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Stability Radii and Lyapunov Exponents*

Fritz Colonius Wolfgang Kliemann

In the state space approach to stability of uncertain systems the concept of stability radius plays a central role. In this paper we use the classical concept of Lyapunov exponents, which describe the exponential growth behavior, in order to define a variety of stability and instability radii for families of linear systems $\dot{x} = [A + u(t)]x$, $u(t) \in U_\rho$, $\rho \geq 0$. Here $\{U_\rho, \rho \geq 0\}$ can denote sets of real or complex matrices, and the perturbation $u(t)$ can be deterministic or belong to different classes of stochastic processes. We analyze the stability radii and their relations, using Lyapunov exponents of associated bilinear control systems. The well developed theory of Lyapunov exponents for stochastic systems provides the background for the discussion of stability radii of systems excited by random noise. The example of the linear oscillator with uncertain restoring force is discussed in detail.

1 Introduction

It is the purpose of this paper to show that the concept of Lyapunov exponents can be used to define and to analyze various stability radii of matrices with respect to deterministic or stochastic (time-varying) perturbations. We hope to demonstrate that the well-developed theory of Lyapunov exponents for stochastic linear, parameter-excited systems [2, 3, 4] and also very recent results on Lyapunov exponents of bilinear control systems [8, 9, 10] (both theories are closely related) are of interest for researchers studying robustness properties of linear systems.

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Consider the family of linear systems in $\mathbb{R}^d$

$$\dot{x}(t) = [A + u(t)]x(t), \quad u(t) \in U_\rho, \rho > 0 \tag{1.1}$$

where $\{U_\rho; \rho \geq 0\}$ is an increasing family of sets of real $d \times d$-matrices. The admissible perturbations (presenting uncertainties) for each ρ with values in U_ρ are denoted by $\mathcal{U}_\rho$. For a stable matrix A, we define a (Lyapunov) stability radius r_L with respect to $\{U_\rho : \rho \geq 0\}$ as the lower bound of those ρ for which there exists $u \in \mathcal{U}_\rho$ such that the system (1.1) has a nonnegative exponential growth rate, i.e. a nonnegative Lyapunov exponent.

For deterministic perturbations $u(\cdot)$ the approach of this paper is an alternative to the one in [13, 16], where instead of Lyapunov exponents, Bohl exponents are used in order to define a stability radius. For the theory of Lyapunov and Bohl exponents of linear time varying differential equations we refer to [11] and [12]. The Bohl exponent is negative iff uniform asymptotic stability holds while negativity of the largest Lyapunov exponent is equivalent to asymptotic stability. Hence, for a fixed time-varying linear differential equation, the Bohl exponent is — in general — larger than the largest Lyapunov exponent. This implies that for a system (1.1) the corresponding Bohl stability radius is not greater than the corresponding Lyapunov stability radius. In fact, Theorem 5, below, states that for real perturbations the Bohl and the Lyapunov stability radii coincide. The (Bohl) stability radius studied in [16, 13] is based on complex perturbations (for special families $\{U_\rho; \rho \geq 0\}$). Hence it presents a more conservative robustness estimate than the radius for real perturbations. It is shown in [16], that the complex (Bohl-) radius coincides with the stability radius $r_{\mathbb{C}}$, which only takes into account constant complex perturbations. Below we will discuss in more detail the relation of our results to those in the literature. Here we remark only that (i) the Lyapunov exponents approach allows for a very general class of real perturbations and (ii) that stochastic perturbations can be analyzed within the same framework. The price for this is that contrary to [13, 15, 16], we do not obtain algebraic expressions, like associated Riccati equations, or explicit formulas, except in very special cases, (see e.g. Theorem 7, (2.1.2) and (2.2), Corollary 9 or the example of the linear oscillator, Example 9). For general systems with many degrees of freedom, reliable numerical methods are required (compare [14] and [18] for numerics in the case of time invariant uncertainties).

Section 2 introduces concepts of stability radii via Lyapunov exponents. In Section 3 a basic idea for the description of Lyapunov exponents of (bi-)linear systems is presented: The separation of the radial and the angular part of the solution, allowing the characterization of Lyapunov exponents on the sphere (resp. the projective space), a compact manifold.

Section 4 presents some basic facts about the maximal and minimal spectral values of *bilinear* control systems, which are crucial for the characterization of Lyapunov (in-)stability radii of *linear* systems. This is discussed in Section 5. Two examples of 2-dimensional systems are presented in Section 6, in particular the linear oscillator with uncertain restoring force: It is a simple, but prototypical example exhibiting much of the (surprising) behaviour of bilinear systems, and thus of the robustness properties of linear systems.

In Section 7 stability radii for stochastic excitations are defined for two classes of stochastic processes: stationary ones and stationary, ergodic, nondegenerate diffusions. While the theory for the first class parallels the results for the deterministic r_L, Markovian uncertainties yield a considerably more complicated behavior: Their radius of stability for all moments agrees with r_L, while the almost sure radius is in general greater. The difference is explained via the theory of large deviations. Finally, Section 8 discusses robust design, based on different information about the uncertainties.

2 Definitions of Deterministic Stability Radii

In this section we define several stability and instability radii using the concept of Lyapunov exponents.

Denote by $M(n,m;\mathbb{K})$ the set of $n \times m$ matrices over a field $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $A \in M(d,d;\mathbb{R})$, let $\{U_\rho, \rho \geq 0\}$ be an increasing family of subsets in $M(d,d;\mathbb{K})$ and denote

$$\mathcal{U}_\rho := \{u : [0,\infty) \to U_\rho; u \text{ measurable and locally integrable}\} \tag{2.1}$$

Define for $u \in \mathcal{U}_\rho$ and $0 \neq x_0 \in \mathbb{R}^d$ the Lyapunov exponent

$$\lambda(x_0,u) = \limsup_{t\to\infty} \frac{1}{t} \log |\varphi(t,x_0,u)| \tag{2.2}$$

where $\varphi(\cdot,x_0,u)$ solves

$$\dot{x}(t) = [A + u(t)]x(t),\ t \geq 0, \quad x(0) = x_0. \tag*{$(2.3)_\rho$}$$

Note that for fixed u, there are at most d different $\lambda(x_0,u)$, $x_0 \neq 0$. Let for $\rho \in \mathbb{R}_+ = [0,\infty)$

$$\kappa_\rho(A) := \sup\{\lambda(x_0,u);\ 0 \neq x_0 \in \mathbb{R}^d,\ u \in \mathcal{U}_\rho\}. \tag{2.4}$$

Thus κ_ρ denotes the supremal Lyapunov exponent of the control system (2.3) with controls u in $\mathcal{U}_\rho$. A corresponding Lyapunov stability radius of $A \in M(d,d;\mathbb{R})$ is given by

$$r_L(A) = \inf\{\rho \geq 0; \kappa_\rho(A) \geq 0\}. \tag{2.5}$$

Naturally, $r_L(A)$ depends crucially on the family $\{U_\rho; \rho \geq 0\}$. For simplicity we suppress this dependence in our notation. For a stable matrix A (i.e. $\mathrm{Re}\,\lambda < 0$ for all λ in the spectrum $\sigma(A)$), $r_L(A)$ is the lower bound of those indices ρ, for which there exists $u \in \mathcal{U}_\rho$ such that the corresponding system $\dot{x} = [A + u]x$ is not asymptotically stable.

Similarly, instability radii can be defined via

$$\begin{aligned} \kappa_\rho^*(A) &= \inf\{\lambda(x_0, u);\ 0 \neq x_0 \in \mathbb{R}^d, u \in \mathcal{U}_\rho\} \qquad (2.6) \\ \kappa_\rho^*(A, x_0) &= \inf\{\lambda(x_0, u);\ u \in \mathcal{U}_\rho\} \qquad (2.7) \end{aligned}$$

and

$$\begin{aligned} r_L^*(A) &= \inf\{\rho \geq 0;\ \kappa_\rho^*(A) \leq 0\} \qquad (2.8) \\ \tilde{r}_L(A) &= \inf\{\rho \geq 0;\ \kappa_\rho^*(A, x_0) \leq 0 \text{ for all } x_0 \neq 0\} \qquad (2.9) \end{aligned}$$

The radius $r_L^*(A)$ makes sense for totally unstable matrices A (i.e. $\mathrm{Re}\,\lambda > 0$ for all $\lambda \in \sigma(A)$) and describes the lower bound of those indices ρ, for which there exists $u \in \mathcal{U}_\rho$, making $\dot{x} = [A + u]x$ asymptotically stable for some $x_0 \neq 0$. The radius r_L^* is the dual of the stability radius r_L in a precise sense, see Proposition 6, below.

On the other hand, the instability radius $\tilde{r}_L(A)$ makes sense for unstable matrices A (i.e. there is $\lambda \in \sigma(A)$ with $\mathrm{Re}\,\lambda > 0$) and describes the lower bound of these indices ρ for which one obtains asymptotic stability for all initial values $x_0 \neq 0$.

The concepts $r_L, r_L^*, \tilde{r}_L$ introduced so far, depend on the choice of the initial value in the following way: For r_L (and r_L^*) only existence of a point x_0 with stable (or unstable) solution is considered, while for $\tilde{r}_L$ the system has to be stable for all x_0, but the corresponding $u(\cdot)$ may still depend on x_0.

Remark 1. One may also introduce the following additional uniform concepts, cp. [9]:

$$\hat{\kappa}_\rho(A) = \sup_{u \in \mathcal{U}_\rho} \inf_{x_0 \neq 0} \lambda(x_0, u), \quad \hat{\kappa}_\rho^*(A) = \inf_{u \in \mathcal{U}_\rho} \sup_{x_0 \neq 0} \lambda(x_0, u) \qquad (2.10)$$

and the corresponding radii

$$\hat{r}_L(A) = \inf\{\rho \geq 0;\ \hat{\kappa}_\rho(A) \geq 0\}, \quad \hat{r}_L^*(A) = \inf\{\rho \geq 0;\ \hat{\kappa}_\rho^*(A) \leq 0\} \quad (2.11)$$

For a stable matrix A, $\hat{r}_L(A)$ describes the lower bound of those indices ρ for which there is $u \in \mathcal{U}_\rho$ making $\dot{x} = [A + u]x$ exponentially unstable for all $x_0 \neq 0$; similarly for $\hat{r}_L^*(A)$. The following inequalities follow directly from the definitions:

$$r_L(A) \leq \hat{r}_L(A), \quad r_L^*(A) \leq \tilde{r}_L(A) \leq \hat{r}_L^*(A) \qquad (2.12)$$

□

The theory of (in-)stability radii in the above set-up relates to specific problems of (open-loop) exponential destabilizability and stabilizability of bilinear control systems: One simply considers u as a control function, which is to be chosen such that $\lambda(x_0, u)$ is maximized or minimized. In the next two sections we recall some related facts from [9].

3 Bogolyubov's Projection and Lyapunov Exponents

In this section, we show how the Lyapunov exponents can be obtained from the projected system on the sphere. This is a standard argument for stochastic differential equations (cp. e.g. [6]). We denote the Euclidean norm on $\mathbb{K}^d = \mathbb{R}^d$ or $\mathbb{C}^d$ by $|\cdot|$.

Any linear differential equation

$$\dot{x} = M(t)x, \quad x(0) = x_0 \in \mathbb{K}^d \tag{3.1}$$

can be described in polar coordinates by its angular and radial part. A straightforward application of the chain rule shows that the angular part $s(t) = \frac{x(t)}{|x(t)|}$ is decoupled and looks like

$$\begin{aligned} \dot{s} &= h(M, s) = \{M(t) - \frac{1}{2}s^*(t)[M(t) + M^*(t)]s(t)\,\mathrm{Id}\}s(t) \\ s(0) &= s_0 = \frac{x_0}{|x_0|} \in S^{d-1} \end{aligned} \tag{3.2}$$

on the sphere $S^{d-1} = \{x \in \mathbb{R}^d; |x| = 1\}$. Note that a linear vector field is homogeneous, thus it suffices to consider the system on projective space $\mathbb{P} := \mathbb{P}^{d-1}$, obtained by identifying opposite points on the sphere $\mathbb{S} = S^{d-1}$. Recall that $\mathbb{S}$ and $\mathbb{P}$ are compact manifolds.

