad \text{for } t \in [0, T_2 - T_1], \quad k \in \mathbb{N}.$$

Then the assertion follows since final segments of optimal solutions are optimal.

We call a solution $(x, \hat{u})$ with the property above finally periodic.

For the reader's convenience, we cite the following classical theorem (see, e.g., Beck [3, Cor. C.23]), which will be used frequently.

JORDAN'S CURVE THEOREM. *Let J be a Jordan curve in* $\mathbb{R}^2$ (*i.e., a homeomorphism from the circle into* $\mathbb{R}^2$). *Then* $\mathbb{R}^2 \backslash \operatorname{Im} J$ *has two components, one of which is bounded* (*called* ins J) *and the other one* (*called* outs J) *is unbounded. Each one has boundary* Im J *and is simply connected.*

Since the orientation does not concern us, we identify J with its image.

LEMMA 4.2. *Let* $(x, u) \in \mathbb{R}_+^2 \times U_{\text{ad}}$ *and suppose that the corresponding trajectory* $\varphi(\cdot, x, u)$ *is nonself-intersecting. Then a local transversal section S has at most one point in common with* $\omega(x, u)$. *For optimal* $\mathbb{R}$*-solutions it follows also that S has at most one point in common with* $\alpha(x, u)$.

Proof. As in the theory of uncontrolled differential equations (see, e.g., [1, Lemma 24.1] or [14]) we prove the following: Let (x_i) be a sequence of points in $\varphi(x, u) \cap S$. If (x_i) is increasing on $\varphi(\cdot, x, u)$, then it is also on S. Now suppose that $y_1, y_2 \in \omega(x, u) \cap S$ and $y_1 \neq y_2$. Let U_j be disjoint neighborhoods of y_j, $j = 1, 2$. Then there exists a sequence $t_k \to \infty$ such that

$$\varphi(t_{2k+1}, x, u) \in U_1 \quad \text{and} \quad \varphi(t_{2k}, x, u) \in U_2, \quad k \in \mathbb{N}.$$

By Theorem 2.16, we may choose $U^j = V_1^j$, where (V_0^j, V_1^j, S) are flow boxes around y_j, $j = 1, 2$, and $V_0^1 \cap V_0^2 = \varnothing$. Then there exists a sequence (s_k), $s_k \to \infty$, with

$$\varphi(s_{2k+1}, x, u) \in U_1 \cap S, \qquad \varphi(s_{2k+1}, x, u) \in U_2 \cap S.$$

This contradicts the assertion above.

The same arguments apply to a α-limit sets of optimal $\mathbb{R}$-solutions.

PROPOSITION 4.2. *Let $(x, u) \in \mathbb{R}_+^2 \times U_{\text{ad}}$ be optimal and suppose that $\varphi(\cdot, x, u)$ is nonself-intersecting. Let (y, v) be an optimal $\mathbb{R}$-solution with $\varphi(\mathbb{R}, y, v) \subset \omega(x, u)$. Then $\omega(y, v)$ and $\alpha(y, v)$ consist of equilibria only or $\varphi(\cdot, y, v)$ intersects itself in a point z possessing a local transversal section S.*

Proof. Suppose that $\omega(y, v)$ contains a point z which is not an equilibrium. Then z possesses a local transversal section S by Lemma 2.11. Using a flow box around z we find that $\varphi(y, v) \cap S \neq \varnothing$. Since $\varphi(y, v)$, $\omega(y, v) \subset \omega(x, u)$ and $z \in \omega(y, v) \cap S \subset \omega(x, u) \cap S$ this implies by Lemma 4.2 that $S \cap \omega(x, u) = \{z\}$, and hence $\{z\} = \varphi(y, v) \cap \omega(y, v)$. Thus there is $T_1 \geqq 0$ such that $\varphi(T_1, y, v) = z$. Since z is not an equilibrium, there is a neighborhood V_0 of z and $s > T_1$ with $\varphi(s, y, v) \notin V_0$. Using a flow box (V_0, V_1, S) around z, we find a $T_2 > s$ with $\varphi(T_2, y, v) \in S$. Hence $\varphi(T_2, y, v) = \varphi(T_1, y, v) = z$. Thus $\varphi(\cdot, y, v)$ intersects itself in z.

We prepare the proof of the next proposition by the following lemma.

LEMMA 4.4. *Let (C_n) be a decreasing sequence of closed sets in $\mathbb{R}^q$. Define $C := \bigcap_n C_n$, let $n_k \to \infty$, and*

$$D := \{y \in \mathbb{R}^q \text{: there exist } x_{n_k} \text{ with } x_{n_k} \in \partial C_{n_k} \text{ and } x_{n_k} \to y\}.$$

Then $\partial C = D$.

Proof. Suppose $y \in D$, i.e., there are (x_{n_k}) with $x_{n_k} \in \partial C_{n_k}$ and $x_{n_k} \to y$. Let $B(y, \varepsilon)$ be the ball with center y and radius $\varepsilon > 0$. Then for k large enough $x_{n_k} \in B(y, \varepsilon)$. Since $x_{n_k} \in \partial C_{n_k}$ there is $y_{n_k} \in B(y, \varepsilon) \setminus C_{n_k} \subset B(y, \varepsilon) \setminus C$. Since $\varepsilon > 0$ is arbitrary, $y \notin \operatorname{int} C$. Since $C_{n_k} \subset C_{n_l}$ for $k > l$ it follows that $x_{n_k} \in C_{n_l}$ for $k > l$ and, since C_{n_l} is closed, $y \in C_{n_l}$ for all l. Hence $y \in C \setminus \operatorname{int} C = \partial C$.

