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Properties of hierarchical Archimedean copulas

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Summary: In this paper we analyse the properties of hierarchical Archimedean copulas. This class is a generalisation of the Archimedean copulas and allows for general non-exchangeable dependency structures. We show that the structure of the copula can be uniquely recovered from all bivariate margins. We derive the distribution of the copula values, which is particularly useful for tests and constructing confidence intervals. Furthermore, we analyse dependence orderings, multivariate dependence measures, and extreme value copulas. We pay special attention to the tail dependencies and derive several tail dependence indices for general hierarchical Archimedean copulas.

1 Introduction

Copulas play an increasingly important role in econometrics. For an arbitrary multivariate distribution, they allow of separating the marginal distributions and the dependency model. As a result we obtain a convenient tool to analyse complex relationships between variables. In particular, all common measures of dependence can be given in terms of the copula function. Modeling using copulas offers wide flexibility in terms of the form of dependence and is often encountered in applications from financial econometrics, hydrology, medicine, etc.

Copulas were first introduced in the seminal paper of [22]. Here we restate Sklar's theorem.

Theorem 1.1 *Let F be a k-dimensional distribution function with marginal distributions $F_1, \dots, F_k$. Then there exist a copula $C : [0,1]^k \to [0,1]$ which satisfies the equality*

$$F(x_1, \dots, x_k) = C\{F_1(x_1), \dots, F_k(x_k)\}, \quad x_1, \dots, x_k \in \mathbb{R}. \tag{1.1}$$

If all $F_1, \dots, F_k$ are continuous, then C is unique; otherwise C is uniquely determined on $\operatorname{Ran} F_1 \times \cdots \times \operatorname{Ran} F_k$. Conversely, if C is a copula and $F_1, \dots, F_k$ are distribution functions, then the function F defined as in (1.1) is a k-dimensional distribution function with margins $F_1, \dots, F_k$.

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As follows from the theorem, the copula function captures the dependency between variables, with the impact of the marginal distributions being eliminated. Sklar's Theorem allows expressing the copula function directly by

$$C(u_1,\dots,u_k) = F\{F_1^{-1}(u_1),\dots,F_k^{-1}(u_k)\}, \quad u_1,\dots,u_k \in [0,1],$$

where $F_1^{-1}(\cdot),\dots,F_k^{-1}(\cdot)$ are the corresponding quantile functions.

If the cdf F belongs to the class of elliptical distributions, then this results in an elliptical copula. Note, however, that this copula cannot be given explicitly, because F and the inverse marginal distributions F_i have only integral representations. One of the alternatives is the important class of Archimedean copulas. The k-dimensional Archimedean copula function $C : [0,1]^k \to [0,1]$ is defined by

$$C(u_1,\dots,u_k) = \phi\{\phi^{-1}(u_1) + \dots + \phi^{-1}(u_k)\}, \quad u_1,\dots,u_k \in [0,1], \tag{1.2}$$

where ϕ with $\phi(0) = 1$ and $\phi(\infty) = 0$ is called the generator of the copula. [15] provide necessary and sufficient conditions for ϕ to generate a feasible Archimedean copula. The generator ϕ is required to be k-*monotone*, i.e., differentiable up to the order $k-2$, with $(-1)^i\phi^{(i)}(x) \geq 0$, $i = 0,\dots,k-2$ for any $x \in [0,\infty)$ and with $(-1)^{k-2}\phi^{(k-2)}(x)$ being nondecreasing and convex on $[0,\infty)$. We consider a stronger assumption that ϕ is a *completely monotone* function, i.e., $(-1)^i\phi^{(i)}(x) \geq 0$ for all $i \geq 0$. The class of feasible generator functions is defined by (see [12, Theorems 1 and 2])

$$\mathcal{L} = \{\phi : [0;\infty) \to [0,1] \mid \phi(0) = 1, \phi(\infty) = 0; (-1)^i\phi^{(i)} \geq 0; i = 1,\dots,\infty\}.$$

A detailed review of the properties of Archimedean copulas can be found in [15]. Table 4.1 of [18] contains a list of common one-parameter generator functions. Throughout the paper we consider only generator functions with a single parameter, however, most of the theory can be easily extended to the case of several parameters.

From Bernstein's Theorem ([2]) it follows that each $\phi \in \mathcal{L}$ is the Laplace transform of some distribution function. This allows us to relate Archimedean copulas to Laplace transforms (see [10]). Let M be the cdf of a positive random variable and ϕ denotes its Laplace transform, i.e., $\phi(t) = \int_0^\infty e^{-tw}\,dM(w)$. For an arbitrary cdf F there exists a unique cdf G such that

$$F(x) = \int_0^\infty G^\alpha(x)dM(\alpha) = \phi\{-\ln G(x)\}.$$

Now consider the class of k-variate cumulative distribution functions with margins F_1, $\dots, F_k$. Then assuming that $G_j = \exp\{-\phi^{-1}(F_j)\}$, the following cdf also belong to that class.

$$\int_0^\infty G_1^\alpha(x_1)\cdots G_k^\alpha(x_k)dM(\alpha) = \phi\left\{-\sum_{i=1}^k \ln G_i(x_i)\right\} = \phi\left[\sum_{i=1}^k \phi^{-1}\{F_i(x_i)\}\right].$$

This implies that the copula obtained from taking the cdf $U[0;1]$ for $F_1,\dots,F_k$, is given by (1.2). The representation of the copula in terms of Laplace transforms is very useful for simulation purposes (see [24], [14], [9], [13]).

We define the family of simple k-dimensional Archimedean copulas by

$$\mathcal{F}_{k1} = \big\{C_{k1} : [0;1]^k \to [0;1] : C_{k1} = \phi[\phi^{-1}(u_1) + \ldots + \phi^{-1}(u_k)], \\ \phi(\cdot,\theta) \in \mathcal{L}, \theta \in \Theta, u_1, \ldots, u_k \in [0;1]\big\},$$

where Θ is the set of allowable parameters θ of the generator ϕ. The elements of Θ could be of any dimension, but in general they are scalars.

Note that the Archimedean copula is symmetric with respect to the permutation of variables, i.e., the distribution is exchangeable. Furthermore, the multivariate dependency structure depends on a single parameter of the generator function ϕ. This is very restrictive and we can use Laplace transforms to derive flexible extensions. First, note that $G_1^\alpha \cdots G_k^\alpha$ can be seen as a product copula of the cumulative distribution functions $G_1^\alpha, \ldots, G_k^\alpha$. Second, note that the whole model depends on a single cumulative distribution function M. Replacing the product copula $G_1^\alpha \cdots G_k^\alpha$ with an arbitrary multivariate copula $C^*(G_1^\alpha, \ldots, G_k^\alpha)$ and replacing $M(\alpha)$ with some k-variate distribution we obtain a more general type of dependency (see [11]). To avoid to much generality we concentrate on the copulas which arise from the following construction. In three dimensions consider the copula

$$C(u_1,u_2,u_3) = \int_0^\infty \ldots \int_0^\infty G_1^{\alpha_1}(u_1) G_2^{\alpha_1}(u_2) dM_1(\alpha_1;\alpha_2) \times G_3^{\alpha_2}(u_3) dM_2(\alpha_2) \tag{1.3}$$

where $G_1 = G_2 = \exp\{-\phi_1^{-1}\}$, $G_3 = \exp\{-\phi_2^{-1}\}$, and $\phi_1, \phi_2 \in \mathcal{L}$. M_1 has the Laplace transform ϕ_1, while $M_2(\cdot;\alpha_2)$ being the distribution with the inverse Laplace transform $[\phi_2 \circ \phi_2]^{-1}(-\alpha_2^{-1} \log z)$. Integration results in

$$C(u_1,u_2,u_3) = \phi_1\{\phi^{-1} \circ \phi_2\{\phi_2^{-1}(u_1) + \phi_2^{-1}(u_2)\} + \phi_2^{-1}(u_3)\}.$$

Other orders of integration and combinations of G_i functions lead to different dependencies. For example, the fully nested (1.3) copula $C(u_1, \ldots, u_k)$ can be rewritten in terms of the generator functions arising from the cumulative distribution functions $M_1, \ldots, M_{k-1}$ as

$$\begin{aligned} &C(u_1, \ldots, u_k) \\ &\quad = \phi_1[\phi_1^{-1} \circ \phi_2\{\ldots[\phi_{k-2}^{-1} \circ \phi_{k-1}\{\phi_{k-1}^{-1}(u_1) + \\ &\qquad + \phi_{k-1}^{-1}(u_2)\} + \phi_{k-2}^{-1}(u_3)] \cdots + \phi_2^{-1}(u_{k-1})\} + \phi_1^{-1}(u_k)] \\ &\quad = \phi_1\{\phi_1^{-1} \circ C_2(u_1, \ldots, u_{k-1}) + \phi_1^{-1}(u_k)\} = C_1\{C_2(u_1, \ldots, u_{k-1}), u_k\}. \end{aligned}$$

Unfortunately there is no simple and straightforward way to define this type of copula in higher dimensions using Laplace transforms. For this reason we concentrate on the formal set theoretic definition based on elements of graph theory.

