# Structure Theory and Duality for Time Varying Retarded Functional Differential Equations

FRITZ COLONIUS*

*Fachbereich Mathematik, Universität Bremen, Bremen, Federal Republic of Germany*

ANDRZEJ MANITIUS†

*Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181*

AND

DIETMAR SALAMON‡

*Department of Mathematics, University of Warwick, Coventry CV4 7AL, Warwickshire, England*

## 1. INTRODUCTION

The aim of this paper is to extend to time-varying linear retarded functional-differential equations (FDEs) the results on structural operators and duality which were obtained in recent years for autonomous linear FDEs [1, 6, 7, 18, 20]. These results have become very useful, as they clarified the structure of the semigroups corresponding to linear FDEs and helped to characterize solutions of control problems associated with FDEs [24] and to develop certain aspects of numerical approximation schemes [16, 17]. Analogous results for general time-varying equations have not been available so far. Only in the special case of constant point delays there were some results in a paper by Delfour [5], but they were not stated in terms of structural operators. In the case of time-varying point delays, even the

* This work was performed while the author was visiting with the Division of Applied Mathematics, Brown University, Providence, RI 02912.

† This work was supported by NSF Grant DMS-8401719.

‡ This work was performed while the author was visiting with the Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, WI 53706.

question of existence of solutions with $L^p$ initial functions did not have an adequate answer.

In this paper the time-varying RFDEs are investigated in the setting of both the state space $C$ of continious functions and the product space $R^n \times L^p$. In the $C$ space two state concepts are investigated: the state given by the initial function and the state given by the forcing term. The duality theory is based upon these two concepts. What distinguishes the present approach from the previously existing ones [12, 13, 14] is the explicit use of two structural operators $F(t)$ and $G(t)$, their relations with the evolution operators and duality, as well as the results on strong continuity of all these operators. The latter results require an additional assumption on the original equation which cannot be omitted as shown by an example.

In the setting of the product space $M^p = R^n \times L^p$ the following questions are investigated: the extendability of structural and evolution operators defined on the space $C$ to product spaces, the strong continuity of these extended operators, the duality theory in product spaces, and the existence of a functional-differential adjoint equation.

The latter problem has particularly interesting aspects. It has been known that in the setting of the $C$ space the adjoint equation is in general an integral equation of Volterra type, which in some special cases can be "differentiated." By using our duality results we establish a link between the extendability of evolution operators from space $C$ to $M^p$ and the existence of a functional differential adjoint equation.

A surprising discovery of this paper is that the solution of all the problems mentioned above depends critically on a certain assumption about the original equation, which we call a "fundamental extendability hypothesis." We give a general formulation of this hypothesis and show how it intervenes in the solution of these problems. We then translate the hypothesis into a number of concrete conditions related directly to equation parameters, in particular to the behavior of time-varying point delays. Some interesting features of this behavior and their relation to the form of the differential adjoint equation are exhibited in a few examples.

In [4], the results of this paper are used in order to characterize solutions of optimal periodic control problems.

## 2. Structure Theory in the State Space of Continuous Functions

### 2.1. *Time-Varying Retarded Systems*

We consider the time-varying linear retarded functional-differential equation (RFDE)

$$\dot{x}(t) = L(t, x_t), \tag{2.1}$$

where $x(t) \in R^n$ and $x_t$ is defined by

$$x_t(\tau) = x(t+\tau), \qquad -h \leqslant \tau \leqslant 0,\ 0 < h < \infty.$$

We assume that

(H1) (i) *there exists a function* $m(\cdot) \in L^1_{\text{loc}}(-\infty, \infty)$ *such that for almost all* $t \in R$ *and all* $\phi \in C[-h, 0; R^n]$

$$|L(t, \phi)| \leqslant m(t)\ \|\phi\|_{C[-h, 0, R^n]}, \tag{2.2}$$

(ii) *for every* $\phi \in C[-h, 0; R^n]$, *the function* $t \to L(t, \phi)$, $t \in R$, *is measurable.*

LEMMA 2.1. *Suppose that L satisfies hypothesis* (H1). *Then there exists a* $n \times n$ *matrix valued function* $\eta(t, \tau)$ *with the following properties:*

(i) *For every* $t \in R$ *the function* $\eta(t, \cdot) \in \text{NBV}[-h, 0; R^n]$, *i.e.,* $\eta(t, \cdot)$ *is of bounded variation and normalized in the sense that* $\eta(t, \tau)$ *is left continuous in* $\tau$ *for* $-h < \tau < 0$, $\eta(t, \tau) = 0$ *for* $\tau \geqslant 0$, *and* $\eta(t, \tau) = \eta(t, -h)$ *for* $\tau \leqslant -h$;

(ii) *For almost all* $t \in R$ *and all* $\phi \in C[-h, 0; R^n]$

$$L(t, \phi) = \int_{[-h, 0]} [d_\tau \eta(t, \tau)]\ \phi(\tau); \tag{2.3}$$

(iii) $\eta(\cdot, \cdot)$ *is measurable on* $R \times R$.

*Proof.* Without loss of generality we can assume that $n = 1$. The Riesz representation theorem implies the existence of $\eta$ with the properties (i) and (ii). Now fix $\tau_0 \in (-h, 0]$ and define the sequence $\phi_k(\cdot) \in C[-h, 0]$ as indicated in Fig. 1. Then for almost every $t \in R$

$$\eta(t, \tau_0) = \lim_{k \to \infty} \int_{[-h, 0]} [d_\tau \eta(t, \tau)]\ \phi_k(\tau) = \lim_{k \to \infty} L(t, \phi_k)$$

and therefore the function $\eta(\cdot, \tau)$ is measurable for every $\tau \in [-h, 0]$. This implies that the function

$$\eta_+(t, \tau) = \underset{[-h, \tau]}{\text{VAR}}\ \eta(t, \cdot)$$

is measurable in $t$ for every fixed $\tau \in [-h, 0]$. Furthermore, this function is monotone in $\tau$ and hence we can apply [2, Chap. V, Exercise 6] and obtain that $\eta_+$ is measurable on $R \times R$. The same arguments apply to $\eta_- = \eta_+ - \eta$ and thus $\eta = \eta_+ - \eta_-$ is measurable in $R \times R$. This proves statement (iii). ∎

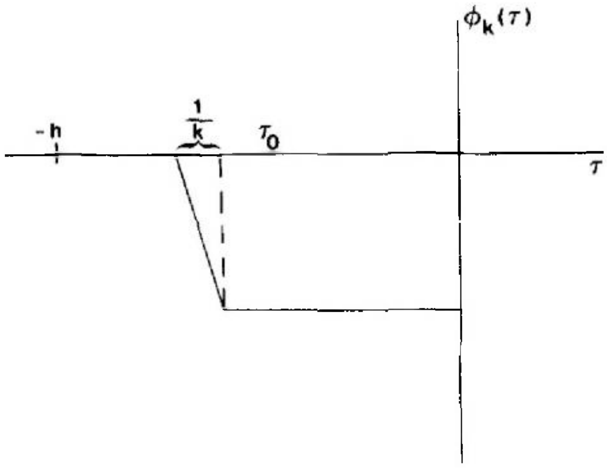



FIGURE 1

*Remark.* Let $L$ satisfy (H1) and let $x(\cdot) \in C[t_0 - h, t_1; R^n]$, $t_0 < t_1$, be given. Then the functions

$$\int_{[-h,0]} [d_\tau \eta(t, \tau)]\, x(t+\tau), \qquad t_0 \leqslant t \leqslant t_1,$$

$$\int_{[-h,t_0-t)} [d_\tau \eta(t, \tau)]\, x(t+\tau), \qquad t_0 \leqslant t \leqslant t_1,$$

$$\int_{[t_0-t,0]} [d_\tau \eta(t, \tau)]\, x(t+\tau), \qquad t_0 \leqslant t \leqslant t_1,$$

are in $L^1[t_0, t_1; R^n]$ and depend in this space continuously on $x(\cdot) \in C[t_0 - h, t_1; R^n]$.

A function $x(\cdot) \in C[t_0 - h, t_1; R^n]$ is said to be a solution of (2.1) if $x(t)$ is absolutely continuous on the inverval $[t_0, t_1]$ and satisfies (2.1) for almost every $t \in [t_0, t_1]$. It has been shown in Hale [12, Chap. 6] that (2.1) admits a unique solution on the interval $[t_0, t_1]$ for every initial condition of the form

$$x(t_0 + \tau) = \phi(\tau), \qquad -h \leqslant \tau \leqslant 0, \tag{2.4}$$

where $\phi \in C[-h, 0; R^n]$, and for an additional forcing term in $L^1[t_0, t_1; R^n]$. For our purposes we need a slightly stronger result.

Integrating Eqs. (2.1), (2.4), we obtain

$$x(t_0 + s) = f^{t_0}(s)$$
$$+ \int_0^s \int_{[-\sigma,0]} [d_\tau \eta(t_0 + \sigma, \tau)]\, x(t_0 + \sigma + \tau)\, d\sigma, \qquad s \geqslant 0, \tag{2.5}$$

where $f^{t_0}(\cdot) \in C[0, t_1 - t_0; R^n]$ is given by

$$f^{t_0}(s) = \phi(0) + \int_0^s \int_{[-h, -\sigma]} [d_\tau \eta(t_0 + \sigma, \tau)] \, \phi(\sigma + \tau) \, d\sigma, \qquad s \geq 0. \quad (2.6)$$

By the Remark after Lemma 2.1 this forcing term $f^{t_0}(\cdot)$ is actually in $W^{1,1}[0, t_1 - t_0; R^n]$ and depends in this space continuously on $\phi \in C[-h, 0; R^n]$. Moreover, note that $f^{t_0}(s)$ is constant for $s \geq h$. Hale's result [12, p. 142] implies that Eq. (2.5) admits a unique solution $x(\cdot) \in W^{1,1}[t_0, t_1; R^n]$ for every $f^{t_0}(\cdot) \in W^{1,1}[0, t_1 - t_0; R^n]$. The next lemma extends this result to arbitrary continuous forcing terms $f^{t_0}(\cdot) \in C[0, t_1 - t_0; R^n]$.

LEMMA 2.2. *Let* (H1) *be satisfied. Then, for every* $f^{t_0}(\cdot) \in C[0, t_1 - t_0; R^n]$, *Eq.* (2.5) *admits a unique solution* $x(\cdot) \in C[t_0, t_1; R^n]$. *This solution satisfies the inequality*

$$|x(t)| \leq \left( \sup_{0 \leq s \leq t - t_0} |f^{t_0}(s)| \right) \exp\left( \int_{t_0}^t m(s) \, ds \right), \qquad t_0 \leq t \leq t_1. \quad (2.7)$$

*Proof.* Let us introduce the space $X = C[0, t_1 - t_0; R^n]$ and let $T \in \mathscr{L}(X)$ be defined by

$$[Tx](s) = \int_0^s \int_{[-\sigma, 0]} [d_\tau \eta(t_0 + \sigma, \tau)] \, x(\sigma + \tau) \, d\sigma$$

for $0 \leq s \leq t_1 - t_0$ and $x(\cdot) \in X$. We have to show that $I - T$ is boundedly invertible. For this sake we introduce on $X$ the equivalent norm

$$\|x(\cdot)\|_\gamma = \sup_{0 \leq s \leq t_1 - t_0} |x(s)| \, e^{-\gamma s},$$

where $\gamma > 0$ is chosen in such a way that the inequality

$$\sup_{t_0 \leq t \leq t_1} \int_{t - \varepsilon}^t m(\sigma) \, d\sigma + e^{-\gamma\varepsilon} \int_{t_0}^{t_1} m(\sigma) \, d\sigma < 1$$

holds for some $\varepsilon > 0$. Then the following inequality holds for every $s \in [0, t_1 - t_0]$:

$$\begin{aligned}
|[Tx](s)|\, e^{-\gamma s} &\leqslant \int_0^s \left| \int_{[-\sigma,0]} [d_\tau \eta(t_0+\sigma,\tau)]\, x(\sigma+\tau) \right| d\sigma\, e^{-\gamma s} \\
&\leqslant \int_0^{s-\varepsilon} m(t_0+\sigma) \sup_{0\leqslant t\leqslant \sigma} |x(t)|\, d\sigma\, e^{-\gamma s} \\
&\quad + \int_{s-\varepsilon}^{s} m(t_0+\sigma) \sup_{0\leqslant t\leqslant \sigma} |x(t)|\, d\sigma\, e^{-\gamma s} \\
&\leqslant \int_{t_0}^{t_0+s-\varepsilon} m(t)\, dt \left( \sup_{0\leqslant t\leqslant s-\varepsilon} |x(t)|\, e^{-\gamma t} \right) e^{-\gamma\varepsilon} \\
&\quad + \int_{t_0+s-\varepsilon}^{t_0+s} m(t)\, dt \sup_{0\leqslant t\leqslant s} |x(t)|\, e^{-\gamma t} \\
&\leqslant \left[ \int_{t_0+s-\varepsilon}^{t_0+s} m(t)\, dt + e^{-\gamma\varepsilon} \int_{t_0}^{t_1} m(t)\, dt \right] \|x(\cdot)\|_\gamma .
\end{aligned}$$

Hence $T$ is a contraction with respect to $\|\cdot\|_\gamma$ and therefore $I-T$ is boundedly invertible.