Conversely suppose that $y \in \partial C$ and note $C = \bigcap_k C_{n_k}$. Then for every $\varepsilon > 0$ there exists $z \in B(y, \varepsilon) \setminus C$. Hence there is n_k such that $y \in B(y, \varepsilon) \setminus C_{n_k}$. Suppose that $B(y, \varepsilon) \cap \partial C_{n_k} = \varnothing$. Then

$$B(y, \varepsilon) = (B(y, \varepsilon) \setminus C_{n_k}) \cup (B(y, \varepsilon) \cap \operatorname{int} C_{n_k}).$$

Since $B(y, \varepsilon)$ is connected and $z \in B(y, \varepsilon) \setminus C_{n_k} \neq \varnothing$ we conclude that $\varnothing = B(y, \varepsilon) \cap \operatorname{int} C_{n_k} = B(y, \varepsilon) \cap C_{n_k} \ni y$. This contradiction shows that there exist $y_{n_k} \in B(y, \varepsilon) \cap \partial C_{n_k}$. Evidently, $\lim y_{n_k} = y$, and hence $y \in D$.

PROPOSITION 4.5. *Let $(x, u) \in \mathbb{R}_+^2 \times U_{\text{ad}}$ be optimal and assume that $\varphi(\cdot, x, u)$ is nonself-intersecting. Suppose that there are $(y, v) \in \hat{\omega}(x, u)$ and $T_2 > T_1$ with*

$$\varphi(T_1, y, v) = \varphi(T_2, y, v) =: z,$$

with z possessing a local transversal section S. Then

$$\omega(x, u) = \varphi([T_1, T_2], y, v).$$

Proof. Since $z \in \omega(x, u)$ we can use a flow box around z to construct inductively a sequence of numbers t_n such that t_{n+1} is the first instant t after t_n with $\varphi(t, x, u) \in S$. Then for all n we have $t_n < t_{n+1}$, $t_n \to \infty$, and $\varphi(t_n, x, n) \to z$. Now define the Jordan arc Γ_n to consist of $\varphi([t_n, t_{n+1}], x, u)$ and the segment on S between $\varphi(t_n, x, n)$ and $\varphi(t_{n+1}, x, u)$. There are two cases.

Case 1. For all n $\operatorname{ins}\Gamma_n \supset \operatorname{ins}\Gamma_{n+1}$.

Case 2. For all n $\operatorname{outs}\Gamma_n \supset \operatorname{outs}\Gamma_{n+1}$.

Let us first consider Case 1. Put $C := \bigcap_n \operatorname{cl}\operatorname{ins}\Gamma_n$. By Lemma 4.4, $\partial C = \omega(x, u)$. Now let $l \in \mathbb{N}$ be arbitrary and consider a flow box (V_0, V_1, S) around z such that V_0 is a ball around z with radius $1/l$. The set V_1 contains a ball around z of positive radius r. Since $(y, v) \in \hat{\omega}(x, u)$, there is $t > 0$ such that

$$|\varphi(t+T_1+s, x, u) - \varphi(T_1+s, y, v)| < r, \qquad 0 \leqq s \leqq T_2 - T_1.$$

By the flow box property we may follow $\varphi(t+T_1+s, x, u)$ starting with $s=0$ and without leaving V_0 until we reach $t+T_1+s = t_{l_n}$. Applying the same argument to the instant $t+T_2-T_1$ we find that the part of $\varphi([t_{n_l}, t_{n_{l+1}}], x, u)$ not contained in $\varphi([t+T_1, t+T_2], x, u)$ is contained in V_0. Hence each $a \in \varphi([t_{n_l}, t_{n_{l+1}}], x, u)$ has a distance less than $1/l$ to some $a' \in \varphi(T_1, T_2], y, v)$. Thus a second application of Lemma 4.4 yields

$$\varphi([T_1, T_2], y, v) = \partial \bigcap_l \operatorname{cl}\operatorname{ins}\Gamma_{n_l} = \partial \bigcap_n \operatorname{cl}\operatorname{ins}\Gamma_n = \omega(x, u).$$

Case 2 can be treated analogously.

The next theorem presents the main result of this paper.