Sufficient conditions on the generator functions which guarantee that C is a copula are given in Theorem 4.4 of [14]. Let $\mathcal{L}^*$ denote the class of functions with a completely monotone first derivative

$$\mathcal{L}^* = \{\omega : [0;\infty) \to [0,\infty) \,|\, \omega(0) = 0, \omega(\infty) = \infty; \\ (-1)^{i-1} \omega^{(i)} \geq 0; i = 1, \ldots, \infty\}.$$

If $\phi_i \in \mathcal{L}$ for $i = 1,\ldots,k-1$ and $\phi_i^{-1} \circ \phi_{i+1} \in \mathcal{L}^*$ has a completely monotone derivative for $i = 1,\ldots,k-2$, then C is a copula. As noted by Lemma 4.1 in [14], the fact that $\phi_i^{-1} \circ \phi_{i+1} \in \mathcal{L}^*$ for $i = 1,\ldots,k-2$ also implies that $\phi_i^{-1} \circ \phi_{i+h} \in \mathcal{L}^*$ for $i = 1,\ldots,k-2$.

Note that generators ϕ_i within a single copula can come either from a single generator family or from different generator families. If the ϕ_is belong to the same family, then the complete monotonicity of $\phi_i^{-1} \circ \phi_{i+1}$ imposes some constraints on the parameters $\theta_1,\ldots,\theta_{k-1}$. Table 4.1 of [18] provides these constrains for different generators. For the majority of copulas, the parameters should decrease from the lowest to the highest level, to guarantee a feasible distribution function. However, if we consider generators from different families within a single copula, the condition of complete monotonicity is not always fulfilled and each particular case should be analysed separately.

Putting these ideas together, we define the class of *hierarchical Archimedean copulas* (HAC) with r nodes, a generalisation of multivariate Archimedean copulas, by

$$\begin{aligned}\mathcal{F}_{kr} = \big\{C_{kr} : [0;1]^k \to [0;1] :\\ &C_{kr} = C\{C_{k_1 r_1}(u_{k_0},\ldots,u_{k_1}),\ldots,C_{k_m-k_{m-1},r_m}(u_{k_{m-1}+1},\ldots,u_{k_m})\},\\ &C \in \mathcal{F}_{k1},\ C_{k_i-k_{i-1},r_i} \in F_{k_i-k_{i-1},r_i}, \forall i = 1,\ldots,m,\ \sum_{i=1}^{m} r_i = r-1\big\},\end{aligned}$$

where $k_0 = 1$, $k_m = k$, and r_i denotes the number of nodes in the ith subcopula and the variables are reordered without loss of generality. The generators at different nodes satisfy $\phi_i^{-1} \circ \phi_{i+h} \in \mathcal{L}^*$ for $i = 1,\ldots,k-2$. If $k_i - k_{i-1} = 1$ then $r_i = 0$ and $C_{11}(u_i) = u_i$. For example, $C = C_1\{C_2(u_1,u_2),u_3\} \in \mathcal{F}_{3,2}$, where $C_1, C_2 \in \mathcal{F}_{2,1}$ are nodes, which are also copulas.

The aim of this paper is to provide the distributional properties of HACs. First we show that if the true distribution is based on an HAC then we can completely recover the true structure of the HAC from all bivariate marginal distributions. This property is helpful in applications, when we estimate the HAC from data. For the normal distribution, for example, the form of the dependency is fixed and only the correlation coefficients must be estimated. For HACs both the structure and the parameters of the generators function are unknown. The established result implies that we can first estimate all bivariate copulas and then recover the tree of the HAC. Alternatively, we are forced to enumerate all possible trees, estimate the corresponding multivariate copulas, and apply goodness-of-fit tests to determine the HAC with the best fit. This approach is computationally much more demanding compared to the aggregation of bivariate copulas.

Furthermore, we derive the distribution of the values of an HAC. This generalises the results of [7] to HACs. We take explicitly into account the hierarchical structure of HACs and provide recursive formulas for the cdf by different types of aggregation. The results given in Section 3 can be used to develop confidence intervals and goodness-of-fit tests. Section 4 summarises the multivariate dependence measures used in the multivariate setup and determines which of them are most convenient for use with an HAC. Section 5 contains results on the dependence orderings of HAC-based distributions. It is shown under which conditions on the generator functions is one HAC more concordant than another one. Finally Section 5 discusses the properties of HACs from the perspective of extreme value theory and provides a detailed analysis of tail dependence. In this section we establish the form of the extreme value copula and provide explicit formulas for two upper and lower tail dependence measures. All proofs are given in the Appendix.

2 Determining the structure

In contrast to other distributional models, in an HAC both the structure and the parameters of the copula must be specified or estimated. [19] consider empirical methods for determining and estimating the structure. If the structure is fixed, we can apply the maximum-likelihood approach to estimate the parameters. However, the choice of the structure itself is not obvious. One possible approach is to enumerate all structures, estimate the parameters, and apply a goodness-of-fit test to determine the best one. This method is, however, unrealistic in higher dimensions. The results established in this section help to overcome this problem. In particular we show that if the true distribution is based on an HAC, then we can completely recover the true distribution from all bivariate margins. This implies that instead of estimating all multivariate structures it suffices to estimate all bivariate copulas and use then to recover the full distribution. This makes the estimation of an HAC particularly attractive in terms of computational effort. The next proposition summarises the result.

Proposition 2.1 *Let F be an arbitrary multivariate distribution function based on an HAC. Then F can be uniquely recovered from the marginal distribution functions and all bivariate copula functions.*

Assuming that the marginal distributions are continuous, from Sklar's Theorem we know that the multivariate distribution function F can be split into margins and the copula function. Therefore, to recover the distribution we need to recover the structure of the HAC. The proof of the proposition has three parts. First, we show that any bivariate margin is a copula with the generator function which is equal to one of the generators of the full structure. Second, we show that for any bivariate copula with a generator function from the full structure, there exists a pair of variables with the same joint bivariate distribution. Third, we suggest an aggregation procedure and show that the recovered HAC is unique.

Let also $\mathfrak{C}_n$ denote the operator which returns a k-dimensional copula given generator functions

$$\mathfrak{C}_n(f)(u_1,\dots,u_k) = f\{f^{-1}(u_1)+\dots+f^{-1}(u_k)\}.$$

Based on this notation, $\mathfrak{C}_2\{\mathcal{N}(C)\} \subset \mathcal{F}_{2,1}$ is the set of all bivariate Archimedean copulas used in the structure of $C \in \mathcal{F}_{kr}$.

Let $\mathcal{N}(C)$ denote the set of generator functions used in the HAC C. Suppose now a k-dimensional HAC $C \in \mathcal{F}_{kr}$ is fixed. The next remark shows that for any pair of variables there exists a bivariate copula with a generator from $\mathcal{N}(C)$ which is the copula of the pair.

Remark 2.2 $\forall i, j = 1,\dots,k,\ i \neq j,\ \exists! C_{ij} \in \mathfrak{C}_2\{\mathcal{N}(C)\} \subset \mathcal{F}_{2,1} : (U_i, U_j) \sim C_{ij}$.

As an example we consider the following four dimensional case with

$$C(u_1,\dots,u_4) \ = \ C_1\{C_2(u_1,u_2), C_3(u_3,u_4)\}$$

with $\mathfrak{C}_2\{\mathcal{N}(C)\} = \{C_1, C_2, C_3\}$. For an arbitrary pair of variables u_i and u_j from $u_1, \dots, u_4$, there exists a copula C_{ij} from $\{C_1, C_2, C_3\}$ such that $(U_i, U_j) \sim C_{ij}$. For example $(U_1, U_3) \sim C_1\{C_2(u_1, 1), C_3(u_3, 1)\} = C_1(u_1, u_3)$. This implies that the bivariate margins use the same generators as the generators in the nodes of the HAC.

The second step of the proof shows the inverse relationship between the bivariate margins and the set of all bivariate copulas with generator function from $\mathcal{N}(C)$. In particular it shows that for any bivariate copula with a generator from $\mathcal{N}(C)$ there exists a pair of variables with the same bivariate distribution.

Remark 2.3 $\forall C_{i,j} \in \mathfrak{C}_2\{\mathcal{N}(C)\} \subset \mathcal{F}_{2,1},\ \exists i^*, j^* = 1, \dots, k :\ (U_{i^*}, U_{j^*}) \sim C_{ij}$.

Next we describe the algorithm which recovers the structure from the bivariate margins. Let C_1 denote a bivariate copula such that each variable belongs to at least one bivariate margin given by C_1. This copula is the top-level copula. From Remark 2.2 if the copula

$$C = C_1\{C_2(u_1, \dots, u_{k_1}), \dots, C_m(u_{k_{m-1}+1}, \dots, u_k)\},$$

then $(U_i, U_j) \sim C_1$, where $i \in [i_1, i_2] \cap \mathbb{N}, j \in ([1, k] \backslash [i_1, i_2]) \cap \mathbb{N}, (i_1, i_2) \in \{(1, k_1), \dots, (k_{m-1} + 1, k)\}$.

At the next step we drop all bivariate margins given by C_1 and identify the sets of pairs of variables with the bivariate distributions given by C_2 to C_m. For the subtrees we proceed in the same way as for C_1. To show that the structure that we recovered is equal to the true one, one needs to explore all bivariate margins. A difference at one of the nodes would imply a change in one or several bivariate margins. But the bivariate marginal distributions coincide by construction.