If $x(\cdot)$ is a solution of (2.5), then the following inequality holds,

$$\sup_{t_0\leqslant \tau\leqslant t} |x(\tau)| \leqslant \sup_{t_0\leqslant \tau\leqslant t} |f^{t_0}(\tau-t_0)| + \int_{t_0}^{t} m(s)\big( \sup_{t_0\leqslant \tau\leqslant s} |x(\tau)|\big)\, ds, \qquad t_0\leqslant t\leqslant t_1,$$

and hence (2.7) follows from the generalization of Gronwall's inequality in Hale [12, p. 15, Lemma 3.1]. ∎

*Remark* 2.3. If the forcing term $f^{t_0}(\cdot)$ of (2.5) is in $W^{1,1}[0, t_1-t_0; R^n]$, then it follows from Remark 2.1 that the solution $x(\cdot)$ is in $W^{1,1}[t_0, t_1; R^n]$ and depends in this space continuously on $f^{t_0}(\cdot)\in W^{1,1}[0, t_1-t_0; R^n]$. In particular, this implies the existence of a unique solution to (2.1), (2.4) for every initial function $\phi\in C[-h, 0; R^n]$.

The rest of this section is devoted to the problem of continuous dependence (of anything under consideration) on the initial time $t_0$. This turns out to be a surprisingly nontrivial problem. First note that the continuous dependence of the solutions of (2.1) and (2.4) on $t_0$ has been shown by Hale [12, p. 41, Theorem 2.2]:

LEMMA 2.4. *Suppose that* (H1) *is satisfied, let* $t_0\in R$, $T>0$, *and* $\phi\in C[-h, 0; R^n]$ *be given, and define* $x(t, t_0, \phi)$, $t_0-h\leqslant t\leqslant t_0+T$, *to be the unique solution of* (2.1). *Then*

$$\lim_{t\to t_0} \big[ \sup_{0\leqslant s\leqslant T} |x(t+s; t, \phi) - x(t_0+s; t_0, \phi)| \big] = 0.$$

The following simple example shows that the continuous dependence on $t_0$ can break down for Eq. (2.5).

EXAMPLE. Consider the RFDE

$$\dot{x}(t) = x(t - h(t)), \tag{2.8}$$

where

$$h(t) = \begin{cases} 0, & t \leqslant 0, \\ t, & 0 \leqslant t \leqslant 1, \\ 1, & t \geqslant 1. \end{cases}$$

In this case assumption (H1) is clearly satisfied. The integrated equation takes the form

$$x(t_0 + s; t_0, f) = f(s) + \int_{t_0}^{t_0+s} x(t - h(t); t_0, f) \cdot \chi_{[t_0, \infty)}(t - h(t))\, dt.$$

The solutions corresponding to $f(s) \equiv x_0$ are given by

$$x(t_0 + s; t_0, f) = \begin{cases} (1+s)\, x_0, & 0 \leqslant s \leqslant 1, \text{ if } t_0 = 0, \\ x_0, & 0 \leqslant s \leqslant 1, \text{ if } 0 < t_0 \leqslant 1 \end{cases}$$

and therefore do not depend continuously on $t_0$.

In order to obtain the desired continuous dependence of the solutions of (2.5) on the initial time $t_0$, we need a further hypothesis. Let $1 \leqslant p < \infty$.

(H2) *For all* $-\infty < t_0 < t_1 < \infty$, *there exists a constant* $K > 0$ *such that*

$$\int_{t_0}^{t_1} |L(t, x_t)|\, dt \leqslant K \left( \int_{t_0 - h}^{t_1} |x(t)|^p\, dt \right)^{1/p} \tag{2.9}$$

*for all* $x \in C[t_0 - h, t_1; R^n]$.

This hypothesis means that the map $x \to (L(t, x_t), t \in R)$ can be continuously extended to a map from $L^p_{\text{loc}}(-\infty, \infty)$ into $L^1_{\text{loc}}(-\infty, \infty)$. It plays a crucial role throughout this paper and will be discussed in detail in Section 3 and 4.

LEMMA 2.5. *Let* (H1) *and* (H2) *be satisfied. Then the following statements hold.*

(i) *For every* $\phi \in C[-h, 0; R^n]$ *the function* $f^{t_0}(\cdot) \in C[0, h; R^n]$ *defined by* (2.6) *depends continuously on* $t_0$.

(ii) *For every* $f(\cdot) \in C[0, T; R^n]$ *the unique solution* $x(t_0 + \cdot; t_0, f)$ *of*

$$x(t_0+s; t_0, f) = f(s) + \int_0^s \int_{[-\sigma, 0]} [d_\tau \eta(t_0+\sigma, \tau)] \times x(t_0+\sigma+\tau; t_0, f)\, d\sigma, \qquad 0 \leqslant s \leqslant T, \tag{2.10}$$

*depends—in* $C[0, T; R^n]$*—continuously on* $t_0$.

*Proof.* (i) Let $f^{t_0}(\cdot) \in C[0, h, R^n]$ be defined by (2.6). Then it follows from the Remark after Lemma 2.1 that $f^{t_0}(\cdot) \in W^{1,1}[0, h; R^n]$ and, since $\mathrm{VAR}_{[-h,0]}\, \eta(t, \cdot) \leqslant m(t)$ we get

$$|f^{t_0}(s)| = \left| \int_{[-h,-s)} [d_\tau \eta(t_0+s, \tau)]\, \phi(s+\tau) \right| \leqslant m(t_0+s)\, \|\phi\|_{C[-h,0;R^n]}.$$

Therefore the functions $f^{t_0}(\cdot)$, $a \leqslant t_0 \leqslant b$, are equicontinuous. Moreover, the following inequality holds for $t \leqslant t'$ and $s \geqslant 0$:

$$\begin{aligned}
&|f^t(s+t'-t) - f^{t'}(s)| \\
&\quad = \left| \int_0^{s+t'-t} \int_{[-h,-q]} [d_\tau \eta(t+\sigma, \tau)]\, \phi(\sigma+\tau)\, d\sigma - \int_0^s \int_{[-h,-\sigma]} [d_\tau \eta(t'+\sigma, \tau)]\, \phi(\sigma, \tau)\, d\sigma \right| \\
&\quad \leqslant \int_0^{t'-t} \left| \int_{[-h,-\sigma]} [d_\tau \eta(t+\sigma, \tau)]\, \phi(\sigma+\tau) \right| d\sigma \\
&\qquad + \int_0^s \left| \int_{[-h,t-t'-\sigma]} [d_\tau \eta(t'+\sigma, \tau)][\phi(\sigma+\tau+t'-t) - \phi(\sigma+\tau)] \right| d\sigma \\
&\qquad + \int_0^s \left| \int_{[t-t'-\sigma,-\sigma]} [d_\tau \eta(t'+\sigma, \tau)]\, \phi(\sigma+\tau) \right| d\sigma \\
&\quad \leqslant \int_t^{t'} m(\tau)\, d\tau \cdot \|\phi\| + \int_{t'}^{t'+s} m(\tau)\, d\tau \sup_{-h \leqslant \tau \leqslant t-t'} |\phi(\tau+t'-t) - \phi(\tau)| \\
&\qquad + K \left( \int_{t-t'}^0 |\phi(\tau)|^p\, d\tau \right)^{1/p}.
\end{aligned}$$

This proves statement (i).

In order to prove statement (ii), suppose that $a \leqslant t_0 \leqslant b$ and note that, by (2.7),

$$\|x(t_0 + \cdot; t_0, f)\|_{C[0,T;R^n]} \leqslant \|f\|_{C[0,T;R^n]} \exp\left(\int_a^{b+T} m(\tau)\, d\tau\right) =: C.$$

Moreover, it follows from the Remark after Lemma 2.1 that $x(t_0 + \cdot; t_0, f) - f(\cdot) \in W^{1,1}[0, T; R^n]$ and

$$\begin{aligned}
&\left|\frac{d}{ds}\,(x(t_0+s; t_0, f) - f(s))\right| \\
&\quad = \left|\int_{[-s,0]} [d_\tau \eta(t+s, \tau)]\, x(t_0 + s + \tau; t_0 f)\right| \\
&\quad \leqslant m(t+s) \cdot \sup_{0 \leqslant \sigma \leqslant s} |x(t_0 + \sigma; t_0, f)| \\
&\quad \leqslant K \cdot m(t+s).
\end{aligned}$$

Therefore, the functions $x(t_0 + \cdot; t_0, f) \in C[0, T; R^n]$, $a \leqslant t_0 \leqslant b$, are equicontinuous. Moreover, the following inequality holds for $a \leqslant t \leqslant t' \leqslant b$ and $0 \leqslant s \leqslant T + t - t'$:

$$\begin{aligned}
&|x(t'+s; t, f) - x(t'+s; t', f)| \\
&\leqslant |f(s+t'-t) - f(s)| \\
&\quad + \left|\int_0^{s+t'-t} \int_{[-\sigma,0]} [d_\tau \eta(t+\sigma, \tau)]\, x(t+\sigma+\tau; t, f)\, d\sigma\right. \\
&\quad \left. - \int_0^s \int_{[-\sigma,0]} [d_\tau \eta(t'+\sigma, \tau)]\, x(t'+\sigma+\tau; t', f)\, d\sigma\right| \\
&\leqslant |f(s+t'-t) - f(s)| + \int_0^{t'-t} \left|\int_{[-\sigma,0]} [d_\tau \eta(t+\sigma, \tau)]\, x(t+\sigma+\tau; t; f)\right| d\sigma \\
&\quad + \int_0^s \left|\int_{[-\sigma,0]} [d_\tau \eta(t'+\sigma, \tau)][x(t'+\sigma+\tau; t, f) - x(t'+\sigma+\tau; t', f)]\right| d\sigma \\
&\quad + \int_0^s \left|\int_{[t-t'-\sigma,-\sigma)} [d_\tau \eta(t'+\sigma, \tau)]\, x(t'+\sigma+\tau; t, f)\right| d\sigma \\
&\leqslant |f(s+t'-t) - f(s)| + C \int_t^{t'} m(\tau)\, d\tau + K\left[\int_t^{t'} |x(\tau; t, f)|^p\, d\tau\right]^{1/p} \\
&\quad + \int_0^s m(t'+\sigma) \cdot \sup_{0 \leqslant \tau \leqslant \sigma} |x(t'+\tau; t, f) - x(t'+\tau; t', f)|\, d\sigma.
\end{aligned}$$

Applying again the generalization of Gronwall's inequality in Hale [12, p. 15, Lemma 3.1], we obtain

$$|x(t'+s;t,f)-x(t'+s;t',f)|$$
$$\leqslant \left[\sup_{0\leqslant\sigma\leqslant s}|f(\sigma+t'-t)-f(\sigma)| + C\int_t^{t'} m(\tau)\,d\tau + K\cdot C\cdot(t'-t)^{1/p}\right]\exp\left(\int_{t'}^{t'+s} m(\tau)\,d\tau\right)$$

for $0\leqslant s\leqslant T+t-t'$. This proves statement (ii). ∎

2.2. *State Concepts and Structural Operator*

The most common way of introducing the state of a retarded functional-differential equation is to specify an initial function of suitable length which describes the past history of the solution. The corresponding state of system (2.1) at time $t$ is the solution segment $x_t\in C[-h,0;R^n]$. The evolution of this state determines a family $\Phi(t,s)$, $t\geqslant s$, of bounded linear operators on $C[-h,0;R^n]$ defined by

$$\Phi(t,t_0)\phi = x_t\in C[-h,0;R^n], \tag{2.11}$$

where $x(t)$, $t\geqslant t_0-h$, is the unique solution of (2.1) and (2.4). It is a direct consequence of Lemmas 2.2 and 2.3 that $\Phi(t,s)$ is a well-defined, strongly continuous evolution operator. More precisely, $\Phi(t,s)$ has the following properties (see also Hale [12]).

