Robust Surveillance of Covariance Matrices Using a Single Observation

Olha Bodnar
Physikalisch-Technische Bundesanstalt, Berlin, Germany
Taras Bodnar
Humboldt-University of Berlin, Berlin, Germany
Yarema Okhrin
University of Augsburg, Augsburg, Germany

Abstract

In this paper a new technique for monitoring shifts in covariance matrices of Gaussian processes is developed. The processes we monitor are obtained from the covariance matrices estimated using a single observation. These processes follow independent Gaussian distribution in the in-control state, thus allowing for application of standard control charts. Furthermore, in contrary to the existing literature, the suggested procedure is asymptotically robust to the shifts in the mean. The explicit out-of-control distribution for an arbitrary moment of the shift is derived. The performance of numerous multivariate control charts is evaluated in an extensive simulation study and applied to monitoring volatilities on financial markets.

AMS (2000) *subject classification.* Primary 62L10; Secondary 62H10.
Keywords and phrases. singular Wishart distribution, CUSUM, EWMA, skew-normal distribution, expected delay.

1 Introduction

The purpose of statistical process control is to detect a structural break in a process as soon as possible after its occurrence. Its methods are widely applied in Engineering and now become increasingly important in other fields, i.e. in Economics, Medicine, Chemistry, and Finance. For a few of most relevant applications see Andersson, Bock and Frisén (2004), Bodnar (2007), Frisén (1992), Golosnoy et al. (2011), Lawson and Kleinman (2005), Messaoud, Weihs and Hering (2008), Schipper and Schmid (2001), Schmid and Tzotchev (2004), Sonesson and Bock (2003), among others. New fields impose new assumptions on the data driven process. In this paper we concentrate on the covariance matrix as a multivariate volatility measure.

Surveillance of volatility has recently become important particularly in finance for monitoring of the riskiness of an asset or of a portfolio.

Control charts are the main tools of statistical process control. A control statistic is derived individually for each quantity of interest and for each type of the chart. The value of the control statistic is updated using the incoming information and compared to a prespecified critical value. If it is exceeded, it is a sign of a potential change in the parameters of the underlying process. The simplest control chart without memory for the mean of an independent univariate Gaussian process was suggested by Shewhart (1931). The memory effect of the chart was exploited in the schemes of Page (1954) and Roberts (1959), called CUSUM and EWMA control charts respectively. The former chart is based on sequential probability ratio test and has some optimality properties (cf. Moustakides, 1986).

Different application areas lead to the extension of the univariate control charts to the multivariate case. Unfortunately, since the de-aggregation of the multivariate quantities to univariate characteristics is not unique, the procedures can be implemented in many alternative ways. A direct multivariate generalization of the Shewhart chart was suggested by Hotteling (1947). The EWMA chart was extended by Lowry et al. (1992). The CUSUM chart depends in the multivariate case on the size and direction of the expected shift. For that reason several authors proposed schemes for the mean which depend only on the magnitude of the expected shift like, Crosier (1988), Ngai and Zhang (2001), Pignatiello and Runger (1990).

In many applications we are interested not only in the mean of the process, but also in the variance. In the univariate case it is technically straightforward to adapt the mentioned procedures to the monitoring of, for example, squared observations. In practice, however, this leads frequently to non-standard distributions of the control statistics and substantially complicates the monitoring process. This becomes even more evident in the multivariate case while monitoring the covariance matrix. Because of this, the discussion is mostly restricted to the special cases of EWMA schemes (see Huwang, Yeh and Wu, 2007; Reynolds and Cho, 2006; Śliwa and Schmid, 2005; Yeh, Huwang and Wu, 2005). Due to the technical difficulties in the derivation the application of CUSUM charts for covariance matrices is rather limited (see Hawkins, 1981, 1991). Among few others, Chan and Zhang (2001) suggested a CUSUM type control chart for the covariance matrix of independent observations based on the projection pursuit method.

A typical problem of control charts for the variance or covariance matrix is the sensitivity of this chart to the changes in the mean vector. Thus a control chart for the volatility would signal a detected shift even if the true variance stays constant, but the mean changes and vise versa (see Morais and Pacheco, 2001). This is particularly important in practical applications, since changes in the mean and in the variance frequently occur together, but should be separated by the monitoring scheme. To our knowledge, only Huwang et al. (2007) suggested a monitoring scheme for the covariance which is robust to shifts in the mean vector.

In this paper we introduce a new technique, which allows us to apply the standard EWMA or CUSUM control charts for the mean directly to monitor the variance of a Gaussian vector. Moreover, the suggested technique is robust to the shifts in the mean of the observed process. This is attained in two steps. First, we eliminate the potential shift in the mean by detrending the observation via the exponentially weighted moving average. Second, the detrended quantities are transformed to follow approximately uncorrelated Gaussian distributions. A shift in the covariance matrix of the original process would lead to shifts in the mean of the transformed process. We derive the explicit out-of-control distribution of the transformed quantities if such a shift occurs. This implies that standard monitoring techniques for the mean vector can be applied to detect a shift in the covariance matrix. Another advantage of the suggested methodology, is the fact that it allows us to use a single multivariate observation to estimate the covariance matrix.

The rest of the paper is organized as follows. In the next section the change point model is presented. Then we specify the transformed process and analyze its properties. Furthermore, we review the multivariate control charts used for monitoring. The simulation study in Section 4 compares the modified charts for the covariance with the charts based on the transformed Gaussian quantities introduced in this paper. An empirical illustration is presented in Section 5. The proofs are given in the Appendix.

2 Models for the Target and the Observed Processes

In the definition of the control problem we distinguish between the target process and the observed process. The target process is defined as a process that fulfills the quality requirement. Usually, it depends on some parameters. These parameters can be set to corresponding target values (industry) or estimated using previous data (economics, finance). The observed process is the actual process that is observed in practice. Our aim is to make a conclusion if the observed and the target processes coincide.

We denote the target process by $\{\mathbf{Y}_t\}$. It is assumed that $\mathbf{Y}_1, ..., \mathbf{Y}_n$ are identical and independently normally distributed with $\mathbf{Y}_i \sim \mathcal{N}_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, $\boldsymbol{\Sigma}_0 = \mathbf{D}_0\mathbf{R}_0\mathbf{D}_0$. The matrices $\mathbf{D}_0$ and $\mathbf{R}_0$ are the diagonal matrix of standard deviations and the correlation matrix respectively. We assume that all parameters of the target process are known. Note, that, usually, this assumption is not fulfilled in practice. Especially in economic applications, the parameters of the target process are unknown and have to be estimated. It leads to an additional estimation risk. The impact of parameter uncertainty in the target process is not the task of our paper. The point has been already discussed by Kramer and Schmid (2000) and Albers and Kallenberg (2004). Without loss of generality we assume that $\boldsymbol{\mu}_0 = \mathbf{0}$.

The observed process we denote by $\{\mathbf{X}_t\}$. To model the observed process we apply the change-point framework given by

$$\mathbf{X}_t \sim \begin{cases} \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma}_0), & t < q \\ \mathcal{N}_p(\boldsymbol{\mu}, \mathbf{D}_\Delta \mathbf{R}_\Delta \mathbf{D}_\Delta), & t \geq q \end{cases}, \tag{2.1}$$

where $\mathbf{D}_\Delta$ and $\mathbf{R}_\Delta$ are the diagonal matrix and the correlation matrix after the change. It is assumed that the target and observed processes coincide, e.e. $\mathbf{X}_t = \mathbf{Y}_t$, for $t < q$. The matrices $\mathbf{D}_\Delta$ and $\mathbf{R}_\Delta$ as well as $q \in I\!N$ are unknown. In case $q < \infty$ we say that there is a change at the time point q. If $q = \infty$, the target process coincides with the observed process for all $t \in \mathbb{N}_+$ and hence the observed process is in control. The in-control covariance matrix $\boldsymbol{\Sigma}_0$ is called the target covariance matrix. If the change occurs the process is said to be out of control. The change can be in the mean vector, in the variances, in the correlations or in any combination of these quantities simultaneously.

In order to simplify the notations the out-of-control covariance matrix is denoted by $\boldsymbol{\Sigma}$. Later on, the expectation calculated with respect to model (2.1) with $q < \infty$ is denoted by $E_q(.)$, while $E_\infty(.)$ stands for the expectation calculated under the assumption that the process is in control and similarly for the variance $Var_q(\cdot)$ vs. $Var_\infty(\cdot)$. Similar notations are used for the variance and covariance. Note that $\mathbf{X}_t = \mathbf{Y}_t$ for $t \leq 0$, e.e. both processes are the same up to time point 0.

3 Control Charts for the Covariance Matrix

In this section, we introduce new control charts for the covariance matrix which are robust to the changes in the mean vector. Our approach is based on the properties of the singular Wishart distribution (see, e.g. Srivastava, 2003) and it is applied to a single observation of the observed process $\mathbf{X}_t$.

The signals produced by any surveillance procedure applied to $\mathbf{X}_t$ give us an indication of some structural change in the process. However, a clear assignment of a given signal to an actual change is hardly possible. For example, a signal coming from a control chart for the covariance matrix can be caused either by a change in the variances or even by a change in the mean behavior. We suggest a monitoring procedure for the covariance matrix, which is robust to changes in the mean. Thus any signal coming form the suggested control scheme can be unambiguously assigned to changes in the covariance matrix.

Let

$$\begin{aligned}\mathbf{Z}_t &= \lambda_Z \mathbf{X}_t + (1-\lambda_Z)\mathbf{Z}_{t-1}\\ &= \lambda_Z \sum_{l=0}^{t-1}(1-\lambda_Z)^l \mathbf{X}_{t-l} + (1-\lambda_Z)\mathbf{Z}_0 = \lambda_Z \sum_{l=0}^{t-1}(1-\lambda_Z)^l \mathbf{X}_{t-l} \qquad (3.1)\end{aligned}$$

with $\mathbf{Z}_0 = \mathbf{0}$ and $\lambda_Z \in (0,1]$. We consider the adjusted observation of the process $\{\mathbf{X}_t\}$ given by

$$\tilde{\mathbf{X}}_t = \mathbf{X}_t - \mathbf{Z}_t\,. \qquad (3.2)$$

Subtracting the EWMA statistics from the original process $\mathbf{X}_t$ corrects the process for potential changes in the mean behavior. Thus any jumps in the mean are eliminated in the long run via this transformation and this allows for tightly focused monitoring of the covariance matrix. In Lemma 3.1, we present the expressions for the mean vector and the covariance matrix of $\tilde{\mathbf{X}}_t$.

LEMMA 3.1. *Let $\mathbf{Y}_1, ..., \mathbf{Y}_n$ be an identically independently distributed Gaussian process with $\mathbf{Y}_i \sim \mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma}_0)$. We assume that the observed process $\{\mathbf{X}_t\}$ is defined according to the model (2.1). Then it follows that*

a) $\boldsymbol{\mu}(q,t) = E_q(\tilde{\mathbf{X}}_t) = (1-\lambda_Z)^{t-q+1}\boldsymbol{\mu}$ *for $q \le t$;*

b) $\mathbf{\Sigma}(q,t) = Var_q(\tilde{\mathbf{X}}_t) = (1-\lambda_Z)^2 \frac{\lambda_Z((1-\lambda_Z)^{2t-2q}-(1-\lambda_Z)^{2t-2})}{2-\lambda_Z}\mathbf{\Sigma}_0 + (1-\lambda_Z)^2\left(1+\frac{\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2t-2q})\right)\mathbf{\Sigma}$ *for $q \le t$;*

c) $$Cov_q(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau}) = -\lambda_Z(1-\lambda_Z)^{\tau+1}\Bigg(\mathbf{\Sigma} - \frac{1-\lambda_Z}{2-\lambda_Z}\Big((1-(1-\lambda_Z)^{2(t-\tau-q)})\mathbf{\Sigma} + ((1-\lambda_Z)^{2(t-\tau-q)} - (1-\lambda_Z)^{2(t-\tau-1)})\mathbf{\Sigma}_0\Big)\Bigg) \text{ for } q \le t-\tau.$$

and $Cov_q(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau}) = -\lambda_Z(1-\lambda_Z)^{\tau+1}\left(1-\frac{1-\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2(t-\tau-1)})\right)\mathbf{\Sigma}_0$ *for $t-\tau < q \le t$.*

The proof of Lemma 3.1 is given in the Appendix. From Lemma 3.1a, it follows that $\lim_{t\to\infty} E_q(\tilde{\mathbf{X}}_t) = \mathbf{0}$, which means that for the larger values of t, the mean vector of $\tilde{\mathbf{X}}$ tends to zero vector. Furthermore, changes in the mean vector of $\mathbf{X}_t$ do not influence the covariance matrix of $\tilde{\mathbf{X}}_t$ as well as the covariance matrix between $\tilde{\mathbf{X}}_t$ and $\tilde{\mathbf{X}}_{t-\tau}$. Moreover, from the proof of Lemma 3.1c) we get for $q > t - \tau$ that $Cov_q(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau}) = Cov_\infty(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau})$, where $Cov_\infty(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau})$ is the in-control covariance matrix between $\tilde{\mathbf{X}}_t$ and $\tilde{\mathbf{X}}_{t-\tau}$ given in Corollary 3.1c) below. We study this point more detailed after Theorem 3.1. In Corollary 3.1 the in-control quantities are presented. The proof of Corollary 3.1 follows from Lemma 3.1 by replacing $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$.

Corollary 3.1. *Let $\mathbf{Y}_1, ..., \mathbf{Y}_n$ be an identically independently distributed Gaussian process with $\mathbf{Y}_i \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma}_0)$. We assume that the observed process $\{\mathbf{X}_t\}$ is defined according to the model (2.1). Then it follows that*

a) $E_\infty(\tilde{\mathbf{X}}_t) = \mathbf{0}$;

b) $Var_\infty(\tilde{\mathbf{X}}_t) = (1-\lambda_Z)^2 \left(1 + \frac{\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2t-2})\right) \boldsymbol{\Sigma}_0$;

c) $Cov_\infty(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau}) = -\lambda_Z(1-\lambda_Z)^{\tau+1} \left(1 - \frac{1-\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2(t-\tau-1)})\right) \boldsymbol{\Sigma}_0$.

We consider

$$\begin{aligned}\mathbf{R}_\infty(\lambda_Z; \tau, t) &= Var_\infty(\tilde{\mathbf{X}}_t)^{-1/2} Cov_\infty(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau}) Var_\infty(\tilde{\mathbf{X}}_t)^{-1/2} \\ &= -\lambda_Z(1-\lambda_Z)^{\tau-1} \frac{1+(1-\lambda_Z)^{2(t-\tau)-1}}{2-\lambda_Z(1-\lambda_Z)^{2t-2}} \mathbf{I} = r(\lambda_Z; \tau, t)\mathbf{I}.\end{aligned}$$

If $\boldsymbol{\Sigma}_0$ is a diagonal matrix, then $\mathbf{R}_\infty(\lambda_Z; \tau, t)$ is an autocorrelation matrix between $\mathbf{X}_t$ and $\mathbf{X}_{t-\tau}$. It can be noted that $\mathbf{R}_\infty(\lambda_Z; \tau, t)$ is independent of $\boldsymbol{\Sigma}_0$. Moreover, $\mathbf{R}_\infty(\lambda_Z; \tau, t) = \mathbf{R}_1(\lambda_Z; \tau, t)$, e.e. the correlation matrix in the in-control and in the out-of-control states are the same if $q = 1$.

Consider now the scalar function $r(\lambda_Z; \tau, t)$. It holds that $r(\lambda_Z; \tau, t) \leq 0$ with $r(\lambda_Z; \tau, t) = 0$ only for $\lambda_Z = 0$ or $\lambda_Z = 1$. Because $0 < \lambda_Z \leq 1$ we obtain that $r(\lambda_Z; \tau, t)$ is a decreasing function in t with the minimum attained at the limit given by $\lim_{t\to\infty} r(\lambda_Z; \tau, t) = r(\lambda_Z; \tau, \infty) = -\lambda_Z(1-\lambda_Z)^{\tau-1}/2$. From Figure 1 we conclude that the autocorrelation in $\tilde{\boldsymbol{X}}_t$ is minor. This holds particularly for small values of λ_Z, e.e. less than 0.1, which are most relevant in practice. Thus the sequence $\tilde{\boldsymbol{X}}_t$ is approximately uncorrelated and the standard procedures for independent process can be applied.

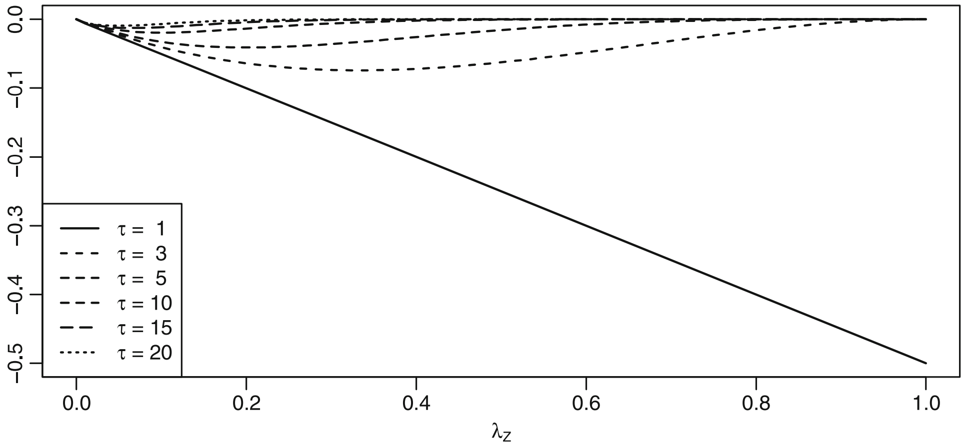

Figure 1: Plot of $r(\lambda_Z; \tau, \infty)$ as a function of λ_Z for different values of the lag τ.