For simplicity let us consider an example:

$$C(u_1, \dots, u_6) = C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}].$$

The bivariate marginal distributions are then given by

$$\begin{array}{lll} (U_1, U_2) \sim C_2(\cdot, \cdot), & (U_2, U_3) \sim C_1(\cdot, \cdot), & (U_3, U_5) \sim C_3(\cdot, \cdot), \\ (U_1, U_3) \sim C_1(\cdot, \cdot), & (U_2, U_4) \sim C_1(\cdot, \cdot), & (U_3, U_6) \sim C_3(\cdot, \cdot), \\ (U_1, U_4) \sim C_1(\cdot, \cdot), & (U_2, U_5) \sim C_1(\cdot, \cdot), & (U_4, U_5) \sim C_4(\cdot, \cdot), \\ (U_1, U_5) \sim C_1(\cdot, \cdot), & (U_2, U_6) \sim C_1(\cdot, \cdot), & (U_4, U_6) \sim C_3(\cdot, \cdot), \\ (U_1, U_6) \sim C_1(\cdot, \cdot), & (U_3, U_4) \sim C_3(\cdot, \cdot), & (U_5, U_6) \sim C_3(\cdot, \cdot). \end{array}$$

In line with Remarks 2.2 and 2.3, the set of bivariate margins is equal to

$$\mathfrak{C}_2\{\mathcal{N}(C)\} = \{C_1(\cdot, \cdot), C_2(\cdot, \cdot), C_3(\cdot, \cdot), C_4(\cdot, \cdot)\}.$$

We observe that each variable belongs to at least one bivariate margin given by C_1. This implies that the distribution of $u_1, \dots, u_6$ has C_1 at the top level. Next we drop all margins given by C_1. We proceed similarly with the rest of the margins, in particular with C_3 since it covers the largest set of variables u_3, u_4, u_5, u_6. This implies that C_3 is at the top level of the subcopula containing u_3, u_4, u_5, u_6. Having information only for the copulas C_1 and C_3 we obtain

$$U_1, \dots, U_6 \sim C_1\{u_1, u_2, C_3(u_3, u_4, u_5, u_6)\}.$$

The remaining copula functions are C_2 and C_4 and they join u_1, u_2 and u_4, u_5 respectively. Summarising, we obtain

$$(U_1, \dots, U_6) \sim C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}]$$

This results in the correct structure. Similarly we can apply inverse procedure by joining variables into pseudo-random variables, using low-level copulas. This problem is related to the multidimensional scaling problem, where having all paired distances between the cities, one has to recover the complete map, see [8].

To be more formal, for each bivariate copula $C^* \in \mathfrak{C}_2\{\mathcal{N}(C)\}$, let $I(C)$ be the set of indices $i \in \{1, \dots, k\}$ such that $(U_i, U_j) \sim C^*$ for at least one $j \in \{1, \dots, k\} \setminus \{i\}$. For instance, for the example presented above we get

$$I(C_1) = \{1, \dots, 6\}; \quad I(C_2) = \{1, 2\}; \quad I(C_3) = \{3, 4, 5, 6\}; \quad I(C_4) = \{4, 5\}$$

The family of sets $I(C^*)$, as C^* ranges over $\mathfrak{C}_2\{\mathcal{N}(C)\}$, is partially ordered by inclusion; in the above example we get

$$I(C_1) \supset \begin{cases} I(C_2), \\ I(C_3) \supset I(C_4). \end{cases}$$

Then the structure of the hierarchical Archimedean copula is completely given by the structure this partial ordering induces on the family of sets $I(C^*)$. For instance, the copula of (U_i, U_j) is C^* if and only if $I(C^*)$ is the minimal set containing $\{i, j\}$.

A similar procedure has been used in practice in the paper [19], where the determination of the structure from the given dataset was discussed. Below we briefly discuss in an example, the procedure of determining the structure.

Example 2.4 Let us consider a four dimensional sample $\{x_{1i}, \dots, x_{4i}\}^\top$ for $i = 1, \dots, n$ and the estimated HAC with binary structure and generators from the same family in the form

$$\hat{C}(u_1, \dots, u_4) = C(C(C(u_1, u_2; \theta_1), u_3; \theta_2), u_4; \theta_3)$$

If $\theta_1 \approx \theta_2$, the structure may be aggregated to

$$\hat{C}(u_1, \dots, u_4) = C(C(u_1, u_2, u_3; \theta_1), u_4; \theta_3).$$

3 Kendall function of an HAC

For testing purposes and the construction of confidence intervals we are interested in the distributions of the empirical and the true copula. Let $V = C\{F_1(X_1), \dots, F_k(X_k)\}$ and let $K(t)$ denote the distribution function (K-distribution) of the random variable V. [7] introduced a nonparametric estimator of K in the case $k = 2$. It is based on the concept of Kendall's process. Suppose given an independent random sample $\mathbf{x}_1 = (x_{11},\dots,x_{1k})^\top$, ..., $\mathbf{x}_n = (x_{n1},\dots,x_{nk})^\top$ of the vector $\mathbf{X} = (X_1,\dots,X_k)^\top$. Let

$$V_{i,n} = \frac{1}{n+1}\sum_{j=1,j\neq i}^{n} \mathbf{I}\{\mathbf{x}_j \le \mathbf{x}_i\}$$

and let K_n denote the empirical distribution function of the $V_{i,n}$. Here the inequality $\mathbf{a} \le \mathbf{b}$ means that all components of the vector $\mathbf{a}$ are less than or equal to those of the vector $\mathbf{b}$. Then the Kendall process is given by

$$\alpha_n(t) = \sqrt{n}\{K_n(t) - K(t)\}.$$

[1] examine the limiting behaviour of the empirical process $\alpha_n(t)$ for $k \ge 2$ and derive explicit formulas for its density $\kappa(t)$ and its distribution function $K(t)$ for general multivariate copulas. The authors provide explicit results for product and multivariate exchangeable Archimedean copulas. The paper of [23] used Kendall's process to determine the copula for failure data. In this section we adopt and extend the results of [1] to find the K-distribution of an HAC.

As the first step, we exploit the hierarchical structure of the HAC. We consider an HAC of the form $C_1\{u_1, C_2(u_2,\dots,u_k)\}$. Let $U_i \sim U[0,1]$ and let $V_2 = C_2(U_2,\dots,U_k) \sim K_2$. In the next theorem we propose a recursive procedure for calculating the distribution function of $V_1 = C_1(U_1,V_2)$ which is based on the knowledge of the distribution function of V_2. This approach is particularly useful when applied to fully nested HACs.

Theorem 3.1 *Let $U_1 \sim U[0,1]$, $V_2 \sim K_2$ and let $P(U_1 \le x, V_2 \le y) = C_1\{x, K_2(y)\}$ with $C_1(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a,b \in [0,1]$. Assume that $\phi : [0,\infty) \to [0,1]$ is strictly decreasing with $\phi(0) = 1$ and $\phi(\infty) = 0$ and that ϕ' is strictly increasing and continuous. Moreover, suppose that K_2 is continuous. Suppose that the random variable V_2 takes values in $[0,1]$. Then the distribution function K_1 of the random variable $V_1 = C_1(U_1,V_2)$ is given for $t \in [0,1]$ by*

$$K_1(t) \;=\; t - \int_0^{\phi^{-1}(t)} \phi'\big(\phi^{-1}(t) + \phi^{-1}\circ K_2 \circ \phi(u) - u\big)du, \qquad (3.1)$$

where "$\circ$" is composition of functions, i.e., $f \circ g(x) = f\{g(x)\}$.

In Theorem 3.1, V_2 is an arbitrary random variable on $[0,1]$ and not necessarily a copula. In the special case that V_2 is uniformly distributed on $[0,1]$, (3.1) reduces to Theorem 4.3.4 of [18] or to the result of [7].

Next we consider a copula of the type $V_3 = C_3(V_4,V_5)$ with $V_4 = C_4(U_1,\dots,U_\ell)$ and $V_5 = C_5(U_{\ell+1},\dots,U_k)$. Making use of the distribution functions of V_4 and V_5, a representation of the distribution function of V_3 is given in the next theorem.

Theorem 3.2 *Let $V_4 \sim K_4$ and $V_5 \sim K_5$ and $P(V_4 \le x, V_5 \le y) = C_3\{K_4(x), K_5(y)\}$ with $C_3(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a,b \in [0,1]$. Assume that $\phi : [0,\infty) \to [0,1]$ is strictly decreasing with $\phi(0) = 1$ and $\phi(\infty) = 0$ and that ϕ' is strictly increasing and continuous. Moreover, suppose that K_4 and K_5 are continuous and that $\phi^{-1}\circ K_4 \circ \phi$ and $\phi^{-1}\circ K_5 \circ \phi$ are of bounded variation on $[0,\phi^{-1}(t)]$. Suppose that the random variables*

V_4 and V_5 take values in $[0,1]$. *Then the distribution function K_3 of the random variable $V_3 = C_3(V_4, V_5)$ is given by*

$$\begin{aligned} K_3(t) &= K_4(t) \qquad (3.2)\\ &\quad -\int_0^{\phi^{-1}(t)} \phi'\{\phi^{-1}[K_5\{\phi(u)\}] + \phi^{-1}\big(K_4[\phi\{\phi^{-1}(t)-u\}]\big)\}\, d\phi^{-1}[K_4\{\phi(u)\}]\\ &= K_5(t) \qquad (3.3)\\ &\quad -\int_0^{\phi^{-1}(t)} \phi'\{\phi^{-1}[K_4\{\phi(u)\}] + \phi^{-1}\big(K_5[\phi\{\phi^{-1}(t)-u\}]\big)\}\, d\phi^{-1}[K_5\{\phi(u)\}] \end{aligned}$$

for $t \in [0,1]$.