PROPOSITION 2.6. *Let* (H1) *be satisfied.*

(i) *$\Phi(t,s)$ is a bounded linear operator on $C[-h,0;R^n]$ for all $t,s\in R$ with $t\geqslant s$.*

(ii) *$\Phi(t,t)=I$ for all $t\in R$.*

(iii) *$\Phi(t,s)\,\Phi(s,\tau)=\Phi(t,\tau)$ for $t\geqslant s\geqslant\tau$.*

(iv) *For every compact interval $[t_0,t_1]\subset R$ there exists a constant $M\geqslant 1$ such that $\|\Phi(t,s)\|\leqslant M$ for $t_0\leqslant s\leqslant t\leqslant t_1$.*

(v) *$\Phi(t,s)\phi$ is a continuous function on the domain $\{(t,s)\in R^2 \mid t\geqslant s\}$ for every $\phi\in C[-h,0;R^n]$.*

An alternative state concept has been introduced by Miller [19] for the description of Volterra integro-differential equations. The basic idea is to define the state of the system to be an additional forcing term of suitable length which determines the future behavior of the solution. This state concept has been introduced independently by Bernier and Manitius [1], Manitius [17], and Delfour and Manitius [7] for time-invariant RFDEs in the product space framework on the basis of a so-called structural operator $F$. Later on, the forcing function state concept was used by Diekmann [9, 10] to describe Volterra integral equations and time invariant RFDEs in the state space of continuous functions. (See also the work by Staffans, e.g., [22, 23].) An extension to neutral system has been developed in Salamon [20] and Delfour and Karrakchou [6]. For time-varying RFDEs the only results in this direction can be found in Delfour [5] for a special class of equations with constant delays.

In this paper we introduce the forcing function state concept for the *integrated* equation (2.5). More precisely, we define $f^{t_0}(\cdot) \in C[0, h; R^n]$ to be the initial state of (2.5). The restriction to the interval $[0, h]$ is justified from the fact that $f^{t_0}(s) = f^{t_0}(h)$ for $s \geq h$ if $f^{t_0}(\cdot)$ is given by (2.6), i.e., results from the initial function $x_{t_0} = \phi \in C[-h, 0; R^n]$ of (2.1). However, we will allow for arbitrary continuous forcing terms $f^{t_0}(\cdot) \in C[0, h; R^n]$ and extend the function to $[0, \infty)$ by defining $f^{t_0}(s) = f^{t_0}(h)$ for $s \geq h$.

The corresponding state at time $t \geq t_0$ can be obtained by applying a time shift to Eq. (2.5). The shifted equation takes the form

$$x(t+s) = f^t(s) + \int_0^s \int_{[-\sigma, 0]} [d_\tau \eta(t+\sigma, \tau)]\, x(t+\sigma+\tau)\, d\sigma, \qquad s \geq 0, \tag{2.12}$$

where $f^t(\cdot) \in C[0, h; R^n]$ is given by

$$f^t(s) = f^{t_0}(t+s-t_0) + \int_0^{t+s-t_0} \int_{[-\sigma, t-\sigma-t_0]} [d_\tau \eta(t_0+s, \tau)]\, x(t_0+\sigma+\tau)\, d\sigma \tag{2.13}$$

for $0 \leq s \leq h$ and again $f^t(s) = f^t(h)$ for $s > h$. Note that the shifted forcing term $f^t(\cdot)$ contains all the information from the past history of the solution at time $t$ which is needed to determine the future behavior of the solution $x(t+s)$, $s \geq 0$. This forcing function $f^t(\cdot) \in C[0, h; R^n]$ is considered to be the state of system (2.5) at time $t \geq t_0$. The evolution of this state determines the family $\Psi(t, s)$, $t \geq s$, of bounded linear operators on $C[0, h; R^n]$ defined by

$$\Psi(t, t_0) f^{t_0} = f^t \in C[0, h; R^n], \tag{2.14}$$

where $x(t)$, $t \geq t_0$, is the unique solution of (2.5) and $f^t(\cdot)$ is defined by (2.13).

PROPOSITION 2.7. *Let* (H1) *be satisfied. Then*

(i) *$\Psi(t, s)$ is a bounded, linear operator on $C[0, h; R^n]$ for all $t, s \in R$ with $t \geq s$,*

(ii) *$\Psi(t, t) = I$ for all $t \in R$,*

(iii) *$\Psi(t_1, t)\, \Psi(t, t_0) = \Psi(t_1, t_0)$ for $t_1 \geqslant t \geqslant t_0$, and*

(iv) *For every compact interval $[t_0, t_1]$ there exists a constant $M \geqslant 1$ such that $\|\Psi(t, s)\| \leqslant M$ for $t_0 \leqslant s \leqslant t \leqslant t_1$.*

(v) *If* (H2) *is satisfied, then $\Psi(t, s)$ is a strongly continuous operator on the domain $\{(t, s) \in R^2 \mid t \geqslant s\}$.*

*Proof.* The statements (i), (iv) follow from the definitions and Lemma 2.2. Statement (v) is an easy consequence of Lemma 2.5.

The relation between the two state concepts can be described by two structural operators $F(t)$: $C[-h, 0; R^n] \to C(0, h; R^n]$ and $G(t)$: $C[0, h; R^n] \to C[-h, 0; R^n]$. The operator $F(t_0)$ maps the initial function $\phi \in C[-h, 0; R^n]$ of (2.1), (2.4) into the corresponding forcing term $f^{t_0}(\cdot) \in C[0, h; R^n]$ of (2.5) which is given by (2.6) and the operator $G(t_0)$ maps this forcing term $f^{t_0}(\cdot)$ into the corresponding solution segment $x_{t_0+h} \in C[-h, 0; R^n]$ of (2.5) at time $t_0 + h$. These two operators can be described explicitly by the formulas

$$[F(t)\phi](s) = \phi(0) + \int_0^s \int_{[-h, -\sigma]} [d_\tau \eta(t + \sigma, \tau)]\, \phi(\sigma + \tau)\, d\sigma, \tag{2.15}$$

$$[G(t)^{-1}\phi](s) = \phi(s - h) - \int_0^s \int_{[-\sigma, 0]} [d_\tau \eta(t + \sigma, \tau)]\, \phi(\sigma + \tau - h)\, d\sigma \tag{2.16}$$

for $0 \leqslant s \leqslant h$ and $\phi \in C[-h, 0; R^n]$. By Lemma 2.2 the operator $G(t)^{-1}$ is boundedly invertible and its inverse is the desired operator $G(t)$.

PROPOSITION 2.8. *Let* (H1) *be satisfied. Then the following statements hold.*

(i) *The operator $G(t)$: $C[0, h; R^n] \to C[-h, 0; R^n]$ is bijective.*

(ii) *If* (H2) *is satisfied, then the operators $F(t)$ and $G(t)$ are strongly continuous.*

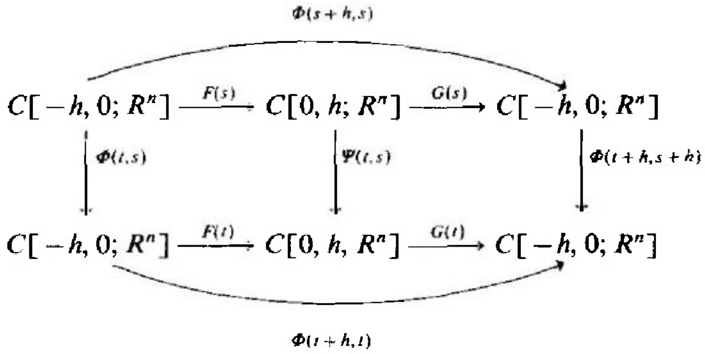
(iii) *$\Phi(t + h, t) = G(t)\, F(t)$, $\Psi(t + h, t) = F(t + h)\, G(t)$.*

(iv) *$F(t)\, \Phi(t, s) = \Psi(t, s)\, F(s)$, $\Phi(t + h, s + h)\, G(s) = G(t)\, \Psi(t, s)$, $t \geqslant s$.*

*Proof.* Statement (i) has been shown above and statement (ii) follows from Lemma 2.5. The first equation in (iii) is an immediate consequence of the definition of the operators $F(t)$ and $G(t)$. The other equation follows by straightforward computation, and (iv) is an easy consequence. The equation $F(t)\, \Phi(t, t_0) = \Psi(t, t_0)\, F(t_0)$ can also be interpreted in the following way. If $f^{t_0}(\cdot)$ is given by (2.6), if $x(t)$, $t \geqslant t_0 - h$, is the unique solution of

(2.1), (2.4), and if $f'(\cdot)$, $t \geqslant t_0$, is defined by (2.13), then $f' = F(t)\, x_t$. This fact can be established directly.

The relations of Proposition 2.8 can be summarized in Scheme 1.




SCHEME 1

Finally, we remark that the operator $F(t)$ is closely related to the "hereditary product" which has been used in Hale [12, p. 151] for a special class of time-varying RFDEs with constant delays. The operator $G(t)$ is related to the operator $\Omega$ in Henry [13] and Hale [12, Chap. 6]. A more detailed discussion of these relationships will be given in Sections 2.3 and 4.

## 2.3. *Duality*

In this section we give an interpretation of the adjoint operators $F^*(t)$, $G^*(t)$, $\Phi^*(t, s)$, $\Psi^*(t, s)$ in terms of a certain transposed system which takes the form of a Volterra integral equation of the second kind. In order to give a concrete representation of the adjoint operators, we identify the dual space of $C[-h, 0; R^n]$ with the space $\text{NBV}[-h, 0; R^n]$ of normalized functions of bounded variation by means of the duality pairing

$$\langle g, \phi \rangle = \int_{-h}^{0} [d_\tau\, g^T(\tau)]\, \phi(\tau),$$

$$g \in \text{NBV}[-h, 0; R^n], \qquad \phi \in C[-h, 0; R^n].$$

The normalization is $g(0) = 0$ and left continuity on the open interval $(-h, 0)$ for $g \in \text{NBV}[-h, 0; R^n]$. Analogously the dual space of $C[0, h; R^n]$ will be identified with the space $\text{NBV}[0, h; R^n]$ via the pairing $\langle \psi, f \rangle = \int_0^h [d_s \psi^T(s)]\, f(s)$, $\psi \in \text{NBV}[0, h; R^n]$. The normalization here is $\psi(h) = 0$ and left continuity on the open interval $(0, h)$.

Let us first establish formulae for the operators $F^*(t)\colon \text{NBV}[0, h\, R^n] \to \text{NBV}[-h, 0; R^n]$ and $G^*(t)\colon \text{NBV}[-h, 0; R^n] \to \text{NBV}[0, h; R^n]$.

