STABILIZATION OF UNCERTAIN LINEAR SYSTEMS

FRITZ COLONIUS AND WOLFGANG KLIEMANN

ABSTRACT. Feedback stabilization of linear, uncertain systems is usually analyzed using quadratic Lyapunov functions that are common to all values in the uncertainty set. In this paper we use the alternative classical concept of Laypunov exponents to characterize the precise (exponential) stability regions for systems with contrained linear output feedback. In particular, we exploit the continuity of the maximal Lyapunov exponent depending on the size of the uncertainty and on the bounds of the feedback gain matrix, to obtain results on the exponential stabilizability radius $r(u)$ as a function of the linear, time invariant feedback u. Several examples show, among other facts, that quadratic Lyapunov functions lead in general to conservative criteria, when compared to the precise exponential stabilizability region.

1. Introduction

Linear systems theory has proved very useful for the analysis of physical systems and for their design. Its success is partially based on different schemes for approximating nonlinear dynamics by linear ones, and also on linearization techniques, which allow the study of local behavior e.g. about rest points. A great advantage of linear systems (with unbounded input) is the use of linear algebra, e.g. through quadratic Lyapunov functions and Riccati equations, which often yields explicit criteria and design principles that are easy to compute.

Over the last decade, some drawbacks of the "precise" linear approach have been removed by considering uncertain systems, where the systems parameters are allowed to vary within given bounds, and design criteria for performance, stability, control, etc. are investigated that work for all systems within the uncertainty bounds. Different approaches to this problem include operator theoretic techniques in H^{∞}-theory (see e.g. Francis (1987)), analysis of sets of stable polynomials via transfer functions, and (quadratic) Lyapunov function criteria for state space representations, see e.g. the recent conference proceedings Milanese et al. (1989) and Hinrichsen and Martensson (1990) for an overview.

1980 *Mathematics Subject Classification* (1985 *Revision*). 93D15, 93D05, 93C05.

Key words and phrases. Lyapunov exponents, stabilization radius, constrained output feedback, quadratic Lyapunov functions.

*Research partially supported by NSF grant no. 8813976 and DFG grants no. Co124/6-1 and Co124/8-1. .

In this paper we concentrate on the problem of stabilization of uncertain linear systems, given in state space form as

$$(1) \qquad \dot{x}(t) = [A + v(t)]\, x(t) + Bu, \quad y = Cx,$$

where $v(t) \in V \subset \mathbb{R}^m$ is an unknown, time varying uncertainty. Generalizations to the situation, where also the input matrix B is disturbed, can be found e.g. in Petersen (1985) or Rotea and Khargonekar (1989). Stabilization of the system (1) is usually described in terms of (quadratic, time invariant) Lyapunov functions and for state feedback without a priori bounds on the gain matrix, compare e.g. Barmish (1985) and Rotea and Khargonekar (1989). Such an approach can be shown to be equivalent to considering the structured complex stability radius as introduced by Hinrichsen and Pritchard (1986^a, 1986^b, 1990^a), compare Section 2. The questions arise, whether the use of quadratic Lyapunov functions leads to precise stabilizability criteria, and how to incorporate given bounds on the gain matrix in the theory, because this is the situation usually encountered in applications.

In Colonius and Kliemann (1990^a) we have suggested a Lyapunov exponents approach to stability and instability radii for linear systems under structured, real, time varying uncertainties. This idea leads to the problem of solving a certain infinite time, optimal control problem to determine the precise stability radii and it turns out that the complex stability radius is in general a conservative lower estimate of the real, time varying radius. It is therefore expected that quadratic Lyapunov functions lead to conservative criteria for the stabilization of system (1), and this is in fact true, as examples below show. Our approach here is based on the analysis of parameter dependence of Lyapunov exponents for (1).

In Section 2. we describe the set up and recall several stabilization concepts for linear, uncertain systems together with their interrelationships. The Lyapunov exponents approach is outlined in Section 3. and some crucial results on the dependence of Lyapunov exponents on parameters are obtained. This leads to the definition of precise (exponential) stabilization radii, and some of their properties. Section 4. is devoted to the comparison of several stabiization concepts, and in Section 5. several conclusions are drawn for the design of linear, uncertain systems.

2. Stabilization via Lyapunov Functions

Throughout this paper, we will use the following set up and notations. Consider a linear uncertain control system of the form

$$(1_\rho) \qquad \dot{x}(t) = (A + v(t))\, x(t) + B\hat{u}, \quad y = Cx$$

where $A \in M(d, d; \mathbb{R})$ (the $d \times d$ matrices over $\mathbb{R}$), $B \in M(d, k; \mathbb{R})$ and $C \in M(\ell, d; \mathbb{R})$ are given. The time varying uncertainties v are of the form

$v \in \mathcal{V}_\rho := \{v: \mathbb{R} \to V_\rho \subset V;$ locally integrable$\}$, with $V \subset M(d,d;\mathbb{R})$ a linear subspace. The uncertainty sets $\{V_\rho;\ \rho \geq 0\}$ are of the following form: Let $K \subset V$ be any compact, connected subset of V with $O \in \mathrm{int}_V K$, and define $V_\rho = \rho \cdot K = \{v \in V;$ there exists $w \in K$ with $\rho w = v\}$ for $\rho \in [0,\infty)$. This includes in particular the cases, where K is defined via a norm in V, e.g. the Euclidean norm or an interval norm.