Based on the distributional properties of $\tilde{\mathbf{X}}_t$ we suggest control charts for detecting changes in the covariance matrix that are robust to changes in the mean vector of $\mathbf{X}_t$. Moreover, from the proof of Lemma 3.1 we conclude that the in-control covariance matrix of $\tilde{\mathbf{X}}_t$ is given by

$$Var_\infty(\tilde{\mathbf{X}}_t) = h(\lambda_Z; t)\mathbf{\Sigma}_0, \qquad \text{with}$$
$$h(\lambda_Z; t) = (1-\lambda_Z)^2 \left(1 + \frac{\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2t-2})\right).$$

Because $\mathbf{\Sigma}_0(t)$ is proportional to $Var_\infty(\tilde{\mathbf{X}}_t)$ we apply the approach of Bodnar, Bodnar and Okhrin (2009), who derived sequential surveillance for the covariance matrix based on the properties of the singular Wishart distribution. These procedures are designed to detect changes in the covariance matrix of the normally distributed random vector which are not proportional to the in-control covariance matrix. To see this, note that $Var_\infty(\tilde{\mathbf{X}}_t)$ is proportional to $\mathbf{\Sigma}_0$ and the coefficient cancels out in the control statistic.

3.1. Transformation of the observed process. As an estimator of the covariance matrix we use the point estimate based on a single observation. At the time point t the covariance matrix is estimated by $\mathbf{V}_t = \tilde{\mathbf{X}}_t \tilde{\mathbf{X}}_t'$, where $\tilde{\mathbf{X}}_t$ is given in (3.2) with $\mathbf{Z}_t$ as in (3.1). The matrix $\mathbf{V}_t$ follows a singular Wishart distribution, i.e.

$$\mathbf{V}_t \sim W_p(1, h(\lambda_Z; t)\mathbf{\Sigma}_0)$$

in the in-control state. In the out-of-control state it holds that

$$\mathbf{V}_t \sim W_p(1, \boldsymbol{\Sigma}(q,t), \boldsymbol{\Omega}(q,t)) \qquad \text{with}$$
$$\boldsymbol{\Omega}(q,t) = \boldsymbol{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t)\boldsymbol{\mu}(q,t)'.$$

The symbol $W_p(1, \boldsymbol{\Sigma})$ denotes the p-dimensional singular Wishart distribution with 1 degree of freedom and the covariance matrix $\boldsymbol{\Sigma}$, while $W_p(1, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$ stands for the p-dimensional non-central singular Wishart distribution with 1 degree of freedom, the covariance matrix $\boldsymbol{\Sigma}$, and the non-centrality matrix $\boldsymbol{\Omega}$ (cf. Bodnar and Okhrin, 2008). The vector $\boldsymbol{\mu}(q,t)$ and the matrix $\boldsymbol{\Sigma}(q,t)$ are given in Lemma 3.1.

The rank of the matrix $\mathbf{V}_t$ is equal to one with probability one. It is not new to apply the unbiased point estimator $\mathbf{V}_t$ for monitoring purposes. Yeh et al. (2005) used $\mathbf{V}_t$ to update the matrix variate EWMA recursion, while Bodnar et al. (2009) obtained control charts for the covariance matrix that are based on the distributional properties of the singular Wishart distribution.

The standard control schemes for the covariance matrix are mainly based on the determinant and cannot be directly applied to the point estimator $\mathbf{V}_t$. In this paper we use the properties of the singular Wishart distribution to transform $\mathbf{V}_t$ to a set of Gaussian vectors. Then the mean charts can be immediately applied to monitoring the shifts in the variance. Let $\sigma_{0;ii}$ denote the (i,i)-th element of the matrix $\boldsymbol{\Sigma}_0$, $i = 1, ..., p$. By $\boldsymbol{\Sigma}_{0;21,i}$ we denote the i-th column of the matrix $\boldsymbol{\Sigma}_0$ without $\sigma_{0;ii}$. Let $\boldsymbol{\Sigma}_{0;22,i}$ denote a quadratic matrix of order $p-1$, which is obtained from the matrix $\boldsymbol{\Sigma}_0$ by deleting the i-th row and the i-th column. Let $\boldsymbol{\Sigma}_{0;22\cdot 1,i} = \boldsymbol{\Sigma}_{0;22,i} - \boldsymbol{\Sigma}_{0;21,i}\boldsymbol{\Sigma}'_{0;21,i}/\sigma_{0;ii}$. In the same way we define $v_{t;ii}$, $\mathbf{V}_{t;21,i}$, $\mathbf{V}_{t;22,i}$, $\mathbf{V}_{t;22\cdot 1,i}$, $\sigma_{ii}(q,t)$, $\boldsymbol{\Sigma}_{21,i}(q,t)$, $\boldsymbol{\Sigma}_{22,i}(q,t)$, and $\boldsymbol{\Sigma}_{22\cdot 1,i}(q,t)$ by partitioning $\mathbf{V}_t$ and $\boldsymbol{\Sigma}(q,t)$ respectively.

For detecting changes in the covariance matrix of the process $\{\mathbf{X}_t\}$, we consider the sequences of $\boldsymbol{\eta}_{i,t}$ expressed as

$$\boldsymbol{\eta}_{i,t} = \boldsymbol{\Sigma}_{0;22\cdot 1,i}^{-1/2}(\mathbf{V}_{t;21,i}/v_{t;ii} - \boldsymbol{\Sigma}_{0;21,i}/\sigma_{0;ii})v_{t;ii}^{1/2}.$$

The exact distributions of $\boldsymbol{\eta}_{i,t}$ in the in-control and out-of-control states are derived in Theorem 3.1. It is shown that the mean of these quantities reacts to the changes in the covariance matrix of the original process $\{\mathbf{X}_t\}$. Due to the transformation of $\mathbf{X}_t$ to $\tilde{\mathbf{X}}_t$ in (3.2) any shift in the mean of $\boldsymbol{\eta}_{i,t}$ can be uniquely assigned to a shift in the covariance matrix of $\mathbf{X}_t$. This implies that $\boldsymbol{\eta}_{i,t}$ has two advantages. First, the standard control schemes for the mean can be used for monitoring the variance. Second, the surveillance is robust to changes in the mean of the original process.

THEOREM 3.1. *Let $\mathbf{Y}_1, ..., \mathbf{Y}_n$ be an identically independently distributed Gaussian process with $\mathbf{Y}_i \sim \mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma}_0)$. We assume that the observed process $\{\mathbf{X}_t\}$ is defined according to the model (2.1). Then it follows that*

a) in the in-control state

$$\boldsymbol{\eta}_{i,t} \sim \mathcal{N}_{p-1}(\mathbf{0}_{p-1}, \mathbf{I}_{p-1})$$

and is independent of $v_{t;ii}$;

b) in the out-control state the density of $\boldsymbol{\eta}_{i,t}$ is given by

$$\begin{aligned}
f_{\boldsymbol{\eta}_{i,t};q}(\mathbf{z}) = & \frac{\sqrt{1+\sigma_{ii}(q,t)\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}{\pi^{-(p-1)/2}2^{-(p-1)/2}|\mathbf{\Xi}_i(q,t)|^{1/2}} \\
& \times \left(exp\left(-\frac{1}{2}\left(\mathbf{z}'\mathbf{\Xi}_i(q,t)^{-1}\mathbf{z}\right.\right.\right. \\
& \left.\left. - \frac{(\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\mathbf{z}+\nu(q,t))^2}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)} - 2g(\mathbf{z};q,t)\right)\right) \\
& \times \Phi\left(\frac{\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\mathbf{z}+\nu(q,t)}{\sqrt{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}\right) \\
& + exp\left(-\frac{1}{2}\left(\mathbf{z}'\mathbf{\Xi}_i(q,t)^{-1}\mathbf{z}\right.\right. \\
& \left.\left. - \frac{(\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\mathbf{z}-\nu(q,t))^2}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)} + 2g(\mathbf{z};q,t)\right)\right) \\
& \left. \times \Phi\left(\frac{\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\mathbf{z}-\nu(q,t)}{\sqrt{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\mathbf{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
g(\mathbf{z};q,t) &= [0 \;\; \mathbf{z}'\mathbf{\Sigma}_{0;22\cdot1,i}^{1/2}]\mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t), \\
\nu(q,t) &= [1 \;\; \mathbf{\Sigma}_{0;21,i}'/\sigma_{0;ii}]\mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t)\,, \\
\boldsymbol{\delta}_i(q,t) &= \mathbf{\Sigma}_{0;22\cdot1,i}^{-1/2}\left(\frac{\mathbf{\Sigma}_{21,i}(q,t)}{\sigma_{ii}(q,t)} - \frac{\mathbf{\Sigma}_{0;21,i}}{\sigma_{0;ii}}\right), \\
\mathbf{\Xi}_i(q,t) &= \mathbf{\Sigma}_{0;22\cdot1,i}^{-1/2}\mathbf{\Sigma}_{22\cdot1,i}(q,t)\mathbf{\Sigma}_{0;22\cdot1,i}^{-1/2}\,,
\end{aligned}$$

and $\Phi(.)$ denotes the distribution function of the univariate standard normal distribution.

For the proof of the first part of the theorem we refer to Bodnar and Okhrin (2008). The proof of the second part is given in the Appendix. If $\boldsymbol{\mu} = \mathbf{0}$ then the out-of-control density is simplified to

$$
\begin{aligned}
f_{\boldsymbol{\eta}_{i,t};q}(\mathbf{z}) = {}& 2\frac{\sqrt{1+\sigma_{ii}(q,t)\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}{\pi^{-(p-1)/2}2^{-(p-1)/2}|\boldsymbol{\Xi}_i(q,t)|^{1/2}} \\
& \times exp\left(-\frac{1}{2}\left(\mathbf{z}'\boldsymbol{\Xi}_i(q,t)^{-1}\mathbf{z}\right.\right. \\
& \left.\left. -\frac{(\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\mathbf{z})^2}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}\right)\right) \\
& \times \Phi\left(\frac{\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\mathbf{z}}{\sqrt{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}\right).
\end{aligned}
$$

Using the identities (cf. Harville, 1997, Theorem 18.1.1 and Theorem 18.2.8)

$$
\begin{aligned}
&\left|\boldsymbol{\Xi}_i(q,t)^{-1} - \frac{\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}\right| \\
&\quad = |\boldsymbol{\Xi}_i(q,t)^{-1}|(\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t))^{-1} \\
&\quad\quad \times (\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t) - \boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)) \\
&\quad = |\boldsymbol{\Xi}_i(q,t)|^{-1}(\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t))^{-1}\sigma_{ii}(q,t)^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\boldsymbol{\Xi}_i(q,t)^{-1} - \frac{\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}\right)^{-1} \\
&\quad = \boldsymbol{\Xi}_i(q,t) + \frac{\frac{\boldsymbol{\delta}_i(q,t)\boldsymbol{\delta}_i(q,t)'}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}{1-\frac{\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\Xi}_i(q,t)\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}} \\
&\quad = \boldsymbol{\Xi}_i(q,t) + \sigma_{ii}(q,t)\boldsymbol{\delta}_i(q,t)\boldsymbol{\delta}_i(q,t)'.
\end{aligned}
$$

we get

$$
\begin{aligned}
f_{\boldsymbol{\eta}_{i,t};q}(\mathbf{z}) = {}& 2\phi_{p-1}(\mathbf{z};\boldsymbol{\Xi}_i(q,t)+\sigma_{ii}(q,t)\boldsymbol{\delta}_i(q,t)\boldsymbol{\delta}_i(q,t)') \\
& \times \Phi\left(\frac{\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\mathbf{z}}{\sqrt{\sigma_{ii}(q,t)^{-1}+\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}}\right),
\end{aligned} \tag{3.3}
$$

where $\phi_m(.;\boldsymbol{\Sigma})$ stands for the density function of the m-dimensional normal distribution with zero mean vector and covariance matrix $\boldsymbol{\Sigma}$. The last equality shows that the random vector $\boldsymbol{\eta}_{i,t}$ in the out-of-control state under the assumption $\boldsymbol{\mu} = \mathbf{0}$ has a multivariate skew normal distribution (cf. Azzalini, 2005; Domínguez-Molina et al., 2007), while it is standard multivariate normally distributed in the in-control state. Moreover, using the properties of the multivariate skew normal distribution we get

$$E_q(\boldsymbol{\eta}_{i,t}) = \sqrt{\frac{2}{\pi}} \frac{(\boldsymbol{\Xi}_i(q,t) + \sigma_{ii}(q,t)\boldsymbol{\delta}_i(q,t)\boldsymbol{\delta}_i(q,t)')\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t)}{\sqrt{\sigma_{ii}(q,t)^{-1}}\sqrt{(1+\sigma_{ii}(q,t)\boldsymbol{\delta}_i(q,t)'\boldsymbol{\Xi}_i(q,t)^{-1}\boldsymbol{\delta}_i(q,t))^2}}$$
$$= \sqrt{\frac{2}{\pi}} \sqrt{\sigma_{ii}(q,t)}\boldsymbol{\delta}_i(q,t)\,.$$

Hence, the out-of-sample mean vector of $\boldsymbol{\eta}_{i,t}$ is not equal to zero iff $\boldsymbol{\delta}_i(q,t) \neq \mathbf{0}$, e.e. a change in the covariance matrix is present such that the out-of-control covariance matrix of $\mathbf{X}_t$ is not proportional to the in-control covariance matrix $\boldsymbol{\Sigma}_0$.

3.2. Asymptotic robustness to shifts in the mean. Next, we analyze the influence of changes in the mean vector on the out-of-control distribution of $\boldsymbol{\eta}_{i,t}$. For simplicity we put $q = 1$ and note that the results with a moderate effort can be generalized to an arbitrary q and to the cases when the shift in the mean does not occur simultaneously with the shift in the covariance matrix. For $q = 1$ we get

$$\boldsymbol{\mu}(1,t) = (1-\lambda_Z)^t \boldsymbol{\mu}\,,$$

$$\boldsymbol{\Sigma}(1,t) = (1-\lambda_Z)^2 \left(1 + \frac{\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2t-2})\right)\boldsymbol{\Sigma} = h(\lambda_Z;t)\boldsymbol{\Sigma}\,,$$

$$\boldsymbol{\delta}_i(1,t) = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\left(\frac{\boldsymbol{\Sigma}_{21,i}(1,t)}{\sigma_{ii}(1,t)} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right) = \boldsymbol{\delta}_i\,,$$

$$\boldsymbol{\Xi}_i(1,t) = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\boldsymbol{\Sigma}_{22\cdot 1,i}(1,t)\boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2} = h(\lambda_Z;t)\boldsymbol{\Xi}_i\,,$$

$$g(\mathbf{z};1,t) = [0\ \ \boldsymbol{\Sigma}_{0;22\cdot 1}^{1/2}\mathbf{z}]\boldsymbol{\Sigma}(1,t)^{-1}\boldsymbol{\mu}(1,t) = \frac{(1-\lambda_Z)^t}{h(\lambda_Z;t)}[0\ \ \boldsymbol{\Sigma}_{0;22\cdot 1}^{1/2}\mathbf{z}]\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$
$$= \frac{(1-\lambda_Z)^t}{h(\lambda_Z;t)} g(\mathbf{z})\,,$$

$$\nu(1,t) = [1\ \ \boldsymbol{\Sigma}_{0;21}/\sigma_{0;11}]\boldsymbol{\Sigma}(1,t)^{-1}\boldsymbol{\mu}(1,t) = \frac{(1-\lambda_Z)^t}{h(\lambda_Z;t)}[1\ \ \boldsymbol{\Sigma}_{0;21}/\sigma_{0;11}]\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$
$$= \frac{(1-\lambda_Z)^t}{h(\lambda_Z;t)}\nu\,.$$