LEMMA 2.9. *Let $\psi \in \text{NBV}[0, h; R^n]$ be given. Then the following equations hold for $-h \leqslant \tau < 0$:*

$$[F^*(t)](\tau) = \psi(0) - \int_0^h [\eta^T(t+s, \tau - s) - \eta^T(t+s, -s)]\, \psi(s)\, ds, \quad (2.17)$$

$$[G^*(t)^{-1}\psi](\tau) = \psi(\tau + h) + \int_\tau^0 \eta^T(t+h+\sigma, \tau - \sigma)\, \psi(\sigma + h)\, d\sigma. \quad (2.18)$$

*Proof.* Follows using the unsymmetric Fubini theorem. ▮

The operators $F^*(t)$ and $G^*(t)$ are related to the transposed equation

$$z(t) - z(t_1) = -\int_t^{t_1+h} [\eta^T(\alpha, t-\alpha) - \eta^T(\alpha, t_1 - \alpha)]\, z(\alpha)\, d\alpha, \qquad t \leqslant t_1. \quad (2.19)$$

This equation is sometimes called the "formal adjoint equation" and has been used in the theory of functional-differential equations for a long time. Equation (2.19) admits a unique solution $z(t)$ in the space $\mathrm{NBV}[T, t_1+h; R^n]$, $T < t$ for every final condition of the form

$$z(t_1 + s) = \psi(s), \qquad 0 \leqslant s \leqslant h, \quad (2.20)$$

where $\psi \in \mathrm{NBV}[0, h; R^n]$ (see, e.g., Hale [12, p. 148, Theorem 3.1]). This motivates the definition of the state of system (2.19) at time $t \leqslant t_1$ to be the solution segment $z^t \in \mathrm{NBV}[0, h; R^n]$ given by

$$z^t(s) = \begin{cases} z(t+s), & 0 \leqslant s < h, \\ 0, & s = h. \end{cases} \quad (2.21)$$

Equations (2.19), (2.20) can be rewritten in the form

$$z(t_1 + \tau) = g^{t_1}(\tau) - \int_\tau^0 \eta^T(t_1 + \sigma, \tau - \sigma)\, z(t_1 + \sigma)\, d\sigma, \qquad \tau < 0, \quad (2.22)$$

where $g^{t_1}(\cdot) \in \mathrm{NBV}[-h, 0; R^n]$ is given by

$$\begin{aligned} g^{t_1}(\tau) &= \psi(0) - \int_0^h [\eta^T(t_1 + s, \tau - s) - \eta^T(t_1 + s, -s)]\, \psi(s)\, ds \\ &= [F^*(t_1)\psi](\tau), \qquad -h \leqslant \tau < 0. \end{aligned} \quad (2.23)$$

This shows that the dual of the forcing term operator is the forcing term operator for the adjoint equation.

A comparison of formulae (2.18) and (2.22) shows that a function $z(\cdot) \in \mathrm{NBV}[t_1 - h, t_1; R^n]$ satisfies (2.22) if and only if

$$z^{t_1 - h} = G^*(t_1 - h)\, g^{t_1}, \quad (2.24)$$

where $z^{t_1-h} \in \mathrm{NBV}[0, h; R^n]$ is given by (2.21). Since $G^*(t_1-h)$ is bijective (Proposition 2.8), this shows that Eq. (2.22) admits a unique solution for every $g^{t_1} \in \mathrm{NBV}[-h, 0; R^n]$. As in Section 2.2, we may now define the forcing term $g^{t_1}$ to be the final state of Eq. (2.22). The corresponding state at time $t \leqslant t_1$ can be obtained by means of a time shift. The shifted equation takes the form

$$z(t+\tau) = g^t(\tau) - \int_\tau^0 \eta^T(t+\sigma, \tau-\sigma)\, z(t+\sigma)\, d\sigma, \qquad \tau < 0, \tag{2.25}$$

where $g^t(\cdot) \in \mathrm{NBV}[-h, 0; R^n]$ is given by

$$\begin{aligned} g^t(\tau) &= g^{t_1}(t-t_1+\tau) \\ &\quad - \int_\tau^0 \eta^T(\alpha, t+\tau-\alpha)\, z(\alpha)\, d\alpha, \qquad -h \leqslant \tau < 0. \end{aligned} \tag{2.26}$$

The forcing term $g^t$ of the shifted equation is now regarded as the state of system (2.22) at time $t \leqslant t_1$.

THEOREM 2.10. (i) *Let* $\psi \in \mathrm{NBV}[0, h; R^n]$ *be given, let* $z(t)$, $t \leqslant t_1 + h$, *be the corresponding solution of* (2.19), (2.20), *and let* $z^t \in \mathrm{NBV}[0, h; R^n]$ *be defined by* (2.21). *Then*

$$z^t = \Psi^*(t_1, t)\psi, \qquad t \leqslant t_1. \tag{2.27}$$

(ii) *Let* $g(\cdot) = g^{t_1} \in \mathrm{NBV}[-h, 0; R^n]$ *be given, let* $z(t)$, $t \leqslant t_1$, *be the corresponding solution of* (2.22), *and let* $g^t$ $\mathrm{NBV}[-h, 0; R^n]$ *be defined by* (2.26). *Then*

$$g^t = \Phi^*(t_1, t)\, g^{t_1}, \qquad t \leqslant t_1. \tag{2.28}$$

*Proof.* Straightforward, but tedious computation. ∎

*Remark* 2.11. (i) The equation

$$F^*(t)\, \Psi^*(t_1, t) = \Phi^*(t_1, t)\, F^*(t_1) \tag{2.29}$$

can now be interpreted in the following way. If $g^{t_1}(\cdot) \in \mathrm{NBV}[-h, 0; R^n]$ is given by (2.23), if $z(t)$, $t \leqslant t_1 + h$, is the unique solution of (2.19) and (2.20), and if $g^t(\cdot) \in \mathrm{NBV}[-h, 0; R^n]$ is given by (2.26), then $g^t = F^*(t) z^t$.

(ii) *Let us introduce the shift operator* $J$: $C[-h, 0; R^n] \to C[0, h; R^n]$

by $[J\phi](s)=\phi(s-h)$, $0\leqslant s\leqslant h$. Then $J^*$: $\mathrm{NBV}[0,h;R^n]\to\mathrm{NBV}[-h,0;R^n]$ is given by $[J^*\psi](\tau)=\psi(\tau+h)$, $-h\leqslant\tau\leqslant 0$. Hence

$$[G^*(t-h)^{-1}\psi](\tau)=\psi(\tau+h)+\int_\tau^0 \eta^T(t+\sigma,\tau-\sigma)\,\psi(\sigma+h)\,d\sigma$$

$$=[(I+\Omega(t))\,J^*\psi](\tau),\qquad -h\leqslant\tau\leqslant 0,$$

where $\Omega(t)\in\mathscr{L}(\mathrm{NBV}[-h,0;R^n])$ is the operator introduced by Henry [13]. Hence

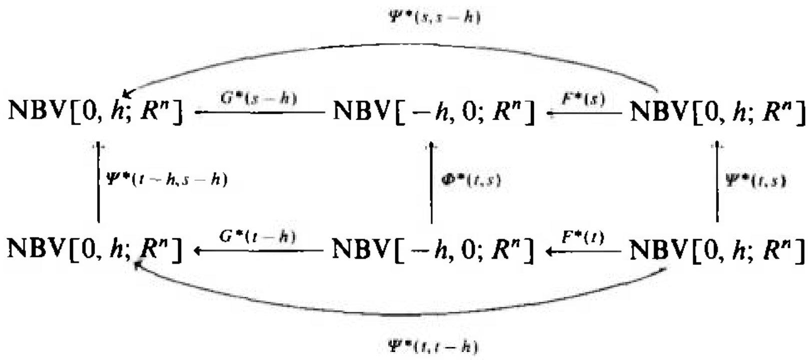
$$I+\Omega(t)=\{(G(t-h)J)^*\}^{-1}. \tag{2.30}$$

This shows that the similarity relation in Henry [13] and Hale [12, p. 152, Thm. 4.1] is nothing more than the intertwining relation

$$G^*(t)\,\Phi^*(t_1+h,t+h)=\Psi^*(t_1,t)\,G^*(t_1), \tag{2.31}$$

which has a natural interpretation in the framework developed above.

We close this section with a commuting diagram summarizing Eqs. (2.29) and (2.31) (see Scheme 2).




$s\leqslant t$

SCHEME 2

## 3. STRUCTURE THEORY IN THE PRODUCT SPACE

The structure and duality theory in Section 2 follows the traditional mode to take the space $C$ of continuous functions as a state space. Though the "adjoint equation" is derived in a natural way, the basic disadvantages of state space theory in $C$ remain untouched. The space $C$ is not reflexive and the adjoint equation is, in general, not a differential-delay equation. Furthermore, the variation-of-constants formula for the inhomogeneous equation can only be understood in a generalized sense. This causes some difficulty, e.g., in control problems where the input operator does not have values in the state space $C$, but in a larger space.

For autonomous systems there is now a well-established way to overcome these problems. The state space $C$ is embedded in the space $M^p$ and the operators describing the evolution of the initial function state $x_t$ and the forcing functions state $f^t$ are shown to generate strongly continuous semigroups. Then the adjoint equation is again a retarded equation of the same type.

In this chapter, we study a general class of time-varying retarded systems, for which both state concepts can be extended to $M^p$ spaces and develop the corresponding structure and duality theory. In the special case of time-invariant systems this theory is equivalent to the known structure theory just mentioned.

In particular, the extendability of $\Phi$ and $\Psi$ to $M^p$-spaces turns out to be equivalent to restrictability of $\Phi^*$ and $\Psi^*$ to spaces of absolutely continuous functions. Thus, differentiability of the adjoint equation is related to the $M^p$-extendability property. In fact, if $M^p$-extendability holds, the adjoint equation is a functional-differential equation, where the right-hand side is given by a linear continuous map $L^*(t)$ on $W^{1,p}[0, h; R^n]$ (instead of $C$, as in (2.1)).

### 3.1. *Extendability to the Product Space*

In this section we consider the functional-differential equation (2.1) in the product space $M^p[-h, 0; R^n]$, that is, we want to allow for initial conditions of the form

$$x(t_0) = \phi^0, \qquad x(t_0 + \tau) = \phi^1(\tau), \qquad -h \leqslant \tau \leqslant 0, \tag{3.1}$$

where $\phi = (\phi^0, \phi^1) \in M^p[-h, 0; R^n]$. For this purpose we need both hypotheses (H1) and (H2) to be satisfied in order to give a meaning to the right-hand side of Eq. (2.1) in the case of discontinuous initial data. We consider $\phi \in M^p[-h, 0; R^n]$ to be the initial state of (2.1) and define the state at time $t \geqslant 0$ to be the pair

$$\hat{x}(t) = (x(t), x_t) \in M^p[-h, 0; R^n]. \tag{3.2}$$

The time evolution of this state of the RFDE (2.1) can be described by an extended evolution operator $\Phi_M(t, t_0)$ on the state space $M^p[-h, 0; R^n]$ as we will see below. Correspondingly we have the natural injection $\iota$ of $C[-h, 0; R^n]$ into $M^p[-h, 0; R^n]$ which maps $\phi$ into $\iota\phi = (\phi(0), \phi)$.