The radial part becomes

$$|x(t, x_0)| = |x_0| \exp\Big[\int_0^t q(M(\tau), s(\tau, s_0))\, d\tau\Big] \tag{3.3}$$

with $q(M, s) := \frac{1}{2}s^*(M + M^*)s = \mathrm{Re}\{s^*Ms\}$. Thus we obtain for the corresponding Lyapunov exponent

$$\lambda(x_0, M) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(M(\tau), s(\tau))\, d\tau. \tag{3.4}$$

Hence in the time-varying case, the Lyapunov exponents are the long term averages of the function q, evaluated along the trajectories. For time-invariant situations, the Lyapunov exponents (i.e. the real parts of the eigenvalues) are the averages of q in the direction of the eigenvectors. This fact allows some simple estimates of stability radii, compare Proposition 1 below. To be more specific, let M be a constant $d \times d$-matrix and denote by E_j, $j = 1, \ldots, k$ the eigenspaces and let $\lambda_j = \mu_j + i\nu_j \in \sigma(M)$ be the corresponding eigenvalues. Let $\mathbb{P}E_j$ be the projection of (the nonzero elements of) E_j onto $\mathbb{P}$. Let $s_j \in \mathbb{P}E_j$ be an eigenvector and denote the solution of the differential equation (3.2) in $\mathbb{P}$ with initial value s_j by $s(\cdot, s_j)$. For $\mathbb{K} = \mathbb{C}$, the corresponding trajectory is always a point in $\mathbb{P}$; for $\mathbb{K} = \mathbb{R}$, the corresponding trajectory may be a point or forms a "circle" in $\mathbb{P}$ depending on whether $\lambda_j \in \mathbb{R}$ or not. In the latter case, $s(\cdot, s_j)$ is periodic, say with period T_j, and defines an occupation measure σ_j on the circle whose density is $\frac{1}{|h(M,s)|}$. If $\lambda_j \in \mathbb{R}$ set $\sigma_j = \delta_{s_j}$, the Dirac measure at s_j. One now has

$$\mu_j = \lambda(s_j, M)$$
$$= \begin{cases} \dfrac{1}{T_j} \displaystyle\int q(M,s)\, d\sigma_j = \dfrac{1}{T_j} \int_0^{T_j} q(M, s(\tau, s_j))\, d\tau & \text{for } s(\cdot, s_j) \text{ a circle} \\ \displaystyle\int q(M,s)\, d\sigma_j = q(M, s_j) & \text{for } s(\cdot, s_j) \text{ a point.} \end{cases}$$

Note that $q(M, s)$ can be positive somewhere on the circle even if μ_j is negative and vice versa. These formulas show that the exponential growth rates of solutions to time-invariant linear equations are obtainable by evaluating q in eigenspaces. Formula (3.4) implies that q is also closely connected with exponential growth rates of solutions to time-varying linear equations. More precisely, define for a stable matrix A and $\{U_\rho : \rho \geq 0\}$ as above

$$\rho_*(A) := \inf\{\rho; \text{there is } u \in U_\rho \text{ with } \max q(A + u, s) \geq 0\}$$

Then by (3.4)

$$\rho_*(A) \leq r_L(A). \tag{3.5}$$

The following proposition establishes a connection between the (unstructured) real and complex stability radii $r_{\mathbb{R}}$ and $r_{\mathbb{C}}$ of a (stable) matrix A — as defined in [14] — and the function q: Recall that for $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$$r_{\mathbb{K}}(A) = \inf\left\{\rho > 0 : \begin{array}{l}\text{there is } D \in M(d,d,\mathbb{K}) \text{ with } \|D\|_2 \leq \rho \text{ such} \\ \text{that } A + D \text{ is not assymptotically stable}\end{array}\right\}.$$

where $\|\cdot\|_2$ denotes the operator norm induced by the Euclidean norm on $\mathbb{K}^d$. Recall also that $\mathbb{S}$ denotes the sphere in $\mathbb{R}^d$, and that $\mathbb{P}$ denotes the corresponding projective space.

Proposition 1. *Let A be a stable matrix in $M(d, d; \mathbb{R})$. Then*

$$0 \leq \min_{s \in \mathbb{S}} |q(A,s)| \leq r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq \max_{s \in \mathbb{S}} |q(A,s)|$$

PROOF. One easily sees (cp.e.g. [15, Lemma 4.1]) that

$$0 < r_{\mathbb{C}} \le r_{\mathbb{R}} \le |\mu_n|,$$

where μ_n is the smallest real part of an eigenvalue of A. Clearly, we also have

$$|\mu_n| \le \max_{s\in\mathbb{P}} |q(A,s)| = \max_{s\in\mathbb{S}} |q(A,s)|.$$

In order to prove the remaining inequality

$$\min |q(A,s)| \le r_{\mathbb{C}},$$

we note that $\min|q(A,s)| = 0$, iff there is $s_0 \in \mathbb{P}$ with $q(A,s_0) = 0$; and for A stable, it suffices to consider the case $q(A,z) \le \delta < 0$ for all z in the complex projective space $\mathbb{P}_{\mathbb{C}}$. We use the characterization of $r_{\mathbb{C}}$ from [15, Proposition 3.3]:

$$r_{\mathbb{C}}^2 = \min_{z\in\mathbb{P}_{\mathbb{C}}}\{\|Az\|^2 + \langle A_u z, z\rangle^2\},$$

where $A_u = \frac{1}{2}(A - A^*)$. But for $z \in \mathbb{P}_{\mathbb{C}}$

$$\|Az\|^2 + \langle A_u z, z\rangle^2 = \langle \frac{1}{2}(A + A^*)z, z\rangle^2.$$

Hence

$$r_{\mathbb{C}} \ge \min_{z\in\mathbb{P}_{\mathbb{C}}} |q(A,z)|.$$

But in $\mathbb{R}^2$ and $\mathbb{C}^2$ one easily checks

$$\min_{s\in\mathbb{P}} |q(A,s)| = \min_{z\in\mathbb{P}_{\mathbb{C}}} |q(A,z)|.$$

Q.E.D.

4 Extremal Lyapunov Exponents and the Corresponding Control Sets

In this section we relate the extremal Lyapunov exponents κ, κ^* etc. of bilinear control systems to controllability properties on the projective space $\mathbb{P}$. This discussion is based on [8, 9, 10].

Consider system $(2.3)_\rho$ as a bilinear control system with controls u in $\mathcal{U}_\rho$, where $U_\rho \subset M(d,d;\mathbb{R})$. Under Bogolyubov's projection, the system projected onto $\mathbb{P}$ can be described by

$$\begin{aligned} \dot{s}(t) &= h(u(t), s(t)) = [A + u(t) - \tfrac{1}{2}s^*(t)(A + u(t))s(t)\cdot \mathrm{Id}]s(t), \\ s(0) &= s = \tfrac{x_0}{|x_0|} \in \mathbb{P}. \end{aligned} \qquad (4.1)_\rho$$

Throughout this section we will use the assumption

$$\dim \mathcal{LA}\{h(u,\cdot) \mid u \in U_\rho\}(s) = d-1 \text{ for all } s \in \mathbb{P}, \text{ all } \rho > 0, \qquad (H)$$

where $\mathcal{LA}$ denotes the Lie algebra of vector fields on $\mathbb{P}$ generated by $\{h(u,\cdot),$ $u \in U_\rho\}$. (H) implies (see e.g. [17]) that the system (4.1) is locally accessible on $\mathbb{P}$, i.e. for all $s \in \mathbb{P}$ and all $T > 0$, the set $\mathcal{O}^+_{\leq T}(s)$ has nonvoid interior in $\mathbb{P}$, where $\mathcal{O}^+_{\leq T}(s)$ is the set of points attainable in the time $0 \leq t \leq T$ from s,

$$\begin{aligned}\mathcal{O}^+_{\leq T} := \{y \in \mathbb{P};\ & \text{there exists a piecewise constant control} \\ & u \in \mathcal{U}_\rho \text{ s.t. the corresponding trajectory} \\ & \text{of (4.1) satisfies } \psi(t,s,u) = y \text{ for some } 0 \leq t \leq T\}.\end{aligned}$$

In fact, for any "reasonable" choice of U_ρ, $(4.1)_\rho$ is an analytic system and thus has maximal integral manifolds through each point $s \in \mathbb{P}$. (H) says that the whole space $\mathbb{P}$ is the maximal integral manifold.

Remark 2. (H) is always satisfied if $\operatorname{int} U_\rho \neq \emptyset$, which covers the unstructured case (the interior here is w.r.t. $M(d,d;\mathbb{R})$). For structured systems, the interplay between the dynamics A and the "uncertainty" U_ρ is crucial, see [3] and [4] for some details. Generically, (H) is satisfied, if U_ρ contains at least two matrices, see [3, Remark 2.3]. Note that, if (H) holds for some $\rho_1 > 0$, then it holds for all $\rho \geq \rho_1$. We remark also that — in the stochastic context — [21] presents a theory of Lyapunov exponents where (H) is violated. □

In general, (H) does not imply complete controllability of (4.1) on $\mathbb{P}$. However, there are topologically nice sets where controllability holds.

Let

$$\mathcal{O}^+(x) := \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x).$$

Definition 1. A set $D \subset \mathbb{P}$ is called a control set for (4.1) if D contains more than one point, $D \subset \overline{\mathcal{O}^+(x)}$ for all $x \in D$ and D is maximal with respect to this property. A control set is called invariant if $\mathcal{O}^+(x) \subset D$ for all $x \in D$.

Theorem 2. *Assume that the projected systems $(4.1)_\rho$ in $\mathbb{P}$ satisfy (H). Then for each $\rho > 0$:*

(i) There is exactly one invariant control set C_ρ; this set is closed, $C_\rho = \bigcap_{s\in\mathbb{P}} \overline{\mathcal{O}^+_\rho(s)}$ and $\operatorname{int} C_\rho \neq \emptyset$, $C_\rho = \overline{\operatorname{int} C_\rho}$.

(ii) There is exactly one open control set C^-_ρ.

(iii) Either $C_\rho = C_\rho^-$ (i.e. $(4.1)_\rho$ is controllable on $\mathbb{P}$) or $C_\rho \cap \mathbb{C}_\rho^- = \emptyset$.

(iv) $C_{\rho_1} \subset C_{\rho_2}$ and $C_{\rho_1}^- \subset C_{\rho_2}^-$ for $\rho_1 \leq \rho_2$.

We call C_ρ^- the minimal and C_ρ the maximal control set of $(4.1)_\rho$. A detailed theory of control sets for bilinear systems on $\mathbb{P}$ is developed in [10], where also the significance of the terms "maximal" and "minimal" becomes apparent. The proof of the theorem above can be found in [19] and [9].

Using the Lyapunov exponents of $(2.3)_\rho$, we define the spectrum of the projected system $(4.1)_\rho$ on the control sets C_ρ and C_ρ^- in the following way.