Thus,

$$a_1(t) = \left(\mathbf{z}'\boldsymbol{\Xi}_i(1,t)^{-1}\mathbf{z} - \frac{(\boldsymbol{\delta}_i(1,t)'\boldsymbol{\Xi}_i(1,t)^{-1}\mathbf{z} \pm \nu(1,t))^2}{\sigma_{ii}(1,t)^{-1} + \boldsymbol{\delta}_i(1,t)'\boldsymbol{\Xi}_i(1,t)^{-1}\boldsymbol{\delta}_i(1,t)} \mp 2g(\mathbf{z};1,t)\right)$$
$$= h^{-1}(\lambda_Z;t)\left(\mathbf{z}'\boldsymbol{\Xi}_i^{-1}\mathbf{z} - \frac{(\boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\mathbf{z} \pm (1-\lambda_Z)^t\nu)^2}{\sigma_{ii}^{-1} + \boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\boldsymbol{\delta}_i} \mp 2(1-\lambda_Z)^t g(\mathbf{z})\right)$$

and

$$a_2(t) = \frac{\boldsymbol{\delta}_i(1,t)'\boldsymbol{\Xi}_i(1,t)^{-1}\mathbf{z} \pm \nu(1,t)}{\sqrt{\sigma_{ii}(1,t)^{-1} + \boldsymbol{\delta}_i(1,t)'\boldsymbol{\Xi}_i(1,t)^{-1}\boldsymbol{\delta}_i(1,t)}}$$
$$= h^{-1/2}(\lambda_Z;t)\frac{\boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\mathbf{z} \pm (1-\lambda_Z)^t\nu}{\sigma_{ii}^{-1} + \boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\boldsymbol{\delta}_i}\,.$$

Because $h(\lambda_Z;t) \to h^*(\lambda_Z) = (1-\lambda_Z)^2\left(1 + \frac{\lambda_Z}{2-\lambda_Z}\right) \neq 0$ we obtain that

$$f_{\boldsymbol{\eta}_{i,t};1}(\mathbf{z}) \to f^*_{i;1}(\mathbf{z}) = 2\frac{\sqrt{1 + \sigma_{ii}\boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\boldsymbol{\delta}_i}}{\pi^{-(p-1)/2}2^{-(p-1)/2}(h^*(\lambda_Z))^{p/2}|\boldsymbol{\Xi}_i|^{1/2}}$$
$$\times\, exp\left(-\frac{1}{2h^*(\lambda_Z)}\left(\mathbf{z}'\boldsymbol{\Xi}_i^{-1}\mathbf{z} - \frac{(\boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\mathbf{z})^2}{\sigma_{ii}^{-1} + \boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\boldsymbol{\delta}_i}\right)\right) \tag{3.4}$$
$$\times\, \Phi\left(\frac{1}{\sqrt{h^*(\lambda_Z)}}\frac{\boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\mathbf{z}}{\sqrt{\sigma_{ii}^{-1} + \boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\boldsymbol{\delta}_i}}\right)$$
$$= 2\phi_{p-1}(\mathbf{z}; h^*(\lambda_Z)(\boldsymbol{\Xi}_i + \sigma_{ii}\boldsymbol{\delta}_i\boldsymbol{\delta}_i'))\Phi\left(\frac{1}{\sqrt{h^*(\lambda_Z)}}\frac{\boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\mathbf{z}}{\sigma_{ii}^{-1} + \boldsymbol{\delta}_i'\boldsymbol{\Xi}_i^{-1}\boldsymbol{\delta}_i}\right),$$

which is independent of $\boldsymbol{\mu}$ and is the same as one given in (3.3) for the case $q = 1$. The last identity shows that the out-of-control distribution of $\boldsymbol{\eta}_{i,t}$ is robust to changes in the mean vector of $\mathbf{X}_t$ for sufficiently large t. Moreover, since only ν and $g(\mathbf{z})$ depend on $\boldsymbol{\mu}$ and both quantities are present in the expression of the density for $\boldsymbol{\eta}_{i,t}$ given in Theorem 3.1b with factor $(1-\lambda_Z)^{t-q+1}$ the density $f(\mathbf{z})$ converges to $f^*(\mathbf{z})$ quite fast especially for larger values of λ_Z. We provide a further investigation of this point in Section 4 where the results of the simulation study are given.

In the first part of Theorem 3.1 it is shown that $\boldsymbol{\eta}_{i,t}$ follows the standard normal distribution for each i and t. In the out-of-control state we observe a change in the mean vector of $\boldsymbol{\eta}_{i,t}$ as soon as $\boldsymbol{\delta}_i(q,t) \neq \mathbf{0}$, e.e. the elements of the i-th column of both matrices are not proportional. This is, however, not the case if we consider the change point model as in (2.1). Thus, the transformation $\boldsymbol{\eta}_{i,t}$ translates the shift in the covariance matrix into a shift in the mean. Note that the covariance matrix and the distribution of $\boldsymbol{\eta}_{i,t}$ change in the out-of-control state too.

4 Simulation study

The normality and the shift in the mean caused by the shift in the covariance matrix allow us to use the standard multivariate EWMA and CUSUM control schemes of Crosier (1988), Lowry et al. (1992), Ngai and Zhang (2001), and Pignatiello and Runger (1990) to monitor changes in the means of the processes $\{\boldsymbol{\eta}_{i,t}\}$. Note that we have p vectors $\boldsymbol{\eta}_{i,t}$ and each of them should be monitored. To cope with this problem we follow the approach of Woodall and Ncube (1985). The simultaneous use of univariate control charts for each i is considered to be a single joint control chart. This control chart gives an alarm if at least one of the individual charts signals a shift. Sections 4.1 and 4.2 contain technical details of the implementation of these charts.

As benchmark we consider the control chart of Huwang et al. (2007) which monitors the covariances of the process directly and is robust to changes in the mean vector. To the best of our knowledge it is the only control chart for the covariance matrix available in literature that is robust to changes in the mean vector. This control scheme is discussed in Section 4.3.

4.1. CUSUM control charts. In the multivariate generalization of the univariate CUSUM by Crosier (1986) the univariate quantities are replaced with vectors. In contrary to the univariate charts, where the control statistic is shrunk against zero by a multiple of the standard deviation, the multivariate control statistic is scaled along some predetermined direction. Here we monitor the processes $\{\boldsymbol{\eta}_{i,t}\}$ using the CUSUM scheme of Crosier (1988). Let $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + ... + a_p^2}$ be the Euclidean norm of the p-dimensional vector $\mathbf{a}$. Let $C_{i,t} = ||\mathbf{S}_{i,t-1} + \boldsymbol{\eta}_{i,t}||$, where

$$\mathbf{S}_{i,t} = \begin{cases} \mathbf{0} & \text{if} \quad C_{i,t} \leq k \\ (\mathbf{S}_{i,t-1} + \boldsymbol{\eta}_{i,t}) \left(1 - \dfrac{k}{C_{i,t}}\right) & \text{if} \quad C_{i,t} > k \end{cases}$$

for $t \geq 1$ with $\mathbf{S}_{i,0} = \mathbf{0}$ and for some predetermined constant $k > 0$, which plays the role of a reference value. Each individual control scheme is characterized by $MCUSUM_{i,t}$, which is equal to the length of the vector $\mathbf{S}_{i,t}$, i.e.

$$MCUSUM_{i,t} = (\mathbf{S}'_{i,t}\mathbf{S}_{i,t})^{1/2} = \max\{0, C_{i,t} - k\}\,.$$

The control statistic of the joint scheme is defined by

$$MCUSUM_t \;=\; max_{1\le i\le p}\{MCUSUM_{i,t}\}.$$

The scheme gives an out-of-control signal if $MCUSUM_t$ exceeds a preselected control limit h, which is determined within a simulation study.

Pignatiello and Runger (1990) proposed two types of multivariate CUSUM charts, namely MC1 and MC2. Let $\mathbf{S}_{i;m,l} = \sum_{j=m+1}^{l} \boldsymbol{\eta}_{i,t}$ for $l, m \ge 0$. The control statistic of the individual control charts is defined as

$$MC1_{i,t} = \max\{\|\mathbf{S}_{i;t-n_{i,t},t}\| - kn_{i,t}, 0\}\ ,\ t \ge 1\ ,$$

where

$$n_{i,t} = \begin{cases} n_{t-1}+1 & \text{if} \quad MC1_{i,t-1} > 0 \\ 1 & \text{if} \quad MC1_{i,t-1} = 0 \end{cases}$$

Similarly as for the MCUSUM chart, the joint MC1 scheme is constructed via the control statistics

$$MC1_t \;=\; max_{1\le i\le p}\{MC1_{i,t}\}.$$

The chart signals an alarm if $MC1_t > h$ for some fixed critical value h. The second considered by Pignatiello and Runger (1990) chart is based on

$$MC2_t \;=\; max_{1\le i\le p}\{MC2_{i,t}\}\,,$$

where

$$MC2_{i,t} = \max\{0, MC2_{i,t-1} + D^2_{i,t} - p - k\}\ ,\ t \ge 1$$

with $D^2_{i,t} = \boldsymbol{\eta}'_{i,t}\boldsymbol{\eta}_{i,t}$ and $MC2_{i,0} = 0$. The control scheme gives an alarm as soon as $MC2_t > h$.

The PPCUSUM chart of Ngai and Zhang (2001) is derived using the projection pursuit approach. Applying this approach to the processes $\boldsymbol{\eta}_{i,t}$ we obtain the following CUSUM statistic

$$PPCUSUM_t \;=\; max_{1\le i\le p}\{PPCUSUM_{i,t}\}$$

with

$$PPCUSUM_{i,t} \;=\; \max\{0, \|\mathbf{S}_{i;t-1,t}\| - k, \|\mathbf{S}_{i;t-2,t}\| \;-\; 2k,\, \ldots\,, \|\mathbf{S}_{i;0,t}\| \;-\; tk\}$$

for $t \ge 1$ with $\mathbf{S}_{i;t-v,t}$ defined above. Similarly as above the signal is given if $PPCUSUM_t > h$.

4.2. MEWMA control charts. An alternative class of more heuristically motivated charts constitute the EWMA schemes. Similarly as for the CUSUM charts, the multivariare EWMA chart is a direct generalization of the univariate scheme. Following Lowry et al. (1992) we define $\mathbf{Z}_{i,t} = r\boldsymbol{\eta}_{i,t} + (1-r)\mathbf{Z}_{i,t-1}$ with $\mathbf{Z}_{i,0} = \mathbf{0}$. For computational efficiency it is common to use variance of $\mathbf{Z}_{i,t}$. For the completeness of the study we consider both the exact and the asymptotic cases. The control statistic of the individual charts is defined by

$$Q_{i,t} = \frac{2-r}{r(1-(1-r)^{2t})}\mathbf{Z}'_{i,t}\mathbf{Z}_{i,t} \qquad \text{and} \qquad Qa_{i,t} = \frac{2-r}{r}\mathbf{Z}'_{i,t}\mathbf{Z}_{i,t}$$

respectively. In the simulation study we use only the asymptotic case. The joint MEWMA statistic is then given by

$$MEWMA_t \;=\; max_{1\leq i\leq p}\{Q_{i,t}\} \quad \text{and} \quad MEWMAa_t \;=\; max_{1\leq i\leq p}\{Qa_{i,t}\}\,.$$

Alternatively, the the EWMA chart can be based on the Mahalanobis distance of $\boldsymbol{\eta}_{i,t}$. This scalar quantity is subsequently monitored by the classical EWMA recursion. The individual control chart in this case is given by

$$QM_{i,t} = rD^2_{i,t} + (1-r)QM_{i,t-1} \quad \text{with} \quad QM_{i,0} = p-1\,.$$

As above the joint scheme is characterized by

$$MEWMAM_t \;=\; max_{1\leq i\leq p}\{QM_{i,t}\}\,.$$

Both EWMA charts signal an alarm as soon as the corresponding control statistic exceeds the preselected control limit.

4.3. Benchmark chart. Huwang et al. (2007) suggested the MEWMV control chart for detecting changes in the covariance matrix of the normally distributed random vector that is insensitive to changes in the mean vector. The control statistics is given by

$$MEWMV_t = r(\mathbf{X}_t - \mathbf{Z}_t)(\mathbf{X}_t - \mathbf{Z}_t)' + (1-r)MEWMV_{t-1},$$

with $\mathbf{Z}_t = \tilde{\lambda}_Z\mathbf{X}_t + (1-\tilde{\lambda}_Z)\mathbf{Z}_{t-1}$ and $MEWMV_0 = (\mathbf{X}_1 - \mathbf{Z}_1)(\mathbf{X}_1 - \mathbf{Z}_1)'$ and $\mathbf{Y}_0 = \mathbf{0}$. We can select either equal or different smoothing parameters in both recursions. In the simulation study we set $\tilde{\lambda}_Z = 0.2$ and optimize the chart with respect to r. This makes the chart comparable with other charts, which depend on a single parameter too. The choice of $\tilde{\lambda}_Z$ is in line with the results of the simulation study by Huwang et al. (2007), where the best results are obtained for $\tilde{\lambda}_Z = 0.2$. The MEWMV chart gives an alarm if $|tr(MEWMV_t) - E(tr(MEWMV_t))|/\sqrt{Var(tr(MEWMV_t))}$ exceeds some preselected level. Note that the MEWMV chart is easier to implement and to calibrate compared to the above mentioned schemes, since only the trace has to be computed and no simultaneous charts are needed.

4.4. Setup of study and summary of the results. The goal of this section is to compare the effectiveness of the control charts derived in the previous sections. The run length of a control chart is a standard building block for most of the performance measures is defined by

$$t_A = inf\{t \in \mathbb{N} : C_t \geq c\},$$

where C_t denotes the control statistic and c denotes the control limit. The run length is equal to the number of observations till the first alarm of the chart.

The most popular performance measure is the average run length (ARL). The ARL measures the average number of observations until the first alarm. All charts are calibrated to provide the same in-control average run length $ARL_\infty = E(t_A)$. We chose $ARL_\infty = 200$. This allows us to determine the control limit c for each chart.

The performance measure we use is the maximum expected delay (MED) (see Frisén, 1992; Knoth, 2003). In the case of the ARL we always assume that the shift occurs at the first moment of time. This is, however, not a feasible assumption in practical applications. The expected delay assumes the shift at an arbitrary time point q and is defined as

$$ED_q(t_A) = E_q(t_A - q + 1 | t_A \geq q)$$

provided $E_q(t_A) < \infty$. ED measures the delay in detecting the shift, if we know that the alarm is given after the shift has occurred. To overcome the dependency on q, Pollak and Siegmund (1975) propose the use of the maximum expected delay

$$MED = \sup_{q \geq 1} ED_q(t_A)$$

which can be considered as the criterion for the worst-case scenario. The chart with the smallest MED is argued to be the best one.

If there is no shift the MED coincides with the in-control ARL. If the shift occurs at $q = 1$ then the MED and the out-of-control ARL are equal. Moreover, the MED coincides with the out-of-control ARL for the Shewhart chart used for detecting changes in the mean of the univariate Gaussian process with independent observations. The application of the MED instead of the ARL is motivated by dealing with the inertia behavior of a control chart, e.e. the tendency of the chart to detect a shift, if it is in-control over a certain period of time (see, e.g., Woodall and Mahmoud, 2005). If a control chart can built a large amount of inertia, then the MED is maximized for larger values of q and it can be much higher than the corresponding ARL (e.g., the MC1 chart). For the control charts which are less inertial, the largest ED is usually obtained for $q = 1$ (e.g., the PPCUSUM chart). In this case the MED is close to the out-of-control ARL. The problem is, however, that MED is very sensitive to $\mathbf{R}_\Delta$ or/and $\mathbf{D}_\Delta$. Therefore, it is rarely possible to determine a chart which uniformly outperforms its alternatives.

Because no explicit formulas for the MED and the optimal control limit are available we use a Monte Carlo study. In our study 10^5 independent realizations of the target process are generated to estimate ARL_∞. The control limits of all charts are determined using the *regula falsi* method (see, e.g., Conte and de Boor, 1981) to achieve $ARL_\infty = 200$. Note that it is impossible to compute the maximum of ED over all possible values of q. For this reason we take only a bounded interval $1 \le q \le Q$. In our study Q is set to 30. To estimate the MED we use 10^6 replications.

The target process is defined to be a four-dimensional Gaussian process with the mean vector $\boldsymbol{\mu}_0 = \mathbf{0}$ and the covariance matrix $\boldsymbol{\Sigma}_0 = (0.3^{|i-j|})_{i,j=1,\ldots,4}$. Consequently, it holds that $\mathbf{D}_0 = \mathbf{I}$ and $\mathbf{R}_0 = \boldsymbol{\Sigma}_0$. In order to compare the control charts two out-of-control situations are considered. The first one is given by generating changes in the variance of the first two variables. In this case it holds that $\sigma_{\boldsymbol{\Delta};11} = d_{11}\sigma_{0;11}$ and $\sigma_{\boldsymbol{\Delta};22} = d_{22}\sigma_{0;22}$ with $d_{11}, d_{22} \in \{0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0\}$. In the second out-of-control situation the changes in the correlation coefficient ρ_{21} are generated by taking $\rho_{\boldsymbol{\Delta};21} \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ with $\rho_{0;21} = 0.3$ being the target value.

The purpose of process monitoring is not only to give an alarm if a shift occurs, but also to provide us with insights about the causes of the alarm. The control charts developed in this paper are aimed to detect shifts in the covariance matrix. From the other side they are robust to shifts in the mean. This simplifies the monitoring, but complicates the diagnostics and the analysis of the causes of the alarm. To assess the impact of additional shifts in the mean vector we perform a study with the out-of-control means given by $\boldsymbol{\mu} = (0.5, 0.5, 0, 0)'$ and $\boldsymbol{\mu} = (1.0, 1.0, 0, 0)'$.