In order to extend the forcing function state concept to the product space we consider the integrated equation (2.5) with the forcing term $f^{t_0}(\cdot) \in L^p_{\mathrm{loc}}[0, \infty; R^n]$ given by

$$f^{t_0}(s) = \phi^0 + \int_0^s \int_{[-h, -\sigma]} [d_\tau \eta(t_0 + \sigma, \tau)]\, \phi^1(\sigma + \tau)\, d\sigma, \qquad s \geqslant 0. \tag{3.3}$$

Here we have extended the function $\phi^1$ to all of $R$ by defining $\phi^1(\tau)=0$ for $\tau \notin [-h, 0]$. Note that the function $f^{t_0}(s)$ defined by (3.3) is absolutely continuous on $[0, h]$ and constant for $s \geqslant h$. We will consider the integrated equation (2.5) with more general forcing terms in $L^p_{\text{loc}}[0, \infty, R^n]$ which are constant for $s \geqslant h$. These can be identified with pairs $f=(f^0, f^1)$ $M^p[0, h; R^n]$ via

$$f^{t_0}(s)=\begin{cases} f^1(s), & 0 \leqslant s < h, \\ f^0, & s \geqslant h. \end{cases} \tag{3.4}$$

We consider the pair $f \in M^p[0, h; R^n]$ to be the initial state of Eqs. (2.5), (3.4). Motivated by the development in Section 2.2 we define the state at time $t \geqslant t_0$ to be the pair

$$\hat{f}(t)=(f^t(h), f^t) \in M^p[0, h; R^n],$$

where $f^t(\cdot) \in L^p_{\text{loc}}[0, \infty; R^n]$ is the forcing term of the shifted equation (2.12) given by

$$f^t(s)=\begin{cases} f^1(t+s-t_0)+\displaystyle\int_0^{t+s-t_0} \int_{[-\sigma, t-\sigma-t_0)} [d_\tau \eta(t_0+\sigma, \tau)] \\ \qquad \times x(t_0+\sigma+\tau)\, d\sigma, \qquad 0 \leqslant s < h, \\ f^0+\displaystyle\int_0^{t+h-t_0} \int_{[-\sigma, t-\sigma-t_0)} [d_\tau \eta(t_0+\sigma, \tau)] \\ \qquad \times x(t_0+\sigma+\tau)\, d\sigma, \qquad s \geqslant h. \end{cases} \tag{3.5}$$

Note that this expression is obtained by inserting (3.4) into (2.13). We will see below that the evolution of the forcing function state $\hat{f}(t)$ of (2.5) can be decribed by an extended evolution operator $\Psi_M(t, t_0)$ on $M^p[0, h; R^n]$. Furthermore, the relation between the initial function $\hat{x}(t)$ and the forcing function state $\hat{f}(t)$ leads naturally to extended structural operators $F_M(t)$ and $G_M(t)$. More precisely, we have the relations

$$\hat{x}(t)=\Phi_M(t, s)\, \hat{x}(s),$$

$$\hat{f}(t)=\Psi_M(t, s)\, \hat{f}(s),$$

$$\hat{f}(t)=F_M(t)\, \hat{x}(t), \qquad \hat{x}(t+h)=G_M(t)\, \hat{f}(t)$$

(cf. Section 2). Of course, we have to make sure that all the expressions in the above equations are well defined and that there exist unique solutions of (2.1), (3.1) or, respectively, (2.5). More precisely we have the following two lemmas. Their proofs are straightforward and omitted here.

LEMMA 3.1. *Suppose that the hypotheses* (H1) *and* (H2) *are satisfied. Then the following statements hold.*

(i) *For every* $\phi \in M^p[-h, 0; R^n]$ *there exists a unique solution* $x(\cdot) \in L^p[t_0 - h, t_1; R^n]$ *of* (2.1), (3.1) *which is absolutely continuous on* $[t_0, t_1]$ *and depends continuously on* $\phi$.

(ii) *For* $t_0 \leqslant s \leqslant t$ *the operators* $F(t)$ *and* $\Phi(t, s)$ *given by* (2.15) *and* (2.11), *respectively, admit unique continuous extensions*

$$F_M(t)\colon M^p[-h, 0; R^n] \to M^p[0, h; R^n]$$

*and*

$$\Phi_M(t, s)\colon M^p[-h, 0; R^n] \to M^p[-h, 0; R^n]$$

*satisfying*

$$\begin{aligned} \iota F(t) &= F_M(t)\iota, \\ \iota\Phi(t, s) &= \Phi_M(t, s)\iota. \end{aligned} \tag{3.6}$$

(iii) *The extended operators are uniformly bounded in the region* $t_0 \leqslant s \leqslant t \leqslant t_1 - h$.

To extend the operators $G(t)$ and $\psi(t, s)$ to the space $M^p[0, h; R^n]$, we consider Eq. (2.5) with arbitrary forcing terms in $L^p_{\mathrm{loc}}[0, \infty; R^n]$.

LEMMA 3.2. *Suppose that the hypotheses* (H1) *and* (H2) *are satisfied. Then the following statements hold*:

(i) *For every* $f^{t_0} \in L^p_{\mathrm{loc}}[0, \infty; R^n]$ *there exists a unique solution* $x(\cdot) \in L^p_{\mathrm{loc}}[t_0, \infty; R^n]$ *of* (2.5) *depending continuously on* $f^{t_0}$.

(ii) *For* $t_0 \leqslant s \leqslant t \leqslant t_1 - h$ *the operators* $G(t)$ *and* $\psi(t, s)$ *given by* (2.14) *and* (2.16), *respectively, admit unique continuous extensions* $G_M(t)$: $M^p[0, h; R^n \to M^p[-h, 0; R^n]$ *and* $\Psi_M(t, s)\colon M^p[0, h; R^n] \to M^p[0, h; R^n]$ *satisfying*

$$\begin{aligned} \iota G(t) &= G_M(t)\iota, \\ \iota\Psi(t, s) &= \Psi_M(t, s)\iota. \end{aligned} \tag{3.7}$$

(iii) *The extended operators are uniformly bounded in the region* $t_0 \leqslant s \leqslant t \leqslant t_1 - h$.

PROPOSITION 3.3. *Suppose that* (H1) *and* (H2) *are satisfied. Then the extended operators* $F_M(t)$, $G_M(t)$, $\Phi_M(t, s)$, $\psi_M(t, s)$ *satisfy properties analogous to those stated in Propositions* 2.6–2.8.

*Proof.* All the statements follow from the fact that $\iota C$ is dense in $M^p$ and all the operators satisfy uniform bounds (Lemmas 3.1 and 3.2). ∎

3.2. *Duality and Differentiability of the Adjoint Equation*

The aim of this section is to show that under hypotheses (H1) and (H2) the integral adjoint equation (2.19) can be transformed into a differential adjoint equation of the form

$$\dot{z}(t) = -L^*(t, z^t), \tag{3.8}$$

where $z(\cdot) \in W^{1,q}[t_0, t_1 + h; R^n]$, $z^t(s) = z(t+s)$ for $s \in [0, h]$, and $L^*(t, \cdot)$ is a bounded linear operator from $W^{1,q}[0, h; R^n]$ into $R^n$, for almost every $t$. This naturally leads us to consider a restriction of the state space $\text{NBV}[0, h; R^n]$ of Eq. (2.19) to the space $W^{1,q}[0, h; R^n]$. More precisely, we consider the injection $\iota^*$: $W^{1,q}[0, h; R^n] \to \text{NBV}[0, h; R^n]$ given by

$$(\iota^*\psi)(s) = \begin{cases} \psi(s), & 0 \leqslant s < h, \\ 0, & s = h. \end{cases} \tag{3.9}$$

An analogous injection can be defined for functions defined on $[-h, 0]$.

We are given the natural duality pairing between the spaces $C[0, h; R^n]$ and $\text{NBV}[0, h; R^n]$ as in Section 2.3 and the injections $\iota$ and $\iota^*$. Requiring that $\iota^*$ be a dual operator of $\iota$ in the functional analytic sense forces us to identify the dual space of $M^p[0, h; R^n]$ with $W^{1,q}[0, h; R^n]$ via the duality pairing

$$\langle \psi, f \rangle_{W^{1,q}, M^p} = -\psi^T(h) f^0 + \int_0^h \dot{\psi}^T(s) f^1(s)\, ds, \qquad p^{-1} + q^{-1} = 1. \tag{3.10}$$

Similarly, we identify the dual space of $M^p[-h, 0; R^n]$ with $W^{1,q}[-h, 0; R^n]$ via the duality pairing

$$\langle g, \phi \rangle_{W^{1,q}, M^p} = -g^T(0)\, \phi^0 + \int_{-h}^0 \dot{g}^T(\tau)\, \phi^1(\tau)\, d\tau. \tag{3.11}$$

These identifications have a tremendous advantage, namely the results of the previous section on extendability to the product spaces can be directly translated via duality into results on restrictability of the adjoint equation to the Sobolev space $W^{1,q}$. Each of the operators $F(t)$, $G(t)$, $\Phi(t, s)$ $\Psi(t, s)$ has a continuous extension to the corresponding product spaces iff their dual operators $F^*(t)$, $G^*(t)$, $\Phi^*(t, s)$, $\Psi^*(t, s)$ restrict to bounded linear operators on the corresponding $W^{1,q}$ spaces. In particular, under Hypotheses (H1) and (H2) we have the existence of $\Phi_M^*(t, s) \in \mathscr{L}(W^{1,q}[-h, 0; R^n])$, $\Psi^*(t, s) \in$

$\mathscr{L}(W^{1,q}[0,h], R^n)$, $F_M^*(t) \in \mathscr{L}(W^{1,q}[0,h;R^n], W^{1,q}[-h,0;R^n])$, $G_M^*(t) \in \mathscr{L}(W^{1,q}[-h,0;R^n], W^{1,q}[0,h;R^n])$ satisfying

$$\iota\Phi_M^*(t,s) = \Phi^*(t,s)\,\iota^*,$$

$$\iota^*\Psi_M(t,s) = \Psi^*(t,s)\,\iota^*,$$

$$\iota^*F_M^*(t) = F^*(t)\,\iota^*,$$

$$\iota^*G_M^*(t) = G^*(t)\,\iota^*.$$

This means that the adjoint equation (2.19) can in fact be studied in the state space $W^{1,q}[0,h;R^n]$.

COROLLARY 3.4. Suppose that the Hypotheses (H1) and (H2) are satisfied with $1 \leqslant p < \infty$; Let $1/p + 1/q = 1$, and let $t_0 \leqslant t$ be given. Then for every $\psi \in W^{1,q}[t_0, t_1+h; R^n]$ the unique solution $z(\cdot)$ of (2.19), (2.20) lies in $W^{1,q}[t_0, t_1+h; R^n]$ and depends in this space continuously on $\psi$.

In order to rewrite the integral equation (2.19) into a differential equation of the form (3.8) let us first assume that this transformation is possible and that $L^*(t,\cdot)$ is a bounded linear map from $C[0,h;R^n]$ into $R^n$ for almost every $t \in [t_0, t_1]$. Then there exists a function $\eta^*(t,\cdot) \in$ NBV$[0,h;R^n]$ such that

$$L^*(t,\psi) = \int_0^h [d_\tau \eta^*(t,\tau)]\,\psi(\tau) \tag{3.12}$$

for $\psi \in C[0,h;R^n]$. We assume that $\eta^*(\cdot,\cdot)$ is a bounded measurable function, that $\eta^*(t,\tau) = 0$ for $\tau \leqslant 0$, $\eta^*(t,\tau) = \eta^*(t,h)$ for $\tau \geqslant h$, and that $\eta^*(t,\cdot)$ is right continuous on the interval $(0,h)$. If $\psi \in W^{1,q}[0,h;R^n]$ then we can rewrite (3.12) in the form

$$L^*(t,\psi) = \eta^*(t,h)\,\psi(0) + \int_0^h [\eta^*(t,h) - \eta^*(t,\tau)]\,\dot\psi(\tau)\,d\tau. \tag{3.13}$$

Every functional on $W^{1,q}[0,h;R^n]$ can be represented in this form but the corresponding function $\eta^*(t,\cdot)$ need only be in $L^p[0,h;R^{n\times n}]$. Inserting (3.13) into (3.8) and integrating the latter equation we get

$$z(t_1+\tau) - z(t_1) = \int_\tau^0 [\eta^*(t_1+\sigma,h)\,z(t_1+\sigma)$$
$$+ \int_0^h [\eta^*(t_1+\sigma,h) - \eta^*(t_1+\sigma,\theta)]\,\dot z(t_1+\sigma+\theta)\,d\theta]\,d\sigma. \tag{3.14}$$

Our aim is to show that under Hypotheses (H1) and (H2) there exists a function $\eta^*(t, \tau)$ such that (3.14) is equivalent to (2.19). A sufficient condition for this equivalence would be that

$$\int_s^{t_1+h} [\eta^T(\alpha, t_1-\alpha) - \eta^T(\alpha, s-\alpha)]\, z(\alpha)\, d\alpha$$

$$= \int_s^{t_1} [\eta^*(\alpha, h)\, z(\alpha) + \int_0^h [\eta^*(\alpha, h) - \eta^*(\alpha, \theta)]\, \dot{z}(\alpha+\theta)\, d\theta]\, d\alpha \qquad (3.15)$$

holds for all $z(\cdot) \in W^{1,q}[t_0, t_1+h; R^n]$. The next lemma characterizes the identity (3.15).