THEOREM 4.6. *Let $(x, u) \in \mathbb{R}_+^2 \times U_{\mathrm{ad}}$ be optimal for* (1.1)–(1.4) *with $\varphi(\cdot, x, u)$ nonself-intersecting and suppose that $\omega(x, u)$ contains only finitely many optimal equilibria. Then one of the following cases occurs*:

(i) *There are $T > 0$ and an optimal $\mathbb{R}$-solution $(y, v) \in \hat{\omega}(x, u)$ such that $y = \varphi(T, y, v)$ and $\omega(x, u) = \varphi([0, T], y, v)$.*

(ii) *There are optimal $\mathbb{R}$-solutions $(y_i, v_i) \in \hat{\omega}(x, u)$ and optimal equilibria e_i^+, e_i^- such that for all i,*

$$e_i^- = \lim_{t \to -\infty} \varphi(t, y_i, v_i), \qquad e_i^+ = \lim_{t \to +\infty} \varphi(t, y_i, v_i), \tag{4.1}$$

$$\omega(x, u) = \bigcup_i \varphi(\mathbb{R}, y_i, v_i) \cup \bigcup_i \{e_i^-, e_i^+\}. \tag{4.2}$$

Proof. Let $y \in \omega(x, u)$. By Proposition 2.10 there is $v \in U_{\mathrm{ad}}$ such that $(y, v) \in \hat{\omega}(x, u)$. If either $\alpha(y, v)$ or $\omega(y, v)$ contain a point that is not an equilibrium, then Propositions 4.3 and 4.5 imply that (i) holds (naturally, we may take $T_1 = 0$).

In the other case, $\alpha(y, v)$ and $\omega(y, v)$ consist of equilibria only. Since $\alpha(y, v)$, $\omega(y, v) \subset \omega(x, u)$, Theorems 3.5 and 3.6 imply that there are optimal equilibria e^- and e^+ with

$$e^- = \lim_{t \to -\infty} \varphi(t, y, v), \qquad e^+ = \lim_{t \to +\infty} \varphi(t, y, v).$$

COROLLARY 4.7. *Let $(x, u) \in \mathbb{R}_+^2 \times U_{\mathrm{ad}}$ be optimal for* (1.1)–(1.4) *and suppose that $\omega(x, u)$ does not contain an optimal equilibrium. Then either there is $\hat{u} \in U_{\mathrm{ad}}$ such that $(x, \hat{u})$ is optimal, finally periodic and $\varphi(x, \hat{u}) \subset \varphi(x, u)$ or there are optimal periodic $(y, v) \in \mathbb{R}_+^2 \times U_{\mathrm{ad}}$ with $\omega(x, u) = \varphi(y, v)$.*

Proof. If $\varphi(\cdot, x, u)$ is self-intersecting the assertion follows from Proposition 4.1. Otherwise Proposition 4.5 implies the existence of optimal $(\bar{y}, \bar{v})$ and $T_2 > T_1 \geqq 0$ with $\varphi(T_1, \bar{y}, \bar{v}) = \varphi(T_2, \bar{y}, \bar{v})$ and $\omega(x, u) = \varphi([T_1, T_2], y, v)$. Applying Proposition 4.1 again, we obtain the assertion.

Remark 4.8. By Remarks 2.12 and 3.7, the results above remain true if the performance criterion (1.1) is replaced by (2.5) provided that (2.6) holds.

5. Application to bioeconomic problems. The crucial assumption in the Poincaré–Bendixson Theorem given above is that the integrand of the performance criterion does not depend explicitly on the control u. In this section we show that the weakened form of this assumption specified in (2.6) can be verified in bioeconomic problems.

Furthermore, we present a specific example where the ω-limit set of an optimal solution consists of an optimal periodic trajectory that is not an optimal equilibrium. Feasibility of this case is a specific feature of the two-dimensional problem compared to the one-dimensional problem.

We will have to ensure that the ω-limit set has empty intersection with $\partial\mathbb{R}_+^2$. This deserves special attention also independently of the question considered here. Hence we give the following definition.

DEFINITION 5.1. A pair $(x, u) \in \operatorname{int} \mathbb{R}_+^n \times U_{\text{ad}}$ leads to extinction if $\omega(x, u) \cap \partial\mathbb{R}_+^n \neq \emptyset$.

PROPOSITION 5.2. *Let $(x, u) \in \operatorname{int} \mathbb{R}_+^n \times U_{\text{ad}}$ be optimal. If (x, u) leads to extinction, then there are optimal (y_k, v_k) $k = 0, 1, 2, \cdots$, such that $y_k \in \operatorname{int} \mathbb{R}_+^n$, $y_k \to y_0 \in \partial\mathbb{R}_+^n$ and* $\sup\{d(z, \partial\mathbb{R}_+^n), z \in \varphi(y_k, v_k)\} \to 0$ *for $k \to \infty$.*

Proof. In the case $\omega(x, u) \subset \partial\mathbb{R}_+^n$ we may choose $y_k := \varphi(t_k, x, u)$, $v_k := u(t_k + \cdot)$, with $t_k \to \infty$. Now assume that $\omega(x, u) \cap \partial\mathbb{R}_+^n \neq \emptyset$, but $\omega(x, u) \not\subset \partial\mathbb{R}_+^n$. For $\varepsilon > 0$ let $B_\varepsilon := \{z \in \mathbb{R}_+^n : d(z, \partial\mathbb{R}_+^n) \leqq \varepsilon\}$. Choose ε small enough such that $\omega(x, u) \not\subset B_\varepsilon$. Then there are $t_l \to \infty$ and $s_l > 0$, $l \in \mathbb{N}$, such that for all l large enough

$$\varphi(t_l, x, u) \in \partial B_\varepsilon, \quad \varphi(t_l + s_l, x, u) \in B_{1/l}, \quad \varphi(t_l + s, x, u) \in B_\varepsilon \quad \text{for } s \in [0, s_l].$$