The control schemes depend on further design parameters. The CUSUM charts depend on the reference values k and the EWMA charts on the smoothing parameter r. In our study k takes values from the set $\{0.0, 0.1, ..,$

1.6, 1.7}. Larger values than 1.7 are not considered because it turned out that they lead to numerically instable results. The smoothing parameter of EWMA charts is taken equal to $r \in \{0.1, 0.2, ..., 1.0\}$. Note that the Shewhart chart is a special case of the EWMA chart if $r = 1.0$. For each type of the chart we choose the optimal value of the design parameter (either k or r) which leads to the smallest out-of-control ARL or to the smallest MED.

The results of the simulation study are illustrated in Tables 1 to 4. Table 1 contains the values of the MED for different shifts d_{11} and d_{22} and without shifts in the mean. Each line in each block corresponds to one of the charts: MCUSUM, MC1, MC2, MEWMA, MEWMAM, MEWMV, PPCUSUM. We conclude that the MED decreases for each chart with increasing d_{11}. The MED also increases if d_{22} increases and is larger than one. The evidence is mixed for $d_{22} < 1$. The chart with the best performance are MC2 and MEWMAM. The first one appears to be better for larger values

Table 1: Maximum expected delay (MED) for different values of the shifts d_{11} and d_{22} and without shifts in the mean vector.

$d_{11} \backslash d_{22}$	0.5	0.75	1.0	1.25	1.5	1.75	2.0
1.25	84.98 (*1.2*)	86.20 (*1.3*)	68.27 (*1.5*)	37.25 (*1.5*)	19.53 (*1.5*)	11.74 (*1.6*)	8.06 (*1.5*)
	82.27 (*0.7*)	86.25 (*1.2*)	71.11 (*1.4*)	40.37 (*1.4*)	20.74 (*1.4*)	12.26 (*1.4*)	8.26 (*1.5*)
	69.74 (*3.0*)	72.82 (*3.0*)	54.15 (*3.0*)	27.00 (*3.0*)	14.54 (*3.0*)	9.36 (*3.0*)	6.82 (*3.1*)
	83.96 (*0.1*)	88.36 (*0.4*)	72.46 (*0.5*)	40.43 (*0.6*)	20.76 (*0.6*)	12.26 (*0.6*)	8.24 (*0.7*)
	68.03 (*0.1*)	71.24 (*0.1*)	53.40 (*0.1*)	25.68 (*0.1*)	14.14 (*0.1*)	9.36 (*0.1*)	6.86 (*0.2*)
	112.13 (*1.0*)	101.38 (*0.9*)	63.34 (*0.2*)	34.10 (*0.2*)	19.71 (*0.3*)	12.64 (*0.5*)	8.89 (*0.7*)
	77.64 (*0.8*)	81.11 (*1.1*)	67.96 (*1.1*)	37.46 (*1.3*)	18.91 (*1.5*)	11.12 (*1.5*)	7.43 (*1.5*)
1.50	29.49 (*1.5*)	29.82 (*1.6*)	26.80 (*1.5*)	19.94 (*1.5*)	13.35 (*1.5*)	9.23 (*1.6*)	6.82 (*1.7*)
	29.24 (*1.4*)	29.95 (*1.4*)	27.80 (*1.4*)	21.37 (*1.4*)	14.33 (*1.5*)	9.75 (*1.6*)	7.10 (*1.6*)
	23.64 (*3.0*)	24.28 (*3.0*)	21.22 (*3.0*)	14.74 (*3.0*)	10.11 (*3.0*)	7.40 (*3.1*)	5.75 (*3.3*)
	30.57 (*0.5*)	31.12 (*0.5*)	28.35 (*0.6*)	21.31 (*0.6*)	14.17 (*0.6*)	9.60 (*0.7*)	7.00 (*0.7*)
	23.03 (*0.1*)	23.65 (*0.1*)	20.52 (*0.1*)	14.33 (*0.1*)	9.95 (*0.1*)	7.42 (*0.1*)	5.81 (*0.2*)
	38.19 (*0.9*)	35.05 (*0.4*)	27.35 (*0.3*)	19.45 (*0.3*)	13.47 (*0.4*)	9.70 (*0.5*)	7.37 (*0.8*)
	26.93 (*1.1*)	27.75 (*1.3*)	25.78 (*1.5*)	19.46 (*1.4*)	12.82 (*1.5*)	8.67 (*1.5*)	6.30 (*1.7*)
1.75	14.96 (*1.5*)	15.03 (*1.5*)	14.17 (*1.5*)	11.94 (*1.5*)	9.28 (*1.6*)	7.15 (*1.7*)	5.63 (*1.7*)
	14.83 (*1.4*)	15.04 (*1.4*)	14.49 (*1.4*)	12.53 (*1.4*)	9.81 (*1.5*)	7.51 (*1.6*)	5.89 (*1.6*)
	12.55 (*3.0*)	12.71 (*3.0*)	11.77 (*3.0*)	9.54 (*3.0*)	7.43 (*3.1*)	5.91 (*3.1*)	4.85 (*3.3*)
	15.50 (*0.6*)	15.55 (*0.6*)	14.77 (*0.6*)	12.54 (*0.7*)	9.70 (*0.7*)	7.38 (*0.8*)	5.76 (*0.8*)
	12.58 (*0.1*)	12.73 (*0.1*)	11.69 (*0.1*)	9.55 (*0.1*)	7.49 (*0.1*)	5.94 (*0.2*)	4.91 (*0.3*)
	18.28 (*0.8*)	17.35 (*0.5*)	15.22 (*0.4*)	12.35 (*0.4*)	9.64 (*0.5*)	7.55 (*0.6*)	5.98 (*0.8*)
	13.57 (*1.3*)	13.83 (*1.3*)	13.12 (*1.5*)	11.30 (*1.5*)	8.76 (*1.7*)	6.63 (*1.7*)	5.19 (*1.7*)
2.00	9.42 (*1.5*)	9.46 (*1.6*)	9.10 (*1.6*)	8.16 (*1.6*)	6.86 (*1.6*)	5.65 (*1.7*)	4.72 (*1.7*)
	9.39 (*1.4*)	9.47 (*1.4*)	9.23 (*1.5*)	8.41 (*1.5*)	7.15 (*1.7*)	5.90 (*1.6*)	4.89 (*1.7*)
	8.31 (*3.0*)	8.38 (*3.0*)	7.93 (*3.1*)	6.89 (*3.1*)	5.79 (*3.3*)	4.86 (*3.3*)	4.15 (*3.6*)
	9.68 (*0.6*)	9.72 (*0.7*)	9.35 (*0.6*)	8.39 (*0.7*)	7.06 (*0.8*)	5.78 (*0.8*)	4.78 (*0.8*)
	8.44 (*0.2*)	8.47 (*0.2*)	7.98 (*0.2*)	6.95 (*0.2*)	5.81 (*0.2*)	4.89 (*0.2*)	4.18 (*0.3*)
	10.98 (*0.8*)	10.64 (*0.8*)	9.82 (*0.8*)	8.62 (*0.7*)	7.21 (*0.7*)	5.97 (*0.7*)	4.94 (*0.8*)
	8.47 (*1.5*)	8.58 (*1.5*)	8.34 (*1.5*)	7.51 (*1.5*)	6.34 (*1.6*)	5.17 (*1.6*)	4.30 (*1.7*)

Note: In the in-control state $d_{11} = d_{22} = 1$. The order of the charts is MCUSUM, MC1, MC2, MEWMA, MEWMAM, MEWMV, PPCUSUM. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

Table 2: Maximum expected delay (MED) for different values of the shifts d_{11} and d_{22} and with shifts in the mean vector ($\boldsymbol{\mu} = (0.5, 0.5, 0, 0)'$).

$d_{11} \backslash d_{22}$	0.5	0.75	1.0	1.25	1.5	1.75	2.0
1.25	84.97 (*1.4*)	85.64 (*1.4*)	68.02 (*1.6*)	37.00 (*1.5*)	19.12 (*1.5*)	11.41 (*1.6*)	7.81 (*1.6*)
	81.94 (*0.1*)	85.69 (*1.4*)	70.62 (*1.4*)	39.86 (*1.4*)	20.46 (*1.4*)	11.99 (*1.4*)	8.13 (*1.4*)
	70.69 (*3.0*)	73.29 (*3.0*)	54.52 (*3.0*)	27.19 (*3.0*)	14.63 (*3.0*)	9.32 (*3.2*)	6.75 (*3.2*)
	83.76 (*0.1*)	87.59 (*0.4*)	71.27 (*0.6*)	39.65 (*0.6*)	20.25 (*0.6*)	11.92 (*0.7*)	8.02 (*0.7*)
	67.19 (*0.1*)	70.46 (*0.1*)	52.24 (*0.1*)	24.78 (*0.1*)	13.57 (*0.1*)	9.00 (*0.1*)	6.62 (*0.2*)
	110.36 (*1.0*)	100.27 (*0.7*)	62.75 (*0.2*)	33.58 (*0.2*)	19.41 (*0.3*)	12.38 (*0.5*)	8.56 (*0.6*)
	77.56 (*0.9*)	81.07 (*1.1*)	67.77 (*1.1*)	37.62 (*1.5*)	18.90 (*1.5*)	11.03 (*1.5*)	7.45 (*1.5*)
1.50	29.25 (*1.5*)	29.35 (*1.5*)	26.29 (*1.5*)	19.48 (*1.5*)	12.97 (*1.6*)	8.86 (*1.6*)	6.58 (*1.6*)
	28.98 (*1.3*)	29.67 (*1.4*)	27.45 (*1.4*)	20.94 (*1.4*)	14.03 (*1.4*)	9.54 (*1.4*)	6.94 (*1.6*)
	23.79 (*3.0*)	24.47 (*3.0*)	21.30 (*3.0*)	14.88 (*3.0*)	10.13 (*3.2*)	7.34 (*3.2*)	5.64 (*3.5*)
	30.32 (*0.6*)	30.57 (*0.6*)	27.86 (*0.6*)	20.82 (*0.6*)	13.79 (*0.7*)	9.33 (*0.7*)	6.80 (*0.7*)
	22.68 (*0.1*)	23.14 (*0.1*)	19.90 (*0.1*)	13.90 (*0.1*)	9.60 (*0.1*)	7.16 (*0.2*)	5.59 (*0.2*)
	37.69 (*1.0*)	34.59 (*0.5*)	27.08 (*0.3*)	19.19 (*0.3*)	13.26 (*0.4*)	9.45 (*0.6*)	7.08 (*0.7*)
	27.03 (*1.1*)	27.74 (*1.2*)	25.70 (*1.5*)	19.47 (*1.5*)	12.74 (*1.5*)	8.68 (*1.6*)	6.29 (*1.5*)
1.75	14.75 (*1.5*)	14.77 (*1.6*)	13.84 (*1.5*)	11.59 (*1.7*)	8.95 (*1.6*)	6.88 (*1.7*)	5.46 (*1.6*)
	14.63 (*1.4*)	14.80 (*1.4*)	14.26 (*1.4*)	12.25 (*1.4*)	9.60 (*1.4*)	7.35 (*1.6*)	5.74 (*1.7*)
	12.58 (*3.0*)	12.73 (*3.1*)	11.83 (*3.2*)	9.55 (*3.1*)	7.34 (*3.2*)	5.80 (*3.2*)	4.71 (*3.5*)
	15.21 (*0.6*)	15.31 (*0.6*)	14.41 (*0.6*)	12.22 (*0.6*)	9.39 (*0.7*)	7.15 (*0.7*)	5.60 (*0.7*)
	12.37 (*0.1*)	12.41 (*0.1*)	11.35 (*0.1*)	9.13 (*0.1*)	7.21 (*0.1*)	5.75 (*0.2*)	4.72 (*0.2*)
	17.88 (*0.7*)	17.01 (*0.6*)	14.84 (*0.5*)	12.05 (*0.5*)	9.33 (*0.5*)	7.29 (*0.7*)	5.81 (*0.7*)
	13.60 (*1.5*)	13.83 (*1.2*)	13.18 (*1.5*)	11.28 (*1.5*)	8.72 (*1.5*)	6.62 (*1.5*)	5.16 (*1.5*)
2.00	9.32 (*1.6*)	9.25 (*1.6*)	8.84 (*1.6*)	7.90 (*1.6*)	6.60 (*1.6*)	5.44 (*1.7*)	4.54 (*1.7*)
	9.29 (*1.4*)	9.32 (*1.4*)	9.06 (*1.4*)	8.24 (*1.5*)	6.99 (*1.6*)	5.76 (*1.6*)	4.77 (*1.7*)
	8.21 (*3.2*)	8.29 (*3.2*)	7.84 (*3.4*)	6.82 (*3.2*)	5.67 (*3.6*)	4.73 (*3.5*)	4.02 (*3.5*)
	9.55 (*0.7*)	9.51 (*0.6*)	9.15 (*0.6*)	8.18 (*0.7*)	6.85 (*0.7*)	5.63 (*0.7*)	4.65 (*0.8*)
	8.29 (*0.2*)	8.29 (*0.2*)	7.79 (*0.2*)	6.73 (*0.2*)	5.63 (*0.2*)	4.74 (*0.2*)	4.05 (*0.3*)
	10.76 (*0.8*)	10.39 (*0.8*)	9.55 (*0.7*)	8.31 (*0.6*)	6.94 (*0.6*)	5.73 (*0.7*)	4.78 (*0.9*)
	8.49 (*1.5*)	8.56 (*1.5*)	8.35 (*1.5*)	7.48 (*1.5*)	6.35 (*1.5*)	5.21 (*1.5*)	4.30 (*1.7*)

Note: In the in-control state $d_{11} = d_{22} = 1$. The order of the charts is MCUSUM, MC1, MC2, MEWMA, MEWMAM, MEWMV, PPCUSUM. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

of d_{22}, while the second one has smaller MED for smaller shifts. Most importantly note that control schemes based on the suggested transformation clearly dominate the benchmark chart of Huwang et al. (2007).

Tables 2 and 3 provide the results for the same shifts in the covariance matrix, but with shifts in the mean. Overall tendency of the MED is similar as in the case of constant mean, e.e. the MC2 and MEWMAM dominate the alternatives. More important is, however, the direct comparison of the results in Table 1 with the results in these tables. We conclude that the shift in the mean has practically no impact on the values of the MED, stressing the robustness of the control charts for the variance to shifts in the mean. A minor reduction in the MED can be observed in Table 3. However, the shift in the mean considered here is large leading to slower convergence in (3.4). The scheme of Huwang et al. (2007) shows also a robust performance, however, with uniformly larger MED values.

The impact of shifts in the correlations is summarized in Table 4. The first block contains the results for no shifts in the mean, while the next two

Table 3: Maximum expected delay (MED) for different values of the shifts d_{11} and d_{22} and with shifts in the mean vector ($\boldsymbol{\mu} = (1.0, 1.0, 0, 0)'$).