We consider the problem to stabilize the system (1) via linear, constant output feedback $\hat{u}$ with given constraints, i.e. $\hat{u} = FCx$ with $F \in \widehat{U}_\sigma \subset \widehat{U}$, where $\widehat{U} \subset M(k,\ell;\mathbb{R})$ is a linear subspace and the family of admissible sets of gain matrices $\{\widehat{U}_\sigma;\ \sigma \geq 0\}$ is again defined via some compact subset $\widehat{K}$ of $\widehat{U}$ as above. Denote $B\widehat{U}_\sigma C = U_\sigma \subset M(d,d;\mathbb{R})$, and system (1) reads

$$(2_\rho) \qquad \dot{x}(t) = (A + v(t))\,x(t) + ux(t) \quad \text{in } \mathbb{R}^d,$$

with $u \in U_\sigma$. System (2) is a linear system with time varying, real uncertainties and time constant output feedback.

The problem can now be reformulated as: With varying ρ and/or σ, find a $u \in U_\sigma$ such that the system (2) is stable for all $v \in \mathcal{V}_\rho$.

Currently, the most prominent approach to tackle this problem for unbounded $F \in \widehat{U}$ is via Lyapunov functions. We present here some of the common concepts (compare e.g. Rotea and Khargonekar (1989)):

Let $P \in M(d,d;\mathbb{R})$ be a positive definite matrix.

(a) The function $s(x) = x^T P x$ is called a control Lyapunov function for (1_ρ), if there exists $\alpha > 0$ such that for all $(x,v) \in \mathbb{R}^d \times V_\rho$ there is a $\hat{u} \in \mathbb{R}^k$ (possibly dependent on (x,v)) such that

$$x^T\left(P(A+v) + (A+v)^T P\right)x + 2x^T PB\hat{u} \leq -\alpha\|x\|^2,$$

where $\cdot^T$ denotes transposition and $\|.\|$ is the Euclidean norm in $\mathbb{R}^d$.

(b) The system (1_ρ) is called quadratically stabilizable if there exists a P, a constant $\alpha > 0$ and a continuous feedback map $p: \mathbb{R}^\ell \times [0,\infty) \to \mathbb{R}^d$ such that for any uncertainty $v \in \mathcal{V}_\rho$ we have

$$x^T\left(P(A+v(t)) + (A+v(t))^T P\right)x + 2x^T PBp((x,t)) \leq -\alpha\|x\|^2.$$

(c) The system (1_ρ) is quadratically stabilizable via linear, constant feedback, if (b) holds with $p(x,t) = FCx$.

(d) The system (1) is quadratically stabilizable via bound-invariant Laypunov functions, if (c) holds with P independent of $\rho \geq 0$.

The connections between the concepts above are as follows: obviously (d) $\Longrightarrow$ (c) $\Longrightarrow$ (b). For the state feedback case, i.e. C is the $d \times d$ identity matrix, more can be said: (b) $\Leftrightarrow$ (c) (see Hollot and Barmish (1980), this result does not hold in general, if B is also uncertain, compare Peterson

(1985)), and (a) $\Leftrightarrow$ (c) (see Rotea and Khargonekar (1989) for an even more general class of systems). The concept of control Lyapunov functions is an open loop approach and was used e.g. in Rotea and Khargonekar (1989). Bound invariant Lyapunov function were defined in Hollot (1987), several criteria for their existence can also be found in Zhou and Khargonekar (1988). It is also worth mentioning that H^∞ techniques and quadratic Laypunov function techniques are mathematically equivalent, if a linear, time-invariant controller has to be designed, see Khargonekar et al. (1987).

In a series of papers Hinrichsen and Pritchard (compare (1990[a])) introduced and analyzed real and complex stability radii for time invariant uncertainties. The complex radius is of interest here and can be defined for system (2) in the following way:

$$(3) \qquad r_{\mathbb{C}}(V,u) = \inf\{\|v\|;\ v \in V_{\mathbb{C}},\ \mathrm{Re}\ \sigma(A+v+u) \cap [0,\infty) \neq \phi\},$$

where $V_{\mathbb{C}} \subset M(d,d;\mathbb{C})$ is the complexification of V, $\|.\|$ is any given operator norm on $V_{\mathbb{C}}$, and Re σ denotes the real part of the spectrum of a matrix. In Hinrichsen and Pritchard (1990[b]) various other stability radii, e.g. for dynamical or time varying uncertainties, are defined, but for complex uncertainties they all turn out to be the same (Theorem 3.11). In this context, stabilization with (unbounded) feedback can be defined as:

(e) The system (2_ρ) is stabilizable with respect to the complex stability radius, if there exists $u \in U$ such that $\rho < r_{\mathbb{C}}(V,u)$.

Hinrichsen and Pritchard (1989, 1990[a]) define stabilizability concepts based on the complex stability radius using (unbounded) state feedback with complex gain matrices F. In general, this will lead to less conservative stabilization criteria than (e).

The concepts (c) and (e) can be characterized via associated families of parametrized Riccati equations, which leads to the following result:

Consider uncertainties of the form $V = D\Delta E$, where $D \in M(d,p;\mathbb{R})$, $E \in M(q,d;\mathbb{R})$ and $\Delta \in M(p,q;\mathbb{C})$. Let $\|.\|_2$ denote the matrix norm in $M(p,q;\mathbb{C})$, induced by the Euclidean norm.