For $X \subset \mathbb{P}$, $s \in \mathbb{P}$ let

$$\begin{aligned} \lambda_\rho(s,X) &= \sup\{\lambda(s,u);\ u \in \mathcal{U}_\rho,\ \psi(t,s,u) \in X \text{ f.a. } t \geq 0\} \\ \lambda_\rho^*(s,X) &= \inf\{\lambda(s,u);\ u \in \mathcal{U}_\rho,\ \psi(t,s,u) \in X \text{ f.a. } t \geq 0\}, \end{aligned}$$

where $\psi(\cdot,s,u)$ denotes the corresponding trajectory of $(4.1)_\rho$, and let

$$\begin{aligned} \alpha_\rho &:= \inf\nolimits_{s\in C_\rho} \lambda^*(s,C_\rho), \qquad & \omega_\rho &:= \sup\nolimits_{s\in C_\rho} \lambda(s,C_\rho), \\ \alpha_\rho^- &:= \inf\nolimits_{s\in C_\rho^-} \lambda^*(s,C_\rho^-), \qquad & \omega_\rho^- &:= \sup\nolimits_{s\in C_\rho^-} \lambda(s,C_\rho^-). \end{aligned}$$

The minimal and maximal spectral interval of the system $(4.1)_\rho$ are given by

$$I_\rho^- := [\alpha_\rho^-, \omega_\rho^-], \quad \text{and} \quad I_\rho := [\alpha_\rho, \omega_\rho]$$

respectively. It turns out that three of the objects α_ρ, ω_ρ, α_ρ^-, ω_ρ^- can be "globally" defined, without a restriction on the whole trajectory. This is the decisive property which allows to associate a stability radius with them.

Proposition 3. *Suppose that the projection $(4.1)_\rho$ of system $(2.3)_\rho$ satisfies (H). Then*

$$\begin{aligned} \textit{(i)}\ \omega_\rho &= \sup\nolimits_{s\in\mathbb{P}} \lambda_\rho(s,\mathbb{P}) = \kappa_\rho \\ \textit{(ii)}\ \alpha_\rho &= \inf\nolimits_{s\in C_\rho} \lambda_\rho^*(s,\mathbb{P}) \\ \textit{(iii)}\ \alpha_\rho^- &= \inf\nolimits_{s\in\mathbb{P}} \lambda_\rho^*(s,\mathbb{P}) = \kappa_\rho^* \end{aligned}$$

Furthermore, for $0 < \rho_1 \leq \rho_2$, one has

$$\omega_{\rho_1} \leq \omega_{\rho_2}, \quad \alpha_{\rho_1} \geq \alpha_{\rho_2}, \quad \alpha_{\rho_1}^- \geq \alpha_{\rho_2}^-.$$

PROOF. (i) is proved in [8], (ii) is obvious from invariance of C and (iii) is proved in [9]; then the last assertion is obvious. Q.E.D.

Remark 3. If $C_\rho \cap C_\rho^- = \emptyset$, then $\alpha_\rho^- \leq \alpha_\rho$, $\omega_\rho^- \leq \omega_\rho$, but $\omega_\rho^- > \alpha_\rho$ may occur; thus the intervals I_ρ and I_ρ^- may overlap. □

Remark 4. If, in the definition of ω_ρ^-, we remove similarly as in (i)–(iii) above the restriction on the trajectory, we obtain $\sup_{s\in C_\rho^-} \lambda(s,\mathbb{P})$ which, however, equals ω_ρ. Hence for ω_ρ^-, this restriction is crucial. □

The proposition above shows in particular, that ω_ρ and α_ρ^- coincide with the previously defined κ_ρ and κ_ρ^*, respectively. We obtain the following result, which establishes the connection to stability radii.

Corollary 4. *Suppose that the projection* $(4.1)_\rho$ *of system* $(2.3)_\rho$ *satisfies* (H). *Then*

$$\begin{aligned} \textit{(i)}\ r_L(A) &= \inf\{\rho \geq 0; \omega_\rho \geq 0\} \\ \textit{(ii)}\ r_L^*(A) &= \inf\{\rho \geq 0; \alpha_\rho^- \leq 0\} \\ \textit{(iii)}\ \tilde{r}_L(A) &= \inf\{\rho \geq 0; \alpha_\rho \leq 0\}. \end{aligned}$$

PROOF. (i) and (ii) are immediate from Proposition 3. (iii) follows from [8, Theorem 4.2 and the proof of Theorem 5.1]. Q.E.D.

5 Some Properties of Lyapunov (In-)Stability Radii

In this section we analyze the Lyapunov stability radius and discuss its relation to other stability radii. In [13], a stability radius based on Bohl exponents for complex perturbations was discussed. Our first result in this section shows that for *real* perturbations the concepts of stability radius based on Bohl and on Lyapunov exponents coincide.

Recall that $\{U_\rho; \rho \geq 0\}$ is an increasing family of subsets of $\mathbb{R}^d$ and $\mathcal{U}_\rho$ is defined by (2.1). For equation (2.3) and fixed $u \in \mathcal{U}_\rho$ define the (upper) Bohl exponent as (cp. [13, Proposition 2.2])

$$k_B(u) = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_u(t,s)\|,$$

where $\Phi_u(t,s)$ is the fundamental matrix of (2.3) with $\Phi_u(s,s) = \mathrm{id}$. Let the Bohl stability radius $r_B(A)$ of (2.3) w.r.t. U_ρ be defined by

$$r_B(A) = \inf\{\rho \geq 0;\ \sup_{u\in\mathcal{U}_\rho} k_B(u) \geq 0\}.$$

Theorem 5. *Suppose that for all* $\rho > 0$ *the projection* $(4.1)_\rho$ *of system* $(2.3)_\rho$ *satisfies* (H). *Then the Bohl and Lyapunov stability radii* $r_B(A)$ *and* $r_L(A)$ *of* A *with respect to* U_ρ *coincide, i.e.*

$$r_B(A) = r_L(A).$$

PROOF. The inequality $r_B(A) \leq r_L(A)$ follows from $k_B(u) \geq \lambda(x_0, u)$ for all $x_0 \neq 0$, $u \in \mathcal{U}$. We will show that the converse inequality follows from Lemma 5.4 in [8].

First observe that for all $t \geq s$ and all $u \in \mathcal{U}$

$$\Phi_u(t,s) = \Phi_{u(\cdot+s)}(t-s,0)$$

where $u(\cdot + s)$ is the shifted control function

$$u(\cdot + s)(\tau) = u(\tau + s), \quad \tau \in \mathbb{R}.$$

Hence for all $u \in \mathcal{U}$

$$\begin{aligned} &\limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_u(t,s)\| \\ = &\limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_{u(\cdot+s)}(t-s,0)\| \\ \le &\limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \sup_{v\in\mathcal{U}} \|\Phi_v(t-s,0)\|. \end{aligned}$$

Hence

$$\sup_{u\in\mathcal{U}} k_B(u) \le \limsup_{t\to\infty} \frac{1}{t} \log \sup \|\Phi_v(t,0)\| =: \delta$$

where the supremum at the right hand side is taken over all piecewise constant $v \in \mathcal{U}$. Now Lemma 5.4 in [8] claims that $\delta = \kappa$, hence

$$\sup_{u\in\mathcal{U}} k_B(u) = \kappa = \sup_{\substack{u\in\mathcal{U} \\ x_0\neq 0}} \lambda(x_0, u),$$

which immediately implies the assertion. Q.E.D.

Next we note the following duality result.

Proposition 6. *Suppose that for all $\rho > 0$ the projection $(4.1)_\rho$ of system $(2.3)_\rho$ satisfies (H). Then the stability radius $r_L(A)$ of A w.r.t. U_ρ satisfies*

$$r_L(A) = r_L^*(-A^T)$$

where $r_L(-A^T)$ is the stability radius of $-A^T$ w.r.t. $-U_\rho^T$, $\rho > 0$.

PROOF. For the extremal Lyapunov exponents we have [9] that ω_ρ of the system (2.3) equals $-\alpha_\rho^-$ for the following time reversed control system

$$\dot{x}(t) = -[A + u(t)]^T x(t), \quad t \ge 0, \quad u \in \mathcal{U}_\rho. \qquad (5.1)_\rho$$

Hence the assertion follows from Corollary 4. Q.E.D.

The result above shows that it suffices to concentrate on r_L and $\tilde{r}_L$, what we will do in the sequel.

Remark 5. The duality concerning the Lyapunov exponents of $(2.3)_\rho$ and $(5.1)_\rho$ referred to above is not valid in general for individual control functions u. It is a property of bilinear control systems satisfying (H), and relies on the fact, established in [8, Theorem 4.2 and Theorem 5.1], that for fixed ρ the supremal exponential growth rate $\kappa_\rho = \omega_\rho$ can approximately be attained by using controls and trajectories having a common period. By the way, note that $r_L(A)$ w.r.t. U_ρ is the same as $r_L(A^T)$ w.r.t. U_ρ^T. □

Remark 6. From Theorem 2(iii) we obtain immediately $r_L^* = \tilde{r}_L$ if $C_\rho \cap C_\rho^- \neq \emptyset$ for $\rho = r_L^*$. We will see later on in Section 6 that $\tilde{r}_L$ indeed depends crucially on the control structure of the projected system $(4.1)_\rho$. □

Next we discuss the effect of unbounded parameter variations. We separate the bounded and the unbounded uncertainties:

$$\dot{x} = [A + u(t) + v(t)]x(t), \quad x(0) = x_0 \in \mathbb{R}^d \tag{$5.2)_\rho$}$$

where $\{U_\rho^b, \rho \geq 0\}$ is an increasing family of compact sets in $M(d,d;\mathbb{R})$, $v(t) = \sum_{i=1}^{p} v_i(t)A_i$, $v_i(t) \in \mathbb{R}$, $A_i \in M(d,d;\mathbb{R})$, $i = 1,\dots,p \leq d^2$. Thus $U_\rho = U_\rho^b \times \{\sum_{i=1}^{p} v_iA_i;\ v_i \in \mathbb{R}$ for $i = 1,\dots,p\}$ (note that we do not allow here "one-sided unbounded" uncertainties). Denote by

$$N_\rho = \{A + u + v;\ u \in U_\rho^b, v = \sum_{i=1}^{p} v_iA_i, v_i \in \mathbb{R} \text{ for } i = 1,\dots,p\}$$

the possible constant right hand sides of $(5.2)_\rho$ and by

$$N^u = \{\sum_{i=1}^{p} v_iA_i, v_i \in \mathbb{R} \text{ for } i = 1,\dots,p\}$$

the unbounded part. For a matrix $M \in M(d,d;\mathbb{R})$ define

$$M^0 := M - \frac{1}{d}\operatorname{trace} M \cdot \mathrm{Id}$$

and let $N_\rho^0 = \{M^0, M \in N_\rho\}$, $N^{u,0} = \{M^0, M \in N^u\}$. Define the systems group and semigroup generated by N_ρ as

$$\begin{aligned}
\mathcal{G}_\rho &= \{e^{t_nB_n}\cdots e^{t_1B_1};\ B_i \in N_\rho,\ t_i \in \mathbb{R},\ i = 1,\dots,n \in \mathbb{N}\} \\
\mathcal{S}_\rho &= \{e^{t_nB_n}\cdots e^{t_1B_1};\ B_i \in N_\rho,\ t_i \geq 0,\ i = 1,\dots,n \in \mathbb{N}\}.
\end{aligned}$$

Analogously denote the groups and semigroups generated by N^u, N_ρ^0, $N^{u,0}$ by $\mathcal{G}^u$, $\mathcal{G}_\rho^0$, $\mathcal{G}^{u,0}$. The following estimates for κ, κ^* are given in [2, Theorem 2.3].

Theorem 7. *Assume that the family of systems* $(5.2)_\rho$ *satisfies* (H) *for all* $\rho > 0$. *Then the following holds:*

1. *If* $\overline{\mathcal{G}}^u$ *is not compact (in* $M(d,d;\mathbb{R})$*), then* $\kappa_\rho^* = -\infty$, $\kappa_\rho = \infty$.