$d_{11} \backslash d_{22}$	0.5	0.75	1.0	1.25	1.5	1.75	2.0
1.25	83.39 (*1.5*)	83.11 (*1.5*)	64.74 (*1.5*)	34.30 (*1.5*)	17.43 (*1.5*)	10.30 (*1.7*)	7.07 (*1.7*)
	81.89 (*0.1*)	85.73 (*1.2*)	71.03 (*1.5*)	39.83 (*1.5*)	20.24 (*1.5*)	11.79 (*1.6*)	7.90 (*1.6*)
	70.80 (*3.0*)	73.18 (*3.0*)	55.02 (*3.0*)	27.05 (*3.0*)	14.64 (*3.0*)	9.39 (*3.1*)	6.73 (*3.5*)
	81.67 (*0.1*)	86.72 (*0.4*)	71.66 (*0.4*)	39.99 (*0.7*)	20.31 (*0.7*)	11.74 (*0.6*)	7.84 (*0.7*)
	64.68 (*0.1*)	66.36 (*0.1*)	47.53 (*0.1*)	21.85 (*0.1*)	11.84 (*0.1*)	7.90 (*0.1*)	5.89 (*0.2*)
	107.23 (*1.0*)	95.55 (*0.9*)	61.90 (*0.2*)	32.78 (*0.3*)	18.23 (*0.4*)	11.37 (*0.5*)	7.77 (*0.7*)
	77.75 (*0.7*)	81.93 (*1.0*)	67.99 (*1.4*)	37.50 (*1.5*)	18.98 (*1.4*)	11.06 (*1.5*)	7.44 (*1.4*)
1.50	28.31 (*1.5*)	27.99 (*1.5*)	24.64 (*1.5*)	17.85 (*1.7*)	11.61 (*1.6*)	7.95 (*1.7*)	5.90 (*1.7*)
	28.57 (*1.3*)	29.43 (*1.3*)	27.29 (*1.4*)	20.83 (*1.5*)	13.78 (*1.5*)	9.27 (*1.6*)	6.74 (*1.5*)
	23.94 (*3.0*)	24.41 (*3.0*)	21.24 (*3.0*)	14.81 (*3.0*)	10.19 (*3.0*)	7.33 (*3.3*)	5.65 (*3.6*)
	29.73 (*0.4*)	30.26 (*0.4*)	27.82 (*0.6*)	20.75 (*0.7*)	13.61 (*0.7*)	9.16 (*0.7*)	6.64 (*0.7*)
	21.49 (*0.1*)	21.48 (*0.1*)	17.90 (*0.1*)	12.14 (*0.1*)	8.43 (*0.1*)	6.29 (*0.2*)	4.98 (*0.2*)
	35.36 (*0.9*)	32.57 (*0.8*)	25.91 (*0.4*)	17.94 (*0.4*)	12.14 (*0.5*)	8.61 (*0.8*)	6.33 (*0.8*)
	27.11 (*1.2*)	28.00 (*1.2*)	25.75 (*1.4*)	19.40 (*1.5*)	12.80 (*1.5*)	8.67 (*1.6*)	6.30 (*1.5*)
1.75	13.99 (*1.5*)	13.83 (*1.5*)	12.82 (*1.7*)	10.47 (*1.7*)	8.02 (*1.6*)	6.10 (*1.7*)	4.89 (*1.7*)
	14.37 (*1.3*)	14.61 (*1.3*)	13.98 (*1.5*)	12.05 (*1.5*)	9.35 (*1.5*)	7.10 (*1.6*)	5.57 (*1.7*)
	12.56 (*3.1*)	12.78 (*3.1*)	11.77 (*3.1*)	9.54 (*3.1*)	7.38 (*3.1*)	5.79 (*3.5*)	4.71 (*3.6*)
	14.93 (*0.5*)	15.00 (*0.6*)	14.23 (*0.6*)	12.01 (*0.7*)	9.27 (*0.7*)	6.99 (*0.7*)	5.43 (*0.7*)
	11.57 (*0.1*)	11.41 (*0.1*)	10.20 (*0.1*)	8.06 (*0.1*)	6.34 (*0.1*)	5.07 (*0.2*)	4.25 (*0.2*)
	16.57 (*0.8*)	15.65 (*0.8*)	13.84 (*0.6*)	11.09 (*0.5*)	8.49 (*0.8*)	6.51 (*0.8*)	5.17 (*0.8*)
	13.55 (*1.4*)	13.76 (*1.3*)	13.19 (*1.5*)	11.20 (*1.5*)	8.76 (*1.6*)	6.63 (*1.6*)	5.20 (*1.5*)
2.00	8.85 (*1.7*)	8.67 (*1.7*)	8.15 (*1.6*)	7.14 (*1.7*)	5.95 (*1.7*)	4.88 (*1.7*)	4.14 (*1.7*)
	9.04 (*1.4*)	9.07 (*1.4*)	8.82 (*1.5*)	7.98 (*1.6*)	6.77 (*1.6*)	5.58 (*1.5*)	4.63 (*1.6*)
	8.19 (*3.3*)	8.29 (*3.4*)	7.84 (*3.4*)	6.84 (*3.5*)	5.69 (*3.5*)	4.73 (*4.0*)	3.98 (*3.9*)
	9.29 (*0.6*)	9.26 (*0.6*)	8.90 (*0.7*)	7.99 (*0.7*)	6.68 (*0.7*)	5.44 (*0.8*)	4.49 (*0.7*)
	7.74 (*0.2*)	7.65 (*0.2*)	7.00 (*0.2*)	5.98 (*0.2*)	5.02 (*0.2*)	4.25 (*0.2*)	3.68 (*0.3*)
	9.87 (*0.8*)	9.52 (*0.8*)	8.69 (*0.8*)	7.50 (*0.8*)	6.24 (*0.8*)	5.12 (*0.9*)	4.29 (*0.8*)
	8.47 (*1.4*)	8.57 (*1.5*)	8.32 (*1.5*)	7.53 (*1.5*)	6.30 (*1.6*)	5.21 (*1.5*)	4.32 (*1.6*)

Note: In the in-control state $d_{11} = d_{22} = 1$. The order of the charts is MCUSUM, MC1, MC2, MEWMA, MEWMAM, MEWMV, PPCUSUM. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

blocks correspond to the shifts both in the correlations and in the mean vector. In contrast to the case with the shifts in the variances, here the MC1 chart clearly outperforms the alternatives. Note that for increasing correlation coefficients some of the charts have the MED value, which is larger than the in-control ARL. This is an unsatisfactory characteristic of these schemes. The shift in the mean has, however, no impact on the performance of the charts. The latter point is studied in Table 5 in detail. Here, we present the MEDs calculated when only shifts in the first two components of the mean vector occur and the covariance matrix stays constant. We observe that the MC1, the MC2, the MEWMA, the MEWMAM, and the PPCUSUM charts are relatively robust to changes in the mean vector. In contrast, the MEWMV reacts slightly stronger, whereas the MCUSUM control scheme is the worst one with respect to the robustness against changes in the mean vector. Thus we conclude that the suggested methodology is in general successful in eliminating the impact of shifts in the mean.

Table 4: Maximum expected delay (MED) for different shifts in the correlation coefficient ρ_{12}.

$\rho_{12} = 0.0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
				No shift in the means					
72.01 (*0.2*)	105.09 (*0.2*)	156.80 (*0.2*)	MCUSUM	168.86 (*0.1*)	116.57 (*0.1*)	81.66 (*0.2*)	58.46 (*0.2*)	45.37 (*0.2*)	36.63 (*0.3*)
68.49 (*0.2*)	101.86 (*0.1*)	154.18 (*0.1*)	MC1	168.00 (*0.1*)	110.07 (*0.1*)	75.28 (*0.1*)	55.52 (*0.2*)	41.06 (*0.2*)	32.77 (*0.2*)
104.01 (*3.0*)	129.69 (*3.2*)	160.48 (*3.2*)	MC2	224.88 (*4.3*)	240.86 (*4.3*)	239.88 (*4.5*)	226.61 (*4.5*)	206.14 (*4.6*)	184.04 (*4.7*)
78.38 (*0.1*)	115.15 (*0.1*)	163.58 (*0.1*)	MEWMA	206.26 (*0.1*)	175.41 (*0.1*)	135.62 (*0.1*)	102.22 (*0.1*)	78.21 (*0.1*)	62.60 (*0.1*)
108.84 (*0.2*)	135.72 (*0.2*)	167.55 (*0.4*)	MEWMAM	220.43 (*1.0*)	227.27 (*1.0*)	222.58 (*1.0*)	207.36 (*1.0*)	188.56 (*1.0*)	168.65 (*1.0*)
114.23 (*0.2*)	138.37 (*0.2*)	167.14 (*0.2*)	MEWMV	214.16 (*1.0*)	216.61 (*1.0*)	207.27 (*1.0*)	191.61 (*1.0*)	159.19 (*0.1*)	124.38 (*0.1*)
74.60 (*0.2*)	109.37 (*0.2*)	159.42 (*0.3*)	PPCUSUM	176.38 (*0.1*)	124.95 (*0.1*)	88.87 (*0.1*)	63.35 (*0.2*)	47.87 (*0.2*)	38.76 (*0.2*)
				With shifts in the mean vector ($\boldsymbol{\mu} = (0.5, 0.5, 0, 0)'$)					
73.19 (*0.2*)	105.87 (*0.2*)	157.96 (*0.2*)	MCUSUM	167.71 (*0.1*)	116.43 (*0.1*)	81.16 (*0.2*)	58.09 (*0.2*)	44.90 (*0.2*)	36.23 (*0.3*)
69.28 (*0.2*)	101.55 (*0.1*)	153.26 (*0.1*)	MC1	166.44 (*0.1*)	109.99 (*0.1*)	75.01 (*0.1*)	55.19 (*0.2*)	40.71 (*0.2*)	32.58 (*0.2*)
105.04 (*3.1*)	130.04 (*3.1*)	161.87 (*3.1*)	MC2	226.14 (*4.5*)	239.99 (*4.5*)	237.66 (*4.5*)	224.43 (*4.7*)	205.30 (*4.7*)	183.42 (*4.7*)
79.20 (*0.1*)	115.33 (*0.1*)	163.43 (*0.1*)	MEWMA	205.13 (*0.1*)	172.94 (*0.1*)	134.24 (*0.1*)	100.92 (*0.1*)	77.20 (*0.1*)	61.70 (*0.1*)
107.04 (*0.2*)	132.85 (*0.2*)	166.52 (*0.2*)	MEWMAM	218.42 (*1.0*)	225.95 (*1.0*)	221.08 (*1.0*)	206.08 (*1.0*)	187.53 (*1.0*)	167.82 (*1.0*)
114.17 (*0.4*)	139.71 (*0.2*)	168.37 (*0.2*)	MEWMV	214.70 (*0.9*)	217.56 (*0.9*)	208.25 (*1.0*)	192.49 (*1.0*)	160.62 (*0.1*)	125.90 (*0.1*)
74.61 (*0.2*)	108.05 (*0.2*)	159.35 (*0.2*)	PPCUSUM	175.33 (*0.1*)	124.66 (*0.1*)	88.68 (*0.1*)	63.08 (*0.2*)	47.75 (*0.2*)	38.66 (*0.2*)
				With shifts in the mean vector ($\boldsymbol{\mu} = (1.0, 1.0, 0, 0)'$)					
75.25 (*0.2*)	107.71 (*0.2*)	159.11 (*0.2*)	MCUSUM	165.87 (*0.1*)	113.34 (*0.1*)	78.90 (*0.2*)	56.40 (*0.2*)	43.62 (*0.2*)	35.08 (*0.3*)
71.86 (*0.2*)	105.72 (*0.2*)	155.51 (*0.1*)	MC1	167.69 (*0.1*)	110.39 (*0.1*)	75.34 (*0.1*)	55.43 (*0.2*)	40.89 (*0.2*)	32.65 (*0.2*)
104.99 (*3.0*)	130.10 (*3.0*)	161.42 (*3.0*)	MC2	225.93 (*4.5*)	238.33 (*4.7*)	236.01 (*4.7*)	223.41 (*4.7*)	203.89 (*4.7*)	181.18 (*4.7*)
79.27 (*0.1*)	114.66 (*0.1*)	161.19 (*0.1*)	MEWMA	202.62 (*0.1*)	172.31 (*0.1*)	133.14 (*0.1*)	100.08 (*0.1*)	76.55 (*0.1*)	61.11 (*0.1*)
103.38 (*0.2*)	129.73 (*0.2*)	163.49 (*0.2*)	MEWMAM	219.22 (*0.9*)	228.56 (*0.9*)	222.34 (*0.9*)	208.75 (*1.0*)	188.43 (*1.0*)	169.51 (*1.0*)
111.93 (*0.3*)	138.09 (*0.3*)	165.74 (*0.8*)	MEWMV	203.44 (*1.0*)	205.68 (*1.0*)	196.29 (*1.0*)	180.28 (*1.0*)	163.57 (*1.0*)	130.89 (*0.1*)
74.69 (*0.2*)	108.71 (*0.2*)	159.74 (*0.2*)	PPCUSUM	175.90 (*0.1*)	124.80 (*0.1*)	88.67 (*0.1*)	63.09 (*0.2*)	47.75 (*0.2*)	38.67 (*0.2*)

Note: MED as a function of the shift in ρ_{12} without (upper block) and with (middle and lower blocks) shifts in the mean vector. The in-control value of ρ_{12} is 0.3. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

Table 5: Maximum expected delay (MED) for different values of the shifts μ_1 and μ_2 in the mean vector (no shifts in the covariance matrix).

$\mu_1 \backslash \mu_2$	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5
-1.5	178.18 (*1.6*)	186.45 (*1.6*)	187.96 (*1.5*)	186.24 (*1.5*)	178.43 (*1.5*)	160.45 (*1.5*)	126.76 (*1.5*)
	196.19 (*0.2*)	196.78 (*0.2*)	197.66 (*0.2*)	196.06 (*0.2*)	196.71 (*0.2*)	197.98 (*0.2*)	196.88 (*0.2*)
	197.71 (*4.3*)	198.51 (*4.3*)	198.01 (*3.2*)	197.66 (*3.2*)	197.77 (*4.3*)	197.61 (*3.2*)	197.56 (*4.3*)
	197.52 (*0.6*)	197.91 (*0.7*)	197.23 (*0.6*)	197.35 (*0.7*)	197.34 (*0.7*)	197.61 (*1.0*)	197.69 (*0.6*)
	196.73 (*0.2*)	197.09 (*0.2*)	197.83 (*0.2*)	197.27 (*0.2*)	197.94 (*0.4*)	196.89 (*0.2*)	197.30 (*0.2*)
	166.99 (*1.0*)	179.41 (*0.9*)	185.93 (*0.9*)	189.49 (*0.9*)	190.15 (*0.9*)	188.98 (*0.9*)	180.81 (*0.9*)
	197.58 (*1.1*)	197.96 (*1.6*)	196.75 (*1.1*)	196.76 (*1.1*)	197.11 (*1.1*)	197.73 (*1.1*)	197.13 (*1.1*)
-1.0	185.64 (*1.6*)	194.07 (*1.5*)	195.69 (*1.6*)	194.35 (*1.5*)	192.31 (*1.5*)	183.11 (*1.6*)	161.40 (*1.5*)
	197.11 (*0.2*)	196.54 (*0.2*)	196.91 (*0.2*)	196.95 (*0.2*)	196.45 (*0.2*)	196.75 (*0.2*)	195.53 (*0.2*)
	198.79 (*4.2*)	198.80 (*4.3*)	198.08 (*3.6*)	198.96 (*3.6*)	198.04 (*4.3*)	198.08 (*3.6*)	197.27 (*4.3*)
	197.12 (*0.6*)	198.31 (*0.7*)	198.22 (*0.4*)	197.31 (*0.6*)	196.82 (*0.6*)	197.32 (*1.0*)	197.21 (*0.6*)
	197.23 (*0.2*)	197.20 (*0.2*)	197.81 (*0.2*)	198.46 (*0.2*)	197.60 (*0.2*)	195.99 (*0.2*)	196.17 (*0.2*)
	179.39 (*0.9*)	188.95 (*0.9*)	193.93 (*0.9*)	196.92 (*0.9*)	197.26 (*0.9*)	194.92 (*0.9*)	188.81 (*0.9*)
	197.79 (*0.5*)	197.94 (*1.7*)	197.84 (*1.1*)	198.10 (*1.1*)	196.45 (*1.7*)	196.68 (*1.1*)	197.14 (*1.1*)
-0.5	188.84 (*1.6*)	194.81 (*1.5*)	196.94 (*1.5*)	198.14 (*0.0*)	196.14 (*1.5*)	192.61 (*1.6*)	179.24 (*1.6*)
	197.35 (*1.0*)	197.07 (*0.2*)	196.21 (*0.2*)	197.68 (*0.2*)	196.12 (*0.2*)	197.20 (*0.2*)	196.38 (*0.2*)
	197.51 (*4.3*)	198.62 (*3.6*)	198.50 (*4.3*)	197.99 (*3.2*)	198.86 (*3.6*)	198.12 (*3.6*)	198.04 (*4.3*)
	198.14 (*0.4*)	197.12 (*0.6*)	197.33 (*0.6*)	197.26 (*0.6*)	197.04 (*0.6*)	197.93 (*1.0*)	197.24 (*0.7*)
	197.02 (*0.2*)	197.39 (*0.2*)	199.08 (*0.4*)	199.18 (*0.2*)	197.94 (*0.2*)	196.48 (*0.2*)	197.47 (*0.2*)
	187.25 (*0.9*)	193.59 (*0.9*)	197.28 (*0.8*)	198.07 (*0.9*)	198.87 (*0.9*)	196.07 (*0.9*)	190.35 (*0.9*)
	196.87 (*1.7*)	198.52 (*1.1*)	197.09 (*1.1*)	197.67 (*1.1*)	197.20 (*1.1*)	197.57 (*1.7*)	197.75 (*1.1*)
0.0	187.10 (*1.6*)	194.55 (*1.5*)	197.64 (*0.0*)	198.15 (*0.0*)	197.81 (*1.5*)	194.96 (*1.5*)	186.29 (*1.6*)
	197.91 (*0.2*)	196.99 (*0.2*)	197.13 (*0.2*)	198.70 (*0.2*)	197.44 (*0.2*)	197.80 (*0.3*)	196.85 (*0.2*)
	198.78 (*4.3*)	198.25 (*4.0*)	198.60 (*4.3*)	198.06 (*3.2*)	198.27 (*3.2*)	198.06 (*4.3*)	198.20 (*4.3*)
	197.60 (*0.7*)	197.61 (*0.6*)	197.82 (*0.6*)	198.14 (*0.6*)	197.57 (*1.0*)	197.59 (*0.6*)	197.77 (*0.7*)
	197.18 (*0.2*)	198.24 (*0.2*)	199.60 (*0.4*)	200.50 (*0.4*)	199.00 (*0.2*)	196.12 (*0.2*)	196.85 (*0.2*)
	190.04 (*0.9*)	195.14 (*0.9*)	198.65 (*0.9*)	199.07 (*0.9*)	198.59 (*0.9*)	196.46 (*0.9*)	189.45 (*0.9*)
	197.86 (*1.7*)	196.53 (*1.1*)	196.72 (*1.1*)	198.62 (*1.1*)	197.11 (*1.1*)	197.61 (*1.1*)	197.04 (*1.6*)
0.5	179.39 (*1.5*)	191.70 (*1.5*)	195.63 (*1.5*)	197.80 (*1.5*)	197.62 (*1.5*)	195.63 (*1.5*)	187.67 (*1.6*)
	196.66 (*0.2*)	197.53 (*0.2*)	197.39 (*0.2*)	196.88 (*0.2*)	197.09 (*0.2*)	196.64 (*0.2*)	196.98 (*0.2*)
	198.49 (*4.3*)	197.65 (*3.2*)	198.34 (*4.3*)	198.47 (*3.2*)	197.91 (*4.3*)	198.45 (*4.3*)	198.46 (*3.6*)
	197.77 (*0.6*)	197.36 (*0.6*)	197.67 (*0.6*)	197.31 (*0.6*)	197.29 (*0.6*)	197.67 (*0.4*)	195.96 (*0.6*)
	197.99 (*0.2*)	197.74 (*0.2*)	198.00 (*0.2*)	199.28 (*0.2*)	199.37 (*0.4*)	196.48 (*0.2*)	196.97 (*0.2*)
	189.88 (*0.9*)	195.42 (*0.9*)	198.97 (*0.9*)	198.88 (*0.9*)	197.60 (*0.2*)	194.42 (*0.9*)	187.15 (*0.9*)
	197.28 (*1.1*)	197.63 (*1.7*)	197.92 (*1.7*)	196.93 (*1.1*)	197.65 (*1.1*)	197.27 (*1.7*)	197.20 (*1.1*)