2.1. Proposition. *In this set up, (1_ρ) is quadratically stabilizable via linear, constant state feedback iff (2_ρ) is stabilizable with respect to the complex stability radius.*

The proof can be given using characterizations in Petersen (1987) and Hinrichsen and Pritchard (1986[b]). Since the complex stability radius does not give exact bounds for stability with real, time varying uncertainties (see Colonius and Kliemann (1990[a])), the quadratic Lyapunov function approach to stabilization should yield conservative estimates as well. Examples will be given in Section 4., after describing the Lyapunov exponents approach to uncertain stabilization.

3. Stabilization via Lyapunov Exponents

Consider again the uncertain feedback system

$$(2_\rho) \qquad \dot{x}(t) = (A + v(t))\, x(t) + ux(t)$$

with the set up introduced in Section 2. The exponential growth behavior of (2) is described by the Lyapunov exponents

$$(4) \qquad \lambda(x, v, u) = \limsup_{t\to\infty} \frac{1}{t} \log |\varphi(t, x, v, u)|,$$

where for $v \in \mathcal{V}_\rho$, $u \in U_\sigma$ the solution is denoted by $\varphi(t, x, v, u)$, with $\varphi(0, x, v, u) = x$. Note that $\lambda(x, v, u) < 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$ means asymptotic stability of (2) under the uncertainty v with feedback u. For the classical concept of Lyapunov exponents for linear differential equations with time varying coefficients see e.g. Hahn (1967).

We will analyze the system, where the uncertainties effect all components of the state vector, i.e. we assume the the systems Lie algebra has full rank:

$$(\mathrm{H}) \qquad \mathcal{LA}\{A + v + u,\ v \in V_\rho\}(x) = \mathbb{R}^d \text{ for all } x \neq 0,\ \rho > 0,\ u \in U_\sigma.$$

The techniques described here work in the general case as well, compare e.g. Colonius and Kliemann (1990[b], 1990[c]), where also criteria for the validity of (H) are given.

Define exponential growth rates of (2):

$$(5) \qquad \begin{aligned} \kappa(v, u) &= \sup_{x\neq 0} \lambda(x, v, u) && \text{for } v \in \mathcal{V},\ u \in U \\ \kappa(\rho, u) &= \sup_{v\in\mathcal{V}_\rho} \kappa(v, u) && \text{for } \rho \geq 0,\ u \in U \\ \kappa(\rho, \sigma) &= \inf_{u\in U_\sigma} \kappa(\rho, u) && \text{for } \rho \geq 0,\ \sigma \geq 0. \end{aligned}$$

Note that if $\kappa(\rho, \sigma) < 0$ then there exists a feedback $u \in U_\sigma$ such that (2) is asymptotically stable for all $v \in \mathcal{V}_\rho$.

3.1. Remark. Under Assumption (H) we have $\kappa(\rho, u) = \sup_{v\in\mathcal{V}_\rho} \lambda(x, v, u)$ for all $x \neq 0$, compare Colonius and Kliemann (1990[a]), Proposition 3. This means that the extremal growth rate $\kappa(\rho, u)$ is uniform in $x \neq 0$ for all $\rho > e0$, $u \in U$.

3.2. Remark. Define the Bohl exponent of (2) for $(v, u) \in \mathcal{V} \times U$ by

$$k(v, u) = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_{v,u}(t, s)\|,$$

where $\Phi_{v,u}(t,s)$ is the fundamental matrix of (2) with $\Phi_{v,u}(s,s) = Id$. Denote $k(\rho, u) := \sup_{v \in \mathcal{V}_\rho} k(v,u)$, then $k(\rho,u) = \kappa(\rho,u)$, i.e. for the uncertain system (2) asymptotic (exponential) stability is equivalent to uniform asymptotic (exponential) stability. The proof follows directly from Theorem 5. in Colonius and Kliemann (1990[a]).

3.3. Remark. The growth rates $\kappa(\rho,u)$ and $\kappa(\rho,\sigma)$, defined in (5), also describe the behavior of (2) under stochastic uncertainties: Denote by $\mathrm{Stat}(V_\rho)$ the stationary stochastic processes with values in V_ρ, and by $\mathrm{Diff}(V_\rho)$ the nondegenerate stationary diffusion processes in V_ρ. Let $\lambda(x,\xi_t,u)$ denote the (stochastic) Lyapunov exponents of (2) with stochastic perturbation $\xi_t \in \mathrm{Stat}(V_\rho)$. Then we have e.g.

$$\sup_{\xi_t \in \mathrm{Stat}(V_\rho)} \sup_{x \neq 0} \lambda(x,\xi_t,u) = \kappa(\rho,u),$$

$$\sup_{\xi_t \in \mathrm{Diff}(V_\rho)} \sup_{x \neq 0} \lambda(x,\xi_t,u) = \kappa(\rho,u),$$

$$\lim_{p \to \infty} \frac{1}{p} \limsup_{t \to \infty} \frac{1}{t} \log E|\varphi(t,x,\xi_t,u)|^p = \kappa(\rho,u) \text{ for all } \xi_t \in \mathrm{Diff}(V_\rho),$$

in particular $\kappa(\rho,u) < 0$ iff (2) is exponetially stable for all stationary process in V_ρ with $u \in U$ iff all moment Lyapunov exponents are negative for some (and hence all) nondegenerate stationary diffusion processes with values in V_ρ, with $u \in U$. For the precise set up and related results on large deviations see Colonius and Kliemann (1990[a]), Section 7.