Without loss of generality we may assume that $t \to \varphi(t_l + t, x, u)$ converges locally uniformly to some $\varphi(\cdot, y, v)$ with $(y, v) \in \hat{\omega}(x, u)$, $y \in \partial B_\varepsilon$. If (s_l) is bounded we may assume that $s_l \to s \in \mathbb{R}_+$. This implies $\varphi(s, y, v) \in \partial\mathbb{R}_+^n$. Hence $\varphi(0, y, v) \subset \partial\mathbb{R}_+^n$. This contradicts $\varphi(t_l, x, u) \in \partial B_\varepsilon$. If (s_l) is unbounded, we may assume that $s_l \to \infty$. This implies $\varphi(y, v) \subset B_\varepsilon$. Choosing a sequence (ε_k) with $\varepsilon_k \to 0$, we obtain (y_k, v_k) satisfying the assertion.

Example 5.3. Maximize

$$V(a, u) = \int_0^\infty e^{-\delta t}\{p_1 x_1(\gamma_{11} u_1 + \gamma_{12} u_2) F^1(x_1) + p_2 x_2(\gamma_{21} u_1 + \gamma_{22} u_2) F^2(x_2) - c_1 u_1 - c_2 u_2\}\, dt$$

(where dependence on t has been dropped) such that

$$\begin{aligned} \dot{x} &= x_1(F_0^1(x) - (\gamma_{11} u_1 + \gamma_{12} u_2) F^1(x_1)), \\ \dot{x}_2 &= x_2(F_0^2(x) - (\gamma_{21} u_1 + \gamma_{22} u_2) F^2(x_2)), \\ (x_1(0), x_2(0)) &= (a_1, a_2) \in \mathbb{R}_+^2, \\ (u_1(t), u_2(t)) &\in \Omega = [0, U_1] \times [0, U_2]. \end{aligned}$$

This example is designed to model resource-harvesting of two resources, the stocklevel of which (at time t) is denoted by $x_1(t)$, respectively, $x_2(t)$. There are two technologies available, such that an effort u_j spent applying technology j results in a catch-rate $\gamma_{ij} u_j F^i(x_i)$ with respect to the species i. γ_{ij} are nonnegative efficiency coefficients; $F^i(x_i)$ is a positive locally Lipschitz continuous function $\mathbb{R}_+^2 \to \mathbb{R}$, which relates effort and catch. There is a more detailed discussion of these "density profiles" in Clark [6].

p_1, p_2 are nonnegative constants to be interpreted as prices per unit biomass. c_1, c_2 are nonnegative constants to be interpreted as cost per unit effort spent applying technology one, respectively, two. Therefore $V(a, u)$ represents the "total discounted net revenue."

The rewriting of $V(a, u)$ *in* $\operatorname{int}\mathbb{R}^2_+$. Note first that

$$\begin{aligned}
\gamma_{11}u_1+\gamma_{12}u_2&=\frac{x_1F_0^1(x)-\dot{x}_1}{x_1F^1(x_1)},\\
\gamma_{21}u_1+\gamma_{22}u_2&=\frac{x_2F_0^2(x)-\dot{x}_2}{x_2F^2(x_2)},\\
p_1x_1(\gamma_{11}u_1+\gamma_{12}u_2)F^1(x_1)&=F_0^1(x)p_1x_1-p_1\dot{x}_1,\\
p_2x_2(\gamma_{21}u_1+\gamma_{22}u_2)F^2(x_2)&=F_0^2(x)p_2x_2-p_2\dot{x}_2.
\end{aligned}$$

We assume that the matrix (γ_{ij}) is invertible. For obvious reasons the special case of $\gamma_{12}=\gamma_{21}=0$ is called "*selective harvesting.*" Suppose first $\gamma_{11}\neq 0$. Then

$$\left(\gamma_{22}-\frac{\gamma_{21}\gamma_{12}}{\gamma_{11}}\right)u_2=\frac{F_0^2(x)}{F^2(x_2)}-\frac{\gamma_{21}F_0^1(x)}{\gamma_{11}F^1(x_1)}-\frac{\dot{x}_2}{x_2F^2(x_2)}+\frac{\gamma_{21}}{\gamma_{11}}\frac{\dot{x}_1}{x_1F^1(x_1)},$$

or with $d=\gamma_{11}\gamma_{22}-\gamma_{12}\gamma_{21}$

$$\begin{aligned}
u_2&=G^2(x)-\frac{\gamma_{11}}{d}\frac{\dot{x}_2}{x_2F^2(x_2)}+\frac{\gamma_{21}}{d}\frac{\dot{x}_1}{x_1F^1(x_1)},\\
u_1&=G^1(x)+\frac{\gamma_{12}}{d}\frac{\dot{x}_2}{x_2F^2(x_2)}-\left(\frac{\gamma_{12}\gamma_{21}}{d\gamma_{11}}+\frac{1}{\gamma_{11}}\right)\frac{\dot{x}_1}{x_1F^1(x_1)}-c_1u_1-c_2u_2\\
&=G^3(x)+\frac{\dot{x}_1}{x_1F^1(x_1)}\gamma_1+\frac{\dot{x}_2}{x_2F^2(x_2)}\gamma_2
\end{aligned}$$