Table 5: Continued

$\mu_1 \backslash \mu_2$	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5
1.0	161.79 (*1.5*)	183.73 (*1.6*)	191.82 (*1.5*)	195.82 (*1.5*)	195.06 (*1.5*)	193.80 (*1.5*)	185.27 (*1.5*)
	196.92 (*0.2*)	196.46 (*0.2*)	198.04 (*0.2*)	197.23 (*0.2*)	198.34 (*0.3*)	196.25 (*0.2*)	197.08 (*0.2*)
	198.37 (*3.6*)	197.72 (*3.6*)	197.82 (*4.3*)	198.52 (*4.3*)	197.51 (*4.3*)	197.82 (*4.3*)	198.18 (*4.3*)
	196.09 (*0.6*)	196.97 (*1.0*)	197.43 (*0.6*)	197.10 (*0.6*)	197.41 (*0.6*)	197.55 (*0.6*)	196.83 (*0.6*)
	197.12 (*0.2*)	196.65 (*0.2*)	197.59 (*0.4*)	197.45 (*0.2*)	197.79 (*0.2*)	196.80 (*0.2*)	196.62 (*0.2*)
	189.28 (*0.9*)	194.86 (*0.9*)	195.65 (*0.9*)	195.93 (*0.9*)	194.40 (*0.9*)	189.12 (*0.9*)	180.31 (*0.9*)
	196.76 (*1.7*)	197.62 (*1.7*)	196.88 (*1.7*)	197.21 (*1.1*)	197.30 (*1.1*)	198.30 (*1.1*)	197.64 (*1.7*)
1.5	127.00 (*1.5*)	161.82 (*1.7*)	178.60 (*1.5*)	187.09 (*1.6*)	188.71 (*1.5*)	186.40 (*1.5*)	177.51 (*1.6*)
	196.89 (*0.2*)	197.19 (*0.2*)	196.92 (*0.2*)	196.55 (*0.2*)	197.27 (*0.2*)	197.02 (*0.2*)	196.17 (*0.2*)
	198.07 (*3.2*)	198.14 (*4.3*)	197.62 (*3.2*)	198.41 (*4.3*)	197.85 (*3.2*)	197.97 (*3.6*)	196.91 (*4.3*)
	196.56 (*0.6*)	197.16 (*0.6*)	197.13 (*0.4*)	197.20 (*0.6*)	196.81 (*0.6*)	198.32 (*1.0*)	197.38 (*0.6*)
	198.15 (*0.2*)	196.65 (*0.2*)	196.52 (*0.2*)	196.73 (*0.2*)	196.67 (*0.2*)	198.09 (*0.2*)	198.63 (*0.4*)
	181.21 (*0.9*)	189.43 (*0.9*)	191.27 (*0.9*)	190.14 (*1.0*)	187.26 (*0.9*)	178.96 (*0.9*)	167.10 (*0.9*)
	197.07 (*1.1*)	196.49 (*1.1*)	198.04 (*1.7*)	197.16 (*1.7*)	197.33 (*1.7*)	197.59 (*1.1*)	197.25 (*1.7*)
ppcusum							

Note: In the in-control state $\mu_1 = \mu_2 = 0$. The order of the charts is MCUSUM, MC1, MC2, MEWMA, MEWMAM, MEWMV, PPCUSUM. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

It is known in the literature that monitoring some types of test statistics based on the EWMA of the sample observations is generally more effective than monitoring the EWMA of some types of statistics calculated from the original data. This holds for CUSUM charts too. Our results point towards a partially opposite evidence, since the MEWMAM and the MC2 charts outperform the MEWMA and the MC1 charts, respectively. This is another indication that the control charts derived from the $\eta_{i,t}$'s are not just merely detecting a shift in the mean vector, but rather a change in the distribution as well and supports the results of Theorem 3.1.[1]

5 Empirical Illustration

The covariance matrices of financial asset returns play a key role as risk measures in portfolio theory and asset allocation. Since the seminal paper of Markowitz (1952), it is well-known that the optimal portfolios in most models depend on the estimated covariance matrix. Thus monitoring the covariance matrices can be considered as monitoring the risk on financial markets and therefore has a direct impact on determining optimal portfolios and financial decision-making.

The covariance matrices are estimated using asset returns, which are computed using closing prices, e.g. the observed price at the end of trading day. Thus, these covariance matrices are an ideal field of application for the suggested methods of monitoring covariance matrices using a single observation. We employ financial indices for four countries in our study: Standard & Poor 500, FTSE 100, DAX 30 and NIKKEI 225. To check whether the assumptions needed for the suggested control schemes are met, we analyze the autocorrelation pattern and normality of the daily returns. Figure 2 shows the empirical autocorrelation function for all four series with the dash horizontal line marking the area of insignificant autocorrelations. Additionally the p-values of the Pierce-Box test for autocorrelation with 20 lags are given in the first row of Table 6. Relying on both results we can conclude that the autocorrelation can be neglected. The assumption of normality is verified for each time series using the Kolmogorov-Smirnov test with the p-values given in the second row of Table 6. We used for estimation the data which precedes the Euro-zone crisis (see below), but for other periods the results are very similar.

Note that the violation of the above-mentioned assumption is not always dangerous: in many examples, where the autocorrelation is only weak,

[1]We thank the anonymous referee for pointing our attention to this fact.

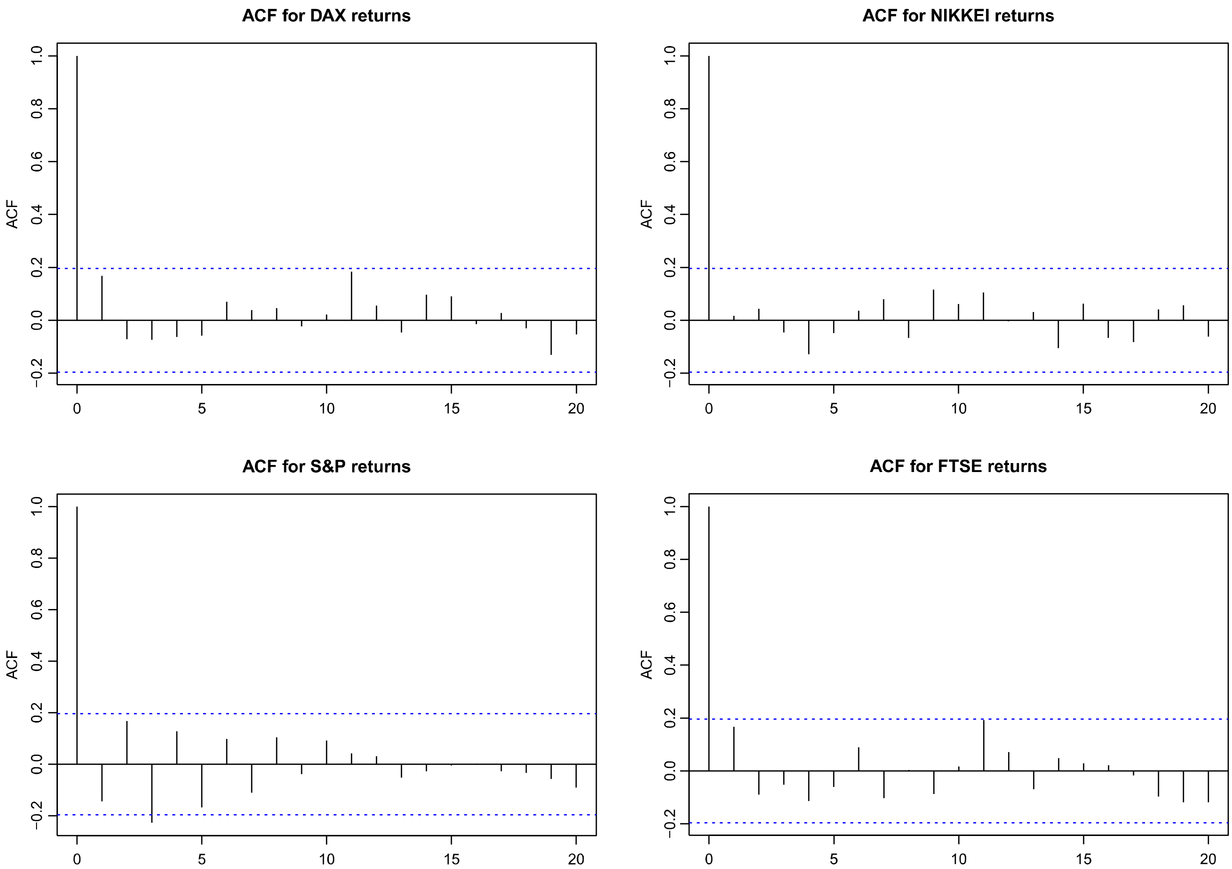

Figure 2: Empirical autocorrelation functions for the returns of the four financial indices.

it has only a minor impact on the performance of the charts (see Okhrin and Schmid, 2008) and will influence all of them in a fairly similar manner. Furthermore, the control charts under consideration remain robust if the assumption of normality is replaced by the assumption of elliptically contoured distributions which include a large number of heavy-tailed distributions. The reason behind this observation is that the control procedures do not detect changes in the covariance matrix which are proportional to the in-control matrix. However, in case of elliptically contoured distributions,

Table 6: p-values of the Pierce-Box test for autocorrelation with $m = 20$ (first row) and p-values of the Kolmogorov-Smirnov test for normality (second row).

Index	DAX	NIKKEI	S&P	FTSE
p-value (PB)	0.8602	0.9683	0.4125	0.6543
p-value (KS)	0.6327	0.7509	0.8032	0.7317

the conditional distribution of the random vector given a random variable, the so-called generating variable, is normal, whereas the realization of the generating variable is given as a multiplicative factor in the formula for the covariance matrix (see, e.g. Gupta, Varga and Bodnar, 2013).

Increases in risk of country indices (or increases in volatility) are usually linked to some important macroeconomic events and frequently lead to increases in the correlation between markets. In our analysis, we concentrate on three such events: the bankruptcy of Lehmann Brothers in September 2008, the Greek crisis of government debt in April 2010 and the Euro-zone crisis in July-August 2011. We use the MC1, the MEWMAM and the MEWMV control schemes to assess how long the charts need to detect changes in financial risk, measured by the covariance matrices.

Determining the in-control state is a difficult problem in non-manufacturing examples, since frequently there is no target value for the process parameters, e.g. expected return or the variances/covariances of returns. We estimated the parameters using the period of 90 days which ends on the first plotted day, e.e. approximately 50 days before the expected financial crash. The purpose of these 50 days is to eliminate any disturbances which precede the crash and might be expected by market participants. The estimation window of 90 is common in finance and corresponds roughly to 3 months of daily data. Taking shorter periods leads to increased estimation risk, while longer periods might cause bias due to structural breaks. Due to specific properties and high volatility of index returns and to multivariate nature of the problem, a statistical search for the period of constant parameters is useless. The estimated in-control covariance matrix is used to transform the returns to have the same in-control parameters as those in the simulation study.

The chart parameters are as follows: MC1 ($\lambda_Z = 0.2$, $k = 0.1$, critical limit 15.2856), MEWMAM ($\lambda_Z = 0.2$, $r = 0.5$, critical limit 6.52985) and MEWMV ($\lambda_Z = 0.2$, $r = 0.5$, critical limit 3.47993). The control statistics for all three charts are plotted for each crisis period in Figure 3. We conclude that MEWMAM and MEWMV show very synchronous performance, giving signals in all three period on the same day. The MC1 chart outperforms the alternatives for the Euro crisis, but has a relatively large delay for the other two periods. The inertia of this type of chart (cf. Woodall and Mahmoud, 2005) provides an explanation for this observation. The main reason is that the truncation of the control statistic to zero does not help much to deal with the inertia problem in the multivariate case as it does in the univariate case. In the multivariate case, the reset of the control statistic to zero means that the norm of the cumulative sum is relatively small. In contrast, the

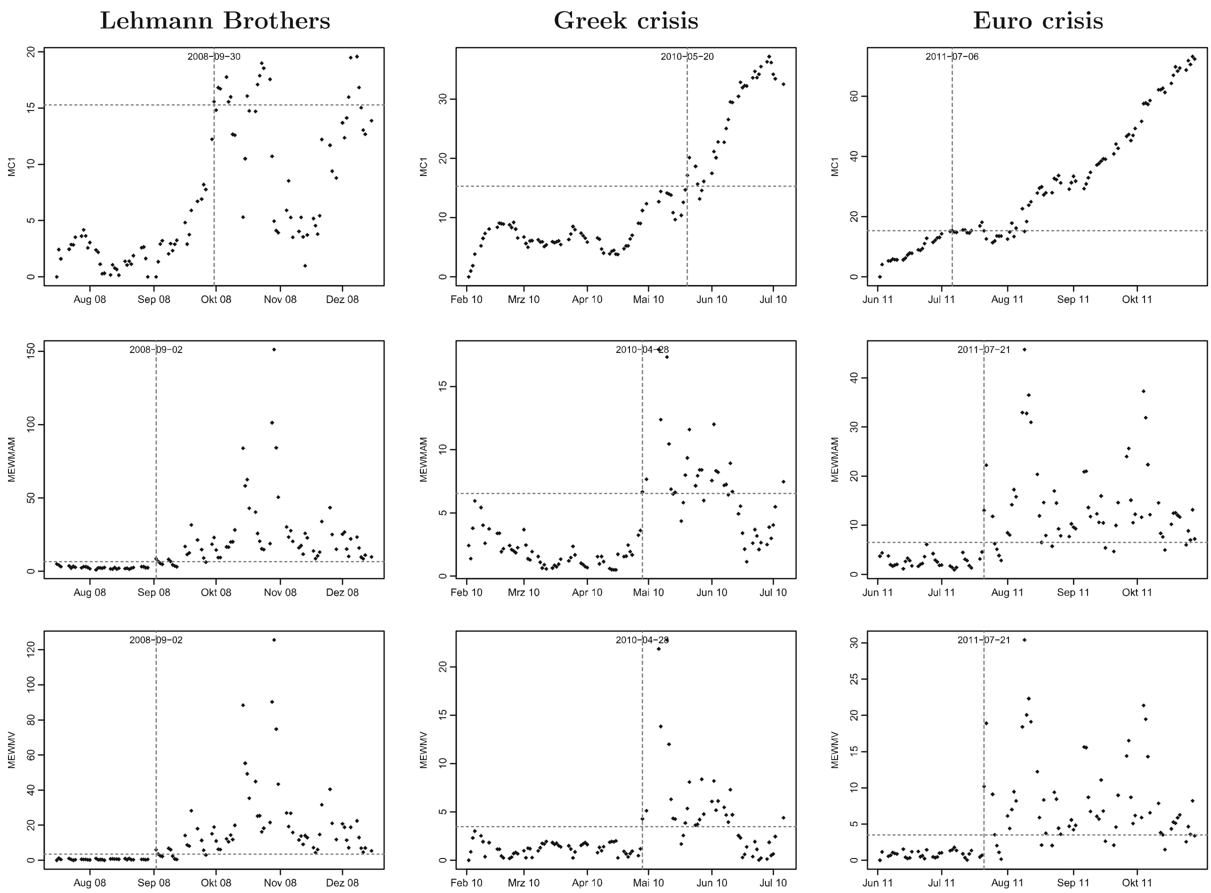

Figure 3: The control statistics for MC1 (top), MEWMAM (middle) and MEWMV (bottom) for three crisis periods of Lehmann Brothers (left), Greek crisis (middle) and Euro crisis (right) with the critical limits (horizontal lines) and the first signals (vertical lines). For the estimation of the in-control parameters the last 90 trading days are used.

test statistic is not obviously set to zero if the norm of the cumulative sum vector is large, but its direction is opposite to the direction of the shift. As a result, in the next period the shift could compensate the previous value. It leads to a small value of the test statistic which is then reset to zero. Since several MC1 control charts are performed simultaneously, the situation may become even worse.