We start our analysis of the exponential growth rates defined in (5) by considering their continuity and monotonicity properties. Recall that the complex stability radius depends continuously on A, while the real (time invariant) radius is lower semicontinuous, see e.g. Hinrichsen and Pritchard (1990[a]), Proposition 2.4. For the (real, time variant) exponential growth rates we have:

3.4. Theorem. *The function $\kappa\colon \mathbb{R}^+ \times U \to \mathbb{R}$, $(\rho,u) \mapsto \kappa(\rho,u)$, is continuous in (ρ,u), and increasing in ρ. The function $\kappa\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $(\rho,\sigma) \mapsto \kappa(\rho,\sigma)$ is continuous in (ρ,σ), increasing in ρ and decreasing in σ.*

The proof of this theorem, which uses parameter dependence of control sets of the projection of system (2) onto the projective space can be found in Colonius and Kliemann (1990[c]).

3.5. Remark. As the examples in Section 4. show, $\kappa(\rho,u)$ need not be monotone in u. Furthermore, $\kappa(\rho,u)$ and $\kappa(\rho,\sigma)$ need not be strictly increasing in ρ, and $\kappa(\rho,\sigma)$ may not be strictly decreasing in σ.

Define the following zero level sets for the functions κ:

$$\Gamma(U) = \{(\rho,u) \in \mathbb{R}^+ \times U,\ \kappa(\rho,u) = 0\}$$
$$\Gamma = \{(\rho,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \kappa(\rho,\sigma) = 0\}. \tag{6}$$

Then we have the following first corollary to Theorem 3.4:

3.6. Corollary. *The zero level sets $\Gamma(u)$ and Γ are closed and connected.*

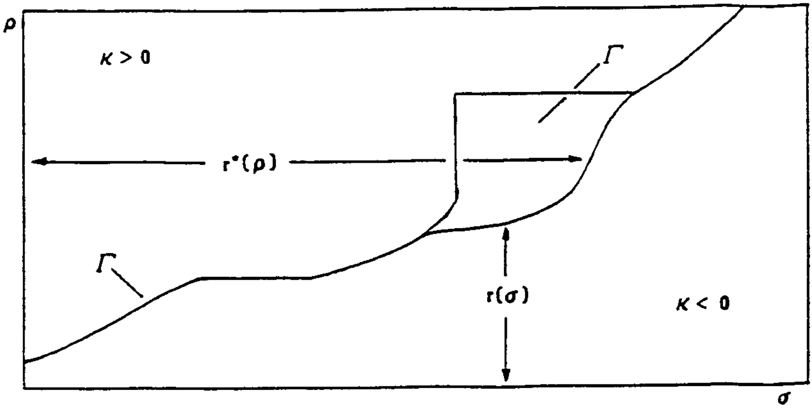

Figure 1: The zero level set Γ, the radii $r(\sigma)$ and $r^*(\rho)$.

Define for a given $A \in M(d,d;\mathbb{R})$, $U \subset M(d,d;\mathbb{R})$ and $V \subset M(d,d,;\mathbb{R})$ the stabilizability radius for $u \in U$: $r(u) = \inf\{\rho \geq 0;\ \kappa(\rho,u) \geq 0\}$, the stabilizability radius for $\sigma \geq 0$: $r(\sigma) = \inf\{\rho \geq 0;\ \kappa(\rho,\sigma) \geq 0\}$, the destabilizability radius for $\rho \geq 0$: $r^*(\rho) = \sup\{\sigma \geq 0;\ \kappa(\rho,\sigma) \leq 0\}$. As a second corollary to Theorem 3.4. we have:

3.7. Corollary. *The radii $r(u)$ and $r(\sigma)$ are lower semicontinuous in u, and in σ respectively. In particular, $r(u) = \inf\{\rho \geq 0;\ \kappa(\rho,u) = 0\}$ and $r(\sigma) = \inf\{\rho \geq 0;\ \kappa(\rho,\sigma) = 0\}$. Furthermore, $r^*(\rho)$ is upper semicontinuous, and $r^*(\rho) = \sup\{\sigma \geq 0;\ \kappa(\rho,\sigma) = 0\}$.*

3.8. Remark. The examples in Section 4. show that $r(u)$, $r(\sigma)$, and $r^*(\rho)$ need not be continuous.

3.9. Remark. Other stability and instability radii were defined and discussed in Colonius and Kliemann (1990[a]). For the question of global stabilization, however, the ones introduced above are the most important quantities.

We are now ready to define stabilizability of uncertain systems of type (1) via output feedback with the help of Lyapunov exponents:

3.10. Definition. The linear system (1) with uncertainties of size $\rho \geq 0$ is stabilizable via the constant output feedback $u \in U$, if $r(u) > \rho$; and stabilizable via constant output feedback of size $\sigma \geq 0$, if $r(\sigma) > \rho$. It is stabilizable via some constant output feedback, if there exists $u \in U$ (or $\sigma \geq 0$) such that $r(u) > \rho$ (or $r(\sigma) > \rho$, respectively).