2. *If* $\overline{\mathcal{G}}^u$ *is compact, then* $-\infty < \kappa_\rho^* \leq \kappa_\rho < +\infty$.

   2.1.1 If $\mathcal{G}_\rho$ *is not compact, and* $\mathcal{G}_\rho^0$ *is not compact, then* $\kappa_\rho^* < \kappa_\rho$.

   2.1.2 If $\mathcal{G}_\rho$ *is not compact, but* $\mathcal{G}_\rho^0$ *is compact then*

   (i) if $\frac{1}{d}\,\mathrm{trace}(A+u) \equiv c$, *then* $\kappa_\rho^* = \kappa_\rho = c$

   (ii) if $\frac{1}{d}\,\mathrm{trace}(A+u)$ *is not constant, then*

$$\begin{aligned}\kappa_\rho^* &= \frac{1}{d}\min_{u\in U_\rho^b}\mathrm{trace}(A+u)\\ &< \frac{1}{d}\max_{u\in U_\rho^b}\mathrm{trace}(A+u) = \kappa_\rho.\end{aligned}$$

   2.2 If $\mathcal{G}_\rho$ *is compact, then* $\kappa_\rho^* = \kappa_\rho = 0$.

Note that the group $\mathcal{G}_\rho \subset \mathrm{Gl}(d,\mathbb{R})$ (the group of invertible $d \times d$-matrices) is compact, if there exists $T \in \mathrm{Gl}(d,\mathbb{R})$ such that $TN_\rho T^{-1} \subset \mathrm{so}(d,\mathbb{R})$, the skew symmetric matrices; similarly for the other groups.

Theorem 7(1) implies in particular that the system $(5.2)_\rho$ has $\kappa_\rho^* = -\infty$, $\kappa_\rho = +\infty$ unless there exists a basis in $\mathbb{R}^d$, in which the unbounded part gives rise only to rotations. If we are in case (2), then in this basis the unbounded part does not contribute to the Lyapunov exponents at all (since $s^*Bs = 0$ for skew symmetric B). Note also that if a system semigroup $\mathcal{S}_\rho$ satisfies $\overline{\mathcal{S}}_\rho$ is compact, then $\overline{\mathcal{G}}_\rho = \overline{\mathcal{S}}_\rho$ and — under (H) — the corresponding system is completely (exactly) controllable on $\mathbb{P}$, i.e. $C_\rho = C_\rho^- = \mathbb{P}$ (cp. [3, Corollary 3.2]). Hence the cases (2.1.2) and (2.2) above describe completely controllable situations on $\mathbb{P}$.

This theorem has some immediate implications for the analysis of r_L.

Corollary 8. *Assume that the family* $(5.2)_\rho$ *satisfies* (H) *for all* $\rho > 0$ *and that the stability radius of* A *with respect to* $\{U_\rho^b; \rho \geq 0\}$ *is positive. Then the following are equivalent:*

(i) $r_L(A) > 0$,

(ii) $\overline{\mathcal{G}}^u$ *is compact,*

(iii) there exists a basis in $\mathbb{R}^d$ *such that all elements of* N^u *are skew symmetric.*

PROOF. By Theorem 7(1), assertion (i) implies (ii), and by the remarks following Theorem 7, assertions (ii) and (iii) are equivalent. Finally (iii) implies (i), since here the unbounded part does not contribute to κ_ρ, in the basis chosen according to (iii). (Note by the way, that $r_L(A)$ w.r.t. $\{U_\rho\}$ is the same as $r_L(TAT^{-1})$ w.r.t. $\{TU_\rho T^{-1}\}$, $T \in \mathrm{Gl}(d, \mathbb{R})$. Q.E.D.

Another consequence of Theorem 7 is the characterization of infinite stability radii: Let $\{U_\rho^b; \rho \geq 0\}$ be defined by a norm, i.e. for given matrices $B \in M(d, m; \mathbb{R})$, $C \in M(k, d, \mathbb{R})$ let $U_\rho^b := \{BDC \in M(d, d; \mathbb{R}); \|D\| \leq \rho\}$, where $\|\cdot\|$ is any norm in $M(m, k; \mathbb{R})$ (compare the structured or unstructured case in [13, 16]). If the assumptions of Corollary 8 are met, then $r_L(A) = \infty$ iff Theorem 7.2.1.2(i) holds with $c < 0$ for all $\rho \in (0, \infty)$.

Next we consider the following "interval" case when the family $\{U_\rho; \rho \geq 0\}$ is given by

$$U_\rho = \{M \in M(d, d; \mathbb{R});\ -\rho \leq m_{ij} \leq \rho,\ (i,j) \in I \qquad (5.3)$$
$$\text{and } m_{ij} = 0,\ (i,j) \in \underline{d} \times \underline{d} \backslash I\}$$

here I is a subset of $\underline{d} \times \underline{d} := \{1, \ldots, d\} \times \{1, \ldots, d\}$ such that m_{ii} is allowed to be nonzero for at least one $i \in \{1, \ldots, d\}$.

Corollary 9. *Assume that A is stable, and for all $\rho > 0$ and U_ρ given by (5.3), $\mathcal{G}_\rho$ is not compact, but $\mathcal{G}_\rho^0$ is compact. Then the corresponding stability radius of A satisfies*

$$r_L(A) = -\frac{1}{l} \operatorname{trace} A.$$

where l is the number of diagonal elements which are allowed non-zero in U_ρ.

PROOF. By Theorem 7 case (2.1.2), it holds for all $\rho > 0$ that

$$\kappa_\rho = \frac{1}{d} \max_{u \in U_\rho} \operatorname{trace}(A + u) = \frac{1}{d}[\operatorname{trace} A + \max_{u \in U_\rho} \operatorname{trace} u].$$

Hence $r_L(A) = \rho$ iff $\max_{u \in U_\rho} \operatorname{trace} u = -\operatorname{trace} A$. Now $\mathcal{G}_\rho^0$ compact and A stable (hence $\operatorname{trace} A < 0$) imply: For each $\rho > 0$ there is at least one $M \in U_\rho$ with $m_{ii} = \rho$ for l diagonal elements m_{ii}. Therefore $r_L(A) = -\frac{1}{l} \operatorname{trace} A$. Q.E.D.

Now let A be an unstable matrix. From Remark 6 and $r_L^* \leq \tilde{r}_L$, we have that $\tilde{r}_L(A) = 0$ if $\overline{\mathcal{G}}^u$ is not compact and $C_\rho \cap C_\rho^- \neq \emptyset$ for all $\rho > 0$.

Remark 7. Proposition 3 and Corollary 4 show that determining the time varying stability radius r_L means solving a parametrized optimal control problem:

One has to find

$$\inf\{\rho > 0;\ \sup_{u\in\mathcal{U}_\rho, s_0} \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(A+u(\tau), s(\tau, s_0, u))\, d\tau \geq 0\}$$

where $s(\tau, s_0, u)$ means the solution of the projected system $(4.1)_\rho$ for initial state s_0 and control u. In [8] it is shown that in fact it suffices to take the supremum over all piecewise constant control functions u and corresponding trajectories having a common period $T > 0$, T arbitrary. Similarly for $\tilde{r}_L$.

□

Thus the problem of finding r_L can considerably be simplified; nevertheless it still requires optimization over function spaces. Thus contrary to what is known for the time invariant radii $r_{\mathbb{C}}$ and $r_{\mathbb{R}}$, we obtain explicit formulae for r_L only in very special cases, compare Theorem 7 and Corollary 9, and Example 8 in the next section. A general strategy is to solve (for fixed $\rho > 0$) the optimal control problem numerically and to extract the point(s) where $\kappa_\rho = \omega_\rho = 0$. This is demonstrated for the linear oscillator with uncertain restoring force in the next section. The computed Lyapunov exponents κ_ρ, $\rho \neq r_L$, may also be used for design purposes, see Section 8.

6 Examples

In this section we will discuss the relations between the Lyapunov stability radius defined above and the stability radius for matrices under constant complex- or real-valued perturbations. In particular, we will consider the linear oscillator with uncertainties in the restoring force.

We use the following set-up. Let $A \in M(d,d;\mathbb{R})$ be stable and let $B \in M(d,m;\mathbb{R})$, $C \in M(k,d;\mathbb{R})$ be given. Denote for $\rho \geq 0$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$$U_\rho^{(2)} := \{BDC; D \in M(m,k,\mathbb{K}) \text{ with } \|D\|_2 \leq \rho\} \subset M(d,d,\mathbb{K})$$
$$U_\rho^{(m)} := \{BDC; D \in M(m,k,\mathbb{K}) \text{ with } \|D\|_m \leq \rho\} \subset M(d,d,\mathbb{K})$$

here $\|\cdot\|_2$ is the operator norm induced by the Euclidean norm on $\mathbb{K}^2$ and $\|\cdot\|_m$ is defined by

$$\|D\|_m = \max_{i,j} |d_{ij}|, \quad D = (d_{ij})_{i,j},$$

corresponding to the case of interval uncertainties. We denote the Lyapunov stability radii of A with respect to the families $\{U_\rho^{(2)}; \rho \geq 0\}$ and $\{U_\rho^{(m)}; \rho \geq$

$0\}$, where $U_\rho^{(2)}$, $U_\rho^{(m)} \subset M(d,d;\mathbb{R})$, by $r_L^{(2)}$ and $r_L^{(m)}$ respectively. We denote the stability radii of A with respect constant complex- and real-valued perturbations in $U_\rho^{(2)}$ and $U_\rho^{(m)}$ by

$$r_{\mathbb{C}}^{(2)}, \quad r_{\mathbb{R}}^{(2)}, \quad r_{\mathbb{C}}^{(m)}, \quad r_{\mathbb{R}}^{(m)}.$$

The objects studied in [13, 16] are $r_{\mathbb{C}}^{(2)}$ and $r_{\mathbb{R}}^{(2)}$; by [16, Proposition 2.2 and Proposition 5.2] $r_{\mathbb{C}}^{(2)}$ is also the Bohl stability radius with respect to complex-valued time-varying perturbations. Hence

$$r_{\mathbb{C}}^{(2)} \leq r_L^{(2)}$$

The inequality $\|D\|_m \leq \|D\|_2$ implies

$$r_{\mathbb{C}}^{(m)} \leq r_{\mathbb{C}}^{(2)}, \quad r_{\mathbb{R}}^{(m)} \leq r_{\mathbb{R}}^{(2)}, \quad r_L^{(m)} \leq r_L^{(2)}.$$

Furthermore the inequalities

$$r_{\mathbb{C}}^{(m)} \leq r_{\mathbb{R}}^{(m)}, \quad r_{\mathbb{C}}^{(2)} \leq r_{\mathbb{R}}^{(2)}, \quad r_L^{(m)} \leq r_{\mathbb{R}}^{(m)}, \quad r_L^{(2)} \leq r_{\mathbb{R}}^{(2)}$$

are obvious from the definitions. The following Examples 8 and 9 show that

$$r_L^{(m)} < r_{\mathbb{C}}^{(2)} = r_L^{(2)} = r_{\mathbb{R}}^{(2)}$$

and

$$r_{\mathbb{C}}^{(m)} = r_{\mathbb{C}}^{(2)} < r_L^{(m)} = r_L^{(2)} < r_{\mathbb{R}}^{(m)} = r_{\mathbb{R}}^{(2)}$$

are possible. We do not know whether $r_L^{(m)} < r_{\mathbb{C}}^{(m)}$ is possible.

We now start the analysis of two-dimensional examples.