where

$$\gamma_1=\frac{c_1}{\gamma_{11}}+\frac{c_1\gamma_{12}\gamma_{21}}{d\gamma_{11}}-\frac{c_2\gamma_{21}}{d},\qquad \gamma_2=\frac{c_1\gamma_{12}}{d}-\frac{c_2\gamma_{11}}{d},$$

and G^1, G^2, G^3 are locally Lipschitz continuous functions of x. Put $G^4(x)=G^3(x)+F_0^1(x)p_1x_1+F_0^2(x)p_2x_2$. Then

$$V(a,u)=\int_0^\infty e^{-\delta t}G^4(x(t))\,dt+\int_0^\infty e^{-\delta t}\left\{\left[\frac{\gamma_1}{x_1F^1(x_1)}-p_1\right]\dot{x}_1+\left[\frac{\gamma_2}{x_2F^2(x_2)}-p_2\right]\dot{x}_2\right\}dt.$$

Put $g_j(y_j)=\int_{z_j}^{y_j}(\gamma_j/\xi F^j(\xi)-p_j)\,d\xi$, where $z=(z_1,z_2)$ is a pair of positive reals fixed once and for all. Now

$$\begin{aligned}
V(a,u)&=\int_0^\infty e^{-\delta t}G^4(x(t))\,dt+\int_0^\infty e^{-\delta t}[g_1'(x_1(t))\dot{x}_1(t)+g_2'(x_2(t))\dot{x}_2(t)]\,dt\\
&=\int_0^\infty e^{-\delta t}G^4(x(t))\,dt+\int_0^\infty e^{-\delta t}\frac{d}{dt}[g_1(x_1(t))+g_2(x_2(t))]\,dt\\
&=\int_0^\infty e^{-\delta t}G^4(x(t))\,dt+e^{-\delta t}[g_1(x_1(t))+g_2(x_2(t))]\Big|_{t=0}^{\infty}\\
&\quad+\int_0^\infty e^{-\delta t}\delta[g_1(x_1(t))+g_2(x_2(t))]\,dt\\
&=r(a)+\int_0^\infty e^{-\delta t}R(x(t))\,dt
\end{aligned}\tag{5.1}$$

with $r(a) := -g_1(a_1) - g_2(a_2)$, $R(y_1, y_2) = G^4(y) + \delta g_1(y_1) + \delta g_2(y_2)$, provided that

$$\lim_{t\to\infty} e^{\delta t}[g_1(x_1(t)) + g_2(x_2(t))] = 0. \tag{5.2}$$

To establish (5.2) let $c(t) := e^{-\delta t} g_j(x_j(t))$ for $j = 1$ or $j = 2$. It suffices to show that $\lim_{t\to\infty} c(t) = 0$. Now

$$\begin{aligned}\dot c(t) &= -\delta c(t) + e^{-\delta t}\left[\frac{\gamma_j}{x_j(t)F^j(x_j(t))} - p_j\right] x_j(t)[F_0^j(x(t)) - (y_{j1}u_1(t) + \gamma_{j2}u_2(t))F^j(x_j(t))] \\ &= -\delta c(t) + h(t),\end{aligned}$$

where h is a measurable and, since $F^j(0) \neq 0$, also bounded function on $\mathbb{R}_+$. By the variation of constant formula

$$|c(t)| \leqq (|c(0)| + \|h\|_\infty/\delta)\, e^{-\delta t}$$

and (5.2), and hence (5.1), follows.

The definition of R given above implies for x_1, $x_2 > 0$ that

$$\lim_{y\to(x_1,0)} R(y) = \pm\infty \quad \text{if } \gamma_2 \lessgtr 0,$$

$$\lim_{y\to(0,x_2)} R(y) = \pm\infty \quad \text{if } \gamma_1 \lessgtr 0.$$

Property (2.6)(α) follows from (5.1) and (5.2).

Analogous arguments can be used if $\gamma_{11} = 0$, and also in the case of nonselective harvesting, where $\gamma_{12} = \gamma_{21} = 0$.

Remark 5.4. We may construct examples of optimal control systems (with $m > n$) where condition (2.6) is not satisfied.

Now we present an example of a predator-prey system where both species are subject to innerspecific competition. Only the predator is harvested and the costs are proportional to the effort. The unharvested system possesses a limit cycle. We will show that there are optimal trajectories tending to an optimal periodic solution as $t \to \infty$. The system equation and the analysis of the uncontrolled system are taken from Sieveking [17].