If $\phi^{-1}[K_4\{\phi(x)\}]$ has a continuous derivative, then (3.2) can be written as

$$K_3(t) = K_4(t) - \int_0^{\phi^{-1}(t)} \frac{\phi'\{\phi^{-1}[K_5\{\phi(u)\}] + \phi^{-1}\big(K_4[\phi\{\phi^{-1}(t)-u\}]\big)\}}{\phi'\{(\phi^{-1}\circ K_4 \circ \phi)^{-1}(u)\}} K_4'\{\phi(u)\}\phi'(u)du.$$

Similarly, if $\phi^{-1}[K_4\{\phi(x)\}]$ has a continuous derivative, then (3.3) is equivalent to

$$K_3(t) = K_5(t) - \int_0^{\phi^{-1}(t)} \frac{\phi'\{\phi^{-1}[K_4\{\phi(u)\}] + \phi^{-1}\big(K_5[\phi\{\phi^{-1}(t)-u\}]\big)\}}{\phi'\{(\phi^{-1}\circ K_5 \circ \phi)^{-1}(u)\}} K_5'\{\phi(u)\}\phi'(u)du.$$

Theorem 3.2 reduces to Theorem 3.1 if V_4 or V_5 are uniformly distributed on $[0,1]$. Moreover, by taking the derivative of the generator function, it can be shown that the expression in (3.2) is symmetric with respect to K_4 and K_5.

Note that using these two results we can determine the distribution function for an arbitrary grouping of the variables at the top level. For example, consider the copula $C_1\{u_1, u_2, C_2(u_3, \dots, u_k)\}$. From the properties of Archimedean copulas, this copula is equivalent to

$$C_1[u_1, C_1\{u_2, C_2(u_3, \dots, u_k)\}],$$

and thus Theorem 3.1 can be applied.

Theorems 3.1 and 3.2 provide recursive presentations for certain copula structures. In the next step we give a corollary with a direct formula for the distribution function of a copula of the form

$$C\{u_1, C_{k-1}(u_2, \dots, u_k)\}.$$

It is an extension of the result of [1]. Here we assume that u_k lies on the top level of the copula. Other cases could be derived for every single form of the copula, but it is difficult to present a general result.

Corollary 3.3 *Consider an HAC of the form*

$$\begin{aligned} C(u_1, \dots, u_k) &= C_1\{u_1, C_2(u_2, \dots, u_k)\}\\ &= \phi_1\left[\phi_1^{-1}(u_1) + \phi_1^{-1}\{C_2(u_2, \dots, u_k)\}\right]. \end{aligned}$$

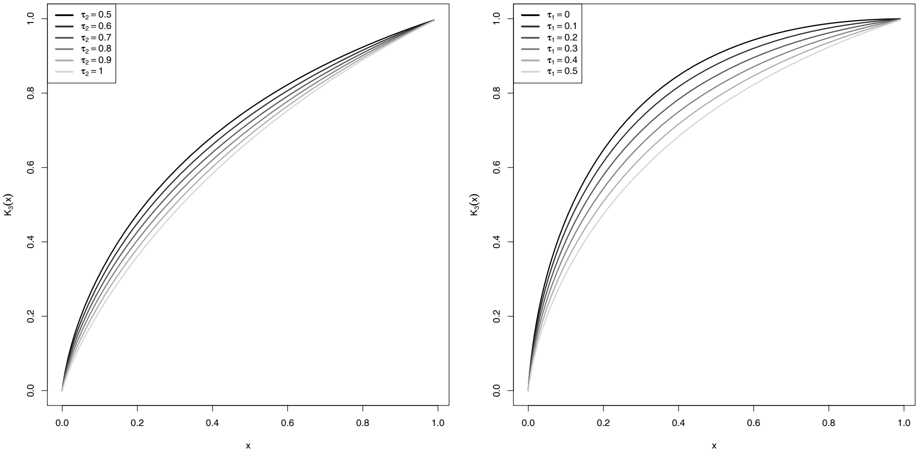

Figure 3.1 K-distribution for fully nested 3-dimensional hierarchical Archimedean copula with Gumbel generator functions.

Assume that $\phi_1 : [0,\infty) \to [0,1]$ *is strictly decreasing and continuously differentiable with* $\phi_1(0) = 1$. *Then the distribution function* K_1 *of* $C(U_1,\dots,U_k)$ *is equal for* $t \in [0,1]$ *to*

$$K_1(t) = \int_0^t k(x)\,dx = \int_0^t \int \cdots \int_{(0,1)^{k-1}} h_k\{x,u_2,\dots,u_k\}\,du_2 \dots du_k\,dx,$$

where

$$\begin{aligned} h_k(t,u_2,\dots,u_k) \;=\; & \frac{\phi_1'\left\{\phi_1^{-1}(t) - \phi_1^{-1} \circ C_2(u_2,\dots,u_k)\right\}}{\phi_1'\{\phi_1^{-1}(t)\}} \\ & \times\; c\left[\phi_1\{\phi_1^{-1}(t) - \phi_1^{-1} \circ C_2(u_2,\dots,u_k)\}, u_2,\dots,u_k\right] \\ & \times\; \mathbf{I}\big\{C_2(u_2,\dots,u_k) > t\big\} \text{ for } (u_2,\dots,u_k) \in [0,1]^{k-1}. \end{aligned}$$

Here, $c(u_1,\dots,u_k)$ *denotes the copula density of* C.

The practical calculation of K_1 using Theorem 3.3 seems to be quite difficult because of multivariate integration. As an example we consider the Gumbel family.

Example 3.4 Here we consider the simplest three-dimensional fully nested Archimedean copula with Gumbel generator functions and apply Theorem 3.1. Below, you find the generator function, the inverse of the generator, and the first derivative.

$$\begin{aligned} \phi_\theta(t) &= \exp(-t^{1/\theta}), \\ \phi_\theta^{-1}(t) &= \{-\log(t)\}^\theta, \\ \phi_\theta'(t) &= -\frac{1}{\theta}\exp(-t^{1/\theta})t^{-1+1/\theta}. \end{aligned}$$

Following [7], the K-distribution for the simple 2-dimensional Archimedean copula with generator ϕ is given by $K(t) = t - \phi^{-1}(t)\phi'\{\phi^{-1}(t)\}$. Thus, the $K_2(t,\theta)$ for the 2-dimensional Gumbel copula is

$$K_2(t,\theta) = t - \frac{t}{\theta}\log(t).$$

In Figure 3.1, we represent the K_3 distribution with respect to different parameters. In the left panel we fix the highest level parameter $\tau_1 = 0.5$, and modify the parameter $\tau_2 = \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$, where the case for $\tau_2 = 0.5$ represents the simple Archimedean copula. On the right panel the lower-level parameter is kept fixed $\tau_2 = 0.5$ and we do the changes in the higher-level parameter $\tau_1 = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. This can be used in the goodness-of-fit tests, discussed in [1].

4 Dependence orderings

Dependence orderings allow us to compare the strength of the dependence imposed by different copula functions. In this section we give some necessary conditions under which one HAC is more concordant than other. By definition ([11], p. 37), C' is more *concordant* than C if

$$C \prec_c C' \Leftrightarrow C(\mathbf{x}) \le C'(\mathbf{x}) \text{ and } \overline{C}(\mathbf{x}) \le \overline{C'}(\mathbf{x}) \ \forall \mathbf{x} \in [0;1]^k,$$

where $\overline{C}$ is the survival copula defined as $\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. This type of ordering is also called *positive quadrant ordering* or *upper orthant ordering* (see [16]). The case of two multivariate normal distributions gives us interesting insights into this ordering. Let $\boldsymbol{X} \sim \mathrm{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{X}' \sim \mathrm{N}_k(\boldsymbol{\mu}', \boldsymbol{\Sigma}')$. If $\mu_i = \mu_i'$, $\sigma_{ii} = \sigma_{ii}'$ for $i = 1, \dots, k$, and $\sigma_{ij} \le \sigma_{ij}'$ for $1 \le i < j \le k$, then $\boldsymbol{X} \prec_c \boldsymbol{X}'$.

In the bivariate case the most concordant is the Fréchet upper bound and the most discordant copula is the Fréchet lower bound. Another peculiarity of the bivariate case is the relationship between the concordance ordering of the dependence measures. It appears that if C_1 and C_2 are two copulas with Kendall's taus τ_1, τ_2, Spearman rhos ρ_1, ρ_2, tail dependence parameters λ_1, λ_2, Blomqvist's betas β_1, β_2 respectively, then $C_1 \prec_c C_2$ implies that $\tau_1 \le \tau_2$, $\rho_1 \le \rho_2$, $\lambda_1 \le \lambda_2$ ([11]) and $\beta_1 \le \beta_2$ ([21]).

Several interesting results can be derived if C is an Archimedean copula. First note that there is no sharp lower bound for the general class of copulas, however [15] construct a sharp lower bound for the class of Archimedean copulas. Thus there is an Archimedean copula C^L such that $C^L \prec_c C$ for any Archimedean copula C. [11] considers in Theorems 4.8, 4.9, and 4.10, three and four dimensional HACs with different fixed structures. The theorems provide conditions on the top level generator functions which guarantee the concordance of the HACs, assuming that the generators at the lower levels are the same. In [11], the author also states that these theorems could be easily extended to any messy structure of the copula. Next we provide a general result for an arbitrary tree. The proof uses explicitly the hierarchical structure of the copula.

Theorem 4.1 *If two hierarchical Archimedean copulas*

$$C^1 = C^1_{\boldsymbol{\phi}_1}(u_1, \dots, u_k) \quad \textit{and} \quad C^2 = C^2_{\boldsymbol{\phi}_2}(u_1, \dots, u_k)$$

differ by only the generator functions at level r as

$$\boldsymbol{\phi}_1 = (\phi_1, \dots, \phi_{r-1}, \phi, \phi_{r+1}, \dots, \phi_p)$$

and

$$\boldsymbol{\phi}_2 = (\phi_1, \dots, \phi_{r-1}, \phi^*, \phi_{r+1}, \dots, \phi_p)$$

with $\phi^{-1} \circ \phi^* \in \mathcal{L}^*$, *then* $C^1 \prec_c C^2$.