Consider the matrix

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \in M(2,2;\mathbb{R}).$$

To obtain explicit formulas for the projected system on $\mathbb{P}$, we write

$$s = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \in \mathbb{P}, \quad \varphi \in [0,\pi).$$

Then one computes

$$\begin{aligned} h(D,\varphi) &= -d_2\sin^2\varphi + d_3\cos^2\varphi + \frac{1}{2}(d_1 - d_4)\sin 2\varphi \\ q(D,\varphi) &= \frac{1}{2}(d_1 + d_4) + \frac{1}{2}(d_1 - d_4)\cos 2\varphi + \frac{1}{2}(d_2 + d_3)\sin 2\varphi, \end{aligned}$$

here h and q are first defined as in (3.2) and (3.3), and then the induced maps with the second argument in $[0, \pi)$ are denoted by the same symbols.

Note that

$$\begin{aligned} &\|D\|_m = \max|d_i| \\ &\leq \|D\|_2 = \tfrac{1}{2}\textstyle\sum_{i=1}^4 d_i^2 + \sqrt{(d_1d_2 + d_3d_4)^2 + \tfrac{1}{4}(d_1^2 + d_3^2 - d_2^2 - d_4^2)^{1/2}} \\ &\leq 2\|D\|_m \end{aligned}$$

Hence it may not come as a surprise that the strict inequality $r_L^{(m)} < r_{\mathbb{C}}^{(2)}$ can occur, as the following example shows.

Example 8. Consider

$$A = \begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix}$$

Clearly $q(A, \varphi) \equiv -a$, and

$$\max_{D \in U_\rho^{(m)}} \max_{\varphi \in \mathbb{P}} q(D, \varphi) = 2\rho.$$

The maximum is attained for $d_i = \rho$, $i = 1, \ldots, 4$ and $\varphi = \pi/4$. The eigenvalues and eigenvectors of

$$D = \begin{pmatrix} d & d \\ d & d \end{pmatrix}$$

are $\lambda_1 = 0$, $\lambda_2 = 2d$, $e_1 = \frac{3}{4}\pi$, $e_2 = \frac{1}{4}\pi$. Since $q(A + D, \varphi) = q(A, \varphi) + q(D, \varphi) = -a + q(D, \varphi)$, we have

$$\max_{D \in \mathcal{U}_\rho} \max_{\varphi \in \mathbb{P}} q(A + D, \varphi) = -a + 2\rho$$

and hence

$$r_L^{(m)} = \frac{a}{2}.$$

On the other hand

$$r_{\mathbb{C}}^{(2)} = r_{\mathbb{R}}^{(2)} = a \quad \text{(see [18, pp. 41])},$$

and hence

$$r_L^{(m)} < r_{\mathbb{C}}^{(2)} = r_L^{(2)} = r_{\mathbb{R}}^{(2)}.$$

Note that for the matrix D above $\|D\|_2 = 2d$. □

The next example shows the more surprising fact that

$$r_{\mathbb{C}}^{(m)} = r_{\mathbb{C}}^{(2)} < r_L^{(m)} = r_L^{(2)} < r_{\mathbb{R}}^{(m)} = r_{\mathbb{R}}^{(2)}$$

can occur.

Example 9 (The linear oscillator with uncertain restoring force).

Consider the linear oscillator described by

$$\ddot{y} + 2b\dot{y} + (1+u)y = 0$$

or with $x = (x_1, x_2)^T = (y, \dot{y})^T$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + u(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x$$

where $u(t) \in [-\rho, \rho]$, $\rho \geq 0$, i.e. $U_\rho = \left\{ \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} ; \alpha \in [-\rho, \rho] \right\}$ and $b \in \mathbb{R}$ is a constant. This yields structured stability radii (cp. [16]) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Since the perturbation is one-dimensional, the equalities $r_\cdot^{(m)} = r_\cdot^{(2)} = r_\cdot$ are obvious.

Projection onto the sphere gives in polar coordinates with $s = (\cos\varphi, \sin\varphi)^T$, $\varphi \in [0, 2\pi)$ (compare for this example [2, Section 6]):

$$\begin{aligned} \dot{\varphi}(t) &= -\sin^2\varphi(t) + [b^2 - 1 - u(t)]\cos^2\varphi(t) \\ q(A+u, \varphi) &= \frac{1}{2}(b^2 - u)\sin 2\varphi - b. \end{aligned}$$

We restrict our attention to the case where $0 \leq b \leq 2$. Then one easily finds

$$\min_\varphi |q(A, \varphi)| = -\frac{1}{2}b^2 + b.$$

Figure 1 presents the different stability radii for $b \in [0, 1]$ (for $b > 1$ we have $r_\mathbb{C} = r_\mathbb{R} = r_L = 1$), as well as two estimates (Note that in the following we suppress the dependence on the matrix A, instead we indicate by the argument b, that all entities depend on the actual value of the damping b).

(a) is the real stability radius $r_\mathbb{R}(b) (\equiv 1)$, which follows from elementary considerations;

(b) is the time varying radius $r_L(b)$, its computation is described in [2] as the level curve $\gamma = 0$ (cp.[9]);

(c) is the complex radius $r_\mathbb{C}(b)$, computed using the program STABRAD, see [18];

(d) is the estimate via $\min_\varphi |q(A, \varphi)| = b - \frac{1}{2}b^2$;

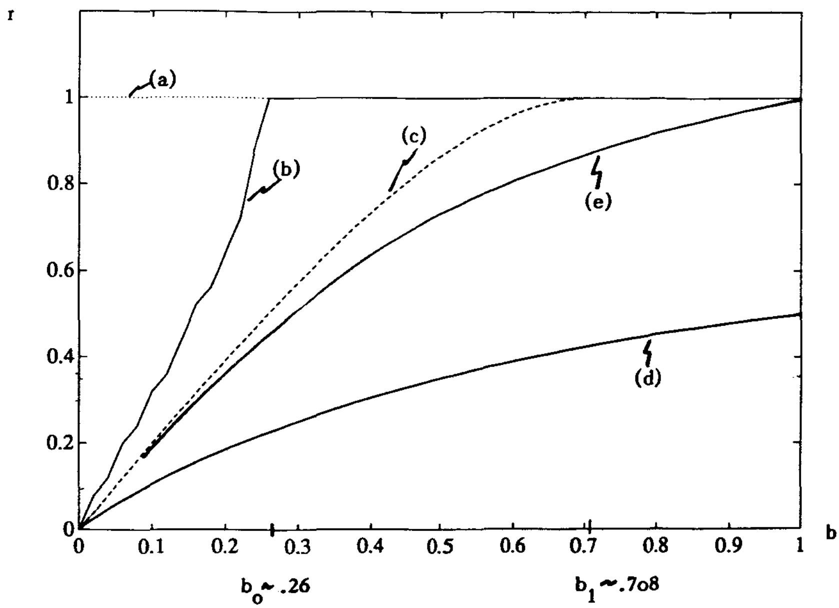

Figure 1. Stability radii for the linear oscillator

(e) is the estimate $\rho_*(b) = 2b - b^2$ according to (3.5).

□

For $\rho \downarrow 0$ and $0 \leq b \leq 1$ we have for the maximal Lyapunov exponent $\kappa_\rho(b)$

$$\kappa_\rho(b) = -b + \frac{1}{\pi\sqrt{1-b^2}}\rho + o(\rho)$$

see [2, Section 6]. Hence for b small one obtains

$$r_L(b) \sim \pi \cdot b$$

and also (by numerical evidence, compare Fig. 1)

$$r_{\mathbb{C}}(b) \sim 2b.$$

We have

$$\begin{aligned} 0 &< r_{\mathbb{C}}(b) < r_L(b) < r_{\mathbb{R}}(b) = 1 \text{ for } 0 < b < b_0 \\ 0 &< r_{\mathbb{C}}(b) < r_L(b) = r_{\mathbb{R}}(b) = 1 \text{ for } b_0 \leq b < b_1 \\ 0 &< r_{\mathbb{C}}(b) = r_L(b) = r_{\mathbb{R}}(b) = 1 \text{ for } b_1 < b. \end{aligned}$$

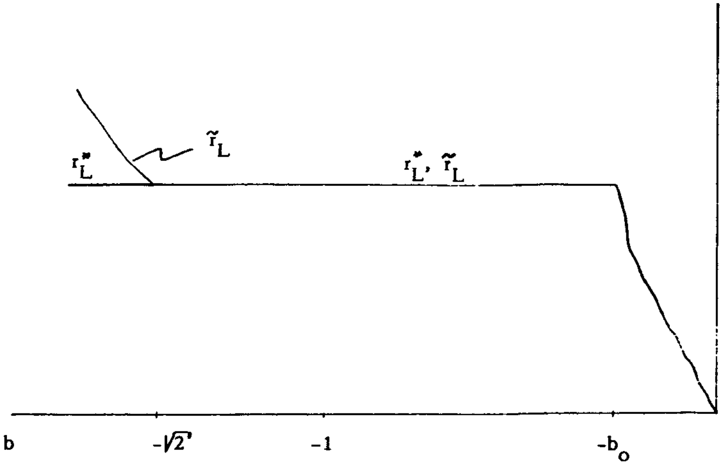

Figure 2. Instability radii for the linear oscillator

The Lyapunov stability radius $r_L(b)$ is monotone increasing in b for $b \geq 0$ (this follows directly from the explicit formulae for $\kappa(b)$ in [2, Section 6]). From the numerics presented in Figure 1, it also seems that $r_{\mathbb{C}}(b)$ is monotone. For the relative sizes we have from the numerical results:

$$\frac{r_{\mathbb{C}}(b)}{r_{\mathbb{R}}(b)} \in (0,1] \quad \text{monotone in } b,$$

$$\frac{r_{\mathbb{C}}(b)}{r_L(b)} \in [\delta, 1] \quad \text{with } 0 < \delta \sim 0.5, \text{ not monotone in } b.$$

For the linear oscillator discussed above we can also compute r^*_L and $\tilde{r}_L$, now of course for $b < 0$. Figure 2 presents the curve $r^*_L(b)$, which is easily obtained from $r_L(b)$, compare Proposition 6. The curve for $\tilde{r}_L$ is more complicated, since it depends on the control structure of the projected system. It turns out that

$$C = C^- = \mathbb{P} \quad \text{iff} \quad \rho + 1 > b^2 \quad \text{cp. [9]}$$

Thus according to Remark 6, $r^*_L = \tilde{r}_L$ if $b \in [-\sqrt{2}, 0]$. For $b < -\sqrt{2}$, one has $r^*_L < \tilde{r}_L$. Figure 2 presents the curve $\tilde{r}_L(b)$.

Finally, we consider the uniform radii $\hat{r}_L$ and $\hat{r}_L^*$ introduced in Remark 1. If A is stable, i.e. $b > 0$, then

$$\hat{\lambda}_\rho(b) = \begin{cases} -b & \text{if } \dfrac{b^2-1}{\rho} < 1 \\ -b - \sqrt{-1-\rho+b^2} & \text{if } 1 \leq \dfrac{b^2-1}{\rho}, \end{cases}$$

hence $\hat{\lambda}_\rho < 0$ for all $\rho > 0$, i.e. $\hat{r}_L = \infty$. Thus uniform destabilization is not possible in this situation: For all $u \in \mathcal{U}_\rho$, $\rho \in (0, \infty)$ there always exists $x_u \neq 0$ such that $\lambda(x_u, u) < 0$.

By duality one gets for unstable A, i.e. $b < 0$, that $\hat{r}_L^* = \infty$.