Example 5.5.

$$\begin{aligned}&\text{Maximize} \quad \int_0^\infty e^{-\delta t}[pqx_2 - c]u\, dt \\ &\text{Subject to} \quad \dot x_1 = x_1[\alpha - \gamma x_2 - h(x_1) - \varepsilon x_1], \\ &\qquad\qquad\quad \dot x_2 = x_2[-\beta + \lambda x_1 - \mu x_2 - qu], \qquad t \in \mathbb{R}_+, \\ &\qquad\qquad\quad (x_1(0), x_2(0)) = x \in \mathbb{R}_+^2, \\ &\qquad\qquad\quad u \in [0, U^1]\end{aligned}$$

where p, q, c, α, β, γ, δ, λ, μ, U^1 are positive constants, and h is defined by

$$h(x_1) = \begin{cases} (x_1 - \beta/\lambda)^2 & \text{for } 0 \leqq x_1 \leqq \beta/\lambda, \\ 0 & \text{for } \beta/\lambda \leqq x_1. \end{cases}$$

The system above is a special case of Example 5.3, and conditions (2.1), (2.6) are satisfied. First we analyze the uncontrolled equation where $u_1 = u_2 = 0$: All trajectories $\varphi(\cdot, x, 0)$ are bounded and for ε, $\mu > 0$, small, the only equilibria are $(0, 0)$, $(\alpha/\varepsilon, 0)$ and a point e near $e^0 = (\beta/\lambda, \alpha/\gamma)$.

The equilibrium e is (locally) asymptotically stable, the points $(0, 0)$ and $(\alpha/\varepsilon, 0)$ are saddles.

For ε, $\mu > 0$, small enough, the equation possesses a limit cycle (applying the Poincaré–Bendixson Theorem to the time-reversed equation, this implies the existence of another—unstable—periodic solution).

In the following, we assume that ε, μ are small enough such that existence of a limit cycle is guaranteed. There exists an initial value $x \in \operatorname{int} \mathbb{R}^2_+$ on the line $x_2 = -\beta/\mu + \lambda/\mu x_1$ such that $\varphi(\cdot, x, 0)$ spirals outward, i.e., there exists a (minimal) time $T_1 > 0$ such that $\varphi(T_1, x, 0)$ lies on the same line above x. Using continuous dependence of solutions on the right-hand side, this implies that for $U^1 > 0$, small enough, also every trajectory $\varphi(\cdot, x, u)$, $u(t) \in [0, U^1]$ almost everywhere, spirals outward. In particular, this is true for an optimal trajectory $\varphi(\cdot, x, u)$. See Fig. 5.1.

The controlled system has exactly two equilibria on $\partial\mathbb{R}^2_+$, namely $(0, 0)$ and $(\alpha/\varepsilon, 0)$.

Next we show that no optimal pair $(x, u) \in \operatorname{int} \mathbb{R}^2_+ \times U_{\mathrm{ad}}$ leads to extinction. For $\xi > 0$, let $A_\xi := [\xi, \alpha/\varepsilon] \times [0, M]$ where $M := \max z_2(t)$ and $z = (z_1, z_2)$ is the unique trajectory in $\operatorname{int} \mathbb{R}^2_+$ of the uncontrolled system with $\lim_{t \to -\infty} z(t) = (\alpha/\varepsilon, 0)$. Then we can show that there exists $\xi > 0$ with the following property: For all $(y, v) \in \operatorname{int} \mathbb{R}^2_+ \times U_{\mathrm{ad}}$ there is $T > 0$ such that for all $t \geqq T$ it follows that $\varphi(t, y, v) \in A_\xi$; furthermore $\varphi(y, v) \subset A_\xi$ for every $(y, v) \in A_\xi \times U_{\mathrm{ad}}$. Hence $\omega(x, u) \cap \{0\} \times \mathbb{R}_+ = \emptyset$. Now suppose that $\omega(x, u) \cap \mathbb{R}_+ \times \{0\} \neq \emptyset$. Then Proposition 5.2 implies the existence of optimal $(y_k, v_k) \in \operatorname{int} \mathbb{R}^2_+ \times U_{\mathrm{ad}}$ with $y_k \to y_0 \in \mathbb{R}_+ \times \{0\}$ and

$$\max \{d(z, \mathbb{R}_+ \times \{0\}) : z \in \varphi(y_k, v_k)\} \to 0 \quad \text{for } k \to \infty.$$

But for x_2 small, we have $pqx_2 - c < 0$. This contradicts the existence of (y_k, v_k) with the properties indicated above.

Conclusion. Suppose that in Example 5.5 the positive constants ε, μ, U^1 are small enough. No optimal pair $(x, u) \in \operatorname{int} \mathbb{R}^2_+ \times U_{\mathrm{ad}}$ leads to extinction and every trajectory is bounded. There are initial values $x \in \operatorname{int} \mathbb{R}^2_+$ such that corresponding optimal trajectories $\varphi(\cdot, x, u)$ spiral outward. Hence, according to Corollary 4.7, there are optimal finally periodic $(x, \hat{u})$ or $\omega(x, u) = \varphi(y, v)$ with (y, v) optimal periodic.

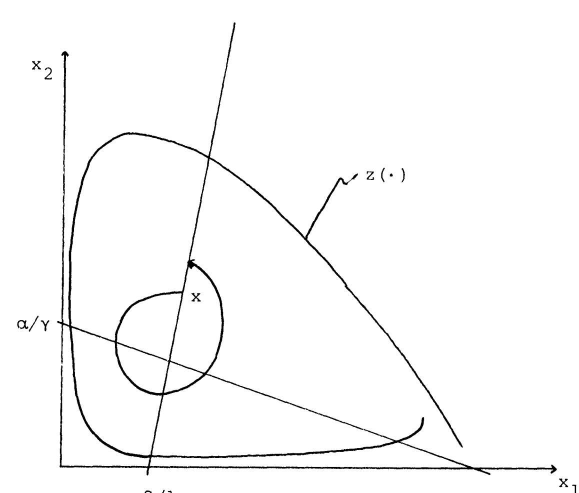

FIG. 5.1.