Note that the condition we impose on the generator function is a sufficient condition to construct an HAC (see Theorem 4.4 of [14]).

5 Extreme value theory and tail dependency

5.1 Extreme value copula

The maximum domain of attraction of the multivariate extreme value distribution can be characterised by the following theorem.

Theorem 5.1 ([5]) *Let* $\{X_{1i}, \dots, X_{ki}\}_{i=1,\dots,n}$ *be a sequence of random vectors with distribution function* F, *marginal distributions* $F_1, \dots, F_k$, *and copula* C. *Let also* $M_j^{(n)} = \max_{1 \le i \le n} X_{ji}$, $j = 1, \dots, k$ *be the componentwise maxima. Then*

$$\lim_{n\to\infty} P\left\{\frac{M_1^{(n)} - a_{1n}}{b_{1n}} \le x_1, \dots, \frac{M_m^{(n)} - a_{kn}}{b_{kn}} \le x_k\right\} = F^*(x_1, \dots, x_k), \quad \forall (x_1, \dots, x_k) \in \mathbb{R}^k$$

with $b_{jn} > 0$, $j = 1, \dots, k$, $n \ge 1$ *if and only if*

1. *for all* $j = 1, \dots, k$ *there exist some constants* a_{jn} *and* b_{jn} *and a non-degenerating limit distribution* F_j^* *such that*

$$\lim_{n\to\infty} P\left\{\frac{M_j^{(n)} - a_{jn}}{b_{jn}} \le x_j\right\} = F_j^*(x_j), \quad \forall x_j \in \mathbb{R};$$

2. *there exists a copula* C^* *such that*

$$C^*(u_1, \dots, u_k) = \lim_{n\to\infty} C^n\big(u_1^{1/n}, \dots, u_k^{1/n}\big).$$

In this case we say that C^* is the *extreme valued copula* and C belongs to the *maximum domain of attraction* of C^* (written $C \in MDA(C^*)$). This implies that a multivariate distribution with all margins being extreme value distributions and with an extreme value copula is a multivariate extreme value distribution. [6] show that the only Archimedean extreme value copula is the Gumbel copula. Thus, most bivariate Archimedean copulas under minor conditions belong to the domain of attraction of the Gumbel copula. Using Proposition 2.1 and the result of [6], we obtain the next theorem.

Theorem 5.2 *If* $C \in \mathcal{F}_{nr_1}$, $C^* \in \mathcal{F}_{nr_2}$, $\forall \varphi_\theta \in \mathcal{N}(C)$, $\partial[\varphi_\ell^{-1}(t)/(\varphi_\ell^{-1})'(t)]/\partial t\big|_{t=1}$ *exists and is equal to* $1/\theta$ *and* $C \in MDA(C^*)$, *then* $r_1 = r_2$, $\forall \phi_\theta \in \mathcal{N}(C^*)$, $\phi_\theta = \exp\{-x^{1/\theta}\}$ *and the structure of* C *is equal to one of the possible structures of* C^*.

Note that an HAC can be attracted by the independence copula. This case is also covered by the theorem, if we recall that the independence copula is a special case of the Gumbel copula.

5.2 Tail dependency

In this section we consider the tail dependence of an HAC. The tail behaviour characterises the tendency of random variables to take extreme values simultaneously. [4] provide a detailed analysis of this aspect for simple Archimedean copulas. Here we extend their results to an HAC. For illustration purposes, consider the upper and lower tail indices of a bivariate vector (X_1, X_2). In the case of a simple Archimedean copula with generator ϕ, the tail indices are defined and given by (see Corollary 5.4.3 of [17])

$$\begin{aligned}
\lambda_U &= \lim_{u\to 1^-} P\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\} \\
&= \lim_{u\to 1^-} \frac{\overline{C}(1-u,1-u)}{u} = 2 - \lim_{w\to 0+} \frac{1-\phi(2w)}{1-\phi(w)}, \\
\lambda_L &= \lim_{u\to 0+} P\{X_2 \le F_2^{-1}(u) \mid X_1 \le F_1^{-1}(u)\} \\
&= \lim_{u\to 0+} \frac{C(u,u)}{u} = \lim_{w\to\infty} \frac{\phi(2w)}{\phi(w)}.
\end{aligned}$$

The definitions are feasible if the corresponding limits exist. However, this is the case for most parametric generator families. The formulas imply that the lower tail index of the Archimedean copula is linked to the index of regular variation at infinity of the generator functions. Recall that a function $\phi : (0,\infty) \to (0,\infty)$ is said to be *regularly varying at infinity with index* $\lambda \in \mathbb{R}$ (written $RV_\lambda(\infty)$) if $\lim_{w\to\infty} \frac{\phi(tw)}{\phi(w)} = t^\lambda$ for all $t > 0$. A function $\phi : (0,\infty) \to (0,\infty)$ is regularly varying at infinity with index $-\infty$ (written $RV_{-\infty}(\infty)$) if

$$\lim_{w\to\infty} \frac{\phi(tw)}{\phi(w)} = \begin{cases} \infty & \text{if } t < 1 \\ 1 & \text{if } t = 1 \\ 0 & \text{if } t > 1 \end{cases}.$$

If ϕ is absolutely continuous with density ϕ', then a sufficient condition for ϕ to be $RV_\lambda(\infty)$ with $\lambda \in \mathbb{R}$ is that $\lim_{w\to\infty} \frac{w\phi'(w)}{\phi(w)} = \lambda$. This condition is necessary as well if ϕ is monotone (cf. [20, Proposition 2.5]). Moreover, it holds for $\lambda \ge 0$ that if $\phi \in RV_{-\lambda}(\infty)$, then $\phi^{-1} \in RV_{-1/\lambda}(0)$, i.e., that $\lim_{v\to 0+} \frac{\phi^{-1}(tv)}{\phi^{-1}(v)} = t^{-1/\lambda}$. This is an immediate consequence of Proposition 2.6 of [20]: the result is obtained if in part (v) of the Proposition the function U is set equal to ϕ^{-1}.

To deal with the upper tail index of an HAC we extend the concept of regular variation to a special case where the function does not converge to zero or infinity. In particular, the function ϕ^{-1} is regularly varying at zero with tail index γ if $\lim_{w\to 0+} \frac{\phi^{-1}(1-tw)}{\phi^{-1}(1-w)} = t^\gamma$.

For the direct function, the tail index equals $1/\gamma$, i.e., $\lim_{w\to 0+}\frac{1-\phi(tw)}{1-\phi(w)} = t^{1/\gamma}$. By the convexity of the generator function, it follows that $\gamma \in [1;\infty]$.

In this section we consider a straightforward extension of the tail dependency measures to the multivariate setup also followed by [4]. In particular we consider the probability that each component of the random vector exceeds or falls below some predetermined threshold conditionally on the fact that a subset of the variables exceeded it.

$$\begin{aligned}\lim_{u\to 0+} \quad & P\{X_i \le F_i^{-1}(u_i u) \text{ for } i \notin \mathcal{S} \subset \mathcal{K} = \{1,\dots,k\} \\ & \qquad \mid X_j \le F_j^{-1}(u_j u) \text{ for } j \in \mathcal{S}\} \\ \lim_{u\to 0+} \quad & P\{X_i > F_i^{-1}(1-u_i u) \text{ for } i \notin \mathcal{S} \subset \mathcal{K} = \{1,\dots,k\} \\ & \qquad \mid X_j > F_j^{-1}(1-u_j u) \text{ for } j \in \mathcal{S}\}.\end{aligned}$$

The above limits can be established via the limits

$$\lambda_L(u_1,\dots,u_k) = \lim_{u\to 0+}\frac{1}{u}C(u_1u,\dots,u_ku) \qquad \text{and} \tag{5.1}$$

$$\begin{aligned}\lambda_U(u_1,\dots,u_k) &= \lim_{u\to 0+}\frac{1}{u}\overline{C}(1-u_1u,\dots,1-u_ku) \\ &= \lim_{u\to 0+}\sum_{s_1\in\mathcal{K}}(-1)^{|s_1|+1}\{1-C_{s_1}(1-u_ju, j\in s_1)\}.\end{aligned} \tag{5.2}$$

Suppose $X_1,\dots,X_k \sim C$ satisfy

$$\begin{aligned}&C(u_1,\dots,u_k)\\ &\quad = C_0\{C_1(u_1,\dots,u_{k_1}),\dots,C_m(u_{k_{m-1}+1},\dots,u_{k_m}),u_{k_m+1},\dots,u_k\},\end{aligned}$$

where C_0 is an Archimedean copula with the generator $\phi_0 \in \mathcal{L}$ and C_i for $i=1,\dots,m$ are Archimedean or hierarchical Archimedean copulas. First we extend Theorem 3.1 of [4]. The case of an HAC is more complex due to the fact that the tail dependency indices of an HAC depend on the interrelation of the generators at different levels. On the other hand, for Archimedean copulas the tail dependency is completely determined by the index of regular variation of the single generator.