Since the precise start of the crises is unknown, it is difficult to assess the delay of the employed control measures. For example, the Lehmann Brothers bankruptcy filing was announced on the 15th September 2008, however, rumors about the bankruptcy were already wide-spread a few weeks before this date. These rumors had already led to a 45% plunge in the share price on the 9th of September. As Lehmann Brothers was an widely-observed international corporation, this plunge has already had an impact on world financial markets and thereby caused signals in the suggested surveillance schemes for market risks. The same holds true for the other two crises, where the exact starting date is not known. Thus the suggested control charts can be successfully used to detect the time point, when some important events manifested themselves in the overall risks on financial markets. Nevertheless, the results of monitoring of financial data have to be interpreted carefully and without definit conclusions as in manufacturing sector. The national financial indices which we consider are influenced by numerous factors, which are linked to national and worldwide affairs. Thus it is not possible to identify the cause of a shift uniquely. For example, determining which event led to the shift on 02.09.2008 is rather difficult and mainly heuristic. The shifts which occur one week later are persistent and much stronger. Those can be assigned with high probability to the collapse of Lehmann Brothers.

To asses the robustness of the chart we also apply the simple MEWMA chart with asymptotic limits (MEWMAa) to the asset returns to detect any potential shifts in the process mean. The smoothing parameter and the critical value are set to 0.1 and 12.73 respectively (see Lowry et al., 1992)

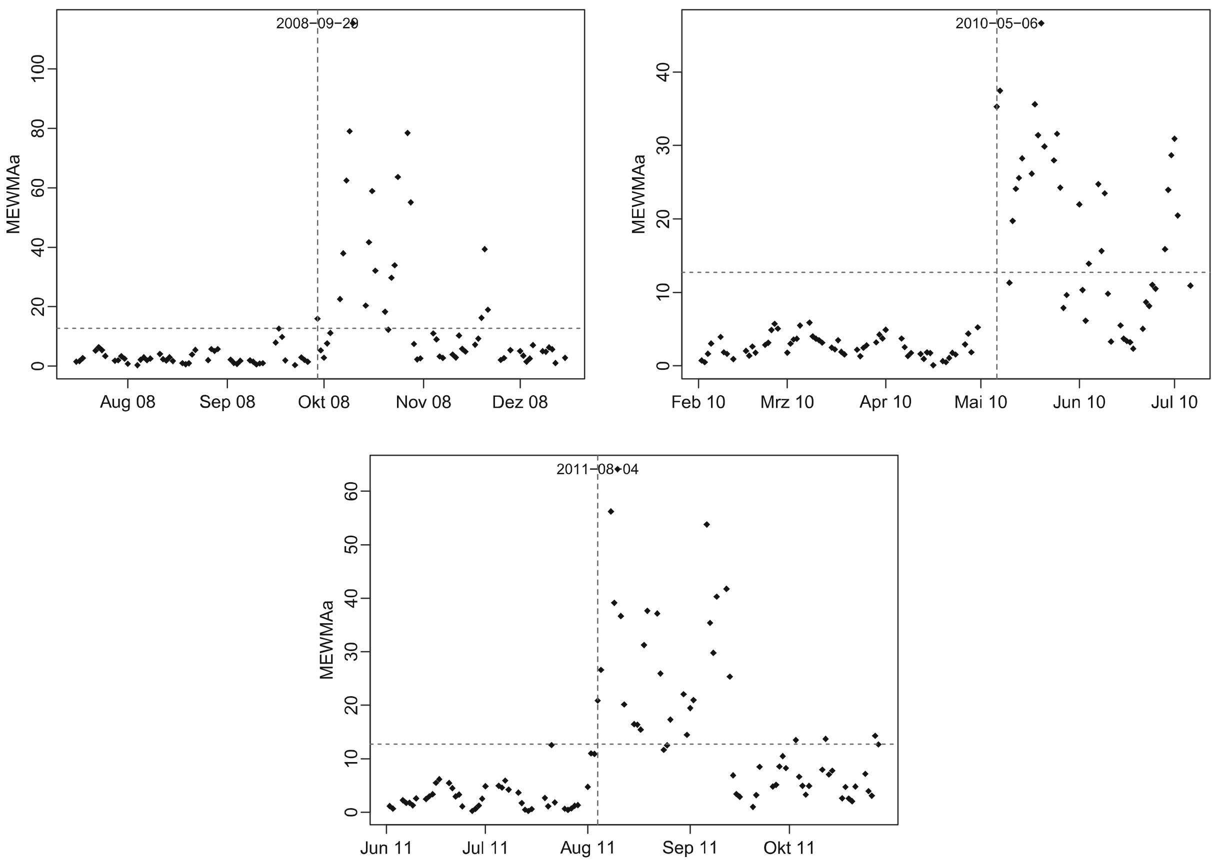

Figure 4: The control statistics of MEWMAa chart applied to detect shifts in the mean of the index returns during the three crisis periods.

and the same setup as in the charts for the variance is used. The chart signals frequently in Figure 4, implying that there are shifts both in the means and in the variances/covariances of the index returns. It is also interesting to note that at the end of 2008 as well as at the end of 2011, only changes in the covariance matrix are present, whereas at the end of the Greek crisis the MEWMAa control chart does not detect any changes in the mean vector.

6 Summary

In this paper we deal with monitoring shifts in the covariance matrices of Gaussian processes. To develop a control scheme robust to the shifts in the mean, the observations are detrended. Subsequently, the covariance matrix, estimated using a single observation, is transformed to follow approximately uncorrelated multivariate Gaussian distribution. This allows us to apply the classical charts for shifts in the mean. Within the simulations study we assess the advantages of the proposed procedure and compare it with a benchmark.

Acknowledgements

The authors appreciate the financial support of the German Science Foundation (DFG), projects BO3521/2 and OK103/1, "Wishart Processes in Statistics and Econometrics: Theory and Applications". The second author is partly supported by the German Science Foundation (DFG) via the Research Unit 1735 "Structural Inference in Statistics: Adaptation and Efficiency". The authors are grateful to the referees and the editor for their suggestions, which have improved the presentation in the paper.

A Appendix

This section contains the proofs.

PROOF OF LEMMA 3.1.

a) It holds that

$$\begin{aligned}
E_q(\tilde{\mathbf{X}}_t) &= E_q(\mathbf{X}_t) - E_q(\mathbf{Z}_t) \\
&= \boldsymbol{\mu} - \lambda_Z \sum_{l=0}^{t-q} (1-\lambda_Z)^l E_q(\mathbf{X}_{t-l}) = \boldsymbol{\mu} - \lambda_Z \sum_{l=0}^{t-q} (1-\lambda_Z)^l \boldsymbol{\mu} \\
&= \left(1 - \lambda_Z \frac{1-(1-\lambda_Z)^{t-q+1}}{1-(1-\lambda_Z)}\right) \boldsymbol{\mu} = (1-\lambda_Z)^{t-q+1} \boldsymbol{\mu}\,.
\end{aligned}$$

b) We get

$$
\begin{aligned}
Var_q(\tilde{\mathbf{X}}_t) &= Var_q(\mathbf{X}_t - \mathbf{Z}_t) \\
&= Var_q\left((1-\lambda_Z)\mathbf{X}_t - \lambda_Z(1-\lambda_Z)\sum_{l=0}^{t-2}(1-\lambda_Z)^l\mathbf{X}_{t-1-l}\right) \\
&= (1-\lambda_Z)^2 Var_q(\mathbf{X}_t) + \lambda_Z^2(1-\lambda_Z)^2 \\
&\quad \times \sum_{l=0}^{t-2}(1-\lambda_Z)^{2l} Var_q(\mathbf{X}_{t-1-l}) \\
&= (1-\lambda_Z)^2\mathbf{\Sigma} + \lambda_Z^2(1-\lambda_Z)^2 \sum_{l=t-q}^{t-2}(1-\lambda_Z)^{2l}\mathbf{\Sigma}_0 \\
&\quad + \lambda_Z^2(1-\lambda_Z)^2 \sum_{l=0}^{t-q-1}(1-\lambda_Z)^{2l}\mathbf{\Sigma} \\
&= (1-\lambda_Z)^2\lambda_Z^2 \frac{(1-\lambda_Z)^{2t-2q} - (1-\lambda_Z)^{2t-2}}{1-(1-\lambda_Z)^2}\mathbf{\Sigma}_0 \\
&\quad + (1-\lambda_Z)^2\left(1 + \lambda_Z^2\frac{1-(1-\lambda_Z)^{2t-2q}}{1-(1-\lambda_Z)^2}\right)\mathbf{\Sigma} \\
&= (1-\lambda_Z)^2 \frac{\lambda_Z((1-\lambda_Z)^{2t-2q} - (1-\lambda_Z)^{2t-2})}{2-\lambda_Z}\mathbf{\Sigma}_0 \\
&\quad + (1-\lambda_Z)^2\left(1 + \frac{\lambda_Z}{2-\lambda_Z}(1-(1-\lambda_Z)^{2t-2q})\right)\mathbf{\Sigma}\,.
\end{aligned}
$$

c) Let $\breve{\mathbf{X}}_t = \mathbf{X}_t - E(\mathbf{X}_t)$. Then it holds that

$$
\begin{aligned}
&Cov_q(\tilde{\mathbf{X}}_t, \tilde{\mathbf{X}}_{t-\tau}) \\
&\quad = E_q\left((\mathbf{X}_t - E(\mathbf{X}_t) - (\mathbf{Z}_t - E(\mathbf{Z}_t)))(\mathbf{X}_{t-\tau} - E(\mathbf{X}_{t-\tau})\right. \\
&\qquad \left. - (\mathbf{Z}_{t-\tau} - E(\mathbf{Z}_{t-\tau})))'\right) \\
&\quad = E_q\Bigg(\left((1-\lambda_Z)\breve{\mathbf{X}}_t - \lambda_Z(1-\lambda_Z)\sum_{l=0}^{t-2}(1-\lambda_Z)^l\breve{\mathbf{X}}_{t-1-l}\right) \\
&\qquad \times \left((1-\lambda_Z)\breve{\mathbf{X}}_{t-\tau} - \lambda_Z(1-\lambda_Z)\sum_{l=0}^{t-\tau-2}(1-\lambda_Z)^l\breve{\mathbf{X}}_{t-\tau-1-l}\right)'\Bigg) \\
&\quad = -\lambda_Z(1-\lambda_Z)E_q\Bigg(\sum_{l=\tau-1}^{t-2}(1-\lambda_Z)^l\breve{\mathbf{X}}_{t-1-l}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left((1-\lambda_Z)\breve{\mathbf{X}}_{t-\tau} - \lambda_Z(1-\lambda_Z) \sum_{l=0}^{t-\tau-2} (1-\lambda_Z)^l \breve{\mathbf{X}}_{t-\tau-1-l} \right)' \Bigg) \\
&= -\lambda_Z(1-\lambda_Z) E_q \Bigg(\Bigg((1-\lambda_Z)^{\tau-1}\breve{\mathbf{X}}_{t-\tau} + \sum_{\tilde{l}=0}^{t-\tau-2} (1-\lambda_Z)^{\tilde{l}+\tau} \breve{\mathbf{X}}_{t-\tau-1-\tilde{l}} \Bigg) \\
&\times \left((1-\lambda_Z)\breve{\mathbf{X}}_{t-\tau} - \lambda_Z(1-\lambda_Z) \sum_{l=0}^{t-\tau-2} (1-\lambda_Z)^l \breve{\mathbf{X}}_{t-\tau-1-l} \right)' \Bigg) \\
&= -\lambda_Z(1-\lambda_Z) \Bigg((1-\lambda_Z)^{\tau}\mathbf{\Sigma} - \lambda_Z(1-\lambda_Z)(1-\lambda_Z)^{\tau} \\
&\times \left(\sum_{l=0}^{t-\tau-q-1} (1-\lambda_Z)^{2l}\mathbf{\Sigma} + \sum_{l=t-\tau-q}^{t-\tau-2} (1-\lambda_Z)^{2l}\mathbf{\Sigma}_0 \right) \Bigg) \\
&= -\lambda_Z(1-\lambda_Z)^{\tau+1} \Bigg(\mathbf{\Sigma} - \lambda_Z(1-\lambda_Z) \Bigg(\frac{1-(1-\lambda_Z)^{2(t-\tau-q)}}{1-(1-\lambda_Z)^2}\mathbf{\Sigma} \\
&+ \frac{(1-\lambda_Z)^{2(t-\tau-q)} - (1-\lambda_Z)^{2(t-\tau-1)}}{1-(1-\lambda_Z)^2}\mathbf{\Sigma}_0 \Bigg) \Bigg) \\
&= -\lambda_Z(1-\lambda_Z)^{\tau+1} \Bigg(\mathbf{\Sigma} - \frac{1-\lambda_Z}{2-\lambda_Z} \Bigg((1-(1-\lambda_Z)^{2(t-\tau-q)})\mathbf{\Sigma} \\
&+ ((1-\lambda_Z)^{2(t-\tau-q)} - (1-\lambda_Z)^{2(t-\tau-1)})\mathbf{\Sigma}_0 \Bigg) \Bigg) .
\end{aligned}
$$

PROOF OF THEOREM 3.1.

a) For the proof of this part we refer to Bodnar and Okhrin (2008).

b) Without loss of generality we present the proof only for the case $i = 1$ and note that for the other values the proof is similar. In order to simply the notation the index i is dropped. From the definition of $\mathbf{V}_t$ we get

$$\mathbf{V}_t \sim W_p(1, \mathbf{\Sigma}(q,t), \mathbf{\Omega}(q,t)) \quad \text{with} \quad \mathbf{\Omega}(q,t) = \mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t)\boldsymbol{\mu}(q,t)' .$$

Since $\mathbf{V}_t$ consists only of two functionally independent components $v_{t;11}$ and $\mathbf{V}_{t;21}$, i.e. $\mathbf{V}_{t;22} = \mathbf{V}_{t;21}\mathbf{V}_{t;12}/v_{t;11}^{-1}$, we can write that

$$\mathbf{V}_t = \left[1 \;\; (\mathbf{V}_{t;21}/v_{t;11})'\right]' v_{t;11} \left[1 \;\; (\mathbf{V}_{t;21}/v_{t;11})'\right] .$$

Thus, the density of $\mathbf{V}_t$, i.e. the density of $v_{t;11}$ and $\mathbf{V}_{t;21}$ is given by (cf. Srivastava, 2003, Corrollary 3.2)

$$f_{\mathbf{V}_t}(\mathbf{C}) = \frac{\pi^{(1-p)/2}2^{-p/2}}{\sqrt{\pi}|\mathbf{\Sigma}(q,t)|^{1/2}}|c_{11}|^{-p/2} \\ \times \exp\left(-\frac{1}{2}tr(\mathbf{\Sigma}(q,t)^{-1}\mathbf{C})\right) {}_0F_1\left(\frac{1}{2};\frac{1}{4}\mathbf{\Omega}(q,t)\mathbf{\Sigma}(q,t)^{-1}\mathbf{C}\right),$$

The symbol ${}_0F_1(.;.)$ denotes the hypergeometric function of the matrix argument (see Muirhead, 1982, p. 258 for the definition and properties).