6. Nonuniqueness. For given initial state, solutions of ordinary differential equations are unique (provided that local Lipschitz continuity prevails). In general, optimal control problems do not share this nice property. In fact in this section we give a nonconstructive criterion implying nonuniqueness for a certain initial value. Furthermore, a simple bioeconomic example is presented with nonunique optimal solutions.

DEFINITION 6.1. An element $x \in \mathbb{R}_+^n$ is called a point of nonuniqueness if there are $u, v \in U_{\text{ad}}$ such that (x, u) and (x, v) are optimal and $\varphi(t, x, u) \neq \varphi(t, x, v)$ for some $t \in \mathbb{R}_+$.

"Nonuniqueness" requires that the trajectories corresponding to u and v do not coincide. Thus "redundancies" in the controls do not lead, in our terminology, to nonuniqueness.

THEOREM 6.2. *Suppose that* $(x, u) \in \mathbb{R}_+^2 \times U_{\text{ad}}$ *are optimal and that there are* $T_2 > T_1 \geqq 0$ *such that* $\varphi(\cdot, x, u)$, $t \in [T_1, T_2]$, *is a Jordan curve. If* $I := \operatorname{cl} \operatorname{ins} \Gamma$ *does not contain any optimal equilibrium, then it contains a point of nonuniqueness.*

Proof. Suppose there is no point of nonuniqueness in I and note that I is positively invariant. Hence for every $y \in I$, there is a unique control $u(y) \in U_{\text{ad}}$ such that $(y, u(y))$ is optimal and $\varphi(y, u(y)) \subset I$. Lemma 2.5 implies that $y \to u(y): I \to U_{\text{ad}}$ is continuous, and hence for every $t \geqq 0$ the map $y \to \varphi(t, y, u(y)): I \to I$ is continuous. By the Schoenfliess Theorem (Beck [3, p. 22]), I is homeomorphic to the closed unit ball in $\mathbb{R}^2$. Hence, by Brouwer's Fixed Point Theorem, there is for every $t \geqq 0$ a fixed point x_t with

$$\varphi(t, x_t, u(y)) = x_t.$$

Let (t_n) be a sequence of numbers with $t_n > 0$ such that $\lim t_n = 0$ and $\lim x_{t_n} = e \in I$ exists. We claim that e is an optimal equilibrium. In fact, for every $n \in \mathbb{N}$, uniqueness of optimal solutions implies that $\varphi(\cdot, x_n, u(x_n))$ is a periodic solution of period t_n. Without loss of generality we may assume that $\varphi(\cdot, x_n, u(x_n))$ converges uniformly to the constant trajectory e, which therefore is an optimal equilibrium contrary to our assumption.

In the following example, nonuniqueness is shown by a different argument.

Example 6.3.

$$\text{Maximize} \quad V(x, u) := \int_0^\infty e^{-\delta t}\{[p - c(x_1(t))]u_1(t)x_1(t) + [p - c(x_2(t))]u_2(t)x_2(t)\}\, dt$$

$$\begin{aligned} \text{Subject to} \quad & \dot{x}_1(t) = x_1(t)[2 - x_1(t) - 2x_2(t) - u_1(t)], \\ & \dot{x}_2(t) = x_2(t)[2 - x_2(t) - 2x_1(t) - u_2(t)], \\ & (u_1(t), u_2(t)) \in \Omega := [0, \tfrac{1}{2}] \times [0, \tfrac{1}{2}], \\ & x_1(0) = x_1, \qquad x_2(0) = x_2, \end{aligned}$$

where $p > 0$ and $c(\cdot)$ is continuous and strictly decreasing on $\mathbb{R}_+$ with $c(\tfrac{2}{3}) = p$.

Assertion. For $\delta > 0$ sufficiently small the point $x = (\tfrac{2}{3}, \tfrac{2}{3})$ is a point of nonuniqueness.

Proof. (See Fig. 6.1.) First note that existence of an optimal solution follows by uniform boundedness of the trajectories, linearity in u and convexity and compactness of Ω. Define

$$S\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}_+^2.$$

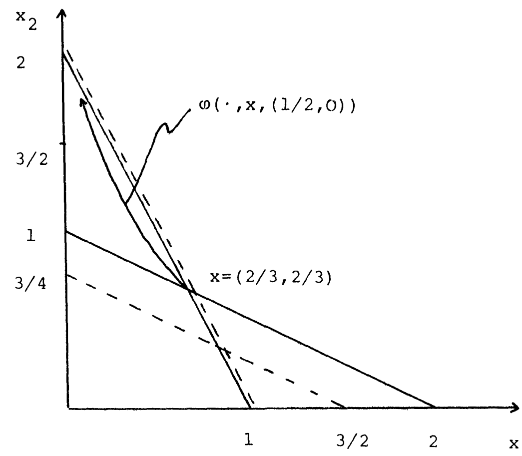

FIG. 6.1. *Illustration of Example* 6.3.