Note that Definition 3.10 makes sense for stable and unstable matrices $A \in M(d,d;\mathbb{R})$: If A is stable, then $r(0) > 0$ by Theorem 3.4 (and possibly

∞, see Proposition 3.12). If A is not stable, then $r(0) = 0$. Furthermore, if the maximal real part of the eigenvalues of A is positive, then there exists a closed interval $[0, \sigma_0]$ for some $\sigma_0 > 0$ such that $r(\sigma) = 0$ for all $\sigma \in [0, \sigma_0]$, again by Theorem 3.4. If the system without uncertainties $\dot{x} = Ax + Bu$, $y = Cx$ is not output stabilizable (compare e.g. Wonham (1979)), then $r(\sigma) = 0$ for all $\sigma \geq 0$. If, however, this system is output stabilizable, then there is a $u \in U$ such that $\kappa(0, u) < 0$, hence $r(u) > 0$ by Theorem 3.4. Therefore the system can be stabilized via u for uncertainties of some size $\rho > 0$.

3.11. Remark. Note that the system (1) with uncertainties of size $\rho \geq 0$ is stabilizable via constant output feedback of size $\sigma \geq 0$ iff $r^*(\rho) < \sigma$. Hence the comments after Definition 3.10 hold, mutatis mutandis, also for $r^*(\rho)$.

The regions described by $r(u)$ in $\mathbb{R}^+ \times U$ and by $r(\sigma)$ in $\mathbb{R}^+ \times \mathbb{R}^+$ are the precise regions of (uniform) asymptotic stabilization of the system (2). For bounded ρ and σ these radii can be computed via solving an infinite time, optimal control problem as described in Colonius and Kliemann (1990[b]). For finite σ and unbounded ρ we have the following result:

3.12. Proposition. *For a fixed $\sigma \geq 0$ we have $r(\sigma) = \infty$ iff there exists $u_0 \in U_\sigma$ such that*

(a) *$\dot{x} = (A + u_0)x$ is exponentially stable, and*
(b) *there exists a nonsingular matrix $T \in M(d, d; \mathbb{R})$ such that $TVT^{-1} \subset so(d; \mathbb{R})$, the skew symmetric $d \times d$ matrices.*

The proof follows from Theorem 7. in Colonius and Kliemann (1990[a]).

3.13. Remark. Note that the conditions of Proposition 3.12. imply the existence of a bound-invariant Lyapunov function, i.e. $r(\sigma) = \infty$ for some $\sigma \geq 0$ implies the stabilization concept (d). An example in Section 4. shows that the converse does not hold.

It remains to find criteria for the following two cases:

- $\lim_{\sigma \to \infty} r(\sigma) < \infty$, i.e. for $r^*(\rho) = \infty$ for some finite $\rho \geq 0$; this situation can occur, as examples in Section 4. show.
- $\lim_{\sigma \to \infty} r(\sigma) = \infty$, while for all $\sigma \geq 0$ we have $r(\sigma) < \infty$, i.e. $\lim_{\rho \to \infty} r^*(\rho) \to \infty$, while $r^*(\rho) < \infty$ for all $\rho \geq 0$.

These questions will be treated elsewhere.

4. Examples

First example shows that a system, which can be stabilized via a bound invariant Lyapunov function, need not have an infinite stabilizability radius $r(\sigma)$ for some $\sigma \geq 0$. In the example, the feedback gain must go to infinity as the bound for the uncertainties goes to infinity (cf. in this context also Remark 2.9. in Zhou and Khargonekar (1988)). We will use the criterion in Section 4. of the cited paper and the result from Proposition 3.12. above.

4.1. Example. Consider the following situation:

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad V = \left\{ \begin{pmatrix} v_1 & 0 \\ v_2 & v_2 \end{pmatrix}, \ v_1, v_2 \in \mathbb{R} \right\}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = Id,$$

i.e. in the notation of (2) we have

$$\text{(7)} \qquad \dot{x}(t) = \left[\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} v_1(t) & 0 \\ v_2(t) & v_2(t) \end{pmatrix} \right] x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} Fx(t).$$

This system does not satisfy the conditions of Proposition 3.12. On the other hand the system is stabilzable via a bound invariant Lyapunov function, if there is a positive definite matrix P such that

(i) $(0 \quad 1)(AP + PA^T)\begin{pmatrix} 0 \\ 1 \end{pmatrix} < 0$, and

(ii) $(1 \quad 1)P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$, compare Lemma 4.4 in Zhou and Khargonekar (1988). It is easy to see that

$$P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{satisfies all these conditions.}$$

The next two examples are concerned with the linear oscillator $\dot{y} + 2b\dot{y} + cy = 0$. To compute the stabilizability radii for various cases of uncertainties and feedbacks, we will use the method of Gonzalez (1990) for 2-dimensional systems with time varying interval uncertainties, adapted to our situation.