LEMMA 3.5. *Let $\eta^*(\cdot, \cdot) \in L^p_{\text{loc}}[[t_0, \infty) \times R, R^{n\times n}]$ be given such that $\eta^*(t, \tau) = 0$ for $\tau \leqslant 0$ and $\eta^*(t, \tau) = \eta^*(t, h)$ for $\tau \geqslant h$. Then* (3.15) *holds for all $t_1 \geqslant s \geqslant t_0$ and all $z(\cdot) \in W^{1,q}[t_0, t_1+h; R^n]$ if and only if*

$$\int_s^t \eta^*(\alpha, t-\alpha)\, d\alpha + \int_s^t \eta^T(\alpha, s-\alpha)\, d\alpha = 0 \qquad (3.16)$$

*for $t_0 \leqslant s \leqslant t$.*

*Proof.* Let us fix $t_1 \geqslant t_0$ and redefine $\eta^*(t, \cdot) \equiv 0$ for $t > t_1$. Then (3.15) is equivalent to

$$\int_s^{t_1+h} \eta^*(\alpha, h)\, d\alpha\, z(t_1+h) - \int_s^{t_1+h} \int_s^\theta \eta^*(\alpha, \theta-\alpha)\, d\alpha\, \dot{z}(\theta)\, d\theta$$

$$= \int_s^{t_1+h} \left[\eta^*(\alpha, h)\, z(\alpha) + \int_\alpha^{t_1+h} [\eta^*(\alpha, h) - \eta^*(\alpha, \theta-\alpha)]\, \dot{z}(\theta)\, d\theta\right] d\alpha$$

$$= \int_s^{t_1} \left[\eta^*(\alpha, h)\, z(\alpha) + \int_0^h [\eta^*(\alpha, h) - \eta^*(\alpha, \theta)]\, \dot{z}(\alpha+\theta)\, d\theta\right] d\alpha$$

$$= \int_s^{t_1+h} [\eta^T(\alpha, t_1-\alpha) - \eta^T(\alpha, s-\alpha)]\, z(\alpha)\, d\alpha$$

$$= \int_s^{t_1+h} [\eta^T(\alpha, t_1-\alpha) - \eta^T(\alpha, s-\alpha)]\, d\alpha\, z(t_1+h)$$

$$- \int_s^{t_1+h} \int_s^\theta [\eta^T(\alpha, t_1-\alpha) - \eta^T(\alpha, s-\alpha)]\, d\alpha\, \dot{z}(\theta)\, d\theta.$$

But this equation holds for all $s \in [t_0, t_1]$ and all $z(\cdot) \in W^{1,q}[t_0, t_1+h; R^n]$ if and only if

$$\int_s^t \eta^*(\alpha, t-\alpha)\, d\alpha + \int_s^t \eta^T(\alpha, s-\alpha)\, d\alpha = \int_s^t \eta^T(\alpha, t-\alpha)\, d\alpha \qquad (3.17)$$

for $t_0 \leqslant s \leqslant t$, and $s \leqslant t \leqslant t_1 + h$. For $t_0 \leqslant s \leqslant t \leqslant t_1$ this identity is equivalent to (3.16) and therefore (3.16) is necessary for (3.15). Conversely, if (3.16) holds for $t_0 \leqslant s \leqslant t$ then we obtain for $t \geqslant t_1$

$$\int_s^{t_1} \eta^*(\alpha, t-\alpha)\, d\alpha + \int_s^t \eta^T(\alpha, s-\alpha)\, d\alpha$$

$$= -\int_{t_1}^t \eta^*(\alpha, t-\alpha)\, d\alpha$$

$$= \int_{t_1}^t \eta^T(\alpha, t_1-\alpha)\, d\alpha$$

$$= \int_s^t \eta^T(\alpha, t_1-\alpha)\, d\alpha.$$

But if $\eta^*(\alpha, \cdot)$ is defined to be identically zero for $\alpha > t_1$ then this identity is equivalent to (3.17) with $t \geqslant t_1$. ∎

The existence of a function $\eta^*(t, \tau)$ which satisfies (3.16) can be obtained as a direct consequence of Hypothesis (H2).

THEOREM 3.6. *Suppose that Hypotheses* (H1) *and* (H2) *are satisfied with* $p = 1$ *and let* $t_0 \in R$ *be given. Then there exists a (unique) locally bounded, measurable function* $\eta^*(t, \tau) \in R^{n \times n}$, $t \in R$, $\tau \in R$, *such that*

(i) $\eta^*(t, \tau) = 0$ *for* $\tau \leqslant 0$,

(ii) $\eta^*(t, \tau) = \eta^*(t, h)$ *for* $\tau \geqslant h$,

(iii) *Eq.* (3.16) *holds for* $s \leqslant t$.

*Proof.* Let us fix $t_1 \geqslant t_0$. Then it follows from Hypothesis (H2) in connection with the Riesz representation theorem that for every $t \in [t_0, t_1]$ there exists a function $K(t)(\cdot) = K_{t_0}(t)(\cdot) \in L^\infty[t_0 - h, t_1; R^{n \times n}]$ such that $K_{t_0}(t)(\alpha) = 0$ for $\alpha \notin [t_0 - h, t]$ and

$$\int_{t_0}^t L(\alpha, x_\alpha)\, d\alpha = \int_{t_0 - h}^t K_{t_0}(t)(\alpha)\, x(\alpha)\, d\alpha \tag{3.18}$$

for all $x(\cdot) \in C[t_0 - h, t_1; R^n]$. It follows also from Hypothesis (H2) that

$$\operatorname*{ess\,sup}_{t_0 - h \leqslant \alpha \leqslant t_1} \|K(t)(\alpha)\|_{n \times n} = \sup_{\|x(\cdot)\|_1 = 1} \left| \int_{t_0 - h}^t K(t)(\alpha)\, x(\alpha)\, d\alpha \right| \leqslant K \tag{3.19}$$

for some constant $K > 0$ which is independent of $t \in [t_0, t_1]$.

Now let us consider the columns of $K(t)(\cdot)$ as functions (in $t$) on the interval $[t_0, t_1]$ with values in $L^2[t_0 - h, t_1; R^n]$. Equation (3.18) shows that these functions are weakly continuous and therefore strongly measurable:

$$K(\cdot) \in L^2[t_0, t_1; L^2[t_0-h, t_1; R^{n\times n}]]$$
$$= L^2[[t_0, t_1]\times[t_0-h, t_1]; R^{n\times n}].$$

The latter identity is standard in the theory of partial differential equations and can be established by using Fubini's theorem in connection with the density of the continuous functions in both spaces. More precisely, there exists a square integrable function $K(\cdot, \cdot)$ on $[t_0, t_1]\times[t_0-h, t_1]$ with values in $R^{n\times n}$ such that

$$K(t, \cdot) = K(t) \in L^2[t_0-h, t_1; R^n]$$

for almost every $t \in [t_0, t_1]$. In particular $K(t, \tau)$ is measurable on the square. Furthermore, it follows from (3.19) that

$$\frac{1}{4\varepsilon^2}\int_{t-s}^{t+\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} |K(\tau, \sigma)|\, d\sigma\, d\tau \leqslant K$$

for all $t \in [t_0, t_1]$, $s \in [t_0-h, t_1]$, $\varepsilon > 0$. Since the Lebesgue points of the function $K(\cdot, \cdot)$ are of full measure in the square $[t_0, t_1]\times[t_0-h, t_1]$ we conclude that $K(\cdot, \cdot)$ is essentially bounded. Our next step is to establish the equation

$$\int_s^t K(t, \alpha)\, d\alpha + \int_s^t \eta(\alpha, s-\alpha)\, d\alpha = 0 \tag{3.20}$$

for $t_0 \leqslant s \leqslant t$. For this purpose we define $\eta(t, \cdot) \equiv 0$ for $t \leqslant t_0$. Then it follows from (3.18) that the following equation holds for $x(\cdot) \in W^{1,1}[t_0-h, t; R^{n\times n}]$ (compare the proof of Lemma 3.5)

$$-\int_{t_0-h}^t \eta(\alpha_1-h)\, d\alpha\, x(t_0-h) - \int_{t_0-h}^t \int_\theta^t \eta(\alpha, \theta-\alpha)\, d\alpha\, \dot{x}(\theta)\, d\theta$$

$$= \int_{t_0-h}^t \left[-\eta(\alpha_1-h)\, x(\alpha) + \int_{-h}^0 [\eta(\alpha_1-h) - \eta(\alpha, \tau)]\, \dot{x}(\alpha+\tau)\, d\tau\right] d\alpha$$

$$= \int_{t_0}^t L(\alpha, x_\alpha)\, d\alpha$$

$$= \int_{t_0-h} K(t, \alpha)\, x(\alpha)\, d\alpha)$$

$$= \int_{t_0-h}^t K(t, \alpha)\, x(t_0-h) + \int_{t_0-h}^t \int_\theta^t K(t, \alpha)\, d\alpha\, \dot{x}(\theta)\, d\theta.$$

This proves Eq. (3.20). Note that, up to a set of measure zero, the function $K(t, \alpha)$ is uniquely determined by (3.20). Furthermore, $K(t, s)$ is independent of $t$ for $t \geq s+h$ or, more precisely,

$$K(t, s) = K(\bar{t}, s), \qquad s+h < t < \bar{t}. \tag{3.21}$$

In fact, it follows from (3.20) that

$$\begin{aligned} &K(\bar{t}, s) - K(t, s) \\ &\quad = \frac{d}{ds} \int_s^{\bar{t}} \eta(\alpha, s-\alpha)\, d\alpha - \frac{d}{ds} \int_s^t \eta(\alpha, s-\alpha)\, d\alpha \\ &\quad = \frac{d}{ds} \int_t^{\bar{t}} \eta(\alpha, s-\alpha)\, d\alpha \\ &\quad = \frac{d}{ds} \int_t^{\bar{t}} \eta(\alpha, -h)\, d\alpha, \qquad \text{since } t > s+h \\ &\quad = 0. \end{aligned}$$

Putting things together we obtain that the function

$$\eta^*(\alpha, \tau) = K_{t_0}^T(\alpha+\tau, \alpha), \qquad \alpha \geq t_0,\ \tau \geq t_0 \tag{3.22}$$

satisfies all the requirements of the theorem. We have to show that $\eta^*(\alpha, \tau)$ is independent of $t_0$. For this purpose we point out that as a consequence of (3.18) we obtain

$$K_{t_0}(t, \alpha) = K_{t_0}(t_1, \alpha) + K_{t_1}(t, \alpha) \tag{3.23}$$

for $t_0 \leq t_1 \leq t$ and $\alpha \in R$. This identity shows that for $\alpha \geq t_1$, $\alpha + \tau \geq t_1 \leq t$, we get

$$K_{t_0}(\alpha+\tau, \alpha) = K_{t_1}(\alpha+\tau, \alpha). \quad \blacksquare$$

In conclusion we see that if (H1) and (H2) hold with $p=1$, then the integral adjoint equation (2.19) is equivalent to the differential adjoint equation (3.8), in which $L^*(t, \psi)$ is defined by (3.13), where $\eta^*$ can be obtained from (3.16).

As a symmetric counterpart of this result, we can also rewrite the original RFDE

$$\begin{aligned} \dot{x}(t) &= L(t, x_t), \qquad & t \geq t_0, \\ x(t_0+\tau) &= \phi(\tau), \qquad & \tau \in [-h, 0] \end{aligned}$$

as an integral equation, namely

$$x(t)-x(t_0)=\int_{t_0}^{t} L(s, x_s)\, ds$$

$$=\int_{t_0-h}^{t} K_{t_0}(t, \alpha)\, x(\alpha)\, d\alpha$$

$$=\int_{t_0-h}^{t} [K_{t_0-h}(t, \alpha)-K_{t_0-h}(t_0, \alpha)]\, x(\alpha)\, d\alpha,$$

where the latter equation follows from (3.23). By using (3.22) we obtain

$$x(t)-x(t_0)=\int_{t_0}^{t} [\eta^*(\alpha, t-\alpha)-\eta^*(\alpha, t_0-\alpha)]^T x(\alpha)\, d\alpha. \tag{3.24}$$

## 4. Characterization of the Fundamental Extendability Hypothesis

In this section we investigate concrete conditions on system matrix $\eta(\cdot, \cdot)$, as well as, in a more specialized situation, conditions on the behavior of point delays $h_i(t)$ which guarantee that the fundamental extendability hypothesis is satisfied. In addition, we write a differential adjoint equation for the special case of equations with finitely many time-varying point delays which satisfy the conditions mentioned above, but do not necessarily make the functions $t-h_i(t)$ strictly increasing.