Theorem 5.3 *Assume that the limits* $\lim_{u\to 0+} u^{-1}C_i(uu_{k_{i-1}+1},\dots,uu_{k_i}) = \lambda_{L,i}(u_{k_{i-1}+1},\dots,u_{k_i})$ *exist for all* $0 < u_{k_{i-1}+1},\dots,u_{k_i} < 1$, $i=1,\dots,m$. *Suppose that* $m+k-k_m \ge 2$. *If* ϕ_0^{-1} *is regularly varying at infinity with index* $-\lambda_0 \in [-\infty,0]$, *then for all* $0 < u_i < 1$, $i=1,\dots,m$,

$$\lim_{u\to 0+}\frac{C(uu_1,\dots,uu_k)}{u} \tag{5.3}$$

$$= \begin{cases}\min\{\lambda_{L,1}(u_1,\dots,u_{k_1}),\dots,\lambda_{L,m}(u_{k_{m-1}+1},\dots,u_{k_m}),u_{k_m+1},\dots,u_k\} \\ \qquad\qquad \textit{if} \quad \lambda_0 = \infty, \\ \left(\sum_{i=1}^m \lambda_{L,i}(u_{k_{i-1}+1},\dots,u_{k_i})^{-\lambda_0} + \sum_{j=k_m+1}^k u_j^{-\lambda_0}\right)^{-1/\lambda_0} \\ \qquad\qquad \textit{if} \quad 0<\lambda_0<\infty, \\ 0 \qquad\qquad \textit{if} \quad \lambda_0 = 0 \textit{ or } \lambda_{L,i}=0 \textit{ for some } i\textit{s}.\end{cases}$$

Note that $0 \le \lambda_{L,i}(u_{k_{i-1}+1}, \dots, u_{k_i}) < 1$ since for any copula, $C(u_1, \dots, u_k) \le \min\{u_1, \dots, u_k\}$.

The results of the theorem can be directly applied to determine more complex dependency indices as in 5.1. This follows from the fact that any multivariate marginal distribution of an HAC is also an HAC. This implies that $C(\underbrace{u_{j_1}u, \dots, u_{j_{|\mathcal{S}|}}u}_{|\mathcal{S}|})$ is an HAC and so Theorem 5.3 can be applied.

Note that for most of the popular generator functions, the index of tail variation is ∞ (see [4]). Theorem 5.3 implies in this case that $C_0(u_1u, \dots, u_ku) = o(u)$ and the benchmark obtained this way is too weak for the assessment of the tail behaviour. To overcome this problem, we rely on the de Haan theory.

In the following let

$$\begin{aligned} C_j^*(u) &= C_j(u_{k_j-1+1}u, \dots, u_{k_j}u) \mid_{u_{k_{j-1}+1}=\dots=u_{k_j}=1}, \\ C^*(u) &= C(u_1u, \dots, u_ku) \mid_{u_1=\dots=u_k=1}, \\ \lambda^*_{L,j}(u, u_{k_{j-1}+1}, \dots, u_{k_j}) &= C_j(u_{k_{j-1}+1}u, \dots, u_{k_j}u)/C_j^*(u). \end{aligned}$$

Note that $0 \le \lambda^*_{L,j}(u, u_{k_{j-1}+1}, \dots, u_{k_j}) \le 1$. Moreover, if

$$\lim_{u\to 0+} u^{-1}C_j(uu_{k_{j-1}+1}, \dots, uu_{k_j}) = \lambda_{L,j}(u_{k_{j-1}+1}, \dots, u_{k_j}) > 0$$

for all $0 < u_{k_{j-1}+1}, \dots, u_{k_j}, \le 1$, then

$$\begin{aligned} \lambda^*_{L,j}(u_{k_{j-1}+1}, \dots, u_{k_j}) &= \lim_{u\to 0+} \frac{C_j(u_{k_j-1+1}u, \dots, u_{k_j}u)/u}{C_j^*(u)/u} \\ &= \frac{\lambda_{L,j}(u_{k_{j-1}+1}, \dots, u_{k_j})}{\lambda_{L,j}(1, \dots, 1)} \end{aligned}$$

Theorem 5.4 *Assume that the limits*

$$\lim_{u\to 0+} \frac{C_i(uu_{k_{i-1}+1}, \dots, uu_{k_i})}{C_i^*(u)} = \lambda^*_{L,i}(u_{k_{i-1}+1}, \dots, u_{k_i})$$

exist for all $0 < u_{k_{i-1}+1}, \dots, u_{k_i} \le 1$, $i = 1, \dots, m$. *Let* $\phi_0^{-1} \in RV_0(0)$ *and let* $\psi(v) = -\phi_0(v)/\phi_0'(v)$ *be regularly varying at infinity with finite tail index* κ. *Then* $\kappa \le 1$ *and for all* $0 < u_i < 1$, $i = 1, \dots, m$,

$$\begin{aligned} \lim_{u\to 0+} \frac{C(uu_1, \dots, uu_k)}{C^*(u)} &= \prod_{j=1}^{m} [\lambda^*_{L,j}(u_{k_{j-1}+1}, \dots, u_{k_j})]^{(m+k-k_m)^{-\kappa}} \\ &\times \prod_{j=k_m+1}^{k} u_j^{(m+k-k_m)^{-\kappa}}. \end{aligned}$$

Next we turn to the upper tail of the HAC. The next theorem establishes the results necessary for analysing the limiting behaviour of the expression in 5.2.

Theorem 5.5 *Assume that the limits*

$$\lim_{u\to 0+} u^{-1}[1-C_i(1-uu_{k_{i-1}+1},\dots,1-uu_{k_i})] = \lambda_{U,i}(u_{k_{i-1}+1},\dots,u_{k_i})$$

exist for all $0 < u_{k_{i-1}+1},\dots,u_{k_i} < 1$, $i = 1,\dots,m$. *Suppose that* $m+k-k_m \geq 2$. *If* $\phi_0^{-1}(1-w)$ *is regularly varying at zero with index* $-\gamma_0 \in [-\infty,-1]$, *then it holds for all* $0 < u_i < 1$, $i = 1,\dots,m$ *that*

$$\lim_{u\to 0+} \frac{1-C(1-uu_1,\dots,1-uu_k)}{u} \tag{5.4}$$
$$= \begin{cases} \min\{\lambda_{U,1}(u_1,\dots,u_{k_1}),\dots,\lambda_{U,m}(u_{k_{m-1}+1},\dots,u_{k_m}),u_{k_m+1},\dots,u_k\} & \text{if } \gamma_0=\infty,\\ \left(\sum_{i=1}^m[\lambda_{U,i}(u_{k_{i-1}+1},\dots,u_{k_i})]^{\gamma_0} + \sum_{j=k_m+1}^k u_j^{\gamma_0}\right)^{1/\gamma_0} & \text{if } 1\leq\gamma_0<\infty, \end{cases}$$

From this it follows that under the hypotheses of Theorem 5.4,

$$\lim_{u\to 0+} \frac{\bar{C}(1-uu_1,\dots,1-uu_k)}{u} = \lim_{u\to 0+} \frac{\sum_{s_1\in\mathcal{K}}(-1)^{|s_1|+1}\{1-C_{s_1}(1-u_j u, j\in s_1)\}}{u}$$
$$= \begin{cases} \min\{\lambda_{U,1}(u_1,\dots,u_{k_1}),\dots,\lambda_{U,m}(u_{k_{m-1}+1},\dots,u_{k_m}),u_{k_m+1},\dots,u_k\} & \text{if } \gamma_0=\infty,\\ \sum_{s_1\in\mathcal{K}}(-1)^{|s_1|+1}\left(\sum_{i=1}^{m_{s_1}}[\lambda_{U,i}(u_j; j\in s_1^c)]^{\gamma_0} + \sum_{j\in s_1\setminus s_1^c} u_j^{\gamma_0}\right)^{1/\gamma_0} & \text{if } 1<\gamma_0<\infty,\\ 0 & \text{if } \gamma_0=1, \end{cases}$$

where s_1 is an arbitrary nonempty subset of $\mathcal{K}$ and $s_1^c \subset s_1$ contains the indices of the variables joined by copulas at lower levels. Since $\gamma_0 \in [1,+\infty]$, this implies that any values of $\lambda_{U,j} \in [0,1]$ do not destroy the behaviour of the last limit. This implies that Theorems 4.3 and 4.6 of [4] can be directly extended to the case of HACs by replacing the u_is with the corresponding upper limits $\lambda_{U,j}$.

A Appendix

Proof of Remark 2.2: We consider here two cases. First let u_i and u_j be on different subnodes of the first level of the copula

$$\begin{aligned} C(u_1,\dots,u_i,\dots,u_j,\dots,u_k) &= C_{kr}\{\dots,C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}(u_{k_{\ell'-1}+1},\dots,u_i,\dots,u_{k_{\ell'}}),\\ &\quad \dots,C_{k_{\ell''}-k_{\ell''-1},r_{\ell''}}(u_{k_{\ell''-1}+1},\dots,u_j,\dots,u_{k_{\ell''}}),\dots\}, \end{aligned}$$

where r is the total number of nodes, $C_{k_1,r_1} \in \mathcal{F}_{k_1,r_1},\dots,C_{k_m-k_{m-1},r_{k_m}} \in \mathcal{F}_{k_m,r_m}$ are the subcopulas on the first level and the root $C_{kr} \in \mathcal{F}_{m,1}$. From the properties of multivariate distributions, the bivariate margin of u_i and u_j is given by

$$\begin{aligned} (u_i,u_j) &\sim C(1,\dots,u_i,\dots,u_j,\dots,1)\\ &= C_{kr}\{\dots,C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}(1,\dots,u_i,\dots,1),\\ &\quad \dots,C_{k_{\ell''}-k_{\ell''-1},r_{\ell''}}(1,\dots,u_j,\dots,1),\dots\}. \end{aligned}$$