4.2. Example. Consider the system $\ddot{y} + 2\,(b + v(t))\,\dot{y} + (1 + u)y = 0$, with $v(t) \in [-\rho, \rho]$, $\rho \geq 0$, and $u \in \mathbb{R}$. In first order from this equation reads

$$\text{(8)} \qquad \dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & -2v(t) \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix} x.$$

Note that the condition (H) is satisfied for this system, except for $u = -1$. Clearly, the system is not exponentially stable for $u \leq -1$ with $\rho = 0$, and for $u > -1$ with $\rho \geq b$. It remains to consider the case $u > -1$ and $\rho \in [0, b)$: According to the algorithm described in Colonius and Kliemann (1990^b), the largest exponent $\kappa(\rho, u)$ occurs for all $u > -1$ at the uncertainty $v(t) \equiv -\rho$. Therefore we have

$$r(u) = \begin{cases} 0 & \text{for } u \leq -1 \\ b & \text{for } u > -1 \end{cases}, \quad r(\sigma) = b \text{ for all } \sigma \geq 0, \quad r^*(\rho) = \begin{cases} 0 & \text{for } \rho < b \\ \infty & \text{for } \rho \geq b \end{cases}$$

and the stabilization radius for time varying uncertainties equals the one for time constant, real uncertainties. This example shows that the stabilizability and the destabilizability radius need not be continuous, they can be constant over intervals and $r(\sigma)$ can be bounded, as $\sigma \rightarrow \infty$. (Proposition 3.12 implies that $r^*(\rho)$ can also be bounded, as $\rho \rightarrow \infty$.)

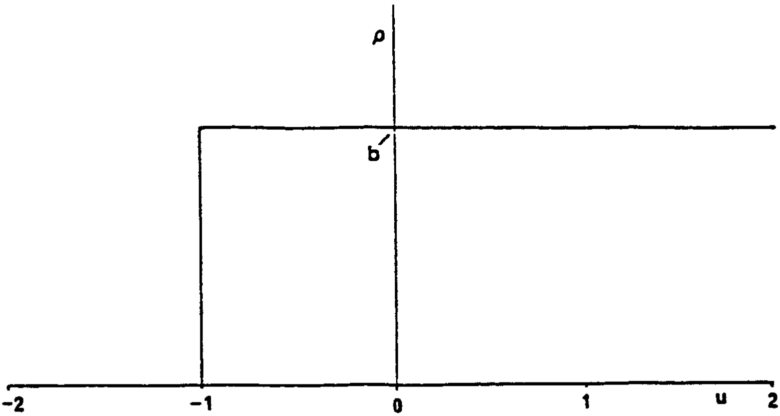

Figure 2: The stabilizability radius $r(u)$ for the system (8).

4.3. Example. Consider the linear oscillator with uncertain restoring force and controlled damping

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & -2u \end{pmatrix} x, \tag{9}$$

where $u \geq 0$ and $v(t) \in [-\rho, \rho]$, $\rho \geq 0$. Clearly, the system is not exponentially stabilizable if $\rho \geq 1$. Hence we concentrate on $0 \leq \rho < 1$. Using Gonzalez' (1990) method, Hinrichsen and Pritchard (1990^b) computed the stabilizability radius $r(u)$ numerically, and Section 6. in Colonius and Kliemann (1990^b) shows, how to compute the Lyapunov exponents $\kappa(\rho, u)$ for $(\rho, u) \in \mathbb{R} \times \mathbb{R}$ explicitly. In particular it turns out that the largest Lyapunov exponent of $\ddot{y} + 2u\dot{y} + (1 + v(t))\, y = 0$ is obtained in the following way:

Projection of (9) onto the projective space $\mathbb{P}$ in $\mathbb{R}^2$ leads to the following equation in polar coordinates $\varphi \in [0, \pi]$

$$\dot{\varphi}(t) = -\sin^2 \varphi(t) - (1 + v(t)) \cos^2 \varphi + u \sin 2\varphi.$$

For the stabilizability radius one only has to consider the following switching:

$$v(t) = \begin{cases} \rho & \text{for } \varphi \in (\frac{\pi}{2}, \pi] \\ -\rho & \text{for } \varphi \in (0, \frac{\pi}{2}] \end{cases},$$

while for the precise maximal Lyapunov exponent one has to choose $v(t)$ for $\varphi \in (0, \frac{\pi}{2}]$ as a constant in $[-\rho, \rho]$ according to an optimization algorithm, see Colonius and Kliemann (1990[b]). In particular it turns out that the maximal exponent $\kappa(\rho, u)$ is attained through a certain periodic, piecewise constant switching of v, which is adapted to the systems dynamics. Faster switchings may even stabilize a system (compare e.g. Bellman et al. (1986), or Arnold et al. (1983)) and do not represent the most important threat to uncertain stability.

In Colonius and Kliemann (1990[a]), Example 9, we have overlooked one case in the computation of the stability radius of the linear oscillator (9). As a result, the radii $r_L(b)$, $r_L^*(b)$, and $\tilde{r}_L(b)$ shown in that paper overestimate the stability of the system for certain values of the damping. (The consequences drawn for a robust design strategy in Section 8.1. remain valid.) Figure 3. below gives the correct values for $r(u) = r(\sigma)$, and Colonius and Kliemann (1990[b]) contains the exact values for the extremal Lyapunov exponents. Note that $r(u) \equiv 1$ for $u \geq \bar{u}$, $\bar{u} \sim 0.405$, and strictly increasing for $u \in [0, \bar{u}]$, with $r(u) \sim \pi \cdot u$ for u small.