It holds that

$$tr(\mathbf{\Sigma}(q,t)^{-1}\mathbf{C}) = tr\left(\sigma_{11}(q,t)^{-1}c_{11} \right. \\ + \left(\frac{\mathbf{C}_{21}}{c_{11}} - \frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)' \mathbf{\Sigma}_{22\cdot1}(q,t)^{-1} \\ \left. \times \left(\frac{\mathbf{C}_{21}}{c_{11}} - \frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right) c_{11}\right).$$

Then the joint density of $v_{t;11}$ and $\mathbf{V}_{t;21}$ is then given by

$$f_{v_{t;11},\mathbf{V}_{t;21}}(c_{11},\mathbf{C}_{21}) = \frac{\pi^{(1-p)/2}2^{-p/2}}{\sqrt{\pi}|\mathbf{\Sigma}(q,t)|^{1/2}}c_{11}^{-p/2}\exp\left(-\frac{\sigma_{11}(q,t)^{-1}c_{11}}{2}\right) \\ \times_0 F_1\left(\frac{1}{2};\frac{1}{4}\mathbf{\Omega}(q,t)\mathbf{\Sigma}(q,t)^{-1}\left[1 \;\; (\mathbf{C}_{21}/c_{11})'\right]' c_{11} \right. \\ \left. \times \left[1 \;\; (\mathbf{C}_{21}/c_{11})'\right]\right) \\ \times \exp\left(-\frac{c_{11}}{2}\left(\frac{\mathbf{C}_{21}}{c_{11}} - \frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)' \mathbf{\Sigma}_{22\cdot1}(q,t)^{-1} \right. \\ \left. \times \left(\frac{\mathbf{C}_{21}}{c_{11}} - \frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)\right).$$

Making the transformation

$$\mathbf{x} = \mathbf{C}_{21}c_{11}^{-1} \qquad \text{and} \qquad y = c_{11}$$

with the Jacobian c_{11}^{k-1} yields

$$f_{v_{t;11},\mathbf{V}_{t;21}/v_{t;11}}(y,\mathbf{x}) = \frac{\pi^{-p/2}2^{-p/2}}{|\mathbf{\Sigma}(q,t)|^{1/2}} y^{\frac{p-2}{2}} \exp\left(-\frac{\sigma_{11}(q,t)^{-1}y}{2}\right) \times {}_0F_1\left(\frac{1}{2};\frac{1}{4}\mathbf{\Omega}(q,t)\mathbf{\Sigma}(q,t)^{-1}[1\ \ \mathbf{x}']'y[1\ \ \mathbf{x}']\right) \times \exp\left(-\frac{y}{2}\left(\mathbf{x}-\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)'\mathbf{\Sigma}_{22\cdot 1}(q,t)^{-1} \times \left(\mathbf{x}-\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)\right).$$

Using the definition of the matrix hypergeometric function and the fact that the non-negative eigenvalues of the matrices

$$\mathbf{\Omega}(q,t)\mathbf{\Sigma}(q,t)^{-1}[1\mathbf{x}']'y[1\ \ \mathbf{x}'] \quad \text{and} \quad [1\ \ \mathbf{x}']\mathbf{\Omega}(q,t)\mathbf{\Sigma}(q,t)^{-1}[1\mathbf{x}']'y$$

coincide, we get

$$f_{v_{t;11},\mathbf{V}_{t;21}/v_{t;11}}(y,\mathbf{x}) = \frac{\pi^{-p/2}2^{-p/2}}{|\mathbf{\Sigma}(q,t)|^{1/2}} y^{\frac{p-2}{2}} \times {}_0F_1\left(\frac{1}{2};\frac{1}{4}[1\ \ \mathbf{x}']\mathbf{\Omega}(q,t)\mathbf{\Sigma}(q,t)^{-1}[1\ \ \mathbf{x}']'y\right) \times \exp\left(-\frac{y}{2}\left(\sigma_{11}(q,t)^{-1}+\left(\mathbf{x}-\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)' \times \mathbf{\Sigma}_{22\cdot 1}(q,t)^{-1}\left(\mathbf{x}-\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)\right)\right).$$

Since

$${}_0F_1\left(\frac{1}{2};z\right) = (\exp(2\sqrt{z})+\exp(-2\sqrt{z}))/2 = cosh(2\sqrt{z})$$

and $\mathbf{\Omega}(q,t) = \mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t)\boldsymbol{\mu}(q,t)'$, it follows that

$$f_{v_{t;11},\mathbf{V}_{t;21}/v_{t;11}}(y,\mathbf{x}) = \frac{\pi^{-p/2}2^{-p/2}}{|\mathbf{\Sigma}(q,t)|^{1/2}} y^{\frac{p-2}{2}} cosh(|[1\ \ \mathbf{x}']\mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t)|\sqrt{y}) \times \exp\left(-\frac{y}{2}\left(\sigma_{11}(q,t)^{-1}+\left(\mathbf{x}-\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)' \times \mathbf{\Sigma}_{22\cdot 1}(q,t)^{-1}\left(\mathbf{x}-\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}\right)\right)\right).$$

Let

$$g(\mathbf{z};q,t)=[0\ \ \mathbf{z}'\mathbf{\Sigma}_{0;22\cdot 1}^{1/2}]\mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t),\quad \nu(q,t)=\left[1\ \ \frac{\mathbf{\Sigma}_{0;21}'}{\sigma_{0;11}}\right]\mathbf{\Sigma}(q,t)^{-1}\boldsymbol{\mu}(q,t)\,,$$

$$\boldsymbol{\delta}(q,t)=\mathbf{\Sigma}_{0;22\cdot 1}^{-1/2}\left(\frac{\mathbf{\Sigma}_{21}(q,t)}{\sigma_{11}(q,t)}-\frac{\mathbf{\Sigma}_{0;21}}{\sigma_{0;11}}\right),\quad \mathbf{\Xi}(q,t)=\mathbf{\Sigma}_{0;22\cdot 1}^{-1/2}\mathbf{\Sigma}_{22\cdot 1}(q,t)\mathbf{\Sigma}_{0;22\cdot 1}^{-1/2}\,.$$

Using the transformation $\mathbf{z}=\mathbf{\Sigma}_{0;22\cdot 1}^{-1/2}(\mathbf{x}-\mathbf{\Sigma}_{0;21}/\sigma_{0;11})\sqrt{y}$ with the Jacobian $|\mathbf{\Sigma}_{0;22\cdot 1}|^{1/2}y^{-(p-1)/2}$ and the fact that $|\mathbf{\Sigma}(q,t)|=\sigma_{11}(q,t)|\mathbf{\Sigma}_{22\cdot 1}(q,t)|$ we get

$$\begin{aligned}f_{v_{t;11},\boldsymbol{\eta}_{1,t}}(y,\mathbf{z}) &= \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{11}(q,t)^{\frac{1}{2}}|\mathbf{\Xi}(q,t)|^{1/2}}y^{-\frac{1}{2}}\,cosh(|g(\mathbf{z};q,t)+\nu(q,t)\sqrt{y}|)\\ &\quad\times\exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_{11}(q,t)}\right.\right.\\ &\qquad\left.\left.+(\mathbf{z}-\boldsymbol{\delta}(q,t)\sqrt{y})'\mathbf{\Xi}(q,t)^{-1}(\mathbf{z}-\boldsymbol{\delta}(q,t)\sqrt{y})\right)\right)\,.\end{aligned}$$

Hence, the density of $\boldsymbol{\eta}_{1,t}$ is given by

$$\begin{aligned}f_{\boldsymbol{\eta}_{1,t}}(\mathbf{z}) &= \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{11}(q,t)^{1/2}|\mathbf{\Xi}(q,t)|^{1/2}}\int_0^\infty y^{-\frac{1}{2}}\,cosh(g(\mathbf{z};q,t)+\nu(q,t)\sqrt{y})\\ &\quad\times\exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_{11}(q,t)}\right.\right.\\ &\qquad\left.\left.+(\mathbf{z}-\boldsymbol{\delta}(q,t)\sqrt{y})'\mathbf{\Xi}(q,t)^{-1}(\mathbf{z}-\boldsymbol{\delta}(q,t)\sqrt{y})\right)\right)dy\,,\end{aligned}$$

where we use that $cosh(x)=cosh(-x)$. The transformation $y=\tau^2$ yields

$$\begin{aligned}f_{\boldsymbol{\eta}_{1,t}}(\mathbf{z}) &= \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{11}(q,t)^{1/2}|\mathbf{\Xi}(q,t)|^{1/2}}\\ &\quad\times\int_0^\infty(exp(g(\mathbf{z};q,t)+\nu(q,t)\tau)+exp(-g(\mathbf{z};q,t)-\nu(q,t)\tau))\\ &\quad\times\exp\left(-\frac{1}{2}\left(\frac{\tau^2}{\sigma_{11}(q,t)}+(\mathbf{z}-\boldsymbol{\delta}(q,t)\tau)'\mathbf{\Xi}(q,t)^{-1}(\mathbf{z}-\boldsymbol{\delta}(q,t)\tau)\right)\right)d\tau\\ &= I_1(\mathbf{z})+I_2(\mathbf{z})\,,\end{aligned}$$

where

$$
\begin{aligned}
I_1(\mathbf{z}) = {} & \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{11}(q,t)^{1/2}|\Xi(q,t)|^{1/2}} \\
& \times \exp\left(-\frac{1}{2}\left(\mathbf{z}'\Xi(q,t)^{-1}\mathbf{z} - \frac{(\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\mathbf{z}+\nu(q,t))^2}{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}\right.\right. \\
& \qquad\qquad \left.\left. - 2g(\mathbf{z};q,t)\right)\right) \\
& \times \int_0^\infty \exp\left(-\frac{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}{2}\right. \\
& \qquad\qquad \left. \times \left(\tau - \frac{\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\mathbf{z}+\nu(q,t)}{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}\right)^2\right) d\tau \\
= {} & \frac{\sqrt{1+\sigma_{11}(q,t)\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}}{\pi^{-(p-1)/2}2^{-(p-1)/2}|\Xi(q,t)|^{1/2}} \\
& \times \exp\left(-\frac{1}{2}\left(\mathbf{z}'\Xi(q,t)^{-1}\mathbf{z} - \frac{(\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\mathbf{z}+\nu(q,t))^2}{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}\right.\right. \\
& \qquad\qquad \left.\left. - 2g(\mathbf{z};q,t)\right)\right) \\
& \times \left(1 - \Phi\left(0; \frac{\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\mathbf{z}+\nu(q,t)}{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)},\right.\right. \\
& \qquad \left.\left.(\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t))^{-1}\right)\right).
\end{aligned}
$$

where $\Phi(.;\mu,\sigma^2)$ denotes the distribution function of the normal distribution with mean μ and variance σ^2. Similarly,

$$
\begin{aligned}
I_2(\mathbf{z}) & \frac{\sqrt{1+\sigma_{11}(q,t)\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}}{\pi^{-(p-1)/2}2^{-(p-1)/2}|\Xi(q,t)|^{1/2}} \\
& \times \exp\left(-\frac{1}{2}\left(\mathbf{z}'\Xi(q,t)^{-1}\mathbf{z} - \frac{(\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\mathbf{z}-\nu(q,t))^2}{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)}\right.\right. \\
& \qquad\qquad \left.\left. + 2g(\mathbf{z};q,t)\right)\right) \\
& \times \left(1 - \Phi\left(0; \frac{\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\mathbf{z}-\nu(q,t)}{\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t)},\right.\right. \\
& \qquad \left.\left.(\sigma_{11}(q,t)^{-1}+\boldsymbol{\delta}(q,t)'\Xi(q,t)^{-1}\boldsymbol{\delta}(q,t))^{-1}\right)\right).
\end{aligned}
$$

Putting the last two expressions together and using the identities $\Phi(0;\mu,\sigma^2) = \Phi(-\mu/\sigma)$ and $1-\Phi(-x)=\Phi(x)$ we get the statement of Theorem 3.1.

References

ALBERS, W. and KALLENBERG, W. (2004). Are estimated control charts in control? *Statistics* **38**, 67–79.

ANDERSSON, E., BOCK, D. and FRISÉN, M. (2004). Detection of turning points in business cycles. *J. Bus. Cycl. Measur. Anal.*, **1**, 93–108.

AZZALINI, A. (2005). The skew-normal distribution and related multivariate families. *Scand. J. Stat.*, **32**, 159–188.

BODNAR, O. (2007). Sequential Procedures for Monitoring Covariances of Asset Returns. In *Advances in Risk Management* (G.N. Gregoriou, ed.). Palgrave, pp. 241–264.

BODNAR, O., BODNAR, T. and OKHRIN, Y. (2009). Surveillance of the covariance matrix based on the properties of the singular wishart distribution. *Comput. Statist. Data Anal.*, **53**, 3372–3385.

BODNAR, T. and OKHRIN, Y. (2008). Properties of the singular, inverse and generalized inverse partitioned Wishart distributions. *J. Multivariate Anal.*, **99**, 2389–2405.

CHAN, L.K. and ZHANG, J. (2001). Cumulative sum control charts for the covariance matrix. *Statist. Sinica*, **11**, 767–790.

CONTE, S. and DE BOOR, C. (1981). *Elementary numerical analysis.* Mc Graw-Hill, London.

CROSIER, R. (1986). A new two-sided cumulative quality control scheme. *Technometrics*, **28**, 187–194.

CROSIER, R. (1988). Multivariate generalizations of cumulative sum quality-control schemes. *Technometrics*, **30**, 291–303.

DOMÍNGUEZ-MOLINA, J., GONZÁLEZ-FARÍAS, G., RAMOS-QUIROGA, R. and GUPTA, A. (2007). A matrix variate closed skew-normal distribution with applications to stochastic frontier analysis. *Comm. Statist. Theory Methods*, **36**, 1671–1703.

FRISÉN, M. (1992). Evaluations of methods for statistical surveillance. *Stat. Med.*, **11**, 1489–1502.

GOLOSNOY, W., OKHRIN, I., RAGULIN, S. and SCHMID, W. (2011). On the application of SPC in finance. *Front. Statist. Qual. Contr.*, **9**, 119–132.

GUPTA, A., VARGA, T. and BODNAR, T. (2013). *Elliptically contoured models in statistics and portfolio theory.* Springer, New-York.

HARVILLE, A. (1997). *Matrix algebra from a statistician perspective.* Springer, New York.

HAWKINS, D. (1981). A CUSUM for a scale parameter. *J. Qual. Technol.*, **13**, 228–231.

HAWKINS, D. (1991). Multivariate quality control based on regressionadjusted variables. *Technometrics*, **33**, 61–75.

HOTTELING, H. (1947). Multivariate Quality Control – Illustrated by the Air Testing of Sample Bombsights. In *Techniques of Statistical Analysis* (C. Eisenhart, M.W. Hastay and W. Wallis, eds.). McGraw Hill, pp. 111–184.

HUWANG, L., YEH, A. and WU, C.-W. (2007). Monitoring multivariate process variability for individual observations. *J. Qual. Technol.*, **39**, 259–278.

knoth, s. (2003). EWMA schemes with nonhomogenous transition kernel. *Sequential Anal.*, **22**, 241–255.

kramer, h. and schmid, w. (2000). The influence of parameter estimation on the ARL of Shewhart-type charts for time series. *Statist. Papers*, **41**, 173–196.

lawson, a. and kleinman, k. (2005). *Spatial & syndromic surveillance.* Wiley, New-York.

lowry, c., woodall, w., champ, c. and rigdon, s. (1992). A multivariate exponentially weighted moving average control chart. *Technometrics*, **34**, 46–53.

markowitz, h. (1952). Portfolio selection. *J. Financ.*, **7**, 77–91.

messaoud, a., weihs, c. and hering, f. (2008). Detection of chatter vibration in a drilling process using multivariate control charts. *Comput. Statist. Data Anal.*, **52**, 3208–3219.

morais, m. and pacheco, a. (2001). Some stochastic properties of upper one-sided $\bar{X}$ and ewma charts for μ in the presence of shifts in σ. *Sequential Anal.*, **20**, 1–12.

moustakides, g. (1986). Optimal stopping times for detecting changes in distributions. *Ann. Statist.*, **14**, 1379–1387.

muirhead, r. (1982). *Aspects of multivariate statistical theory.* Wiley, New York.

ngai, h. and zhang, j. (2001). Multivariate cumulative sum control charts based on projection pursuit. *Statist. Sinica*, **11**, 747–766.

okhrin, y. and schmid, w. (2008). Surveillance of Univariate and Multivariate Linear Time Series. In *Financial Surveillance* (M. Frisén, ed.). Wiley.

page, p. (1954). Continuous inspection schemes. *Biometrika*, **41**, 100–115.

pignatiello, j. and runger, g. (1990). Comparisons of multivariate CUSUM charts. *J. Qual. Technol.*, **22**, 173–186.

pollak, m. and siegmund, d. (1975). Approximations to the expected sample size of certain sequential tests. *Ann. Statist.*, **3**, 1267–1282.

reynolds, m.j. and cho, g.-y. (2006). Multivariate control charts for monitoring the mean vector and covariance matrix. *J. Qual. Technol.*, **38**, 230–253.

roberts, s. (1959). Control chart tests based on geometric moving averages. *Technometrics*, **1**, 239–250.

schipper, s. and schmid, w. (2001). Sequential methods for detecting changes in the variance of economic time series. *Sequential Anal.*, **20**, 235–262.

schmid, w. and tzotchev, d. (2004). Statistical surveillance of the parameters of a one-factor Cox-Ingersoll-Ross model. *Sequential Anal.*, **23**, 379–412.

shewhart, w. (1931). *Economic control of quality of manufactured product.* van Nostrand, Toronto.

śliwa, p. and schmid, w. (2005). Monitoring the cross-covariances of a multivariate time series. *Metrika*, **61**, 89–115.

sonesson, c. and bock, d. (2003). A review and discussion of prospective statistical surveillance in public health. *J. Roy. Statist. Soc. A*, **166**, 5–21.

srivastava, m. (2003). Singular Wishart and multivariate beta distributions. *Ann. Statist.* **31**, 1537–1560.

woodall, w. and mahmoud, m. (2005). The inertial properties of quality control charts. *Technometrics*, **47**, 425–436.

WOODALL, W. and NCUBE, M. (1985). Multivariate CUSUM quality control procedures. *Technometrics*, **27**, 285–292.

YEH, A., HUWANG, L. and WU, C.-W. (2005). A multivariate EWMA control chart for monitoring process variability with individual observations. *IIE Trans.*, **37**, 1023–1035.

Olha Bodnar
Physikalisch-Technische Bundesanstalt
Abbestrasse 2-l2, 10587 Berlin, Germany
E-mail: olha.bodnar@ptb.de

Taras Bodnar
Department of Mathematics
Humboldt-University of Berlin
10099 Berlin, Germany
E-mail: bodnar@math.hu-berlin.de

Yarema Okhrin
Department of Statistics
University of Augsburg
86159 Augsburg, Germany
E-mail: yarema.okhrin@wiwi.uni-augsburg.de