Since $C(1,\dots,1,u,1,\dots,1) = u$ and $C(1,\dots,1) = 1$, it follows that

$$\begin{aligned}(u_i,u_j) &\sim C(1,\dots,u_i,\dots,u_j,\dots,1) = C_{kr}(1,\dots,u_i,\dots,u_j,\dots,1)\\ &= \phi\{\phi^{-1}(1)+\dots+\phi^{-1}(u_i)+\dots+\phi^{-1}(u_j)+\dots+\phi^{-1}(1)\}\\ &= \phi\{\phi_n^{-1}(u_i)+\phi^{-1}(u_j)\} = C_{n,2}(u_i,u_j),\end{aligned}$$

where $C_{k2} \in \mathcal{F}_{2,1}$ and $C_{k2} \in \mathfrak{C}\{\mathcal{N}(C)\}$ by construction. Thus we showed that if two variables lie in different subcopulas of the top level, their bivariate distribution is given by the top level copula. In the second case, when u_i and u_j are on different subnodes of the second level in the copula, then

$$\begin{aligned}&C(u_1,\dots,u_i,\dots,u_j,\dots,u_k)\\ &\quad= C_{kr}\{\dots,C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}(u_{k_{\ell'-1}+1},\dots,u_i,\dots,u_j,\dots,u_{k_{\ell'}}),\\ &\qquad\qquad \dots,C_{k_{\ell''}-k_{\ell''-1},r_{\ell''}}(u_{k_{\ell''-1}+1},\dots,\dots,u_{k_{\ell''}}),\dots\}.\end{aligned}$$

Proceeding with the copula $C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}$ with generator ϕ_2 in the same way as with the original copula C in the first part of the remark, we obtain $(u_i,u_j) \sim \phi_2\{\phi_2^{-1}(u_i)+\phi_2^{-1}(u_j)\}$. Continuing recursively we complete the proof. □

Proof of Remark 2.3: The proof is similar to the proof of Remark 2.2. Let us fix a bivariate copula $C_2^* \in \mathfrak{C}_2\{\mathcal{N}(C)\}$. We may assume, without loss of generality, that the generator $\phi = \mathcal{N}(C_2^*)$ is used to construct the subcopula at the second level of the original copula C. After reordering the variables, if necessary, we have then

$$\begin{aligned}C(u_1,\dots,u_k) &= C_{kr}\{\dots,C_2^*(u_{1^*},\dots,u_{m^*}),\dots\}\\ &= C_{kr}[\dots,\phi_1\{\phi^{-1}\circ\widetilde{C}_1(\cdot),\dots,\phi^{-1}\circ\widetilde{C}_p(\cdot)\},\dots].\end{aligned}$$

Now we proceed as in Remark 2.2 by taking two variables from different subcopulas of the second level of C_2^*. We may assume, without loss of generality, that we may take one variable u from $\widetilde{C}_2$ and another v from $\widetilde{C}_p$. This shows that there exists a pair of random variables (u,v) with joint bivariate distribution function $C_2^*(u,v)$. □

Proof of Theorem 3.1: Fix $t \in [0,1]$.

a) Then $\{V_1 \le t\} \wedge \{U_1 \le t\} = \{U_1 \le t\}$, since $U_1 \le t$ is a subset of $V_1 \le t$. Moreover,

$$\{V_1 \le t\} \wedge \{U_1 > t\} = \{V_2 \le g_t(U_1)\} \wedge \{U_1 > t\}$$

with $g_t(x) = \phi(\phi^{-1}(t)-\phi^{-1}(x))$ for $x \in [t,1]$. The function $g_t(.) : [0,1] \to [0,1]$ with $g_t(x) = 1$ for $x < t$ is strictly decreasing with $g_t(t) = 1$.
Consequently,

$$\begin{aligned}P(V_1 \le t) &= P(V_1 \le t \wedge U_1 \le t) + P(V_1 \le t \wedge U_1 > t) && \text{(A.1)}\\ &= t + P\{V_2 \le g_t(U_1) \wedge U_1 > t\}, && \text{(A.2)}\end{aligned}$$

since $P(U_1 \le t) = t$.

b) In order to calculate the second summand of the right hand side of (A.2) we consider a partition $t = t_0 < t_1 < \cdots < t_N = 1$ of the interval $[t,1]$. Then

$$P\{V_2 \le g_t(U_1) \wedge U_1 > t\} = \sum_{i=1}^{N} P\{V_2 \le g_t(U_1) \wedge t_{i-1} < U_1 \le t_i\}$$

$$\begin{cases} \le \sum_{i=1}^{N} P(V_2 \le g_t(t_{i-1}) \wedge t_{i-1} < U_1 \le t_i) \\ \ge \sum_{i=1}^{N} P(V_2 \le g_t(t_i) \wedge t_{i-1} < U_1 \le t_i) \end{cases}$$

since $g_t(t_i) \le g_t(U_1) \le g_t(t_{i-1})$ if $t_{i-1} < U_1 \le t_i$.

c) First we consider the upper bound.

$$\begin{aligned} & P\{V_2 \le g_t(t_{i-1}) \wedge t_{i-1} < U_1 \le t_i\} \\ &\quad = P\{U_1 \le t_i \wedge V_2 \le g_t(t_{i-1})\} - P\{U_1 \le t_{i-1} \wedge V_2 \le g_t(t_{i-1})\} \\ &\quad = C_1[t_i, K_2\{g_t(t_{i-1})\}] - C_1[t_{i-1}, K_2\{g_t(t_{i-1})\}] \\ &\quad = \phi\big(\phi^{-1}(t_i) + \phi^{-1}[K_2\{g_t(t_{i-1})\}]\big) - \phi\big(\phi^{-1}(t_{i-1}) + \phi^{-1}[K_2\{g_t(t_{i-1})\}]\big). \end{aligned}$$

Now we determine the partition by choosing $t_{N-i} = \phi(iw/N)$ for $i = 0,\ldots,N$ with $w = \phi^{-1}(t)$. This choice fulfills the requirement that $t = t_0 < t_1 < \cdots < t_N \le 1$. Moreover, $g_t(t_i) = t_{N-i}$. Putting $\zeta(x) = \phi^{-1}[K_2\{\phi(x)\}] - x$, there exists a $\xi_{N,i}$ such that

$$\begin{aligned} & P\{V_2 \le g_t(t_{i-1}) \wedge t_{i-1} < U_1 \le t_i\} \\ &\quad = \phi\left\{w - \frac{w}{N} + \zeta\left(\frac{i-1}{N}w\right)\right\} - \phi\left\{w + \zeta\left(\frac{i-1}{N}w\right)\right\} \\ &\quad = -\frac{w}{N}\,\phi'\left\{w + \zeta\left(\frac{i-1}{N}w\right) - \xi_{N,i}\right\} \end{aligned}$$

with $0 \le \xi_{N,i} \le w/N$. Now let $\delta > 0$. Since ϕ is strictly decreasing,

$$P\{V_2 \le g_t(t_{i-1}) \wedge t_{i-1} < U_1 \le t_i\} \le -\frac{w}{N}\,\phi'\left\{w - \delta + \zeta\left(\frac{i-1}{N}w\right)\right\}$$

for $N \ge N_0$. Since

$$\lim_{N\to\infty} \sum_{i=1}^{N} -\frac{w}{N}\,\phi'\left\{w - \delta + \zeta\left(\frac{i-1}{N}w\right)\right\} = -\int_0^w \phi'\{w - \delta + \zeta(x)\}dx,$$

it follows that

$$\begin{aligned} P(V_1 \le t \wedge U_1 > t) &\le \liminf_{\delta\to 0}\left[-\int_0^w \phi'\{w - \delta + \zeta(x)\}dx\right] \\ &= -\int_0^w \phi'\{w + \zeta(x)\}dx. \end{aligned}$$

d) Next we consider the lower bound. We obtain by analogy to c) and with $\xi_{N,i}$ as above that

$$P\{V_2 \le g_t(t_i) \wedge t_{i-1} < U_1 \le t_i\} = -\frac{w}{N}\,\phi'\left\{w + \zeta\left(\frac{i}{N}w\right) - \xi_{N,i}\right\}$$
$$\ge -\frac{w}{N}\,\phi'\left\{w + \zeta\left(\frac{i}{N}w\right)\right\}.$$

Consequently,

$$P(V_1 \le t \wedge U_1 > t) \ge -\int_0^w \phi'\{w + \zeta(x)\}dx.$$

Because the upper and the lower bound are the same, this completes the proof. □

Proof of Theorem 3.2: The proof is based on a similar argument as the proof of Theorem 3.1. I.e., we have

$$P(V_3 \le t) = K_4(t) + P(V_3 \le t \wedge V_4 > t)$$

and

$$P(V_3 \le t \wedge V_4 > t) \begin{cases} \le \sum_{i=1}^{N} P\{V_5 \le g_t(t_{i-1}) \wedge t_{i-1} < V_4 \le t_i\} \\ \ge \sum_{i=1}^{N} P\{V_5 \le g_t(t_i) \wedge t_{i-1} < V_4 \le t_i\} \end{cases}.$$

a) Moreover,

$$\begin{aligned} &P\{V_5 \le g_t(t_{i-1}) \wedge t_{i-1} < V_4 \le t_i\} \\ &\quad = P\{V_4 \le t_i \wedge V_5 \le g_t(t_{i-1})\} - P\{V_4 \le t_{i-1} \wedge V_5 \le g_t(t_{i-1})\} \\ &\quad = C_3[K_4(t_i), K_5\{g_t(t_{i-1})\}] - C_3[K_4(t_{i-1}), K_5\{g_t(t_{i-1})\}] \\ &\quad = \phi\big(\phi^{-1}\{K_4(t_i)\} + \phi^{-1}[K_5\{g_t(t_{i-1})\}]\big) \\ &\qquad - \phi\big(\phi^{-1}\{K_4(t_{i-1})\} + \phi^{-1}[K_5\{g_t(t_{i-1})\}]\big) \\ &\quad = \phi\left\{\zeta_4\left(w - \frac{i}{N}w\right) + \zeta_5\left(\frac{i-1}{N}w\right)\right\} \\ &\qquad - \phi\left\{\zeta_4\left(w - \frac{i-1}{N}w\right) + \zeta_5\left(\frac{i-1}{N}w\right)\right\} \end{aligned}$$