Next we obtain the stabilizability regions of (9) that are achievable via common quadratic Lyapunov functions, see the concepts (c) and (e) above, and Proposition 2.1.

Consider the equation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -w & -2u \end{pmatrix} x =: A(w, u)x$$

with $u \geq 0$, $w = 1 + v \in (0, 2)$. According to Barmish (1985) quadratic stabilizability is equivalent to finding a common Lyapunov function for all $v \in [-\rho, \rho]$, $0 \leq \rho < 1$, i.e. we have to solve the following problem:

Find a positive definite matrix $P = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ such that $Q = A(w, u)^T P + PA(w, u)$ is positive definite for all $v \in [-\rho, \rho]$. Computing the Sylvester equations (compare e.g. Hahn (1967), p. 100) for P and Q yields:

(a) $\alpha > 0$,
(b) $\alpha\gamma - \beta^2 > 0$,
(c) $2w\beta > 0$, i.e. $\beta > 0$ since $w \in (0, 2)$,
(d) $-(\alpha - w\gamma)^2 + 4u\beta(\alpha + w\gamma) - 4\beta^2(w + u^2) > 0$.

Setting, without loss of generality, $\beta = 1$, (d) reads

(d′) $f(\alpha, \gamma) = -\alpha^2 + 2\alpha(2u + w\gamma) - 4(w + u^2) + 4wu\gamma - w^2\gamma^2 > 0$.

We solve for the zeros in α of $f(\alpha, \gamma)$ and obtain

(e) $\alpha_{1,2} = 2u + w\gamma \pm 2\sqrt{w}\,\sqrt{2u\gamma - 1}$.

Note that $2u\gamma - 1 \geq 0$ iff $u\gamma \geq \frac{1}{2}$, and for $u\gamma = \frac{1}{2}$ we have $\alpha_{1,2} = 2u + \frac{w}{2u}$.

For each $(w, u) \in (0, 2) \times \mathbb{R}^+$ the equation (e) describes a parabola in the $\alpha - \gamma$ plane, and (d′) is satisfied in the interior of this parabola. Note that for each $u \geq 0$, the parabolas are monotone in $w \in (0, 2)$. In order to obtain a common Lyapunov function for all $v \in [-\rho, \rho]$, we need that the parabolas corresponding to $w_1 = 1 + \rho$ and $w_2 = 1 - \rho$ intersect. Denote by $d(w_1, w_2)$ the difference between the lower branch (corresponding to α_2) for w_1 and the upper branch (corresponding to α_1) for w_2, as a function of γ:

$$d(w_1, w_2) = 2\rho\gamma - 2\sqrt{2u\gamma - 1}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right).$$

The minimum of $d(w_1, w_2)$ is attained at $\gamma = \frac{1}{2u} + \frac{u}{2\rho^2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right)^2$ and has the value $m(\rho, u) = \frac{\rho}{u} - \frac{2u}{\rho}\left(1 + \sqrt{1-\rho^2}\right)$.

Now we have to find for each $u \geq 0$ the largest $\rho(u) \in [0, 1)$ such that $m(\rho, u) \leq 0$. This value is given by

$$\rho(u) = 2u\sqrt{1-u^2} \quad \text{for } 0 \leq u \leq \frac{1}{\sqrt{2}}.$$

Thus, we obtain for the stabilizability radius $r_{Lf}(u)$ of (9) via quadratic Lyapunov functions

$$r_{Lf}(u) = \begin{cases} 2u\sqrt{1-u^2} & 0 \leq u \leq \frac{1}{\sqrt{2}} \\ 1 & u \geq \frac{1}{\sqrt{2}}. \end{cases}$$

We have $r_{Lf}(u) < r(u)$ for $u \in \left(0, \frac{1}{\sqrt{2}}\right)$, and $r_{Lf}(u) \sim 2u$ for u small, while $r(u) \sim \pi u$ around 0. In this example quadratic stabilization leads to a conservative criterion, if feedback in the damping with values in $\left[0, \frac{1}{\sqrt{2}}\right)$ is considered.

In Figure 1. of Colonius and Kliemann (1990[a]) the curve $r_C(b)$, which is the curve $r_{Lf}(u)$ by Proposition 2.1, was obtained numerically by solving a family of parametrized Riccati equations. It agrees quite well with the analytical result above.

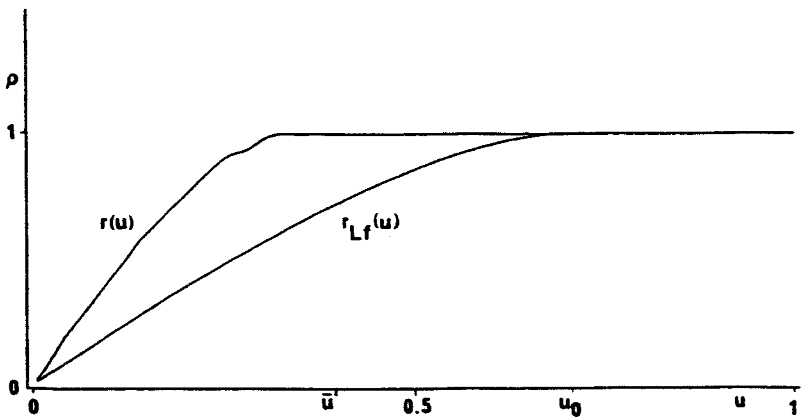

Figure 3: The stabilizability radii $r(u)$ and $r_{Lf}(u)$ for the system (9).