7 Stochastic Stability Radii and Large Deviations

In this section we introduce two stochastic stability radii and discuss their relation to r_L. We also analyze how a stochastic system behaves, if its stochastic stability radius is larger than r_L: This indicates the "presence of large deviations". Since the case with unbounded uncertainty can be reduced to the bounded case, using Theorem 7, we restrict ourselves to bounded situations. In order to be able to compare stochastic stability radii with r_L, we use the following set-up:

Let M be a finite dimensional compact, connected (Riemannian) real analytic manifold and

$$D_\rho : M \to M(d, d; \mathbb{R})$$

a family of (real-) analytic maps for $\rho \geq 0$. Assume again that the family $U_\rho := D_\rho(M)$ is increasing. On M we consider two different classes of stochastic processes:

Stat: the stationary processes with values in M, i.e.
Denote by $D(M)$ the Skorohod space of piecewise continuous functions $\xi : [0, \infty) \to M$, right continuous with left hand limits, and by $\mathcal{B}(M)$ the Borel σ-algebra of $D(M)$. Let $\theta_t : D(M) \to D(M)$ be the shift on $D(M)$ for each $t \geq 0$, i.e. $\theta_t \xi(\cdot) = \xi(t + \cdot)$. Then every (strictly) stationary process with values in M and right continuous trajectories with left hand limits is given by a θ_t-invariant probability measure ν on $(D(M), \mathcal{B}(M))$, i.e. $\theta_t \nu = \nu$ for all $t \geq 0$. (This fact is known as Kolmogorov's construction.) In particular, if $\xi \in D(M)$ is a periodic function with period T, then $\eta(t) = \xi(t + \tau)$ is a stationary process with τ uniformly distributed in $[0, T]$.

Diff: the stationary, ergodic, nondegenerate diffusion processes on M, i.e.

$$d\xi_t = X_0(\xi_t)dt + \sum_{i=1}^{r} X_i(\xi_t) \circ dW_i,$$

where $X_0, \ldots, X_r$ are real analytic vector fields on M with

$$\dim \mathcal{LA}\{X_1, \ldots, X_r\}(m) = \dim M$$

for all $m \in M$. "$\circ$" denotes the symmetric (Stratonovič) stochastic integral.
Again the process ξ_t can be viewed as a flow Φ_t with flow-invariant ergodic measure ν, on the space $\Omega \times M$, where $(\Omega, \mathcal{B}(\Omega), P)$ is the Wiener space of continuous functions $\omega: [0, \infty] \to \mathbb{R}^r$, with its Borel σ-algebra and the Wiener measure P. Here Φ_t is defined as $\Phi_t: \Omega \times M \to \Omega \times M$, $\Phi_t(\omega, p) = (\hat{\theta}_t\omega, \varphi(t, \omega, p))$, with $\hat{\theta}_t: \Omega \to \Omega$, $\hat{\theta}_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)$, the Wiener shift, and $\varphi(t, \omega, p)$ the (pathwise) solution of the stochastic differential equation at time t with initial value $p \in M$.

For background information on differential equations driven by stationary processes see e.g. the survey paper [20].

For a matrix $A \in M(d, d; \mathbb{R})$ we now consider the systems

$$\dot{x} = [A + D_\rho(\xi_t)]x, \quad x(0) = x_0 \in \mathbb{R}^d, \tag{7.1$_\rho$}$$

again with assumption (H) as in Section 4 for the class Stat, and with the assumption

$$\begin{aligned} \dim \mathcal{LA}\{X_0 + h, X_1, \ldots, X_r\}(m, s) &= \dim M + d - 1 \\ \text{for all}(m, s) &\in M \times \mathbb{P}^{d-1} \end{aligned} \tag{H'}$$

for the class Diff.

We define stochastic stability radii for the classes Stat and Diff in the following way:

The Lyapunov exponents of $(7.1)_\rho$ are

$$\lambda_\rho(x_0, \xi_t) = \lim_{t \to \infty} \frac{1}{t} \log |\varphi(t, x_0, \xi_t)|$$

where φ solves $(7.1)_\rho$.

By Oseledeč's Theorem these random numbers exist with probability 1. Define

$$\kappa_\rho(\xi_t) := \operatorname{ess\,sup} \sup_{x_0 \neq 0} \lambda_\rho(x_0, \xi_t)$$

where the essential supremum is taken w. r. t. the measure ν associated with ξ_t, and let the stability radius with respect to ξ_t be

$$r(\xi_t) := \inf\{\rho \geq 0;\ \kappa_\rho(\xi_t) \geq 0\}.$$

Define the stability radii w.r.t. the classes Stat and Diff

$$r_{\text{Stat}} := \inf\{\rho \geq 0;\ \sup_{\xi_t \in \text{Stat}} \kappa_\rho(\xi_t) \geq 0\}$$
$$r_{\text{Diff}} := \inf\{\rho \geq 0;\ \sup_{\xi_t \in \text{Diff}} \kappa_\rho(\xi_t) \geq 0\}.$$

Obviously $r_{\text{Stat}} \leq r_{\text{Diff}}$. Similarly, stochastic instability radii can be defined

$$\kappa_\rho^*(\xi_t) := \operatorname{ess\,inf} \inf_{x_0 \neq 0} \lambda_\rho(x_0, \xi_t)$$
$$\kappa_\rho^*(\xi_t, x_0) := \operatorname{ess\,inf} \lambda_\rho(x_0, \xi_t),$$

and

$$r^*(\xi_t) := \inf\{\rho \geq 0;\ \kappa_\rho^*(\xi_t) \leq 0\}$$
$$\tilde{r}(\xi_t) := \inf\{\rho \geq 0;\ \kappa_\rho^*(\xi_t, x_0) \leq 0 \text{ for all } x_0 \neq 0\},$$

and analogously for r^* and $\tilde{r}$ w.r.t. Stat and Diff using the inf's.

Remark 10. We have defined stochastic stability radii in complete analogy to deterministic ones, using the unifying concepts of Lyapunov exponents. It is important to notice that Lyapunov exponents of a stochastic, stationary process ξ_t do not only depend on the statistics of ξ_t. In fact, if e.g. the generator $L = X_0 + \frac{1}{2}\sum X_i^2$ of a stationary, ergodic, nondegenerate diffusion process ξ_t is given (and therefore the statistics are determined), there are in general many ways to take square roots of the second order part of L, and hence different choices for the vector fields in the associated stochastic differential equation. The Lyapunov exponents of ξ_t depend on the associated stochastic flow (the *multipoint motion* as described in the definition of the classes Stat and Diff), i.e. for diffusion processes on the vector fields $X_0, \ldots, X_r$. It is Oseledeč Multiplicative Ergodic Theorem, which shows that for stationary stochastic processes a theory of Lyapunov exponents can be developed in complete analogy to the deterministic Lyapunov regular case — with probability one, see [20] for more details and examples concerning this point. □

The stability radii r_{Stat} and r_{Diff} are defined via the almost sure Lyapunov exponents of the system $(7.1)_\rho$. We will also consider the Lyapunov exponents of the p-th moment $(0 < p < \infty)$

$$g_\rho(x_0, \xi_t, p) = \lim_{t\to\infty} \frac{1}{t} \log E(|\varphi(t, x_0, \xi_t)|^p), \quad \varphi \text{ solves } (7.1)_\rho,$$
$$g_\rho(\xi_t, p) = \sup_{x_0 \neq 0} g_\rho(x_0, \xi_t, p),$$

and the corresponding stability radii

$$\begin{aligned} r^{[p]}(\xi_t) &= \sup\{\rho \geq 0;\ g_\rho(\xi_t, p) \leq 0\},\ 0 < p < \infty \\ r^{[\infty]}(\xi_t) &= \sup\{\rho \geq 0;\ g_\rho(\xi_t, p) \leq 0 \text{ for all } 0 < \rho < \infty\} \end{aligned}$$

and for the classes Stat and Diff, $0 < p < \infty$

$$\begin{aligned} r^{[p]}_{\text{Stat}} &= \sup\{\rho \geq 0,\ \sup_{\xi_t \in \text{Stat}} g_\rho(\xi_t, p) \leq 0\} \\ r^{[p]}_{\text{Diff}} &= \sup\{\rho \geq 0,\ \sup_{\xi_t \in \text{Diff}} g_\rho(\xi_t, p) \leq 0\}, \end{aligned}$$

analogously for $r^{[\infty]}_{\text{Stat}}$, $r^{[\infty]}_{\text{Diff}}$. In particular, in the engineering literature, p-th moment stability is often considered.

The situation for the class Stat is relatively simple:

Theorem 10. *Let $A \in M(d, d; \mathbb{R})$ be stable. Then under the assumptions above*

$$\frac{1}{p} r^{[p]}_{\text{Stat}} \geq r^{[\infty]}_{\text{Stat}} = r_{\text{Stat}} = r_L \textit{ for all } p > 0,$$

where r_L is taken w.r.t. $U_\rho = D_\rho(M)$. If the initial values x_0 can be taken as bounded random variables, then furthermore $\frac{1}{p} r^{[p]}_{\text{Stat}} = r^{[\infty]}_{\text{Stat}}$ for all $p > 0$.

Proof. The key for the proof of this theorem is the fact that the extremal Lyapunov exponent ω_ρ (as well as $\alpha_\rho, \alpha^-_\rho$) can be approximated by Lyapunov exponents of periodic solutions of $(4.1)_\rho$, compare [8, Theorem 5.1], and [9, Theorem 4.2]. Now a periodic solution $s^0(t, u^0)$ gives rise to a stationary process ξ^0_t, as outlined in the description of the class Stat above, and $\lambda(s^0) = \kappa(\xi^0_t)$ by Corollary 4.4 in [2]. The equality $r_{\text{Stat}} = r_L$ follows from this.

Next, consider for given $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$ and ξ_t in Stat the function $p \mapsto \frac{1}{p} g_\rho(x_0, \xi_t, p)$. This function is continuous and increasing on $(0, \infty]$. Denote its limit at $+\infty$ by $g_\rho(x_0, \xi_t, \infty)$, and let $g_\rho(\xi_t, \infty) = \sup_{x_0 \neq 0} g_\rho(x_0, \xi_t, \infty)$. Then we have by definition $g_\rho(x_0, \xi_t, p) \leq 0$ for all $0 < p < \infty$ iff $g_\rho(\xi_t, \infty) \leq 0$. Corollary 4.4 in [2] now says $r^{[\infty]}_{\text{Diff}} = r_{\text{Stat}}$, and the inequalities $r^{[\infty]}_{\text{Diff}} \geq r^{[\infty]}_{\text{Stat}}$ and $r^{[\infty]}_{\text{Stat}} \geq r_{\text{Stat}}$ are obvious, hence $r^{[\infty]}_{\text{Stat}} = r_{\text{Stat}}$.

The inequality $\frac{1}{p} r^{[p]}_{\text{Stat}} \geq r^{[\infty]}_{\text{Stat}}$ follows directly from the monotonicity of $\frac{1}{p} g_\rho(x_0, \xi_t, p)$ for $p > 0$.

Finally, if x_0 can be taken as a random variable, then we can associate with the periodic solution $s^0(t, u^0)$ a stationary ξ^0_t and x_0 such that

$$E(|\varphi(t, x_0, \xi_t)|^p)^{\frac{1}{p}} = |\varphi(t, x_0, \xi_t)| \quad \text{w.p.1}$$

by defining x_0 accordingly to the uniform distribution on $[0, T]$, where T is the common period of u^0 and s^0. This proves the last statement of the theorem. Q.E.D.