The symmetry of the system equation and $x = Sx$ imply for $u \in U_{\text{ad}}$ and $t \geqq 0$

$$S\varphi(t, x, u) = \varphi(t, x, Su).$$

Furthermore,

$$V(x, Su) = V(x, u).$$

Thus, if (x, u) is optimal, also (x, Su) is optimal. If the optimal solution is unique, it follows that

$$\varphi_1(t, x, u) = \varphi_2(t, x, u) \quad \text{for all } t \geqq 0.$$

Looking at the system equation we find that this implies

$$\varphi_1(t, x, u) \leqq \tfrac{2}{3}, \quad \varphi_2(t, x, u) \leqq \tfrac{2}{3} \quad \text{for all } t > 0.$$

Hence

$$p - c(\varphi_1(t, x, u)) \leqq 0, \quad p - c(\varphi_2(t, x, u)) \leqq 0 \quad \text{for all } t > 0$$

and

$$V(x, u) \leqq 0.$$

Thus, in case of uniqueness, the only candidate for an optimal control is $u_1 \equiv u_2 \equiv 0$, which leaves $x = (\frac{2}{3}, \frac{2}{3})$ fixed and

$$V(x, u) = 0.$$

Thus it suffices to construct $v \in U_{\text{ad}}$ with

$$V(x, v) > 0.$$

Consider first the control $\bar{v} = (\bar{v}_1, \bar{v}_2)$

$$\bar{v}_1(t) \equiv \tfrac{1}{2}, \qquad \bar{v}_2(t) \equiv 0.$$

A phase plane analysis (cf. Fig. 6.1) yields that for t increasing, $\varphi_1(t, x, \bar{v})$ decreases and $\varphi_2(t, x, \bar{v})$ increases, with

$$\lim_{t \to \infty} \varphi(t, x, \bar{v}) = (0, 2). \tag{6.1}$$

Now consider the system

(6.2) $\quad \dot{x}_1(t)=x_1(t)[2-x_1(t)-2x_2(t)], \qquad \dot{x}_2(t)=x_2(t)[2-x_2(t)-2x_1(t)-\frac{1}{2}].$

For this system the point $(0,\frac{3}{2})$ is (locally) asymptotically stable. In fact, the Jacobian at this point is

$$\begin{pmatrix} -1 & 0 \\ -3 & -\frac{3}{2} \end{pmatrix}.$$

Since the region of attraction of an asymptotically stable point is always open and $(0,2)$ is attracted by $(0,\frac{3}{2})$, it follows from (6.1) that there is $t_1>0$ such that in the system (6.2), $\varphi(t_1,x,\bar{v})$ is attracted by $(0,\frac{3}{2})$. Define

$$v(t):=\begin{cases} \bar{v}(t)=(\frac{1}{2},0), & t\in[0,t_1],\\ (0,\frac{1}{2}), & t\in(t_1,\infty).\end{cases}$$

Then

$$\lim_{t\to\infty}\varphi(t,x,v)=(0,\tfrac{3}{2}).$$

By continuity of c, there is $M_1>0$ with

$$(p_1-c(\varphi_1(t,x,\bar{v})))\varphi_1(t,x,\bar{v})\tfrac{1}{2}\geqq -M_1, \qquad 0\leqq t\leqq t_1.$$

Thus

$$\begin{aligned}\delta V(x,v)&=\delta\int_0^\infty e^{-\delta t}\{[p-c(\varphi_1(t,x,v))]\varphi_1(t,x,v)v_1(t)\\ &\qquad\qquad +[p-c(\varphi_2(t,x,v))]\varphi_2(t,x,v)v_2(t)\}\,dt\\ &=\delta\int_0^{t_1}\cdots+\delta\int_{t_1}^\infty\cdots\\ &\geqq -\delta M_1\frac{1}{2}\int_0^{t_1}e^{-\delta t}\,dt+\delta\int_{t_1}^\infty e^{-\delta t}[p-c(\varphi_2(t,x,\bar{v}))]\frac{1}{2}\,dt;\end{aligned}$$

without loss of generality we may assume

$$\varphi_2(t,x,v)\geqq 1 \quad \text{for all } t\geqq t_1.$$

Since there is $M_2>0$ with

$$p-c(y)>M_2 \quad \text{for } y\geqq 1$$

it follows that

$$(p-c(\varphi_2(t,x,v)))\varphi_2(t,x,v)\tfrac{1}{2}\geqq\tfrac{1}{2}M_2 \quad \text{for } t\geqq t_1.$$

Together we get

$$V(x,v)\geqq -\tfrac{1}{2}M_1(1-e^{-\delta t_1})+\tfrac{1}{2}M_2\,e^{-\delta t_1}.$$

For $\delta\to 0$, the right-hand side of this inequality tends to $\frac{1}{2}M_2$. Hence $V(x,\bar{v})>0$ for $\delta>0$, sufficiently small. This proves the assertion.

The idea for this example may be sketched as follows. We start in an equilibrium point x, where two competing species coexist, and where the net revenue $p-c(x)$ is zero. Catching *one* of these species we have a temporary loss. On the other hand, the other species increases until it gets into a domain where it can be caught continually, yielding positive net revenue. The initial loss is, for sufficiently small discount rate $\delta>0$, less than the later revenue.

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