### 4.1. *A Condition for General Systems*

Proposition 4.1. *Let* $1 \leqslant p < \infty$ *and suppose that there exists* $k \in L^q_{\text{loc}}(-\infty, \infty)$, $1/p+1/q=1$, *such that for all* $-\infty < t_0-h \leqslant a < b \leqslant t_1 < \infty$

$$\int_{t_0}^{t_1} |\eta(t, b-t)-\eta(t, a-t)|\, dt \leqslant \int_a^b k(t)\, dt. \tag{4.1}$$

*Then the fundamental extendability hypothesis* (H2) *is satisfied.*

*Proof.* Observe that for $x \in C[t_0-h, t_1; R^n]$, $t \in [t_0, t_1]$,

$$f(t) := \int_{-h}^{0} d_\tau \eta(t, \tau)\, x(t+\tau)$$

$$=\int_{t-h}^{t} d_\tau \eta(t, \tau-t)\, x(\tau)$$

$$=\int_{t_0-h}^{t_1} d_\tau \eta(t, \tau-t)\, x(\tau).$$

Now let $\{a_j^N\}_{j=0}^N$ be a subdivision of $[t_0-h, t_1]$. Then the Riemann–Stieltjes sums $f^N := \sum_{j=1}^N [\eta(t, a_j^N - t) - \eta(t, a_{j-1}^N - t)]\, x(a_j^N)$ converge for almost every $t \in [t_0, t_1]$ to $f(t)$.

Since $|f^N(\cdot)|$ are measurable, nonnegative extended real valued functions, Fatou's Lemma implies that

$$\int_{t_0}^{t_1} |f(t)|\, dt \leqslant \liminf \int_{t_0}^{t_1} |f^N(t)|\, dt. \tag{4.2}$$

However, using assumption (4.1) and Holder's inequality, we find

$$\begin{aligned}
\int_{t_0}^{t_1} |f^N(t)|\, dt &\leqslant \int_{t_0}^{t_1} \sum_{j=1}^N |[\eta(t, a_j^N - t) - \eta(t, a_{j-1}^N - t)]\, x(a_j^N)|\, dt \\
&\leqslant \sum_{j=1}^N |x(a_j^N)| \int_{t_0}^{t_1} |\eta(t, a_j^N - t) - \eta(t, a_{j-1}^N - t)|\, dt \\
&\leqslant \sum_{j=1}^N |x(a_j^N)| \int_{a_{j-1}^N}^{a_j^N} k(t)\, dt \\
&\leqslant \int_{t_0-h}^{t_1} \left( \sum_{j=1}^N |x(a_j^N)|\, \chi_{[a_{j-1}^N, a_j^N]}(t) \right) k(t)\, dt \\
&\leqslant \left\| \sum_{j=1}^N |x(a_j^N)|\, \chi_{[a_{j-1}^N, a_j^N]}(\cdot) \right\|_p \|k(\cdot)\|_q.
\end{aligned}$$

In the limit for $N \to \infty$, we get from (4.2) the desired inequality

$$\int_{t_0}^{t_1} \left| \int_{-h}^{0} d_\tau \eta(t, \tau)\, x(t+\tau) \right| dt \leqslant \|x\|_p\, \|k(\cdot)\|_q. \quad \blacksquare$$

### 4.2. *Composition and Integrability*

In this subsection we study conditions on the functions $\tau: [t_0, t_1] \to [\tau_0, \tau_1]$ and $\alpha: [t_0, t_1] \to R$ under which the map

$$x \to \alpha \cdot x \circ \tau$$

defines a bounded linear operator from $L^p[\tau_0, \tau_1]$ into $L^p[t_0, t_1]$. With these results we prepare the next section in which we investigate systems with point delays.

THEOREM 4.2. *Let $k > 0$ and $1 \leqslant p < \infty$ be given and suppose that $\tau: [t_0, t_1] \to [\tau_0, \tau_1]$, $\alpha: [t_0, t_1] \to R$ are measurable functions. Then the following are equivalent:*

(i) *If* $x(\cdot) \in L^p[\tau_0, \tau_1]$ *then* $\alpha \cdot x \circ \tau \in L^p[t_0, t_1]$ *and*

$$\int_{t_0}^{t_1} |\alpha(t)\, x(\tau(t))|^p\, dt \leqslant K^p \int_{\tau_0}^{\tau_1} |x(\tau)|^p\, d\tau. \tag{4.3}$$

(ii) *For every open interval* $I \subset [\tau_0, \tau_1]$ *the following inequality holds:*

$$\int_{\tau^{-1}(I)} |\alpha(t)|^p\, dt \leqslant K^p \lambda(I). \tag{4.4}$$

*Proof.* Condition (4.4) follows from (4.3) by choosing $x(\cdot) = \chi_I(\cdot)$, the characteristic function of the interval $I$. In order to prove the converse implication let $A$ be a measurable set. Then

$$\int_{\tau^{-1}} |\alpha(t)|^p\, dt \leqslant K^p \lambda(A). \tag{4.5}$$

For every $\varepsilon > 0$ there exists an open set $\theta \supset A$ such that $\lambda(\theta \backslash A) \leqslant \varepsilon$. Since $\theta$ can be represented as a disjoint union of open intervals $\theta = \bigcup_{i=1}^{\infty} I_i$, we get

$$\int_{t^{-1}(A)} |\alpha(t)|^p\, dt \leqslant \int_{\tau^{-1}(\theta)} |\alpha(t)|^p\, dt = \sum_{i=1}^{\infty} \int_{\tau^{-1}(I_i)} |\alpha(t)|^p\, dt$$
$$\leqslant \sum_{i=1}^{\infty} K^p \lambda(I_i) = K^p \lambda(\theta) \leqslant K^p \lambda(A) + K^p \varepsilon.$$

Hence (4.4) follows.

As a consequence we have for every set $N \subset [\tau_0, \tau_1]$ with $\lambda[N] = 0$ that

$$\int_{\tau^{-1}(N)} |\alpha(t)|^p\, dt = 0$$

and hence $\alpha(t) = 0$ for a.e. $t \in \tau^{-1}(N)$.

The proof can be completed by proving (4.3) first for every simple functions and then for every element $x$ of $L^p$. ∎

Our next result is a necessary condition for (4.4) in the case that the function $\tau$ is absolutely continuous.

Lemma 4.3. *Let the functions* $\tau\colon [t_0, t_1] \to [\tau_0, \tau_1]$ *and* $\alpha\colon [t_0, t_1] \to R$ *be given and suppose that* $\tau$ *is absolutely continuous and that* (4.4) *holds for every open interval* $I \subset [\tau_0, \tau_1]$. *Then the inequality*

$$|\dot{\tau}(t)| \geqslant \frac{1}{K^p} |\alpha(t)|^p \tag{4.6}$$

*holds for almost every* $t \in [t_0, t_1]$.

*Proof.* For every $\varepsilon > 0$ we have

$$\int_t^{t+\varepsilon} |\dot{\tau}(s)|\, ds = \operatorname*{VAR}_{[t,t+\varepsilon]} \tau$$
$$\geqslant \lambda(\tau([t, t+\varepsilon]))$$
$$\geqslant \frac{1}{K^p} \int_{\tau^{-1}(\tau[t,t+\varepsilon])} |\alpha(s)|^p\, ds$$
$$\geqslant \frac{1}{K^p} \int_t^{t+\varepsilon} |\alpha(s)|^p\, ds$$

and therefore (4.6) follows if we divide this inequality by $\varepsilon$ and let $\varepsilon$ tend to zero. ∎

The following example shows that condition (4.6) is not sufficient to guarantee (4.4).

EXAMPLE 4.4. Let $\alpha(t) \equiv 1$ and define $\tau$: $[0, 1] \to [-1, 1]$ by piecewise linear extension of $\tau(2^{-n}) = (-1)^n 2^{-n}$ for $n = 0, 1, 2, \ldots$ (Fig. 2). Then $|\dot{\tau}(t)| = 3$ for almost every $t \in [0, 1]$ and therefore (4.6) holds. But for every $\varepsilon > 0$ the inequality

$$\lambda(\tau^{-1}(-\varepsilon, \varepsilon)) \geqslant \sum_{2^{-n} > \varepsilon} 2\varepsilon/3$$

holds and hence (4.4) is violated.

In the next lemma, we present a sufficient condition for (4.4) to hold under slightly stronger assumptions on the function $\tau$.

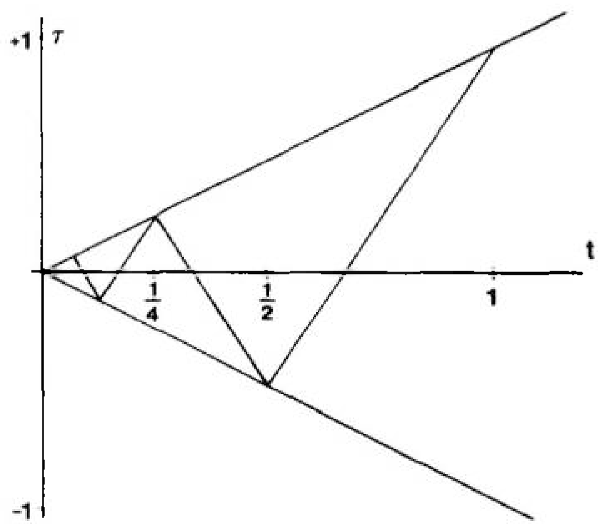



FIGURE 2

LEMMA 4.5. *Let the functions* $\tau: [t_0, t_1] \to [\tau_0, \tau_1]$, $\alpha: [t_0, t_1] \to R^n$ *be given, and suppose that* $\tau$ *is absolutely continuous and that* $\dot{\tau}(t)$ *satisfies the inequality* (4.6) *for almost every* $t \in [t_0, t_1]$. *Furthermore, suppose that* $\dot{\tau}$ *changes its sign only finitely many times. Then* (4.4) *holds for every open interval* $I \subset [\tau_0, \tau_1]$ (*with a different constant* $k$).

*Proof.* Without loss of generality we can assume that $\dot{\tau}(t) \geqslant 0$ for almost every $t \in [t_0, t_1]$ and hence, by (4.6), $\dot{\tau}(t) \geqslant K^{-p} |\alpha(t)|^p$. Then the following inequality holds for $\tau_0 \leqslant a \leqslant b \leqslant \tau_1$:

$$\int_{\tau^{-1}(a)}^{\tau^{-1}} |\alpha(t)|^p \, dt \leqslant K^p \int_{\tau^{-1}(a)}^{\tau^{-1}(b)} \dot{\tau}(t) \, dt = K^p(b-a).$$

This proves (4.4). ∎

Our final counterexample shows that $\tau$ need not be piecewise monotone in order to establish the inequality (4.4) even if $\alpha(t) \equiv 1$.