Our final examples show that the stabilizability radii $r(u)$ and $r_{Lf}(u)$ need not be monotone in u, although $r(\sigma) = \sup_{u \in U_\sigma} r(u)$ and $r_{Lf}(\sigma) = \sup_{u \in U_\sigma} r_{Lf}(u)$ are of course nondecreasing in σ. This shows that a high gain approach to stabilization will, in general, not yield the best results, and that the set up with feedback constraints is adequate for the stabilization of uncertain systems. Of course in practical applications, these constraints are often dictated by technical necessities.

4.4. Example. Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x + v(t)x + \begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix} x \tag{10}$$

with $v(t) \in M(2,2;\mathbb{R})$ and $u \geq 0$. Using the quadratic Lyapunov function approach, we obtain using the results of Kelb (1989) and Proposition 2.1:

$$r_{Lf}(u) = \begin{cases} s_2(A) & \text{for } u \in [0, u_0] \\ \frac{2\sqrt{u}}{1+u} & \text{for } u \geq u_0, \end{cases}$$

where $s_2(A)$ denotes the second singular value of $A = \begin{pmatrix} -1 & 1 \\ -u & -1 \end{pmatrix}$, and u_0 is the unique zero of $u^3 + u^2 + 3u - 1 = 0$, i.e. $u_0 \sim 0.296$. $r_{Lf}(u)$ has a unique maximum at $u = 1$ with $r_{Lf}(1) = 1$, and decreases for $u > 1$ with $\lim_{u \to \infty} r_{Lf}(u) = 0$.

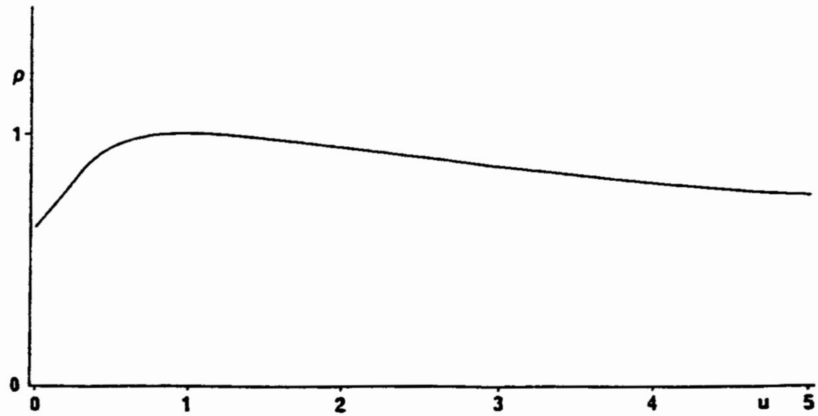

Figure 4: The quadratic stabilization radius $r_{Lf}(u)$ for the system (10).

4.5. Example. Consider again the linear oscillator of equation (9). Figure 5. shows the level curves of $\kappa(\rho, u)$ for the levels $-\frac{3}{4}$, $-\frac{1}{2}$, $-\frac{1}{4}$, and 0. Adding the constant diagonal matrix $A_1 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}$ yields the system

$$\dot{x} = \begin{pmatrix} 1/4 & 1 \\ -1 & 1/4 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -v(t) & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & -2u \end{pmatrix} x, \tag{11}$$

whose exponential stabilizability radius $r(u)$ is the level curve for the level $-\frac{1}{4}$ in Figure 5. In particular, $r(u) > 0$ iff $u \in (\frac{1}{4}, \frac{17}{8})$.

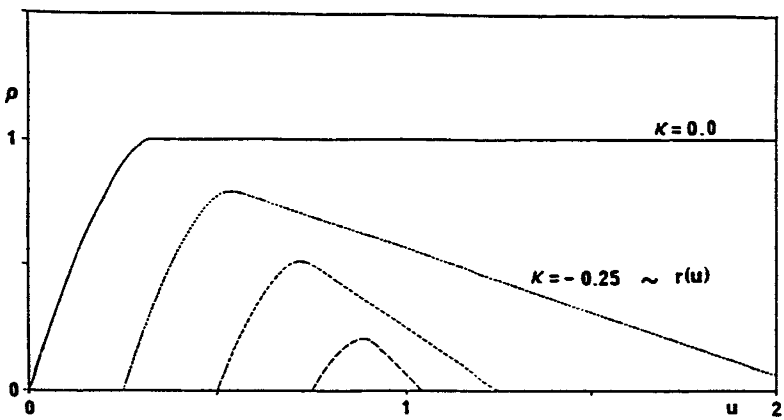

Figure 5: The exponential stabilization radius $r(u)$ for the system (11).

Acknowledgement:The authors would like to thank D. Hinrichsen and A. J. Pritchard for pointing out the reference Gonzalez (1990) to us. The figures were produced by A. Carriquiry on the NAS of Iowa State University using SAS/Graph.

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FRITZ COLONIUS
INSTITUT FÜR MATHEMATIK
UNIVERSITÄT AUGSBURG,
8900 AUGSBURG, FR OF GERMANY

WOLFGANG KLIEMANN
DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA 50011, USA