Remark 11. In order to characterize the stochastic instability radii, we define the Lyapunov exponents of the p-th moment for $p < 0$ as $g_\rho^*(\xi_t, p) = \inf_{x_0 \neq 0} g_\rho(x_0, \xi_t, p)$, and

$$r^{[-\infty]}(\xi_t) = \inf\{\rho \geq 0;\ g_\rho^*(\xi_t, p) \leq 0 \text{ for all } -\infty < p < 0\},$$

similarly for the classes Stat (and Diff). Then the same techniques and the corresponding references as in the proof of Theorem 10 yield

$$\begin{aligned}
&\tilde{r}_{\text{Stat}} = \tilde{r}_L, \\
&\frac{1}{p} r^{[p]}_{\text{Stat}} \geq r^{[-\infty]}_{\text{Stat}} = r^*_{\text{Stat}} = r^*_L \text{ for all } p < 0,
\end{aligned}$$

where the equality holds again if x_0 can be chosen to be a (bounded) random variable. A p-th moment characterization of $\tilde{r}$ is obtained in the following way: Recall that by Proposition 3, the instability radius $\tilde{r}_L$ (and hence $\tilde{r}_{\text{Stat}}$) are obtained over the invariant control set $C \subset \mathbb{P}$. Define $g_\rho^*(\xi_t, p; C) = \inf_{x_0 \in C} g_\rho(x_0, \xi_t, p)$, and similarly $r^{[-\infty]}(\xi_t; C)$, $r^{[-\infty]}_{\text{Stat}}(C)$. Then

$$\frac{1}{p} r^{[p]}_{\text{Stat}}(C) \geq r^{[-\infty]}_{\text{Stat}}(C) = r^*_{\text{Stat}}(C) = \tilde{r}_{\text{Stat}} \text{ for all } p < 0,$$

where the equality holds under the same condition as above. □

The theory for the class Diff is considerably more complicated (and interesting), because we cannot use the periodic solutions of (4.1) as diffusion process that satisfy our assumptions for Diff. In fact, the Lie algebra assumption yields that every $\xi_t \in$ Diff has an invariant probability measure with C^∞ density, which is strictly positive on all of M. This implies (see e.g. [19, Proposition 3.2 and 3.4]) that all solutions of $(7.1)_\rho$, projected onto $\mathbb{P}$, enter the maximal control set C_ρ in finite time w.p.1 (even with finite expectation). Therefore, the Lyapunov exponents for $\xi_t \in$ Diff are independent of the initial value and they are attained in C_ρ. Furthermore, for each ξ_t and ρ the Lyapunov exponents are a.s. constant, and depend only on the (unique) invariant probability of the pair process (ξ_t, s_t), where s_t is the projection of $(7.1)_\rho$ onto $\mathbb{P}$ (see Hasminskii's formula (4.1) in [3]). We thus have for $\xi_t \in$ Diff

$$\lambda_\rho(x_0, \xi_t) = \lambda_\rho(\xi_t) \in \mathbb{R} \text{ for all } x_0 \neq 0, \text{w.p.1}$$

and

$$\lambda_\rho(\xi_t) \in I_\rho = [\alpha_\rho, \omega_\rho],$$

see [3] for proofs of these statements. We thus have

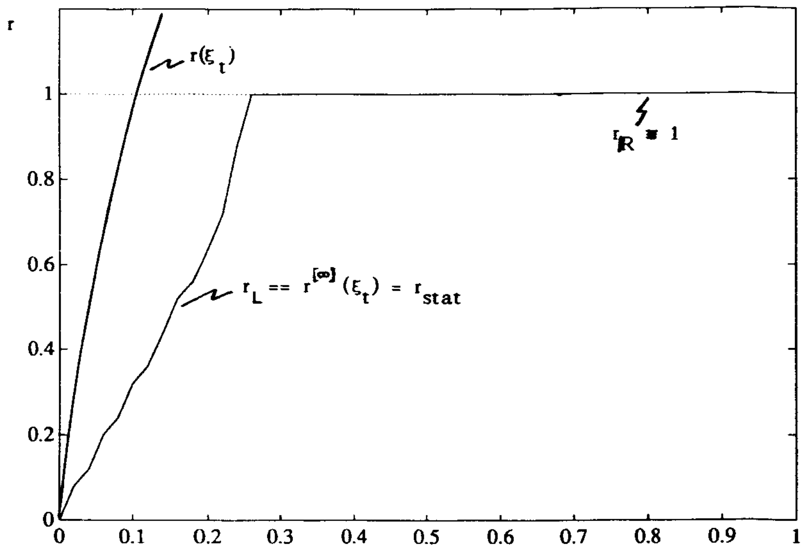

Figure 3. Stochastic stability radii for the linear oscillator

Theorem 11. *Let $\xi_t \in$ Diff, denote $U_\rho = D_\rho(M)$. Then under the assumptions above*

(i) $r(\xi_t) \geq r_{\text{Diff}} \geq r_{\text{Stat}} = r_L$ *for A stable.*

(ii) $\tilde{r}(\xi_t) = r^*(\xi_t) \geq \tilde{r}_{\text{Diff}} = r^*_{\text{Diff}} \geq \tilde{r}_{\text{Stat}} = \tilde{r}_L$ *for A unstable.*

PROOF. (i) is clear from the definitions and Theorem 10. To show (ii), note first that for unstable matrix A and a given $\xi_t \in$ Diff we have $\tilde{r}(\xi_t) = r^*(\xi_t)$, because the Lyapunov exponents of $(7.1)_\rho$ are independent of the initial value x_0. This also means $\tilde{r}_{\text{Diff}} = r^*_{\text{Diff}}$. The rest follows from Remark 11 Q.E.D.

Remark 12. We conjecture that $r_{\text{Diff}} = r_L$ and $r^*_{\text{Diff}} = \tilde{r}_L$. For a characterization of $r(\xi_t) = r_L$, $\xi_t \in$ Diff, see Theorem 12 below. □

Theorem 11 and Example 9 show that $r_{\mathbb{C}}$ is in general not a good estimate for a stochastic stability radius. The linear oscillator shows that the real radius $r_{\mathbb{R}}$ is not suitable either:

Example 13. Consider again the linear oscillator from Example 9

$$\ddot{y}(t) + 2b\dot{y}(t) + [1 + D_\rho(\xi_t)]y(t) = 0$$

where $M = \mathbb{S}^1$, the one dimensional Euclidean sphere in $\mathbb{R}^2$, $D_\rho : \mathbb{S}^1 \to M(d, d; \mathbb{R})$ given by

$$D_\rho(m) = \rho \begin{pmatrix} 0 & 0 \\ -\cos m & 0 \end{pmatrix},$$

where ξ_t is the standard Brownian motion on $\mathbb{S}^1$. □

Figure 3 presents the stability radius $r(\xi_t)$ for this system, computed numerically via the law of large numbers (see "Hasminskii's formula", (4.1) in [3]). For b small, $r_L < r(\xi_t) < r_{\mathbb{R}}$, while for large b one has $r_{\mathbb{R}} < r(\xi_t)$. Note that for small b, $r(\xi_t, b)$ increases like const $\cdot\sqrt{b}$, compare [5].

We now turn to the analysis of the p-th moment stability radii. Throughout the rest of this section, which is based on the results in [2], we will assume that the following hypothesis (H''), which is slightly stronger than (H') holds:

$$\begin{aligned} \dim \mathcal{LA}\{X_0 + h + \tfrac{\partial}{\partial t}, X_1, \ldots, X_r\}(m, s, t) = \dim M + d \\ \text{for all } (m, s, t) \in M \times \mathbb{P}^{d-1} \times \mathbb{R}. \end{aligned} \qquad (H'')$$

(H') and (H'') are equivalent e.g. if $M \times \mathbb{P}^{d-1}$ has a compact universal covering space (which is true e.g. if M is simply connected and $d > 2$).

First of all let us mention that

— $g_\rho(x_0, \xi_t, p) = g_\rho(\xi_t, p)$ for all $x_0 \neq 0$, $0 < p < \infty$,

— $g_\rho(\xi_t; \cdot) : \mathbb{R}_+ \to \mathbb{R}$ is convex, analytic with $g_\rho(\xi_t, 0) = 0$, $g'_\rho(\xi_t, 0) = \lambda_\rho(\xi_t)$.

Theorem 12. *Let $\xi_t \in$ Diff and suppose that A is stable. Then*

(i) $r(\xi_t) = \frac{1}{p} r^{[p]}(\xi_t) = r_L = r^{[\infty]}(\xi_t)$ *iff* $g_\rho(\xi_t, p) = p \cdot \lambda_\rho(\xi_t)$ *for all* $0 < p < \infty$. *This situation occurs exactly in the cases 2.1.2(i) and 2.2 of Theorem 7.*

(ii) $r^{[\infty]}(\xi_t) = r_L$ *for all* $\xi_t \in$ Diff.

The typical situation for deterministic and stochastic stability radii is characterized in Figure 4. (The reader should refer to the text for conditions, under which the strict inequalities hold. Note that $r_{\mathbb{C}}$, as defined in Section 6, is positive, if A is stable.) The corresponding moment Lyapunov function $g_\rho(\xi_t, \cdot)$ is indicated in Figure 5.

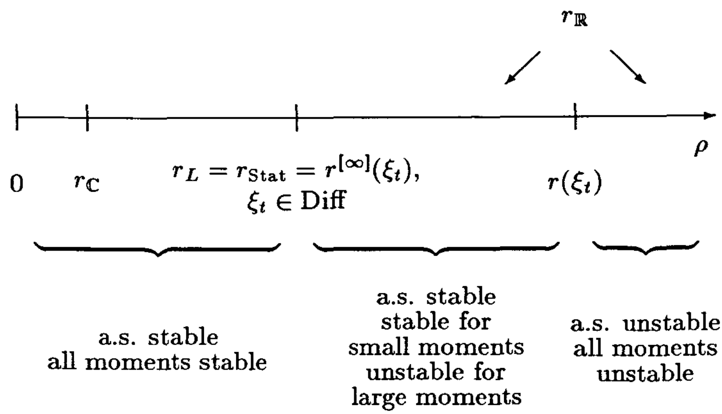

Figure 4. Typical order of deterministic and stochastic stability radii

If $r_L < \rho < r(\xi_t)$, then the system $(7.1)_\rho$ may "exhibit large deviations": Define the rate function $I_\rho(r,\xi_t) := \sup_{p\in\mathbb{R}}\{rp - g_\rho(\xi_t,p)\}$, denote

$$\gamma_\rho(\xi_t,p) = \begin{cases} \frac{1}{p} g_\rho(\xi_t,p) & \text{for } p \neq 0 \\ \lambda_\rho(\xi_t) & \text{for } p = 0 \end{cases}$$

and let

$$\gamma_\rho^*(\xi_t) = \lim_{p\to-\infty} \gamma_\rho(\xi_t,p), \quad \gamma_\rho(\xi_t) = \lim_{p\to+\infty} \gamma_\rho(\xi_t,p).$$

If $\gamma_\rho^*(\xi_t) < \gamma_\rho(\xi_t)$, then $I_\rho \geq 0$, finite exactly in the interval $(\gamma_\rho^*(\xi_t), \gamma_\rho(\xi_t))$ and strictly convex and analytic in this interval. The following is a consequence of a large deviations principle, suitable for our purposes:

Theorem 13. *Let $\xi_t \in$ Diff with (H'') holding. Then for all $\rho > 0$, the family of measures*

$$P_{t,\rho}(\cdot) = P\{\frac{1}{t}\log\frac{|\varphi(t,x_0,\xi_t)|}{|x_0|} \in \cdot\}$$

obeys a large deviations principle with rate function $I_\rho(r,\xi_t)$ (here $\varphi(t,x_0,\xi_t)$ is the solution of $(7.1)_\rho$). In particular for A stable:

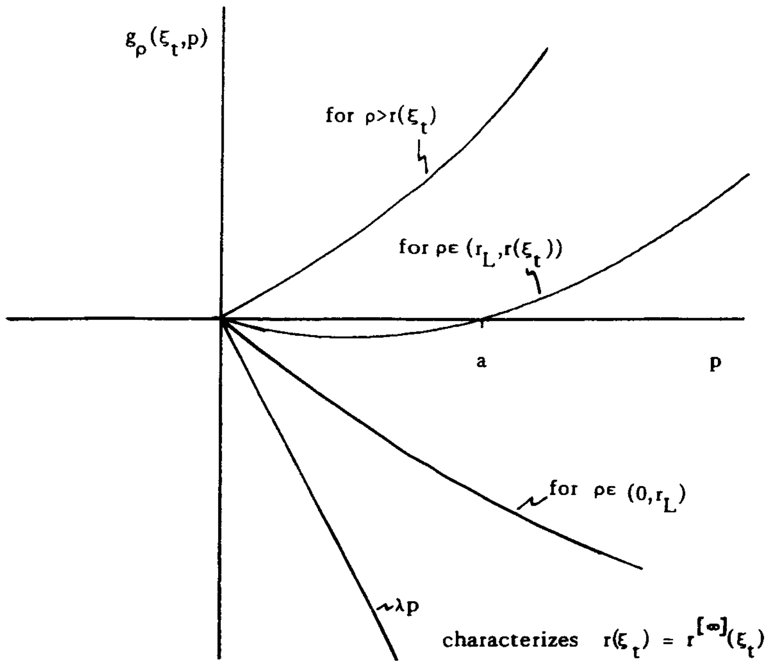

Figure 5. Typical moment Lyapunov functions for different ρ-values

(i) *If $\rho \in (r_L, r(\xi_t))$, there exist constants $a > 0$, $k \geq 1$ such that for all $c > 0$ and all $0 < |x_0| < c$*

$$\frac{1}{k}\left(\frac{|x_0|}{c}\right)^a \leq P\{\sup_{t\geq 0} |\varphi_\rho(t, x_0, \xi_t)| \geq c\} \leq k\left(\frac{|x_0|}{c}\right)^a .$$

(ii) *If $\rho \in (0, r_L)$, then there is a constant $K \geq 1$ such that for all $x_0 \neq 0$*

$$\sup_{t\geq 0} |\varphi(t, x_0, \xi_t)| \leq K|x_0|$$

with probability 1.

A precise formulation of a large deviations principle is given in [2].

This result can be interpreted as follows: If $\rho \in (r_L, r(\xi_t))$ then the system $(7.1)_\rho$ exhibits large deviations, i.e. for all $c > 0$ and all $t > 0$, we have $P\{\frac{1}{|x_0|}|\varphi(t, x_0, \xi_t)| \geq c\}$ is positive (and decays exponentially for $t \to \infty$), while for $0 < \rho < r_L$ this probability decays faster than any exponential function, in fact in this case $|\varphi(t, x_0, \xi_t)|$ is bounded for all

$t \geq 0$. Thus the theorem shows that for $\rho \in (r_L, r(\xi_t))$ the system is still stable with probability one, however arbitrarily large deviations in this range of ρ occur with positive probability.

Remark 14. It is worth noting that $r^{[\infty]}(\xi_t)$ does depend neither on the statistics nor on the dynamics of the diffusion process $\xi_t \in \mathrm{Diff}$, since it always equals r_L and hence depends only on $U_\rho = D_\rho(M)$. □

Remark 15. For an unstable matrix A we know from Theorem 11 that $r^*_{\mathrm{Diff}} = \tilde{r}_{\mathrm{Diff}}$. A large deviations theory that ties the instability radius to the moment Lyapunov functions for $p < 0$ can be developed in complete analogy to Theorems 12 and 13, compare [2], in particular Section 3 therein. □

8 Remarks on Robust Design

In this section we draw some conclusions from the theory presented so far for a robust design technique, in which stability radii and Lyapunov exponents play a central role.

Stability radii can be viewed as a measure of robustness for the stability of a given system $\dot{x} = Ax$, depending on the nature of the uncertainty. Most of the results above deal with r_L, the Lyapunov radius for time varying deterministic disturbances, for a family of sets U_ρ of uncertainty matrices. This concept of stability radius appears to be important because

- information about uncertainties is usually given in terms of the size of possible variations of parameters in A,
- structured variations, i.e. dependent or independent disturbances in the parameters of A are covered by this model,
- complete ignorance about a parameter can be included as unbounded perturbation,
- no prior knowledge about the time dependence of the variation is assumed,
- stochastic, stationary perturbances lead to the same stability radii,
- stochastic, Markovian variations can be included in the theory in a consistent fashion, in particular, their stability radius for all moments leads to the same radius.

The disadvantage of r_L is that, in general, it can only be computed for a given system using numerical methods, i.e. by solving a parametrized optimal control problem for periodic trajectories.

The theory developed above, allows to incorporate information into a robust design which has two aspects:

- How large can the disturbances of A be, such that the system is still stable (this is covered e.g. by looking at stability radii),
- how fast does the system stabilize under a "typical variation" $D \in U_\rho$.

The second aspect is particularly important for the design, if different parameter constellations lead to the same stability radius. One then would choose the parameters in such a way that perturbations are damped in a maximal way. The notion of stability radii covers only the first aspect above. For the remainder of this section we discuss some strategies for this problem.

To outline the ideas let us separate the system parameters (i.e. the entrances of A) into three categories:

- structure parameters (not affected by uncertainties, not tunable by the designer, e.g. the first row of A in Example 9),
- tunable parameters (not affected by uncertainties, e.g. the damping b in Example 9),
- uncertain parameters (e.g. the restoring force in Example 9).

Note that the set of tunable parameters, say denoted by Θ, consists of constants, while the uncertain parameters are allowed to be time varying here. Let the range of the uncertainties as usual be given by U_ρ. Then the design problem may be formulated as follows: Given a family of systems

$$\dot{x} = A(\theta, u)x, \quad \theta \in \Theta, \quad u \in U_\rho, \quad \rho > 0,$$

find all parameters $\theta \in \Theta$ such that the stability radius (with respect to U_ρ) $r_L(\theta) \geq r_{\text{crit}}$, a critical radius, or such that $r_L(\theta)$ becomes maximal, if θ is chosen in Θ.

Additional design criteria can be given e.g. by deterministic or stochastic information on the uncertainty. Then one would not only maximize the (deterministic or stochastic) stability radius, determined by the corresponding Lyapunov exponents; one would also like to minimize Lyapunov exponents of "typical" perturbations.

As an illustration of these general remarks consider again the linear oscillator from Examples 9, 13:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-u(t) & -2b \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Consider $b \in [0,2] =: \Theta \subset \mathbb{R}$ as a tunable parameter and u as an uncertain parameter, $u(t) \in [-\rho, \rho]$, $\rho \geq 0$, i.e. $U_\rho = \left\{ \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}, \ \alpha \in [-\rho, \rho] \right\}$.

8.1 A strategy motivated by deterministic considerations

Figure 6 shows the level curves of the Lyapunov exponents $\kappa_\rho(b)$ as described in Example 9. Again, from the explicit formulae in [9], Section 6,

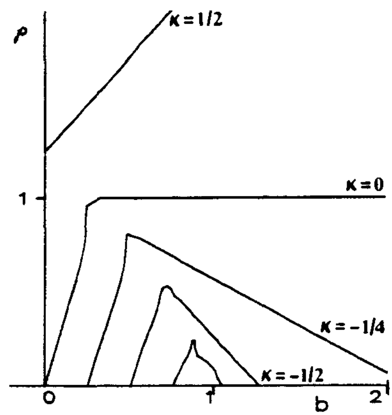

Figure 6. Level curves of κ for the linear oscillator

we obtain: For each $\rho \in (0,1)$, the curve $\kappa_\rho(b)$ has a unique minimum b_ρ, i.e. there is a unique damping value b_ρ such that (timevarying) uncertainties of size $U_\rho = [-\rho, \rho]$ are exponentially damped in an optimal way for b_ρ. Figure 7 shows the location of these values. Note that increased damping with b does not necessarily lead to faster damping of disturbances (overdamping) for fixed ρ. Note also that this strategy leads to optimal b_ρ values for $b_\rho \in (b_0, b_1)$, where the complex stability radius $r_{\mathbb{C}}$ is less than 1; i.e. maximization of $r_{\mathbb{C}}$ does not lead to optimality in this sense.

It follows from Section 7 that the same design is optimal, if one wants to dampen out, at a maximal rate, all moments of the solution $\varphi(t, x_0, \xi_t)$, caused by stationary ergodic nondegenerate diffusions with values in U_ρ or all stationary processes with values in U_ρ. We would like to mention that for $\rho > 1$, $\kappa_\rho(b)$ is strictly increasing in b.

Note that in this example the choice of the "tunable parameter" b may

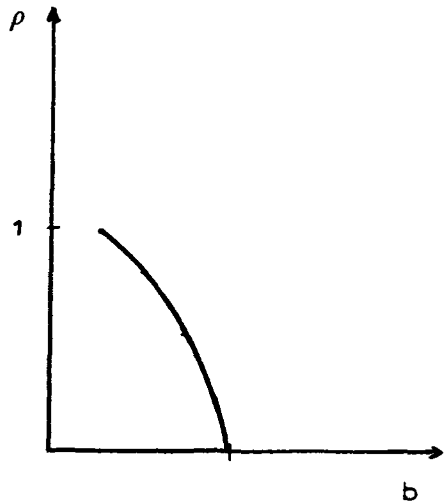

Figure 7. Optimal damping with timevarying uncertainties

be interpreted as the choice of a certain feedback: Define

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1-u & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad C = (0 \ \ 1)$$

Then the choice of the tunable parameter b means choosing a feedback $F = b$ for the system

$$\dot{x} = A_0 x + Bu, \quad y = Cx$$

such that

$$\dot{x} = (A_0 + BFC)x = \begin{pmatrix} 0 & 1 \\ -1-u & -2b \end{pmatrix} x$$

has certain desired properties.

8.2 A strategy motivated by the stochastic a.s. Lyapunov exponent

Figure 8 presents typical level curves of a.s. Lyapunov exponents $\lambda_\rho(x_0, \xi_t; b)$ for a specific excitation ξ_t (see Example 13). Of course, these values were again obtained numerically via the law of large numbers. Recall that for $\xi_t \in \mathrm{Diff}$ we have $\lambda_\rho(x_0, \xi_t; b) = \kappa_\rho(\xi_t, b)$, and thus we see from the numerical results, that the stochastic stability radius $r(\xi_t, b)$ increases to ∞ for $b \to \infty$ and for each $\rho \in (0, \infty)$ the curve $\kappa_\rho(\xi_t, b)$ has a unique minimum

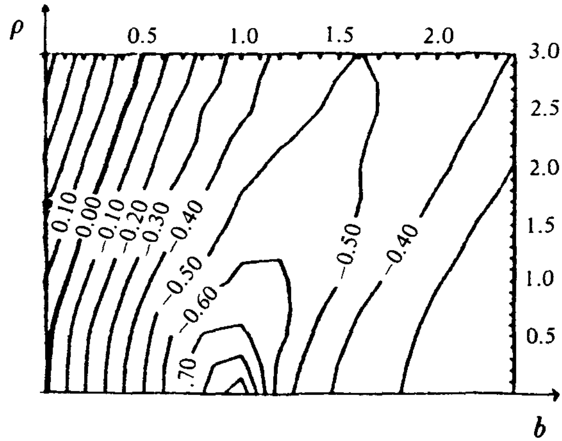

Figure 8. Level curves of a.s. Lyapunov exponents

Figure 9. Optimal damping with stochastic uncertainties

$b_\rho(= b_\rho(\xi_t))$. These (numerical) findings are supported by the asymptotic expansions of stochastic Lyapunov exponents for small and large ρ's, see e.g. [5]. (For numerical methods for the computation of stochastic Lyapunov exponents we refer the reader to the work of Talay, e.g. [22].) The location of the optimal values b_ρ is shown in Figure 9. Obviously, the design strategies in 8.1 and 8.2 above do not agree. An optimal robust design depends not only on the requirements imposed on the system, but also on the prior information available on the uncertainties inherent in the system parameters.

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