EXAMPLE 4.6. Let $\alpha(t) \equiv 1$ and define $\tau: [0, 1] \to [-1, 1]$ by piecewise linear extension of $\tau(2^{-n}) = (-1)^n \sqrt{2^{-n}}$ for $n = 0, 1, 2, \ldots$ (Fig. 3). On the interval $[2^{-n-1}, 2^{-n}]$ the slope of this function $\tau$ has the absolute value

$$\left| \frac{\tau(2^{-n}) - \tau(2^{-n-1})}{2^{-n} - 2^{-n-1}} \right| = \frac{\sqrt{2^{-n}} + \sqrt{2^{-n-1}}}{2^{-n-1}} = (2 + \sqrt{2}) \sqrt{2^n}.$$

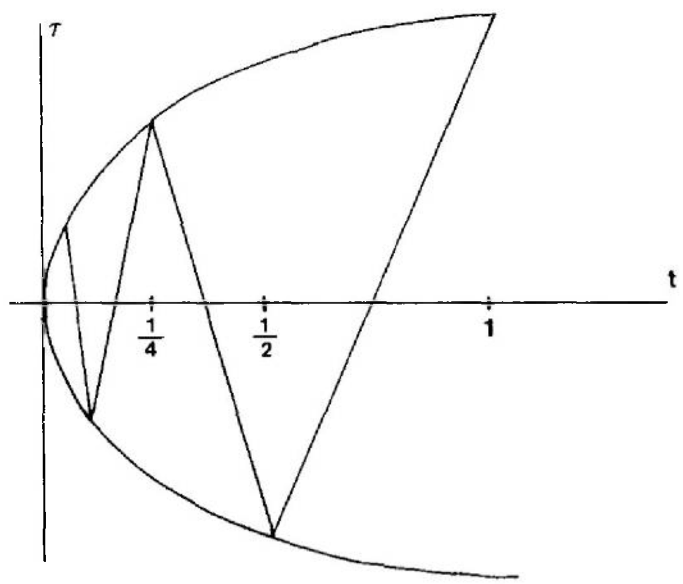



FIGURE 3

Hence the following inequality holds for every open interval $I \subset [-1, 1]$:

$$\lambda(\tau^{-1}(I)) \leqslant \frac{\lambda(I)}{2+\sqrt{2}} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \lambda(I).$$

Therefore (4.4) holds with $K = 1$.

### 4.3. *Systems with Time-Varying Point Delays*

In this section we make use of the results in the previous section in order to investigate retarded systems with point delays described by

$$\dot{x}(t) = \sum_{j=1}^{N} A_j(t)\, x(t - h_j(t)), \qquad t \geqslant t_0, \tag{4.7}$$

where $I_j \subset [t_0, t_1]$ are closed intervals, the coefficient matrices $A_j: I_j \to R^{n \times n}$ are integrable, and the delays $h_j: I_j \to [0, h]$ are Lipschitz continuous and satisfy the inequality

$$\varepsilon < |1 - \dot{h}_j(t)| < 1/\varepsilon \tag{4.8}$$

for some $\varepsilon > 0$ and almost every $t \in I_j$. Furthermore we assume that the functions $1 - \dot{h}_j(t)$ do not change their sign on $I_j$, that is

$$(1 - \dot{h}_j(t))(1 - \dot{h}_j(s)) \geqslant 0, \qquad t, s \in I_j. \tag{4.9}$$

Then the requirements of Lemma 4.5 are satisfied and therefore the RFDE (4.7) satisfies Hypotheses (H1) and (H2) of Section 2. In order to derive the adjoint equation in the context of Section 3, let us first define the closed intervals

$$I_j^* = \{t - h_j(t) \mid t \in I_j\} \cap [t_0 - h, t_1]. \tag{4.10}$$

LEMMA 4.7. *For every* $j \in \{1, ..., N\}$ *there exists a unique function* $h_j^*: I_j^* \to [0, h]$ *such that*

$$h_j^*(t - h_j(t)) = h_j(t), \qquad t \in I_j. \tag{4.11}$$

*These functions also satisfy*

$$h_j(s + h_j^*(s)) = h_j^*(s), \qquad s \in I_j^*, \tag{4.12}$$

$$I_j = \{s + h_j^*(s) \mid s \in I_j^*\}, \tag{4.13}$$

$$1 + \dot{h}_j^*(s) = \frac{1}{1 - \dot{h}_j(s + h_j{}^*(s))} \in [\varepsilon, 1/\varepsilon]. \tag{4.14}$$

*Proof.* It follows from (4.8) that the function $f_j(t) = t - h_j(t)$ from $I_j$ into $I_j^*$ is strictly monotone and therefore continuously invertible. The functions

$$h_j^*(s) = f_j^{-1}(s) - s, \qquad s \in I_j^*,$$

of course satisfy

$$I_j = \{f_j^{-1}(s) \mid s \in I_j^*\} = \{s + h_j^*(s) \mid s \in I_j^*\},$$
$$h_j^*(s) = f_j^{-1}(s) - f_j(f_j^{-1}(s)) = h_j(f_j^{-1}(s)) = h_j(s + h_j^*(s)) \qquad [0, h],$$
$$h_j^*(t - h_j(t)) = h_j^*(f_j(t)) = f_j^{-1}(f_j(t)) - f_j(t) = h_j(t).$$

Differentiating the identity $h_j^*(s) = h_j(s + h_j^*(s))$ we get

$$\dot{h}_j^*(s) = \dot{h}_j(s + h_j^*(s))(1 + \dot{h}_j^*(s))$$

and hence

$$(1 + \dot{h}_j^*(s))(1 - \dot{h}_j(s + h_j^*(s))) = 1, \tag{4.15}$$

which proves (4.14). ∎

Associated with the RFDE (4.7) are the functions

$$\eta_j(t, \tau) = \begin{cases} -A_j(t), & \tau \leqslant -h_j(t),\ t \in I_j, \\ 0, & \text{otherwise} \end{cases}$$

of bounded variation in $\tau$ in the sense of Lemma 2.1. In order to construct functions $\eta_j^*(t, \tau)$ of bounded variation in $\tau$ which satisfy (3.20) we differentiate this expression and define

$$\eta_j^*(s, t - s) = \frac{d}{ds} \int_s^t \eta_j^T(\alpha, s - \alpha)\, d\alpha \tag{4.16}$$

for almost every $s \leqslant t$ and $\eta_j^*(s, \tau) = 0$ for $\tau \leqslant 0$. Then the next result shows that $\eta_j^*(s, \tau)$ is in fact of bounded variation in $\tau$ and therefore satisfies the requirements of Lemma 3.6. In particular it determines the differential adjoint equation.

LEMMA 4.8.

$$\eta_j^*(s, \tau) = \begin{cases} A_j^T(s + h_j^*(s))(1 + \dot{h}_j^*(s)), & \tau \geqslant h_j^*(s),\ s \in I_j, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* First consider the case that $t-h_j(t)$ is increasing and note that

$$\eta_j^*(\alpha, s-\alpha)=\begin{cases}-A_j^T(\alpha), & s\leqslant \alpha-h_j(\alpha),\ \alpha\in I_j,\\ 0, & \text{otherwise.}\end{cases}$$

Now let $\alpha\in I_j$ and $s\in I_j^*$. Then $s\leqslant \alpha-h_j(\alpha)$ if and only if $s+h_j^*(s)\leqslant\alpha$. This implies

$$\int_s^t \eta_j^T(\alpha, s-\alpha)\,d\alpha=\begin{cases}-\int_{s+h_j^*(s)}^t A_j^T(\alpha)\,\chi_{I_j}(\alpha)\,d\alpha, & s+h_j^*(s)\leqslant,\ s\in I_j^*,\\ \text{locally independent of } s, & \text{otherwise.}\end{cases}$$

Hence we obtain from (3.16) that

$$\eta_j^*(s, t-s)=\begin{cases}A_j^T(s+h_j^*(s))(1+\dot h_j^*(s)), & t-s\geqslant h_j^*(s),\ s\in I_j^*,\\ 0, & \text{otherwise.}\end{cases}$$

This proves the statement of the lemma in the case that $t+h_j(t)$ is increasing. The case of a decreasing function $t+h_j(t)$ can be treated analogously. ∎

Combining the previous result with Lemma 3.5 we obtain that the adjoint equation of (4.7) is described by the differential delay equation

$$\dot z(s)=-\sum_{j=1}^N A_j^T(s+h_j^*(s))(1+\dot h_j^*(s))\,z(s+h_j^*(s)),\qquad s\in t_1. \tag{4.17}$$

This formula has also been derived in [12], with different methods.

*Remark* 4.9. It remains an open problem to find necessary and sufficient conditions for the existence of an evolutionary system on $M^p$, analogous to Delfour's result for autonomous systems [25].

## References


1. C. Bernier and A. Manitius, On semigroups in $R^n\times L^p$ corresponding to differential equations with delays, *Canad. J. Math.* **30** (1978), 897–914.
2. N. Bourbaki, "Integration," Vol. V, Hermann, Paris, 1968.
3. J. A. Burns and T. L. Herdman, Adjoint semigroup theory for a class of functional differential equations, *SIAM J. Math. Anal.* **7** (1976), 729–745.
4. F. Colonius, "Optimal Periodic Control," *Lecture Notes in Math.*, Springer-Verlag, New York/Berlin, 1988.
5. M. C. Delfour, State theory of linear, hereditary differential systems, *J. Math. Anal. Appl.* **60** (1977), 8–35.

6. M. C. Delfour and J. Karrakchou, "State Space Theory of Linear Time Invariant Systems with Delays in State, Control and Observation Variables," Pts. I, II, Report CRMA 1223, Universite de Montreal, 1984.
7. M. C. Delfour and A. Manitius, The structural opeator $F$ and its role in the theory of retarded systems, Part 1, *J. Math. Anal. Appl.* **73** (1980), 466–490; Part 2, *J. Math. Anal. Appl.* **74** (1980), 359–381.
8. M. C. Delfour and S. K. Mitter, Hereditary differential systems with constant delays, Part 1: General case, *J. Differential Equations* **12** (1972), 213–235; Part 2: A class of affine systems and the adjoint problem, *J. Differential Equations* **18** (1975), 18–28.
9. O. Diekmann, Volterra integral equations and semigroups of operators, preprint, Mathematisch Centrum Report TW 197/80, Amsterdam, 1980.
10. O. Diekmann, A duality principle for delay equations, preprint, Mathematisch Centrum Report TN 100/81, Amsterdam, 1981.
11. N. Dunford and J. T. Schwartz, "Linear Operators, Part I: General Theory," Interscience, New York, 1957.
12. J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York/Berlin, 1977.
13. D. Henry, The adjoint of a linear functional differential equation and boundary value problems, *J. Differential Equations* **9** (1971), 55–66.
14. D. Henry, "The Adjoint of a Linear Functional Differential Equation of Neutral Type in the Sobolev Space $W_2^{(1)}$," Technical Report, Department of Mathematics, University of Kentucky, Lexington, KY, 1970.
15. E. Hewitt and K. Stromberg, "Real and Abstract Analysis," Springer-Verlag, New York/Berlin, 1976.
16. I. Lasiecka and A. Manitius, "Differentiability and Convergence Rates of Approximating Semigroups for Retarded Functional Differential Equations," *SIAM J. Numer. Anal.* **25** (1988), 883–907.
17. A. Manitius, Completeness and $F$-completeness of eigenfunctions associated with retarded functional differential equations, *J. Differential Equations* **35** (1980), 1–29.
18. A. Manitius, $F$-controllability and observability of linear retarded systems, *Appl. Math. Optim.* **9** (1982), 73–95.
19. R. K. Miller, Linear volterra integro-differential equations as semigroups, *Funkcial. Ekvac.* **17** (1974), 39–55.
20. D. Salamon, "Control and Observation of Neutral Systems," Pitman, Boston, 1984.
21. D. Salamon, Structure and stability of finite dimensional approximations for functional differential equations, *SIAM J. Control Optim.* **23** (1985), 928–951.
22. O. J. Staffans, Semigroups generated by a convolution equation, *in* "Infinite Dimensional Systems" (F. Kappel and W. Schappacher, Eds.), pp. 209–226, Springer-Verlag, New York/Berlin, 1984.
23. O. J. Staffans, Extended initial and forcing function semigroups generated by a functional equation, *SIAM J. Math. Anal.* **16** (1985), 1034–1048.
24. R. B. Vinter and R. H. Kwong, The infinite time quadratic control problem for linear systems with state and control delays: An evolution equation approach, *SIAM J. Control Optim.* **19** (1981), 139–153.
25. M. C. Delfour, The largest class of hereditary systems defining a $C_0$-semigroup on the product space, *Canad. J. Math.* **32** (1980), 969–978.