with $\zeta_4(x) = \phi^{-1}[K_4\{\phi(x)\}]$ and $\zeta_5(x) = \phi^{-1}[K_5\{\phi(x)\}]$. Then

$$\begin{aligned} &P\{V_5 \le g_t(t_{i-1}) \wedge t_{i-1} < V_4 \le t_i\} \\ &\quad = \left\{\zeta_4\left(w - \frac{i}{N}w\right) - \zeta_4\left(w - \frac{i-1}{N}w\right)\right\} \\ &\qquad \times \phi'\left\{\zeta_5\left(\frac{i-1}{N}w\right) + \zeta_4\left(w - \tilde{\xi}_{N,i}w\right)\right\} \end{aligned}$$

with $(i-1)/N \le \tilde{\xi}_{N,i} \le i/N$. Since $\zeta_5\{w(i-1)/N\} \le \zeta_5(w\tilde{\xi}_{N,i})$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{N} P\{V_5 \le g_t(t_{i-1}) \wedge t_{i-1} < V_4 \le t_i\} \\
&\quad \le \sum_{i=1}^{N} \left\{\zeta_4\left(w - \frac{i}{N}w\right) - \zeta_4\left(w - \frac{i-1}{N}w\right)\right\} \\
&\qquad \times \phi'\left\{\zeta_5(w\tilde{\xi}_{N,i}) + \zeta_4(w - w\tilde{\xi}_{N,i})\right\} \\
&\quad \to -\int_0^1 \phi'\{\zeta_5(tw) + \zeta_4(w-tw)\}\, d\zeta_4(w-tw) \\
&\quad = -\int_0^w \phi'\{\zeta_5(u) + \zeta_4(w-u)\}\, d\zeta_4(u).
\end{aligned}$$

b) For the lower bound we get

$$\begin{aligned}
&P\{V_5 \le g_t(t_i) \wedge t_{i-1} < V_4 \le t_i\} \\
&\quad = \phi\left\{\zeta_4\left(w - \frac{i}{N}w\right) + \zeta_5\left(\frac{i}{N}w\right)\right\} - \phi\left\{\zeta_4\left(w - \frac{i-1}{N}w\right) + \zeta_5\left(\frac{i}{N}w\right)\right\} \\
&\quad = \left\{\zeta_4\left(w - \frac{i}{N}w\right) - \zeta_4\left(w - \frac{i-1}{N}w\right)\right\} \times \phi'\left\{\zeta_5\left(\frac{i}{N}w\right) + \zeta_4\left(w - w\xi^*_{N,i}\right)\right\}
\end{aligned}$$

with $(i-1)/N \le \xi^*_{N,i} \le i/N$. Since $\zeta_5(w\xi^*_{i,N}) \le \zeta_5(wi/N)$ we obtain that

$$\begin{aligned}
&\sum_{i=1}^{N} P\{V_5 \le g_t(t_{i-1}) \wedge t_{i-1} < V_4 \le t_i\} \\
&\quad \ge \sum_{i=1}^{N} \left\{\zeta_4\left(w - \frac{i}{N}w\right) - \zeta_4\left(w - \frac{i-1}{N}w\right)\right\} \\
&\qquad \times \phi'\left\{\zeta_5(w\xi^*_{N,i}) + \zeta_4(w - w\xi^*_{N,i})\right\}
\end{aligned}$$

and thus the result follows as in a). □

Proof of Corollary 3.3: we follow the idea of the proof of Theorem 2 of [1]. The copula is given by

$$\begin{aligned}
C(u_1,\dots,u_k) &= \phi_1\{\phi_k^{-1}(u_k) + \phi_1^{-1} \circ C_{k-1}(u_2,\dots,u_k)\} \\
&= P\{F_1(X_1) \le u_1,\dots,F_k(X_k) \le u_k\}.
\end{aligned}$$

Since $\phi_1^{-1}(1) = 0$, we have $C(1,u_2,\dots,u_k) = C_2(u_2,\dots,u_k)$. Differentiating C with respect to u_1, we get

$$\frac{\partial C(u_1,\dots,u_k)}{\partial u_1} = \frac{\phi_1'[\phi_1^{-1}(u_1) + \phi_1^{-1}\{C_2(u_2,\dots,u_k)\}]}{\phi_1'\{\phi_1^{-1}(u_1)\}} > 0.$$

Next consider the conditional copula

$$P\{F_1(X_1) \le u_1 | F_2(X_2) \le u_2, \dots, F_k(X_k) \le u_k\},$$

which we denote by $C(u_1|u_2,\dots,u_k) = \frac{C_k(u_1,\dots,u_k)}{C_{k-1}(u_1,\dots,u_{k-1})}$. We have

$$\begin{aligned} P\{C(u_1,\dots,u_k) \le t\} &= P\{C(u_1|u_2,\dots,u_k)\cdot C_2(u_2,\dots,u_k) \le t\} \\ &= P\left\{C(u_1|u_2,\dots,u_k) \le \frac{t}{C_2(u_2,\dots,u_k)}\right\}. \end{aligned}$$

Let us consider the following function

$$Q(t) = \inf\{u_1 \in [0,1] : C(u_1|u_2,\dots,u_k) \ge t\}$$

for $t \in [0,1]$. It follows, as in the proof of Theorem 2 of [1], that

$$\begin{aligned} K_1(t) &= \int_{(0,1)^{k-1}:C_2(u_2,\dots,u_k)\ge t} \frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\} \\ &\quad \times c_k\left[Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}, u_2,\dots,u_k\right] du_2 \dots du_k. \end{aligned}$$

Next we compute $\frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}$. Note that by definition of Q,

$$C\left[Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}, u_2,\dots,u_k\right] = t.$$

Differentiation with respect to t leads to

$$1 = \left.\frac{\partial C(u_1,\dots,u_k)}{\partial u_1}\right|_{u_1 = Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}} \cdot \frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}.$$

Thus using the fact that C is an HAC with the generator ϕ at the highest level, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\} &= \left[\left.\frac{\partial C(u_1,\dots,u_k)}{\partial u_1}\right|_{u_1 = Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}}\right]^{-1} \\ &= \left[\left.\frac{\phi_1'\{\phi_1^{-1}\circ C_2(u_2,\dots,u_k) + \phi_1^{-1}(u_1)\}}{\phi_1'\{\phi_1^{-1}(u_1)\}}\right|_{u_1 = Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\}}\right]^{-1} \\ &= \frac{\phi_1'\{\phi_1^{-1}(t)\}}{\phi_1'\left[\phi_1^{-1}\circ Q\left\{\frac{t}{C_2(u_2,\dots,u_k)}\right\} + \phi_1^{-1}\circ C_2(u_2,\dots,u_k)\right]} \\ &= \frac{\phi_1'\{\phi_1^{-1}(t)\}}{\phi_1'\circ\phi_1^{-1}\circ C\left[Q\left\{\frac{t}{C_{k-1}(u_1,\dots,u_{k-1})}\right\}, u_2,\dots,u_k\right]}. \end{aligned}$$

Using the following algebraic and probabilistic transformations

$$
\begin{aligned}
&Q_{u_2,\dots,u_k}\left\{\frac{t}{C(1,u_2,\dots,u_k)}\right\}\\
&= \inf\left\{u_1 \in [0,1] : C(u_1|u_2,\dots,u_k) \geq \frac{t}{C(1,u_2,\dots,u_k)}\right\}\\
&= \inf\{u_1 \in [0,1] : P\{F_1(X_1) \leq u_1 \,|\, F_2(X_2) \leq u_2;\dots;F_k(X_k) \leq u_k\}\\
&\qquad \times P\{F_2(X_2) \leq u_2;\dots;F_k(X_k) \leq u_k\} \geq t\}\\
&= \inf\{u_1 \in [0,1] : P\{F_1(X_1) \leq u_1;\dots;F_k(X_k) \leq u_k\} \geq t\}\\
&= \inf\{u_1 \in [0,1] : C(u_1,\dots,u_k) \geq t\}\\
&= \inf\left\{u_1 \in [0,1] : \phi_1\{\phi_1^{-1}(u_1) + \phi_1^{-1} \circ C_2(u_2,\dots,u_k)\} \geq t\right\}\\
&= \inf\left\{u_1 \in [0,1] : u_1 \geq \phi_1\{\phi_1^{-1}(t) - \phi_1^{-1} \circ C_2(u_2,\dots,u_k)\}\right\}\\
&= \phi_1\{\phi_1^{-1}(t) - \phi_1^{-1} \circ C_2(u_2,\dots,u_k)\}
\end{aligned}
$$

we get the final form of h_t:

$$
\begin{aligned}
h_k(t,u_2,\dots,u_k) = &\ \frac{\phi_1'\left\{\phi_1^{-1}(t) - \phi_1^{-1} \circ C_2(u_2,\dots,u_k)\right\}}{\phi_1'\{\phi^{-1}(t)\}}\\
&\times c\left[\phi_1(\phi_1^{-1}(t) - \phi_1^{-1} \circ C_2(u_2,\dots,u_k)), u_2,\dots,u_k\right]\\
&\times \mathbf{I}\left\{C_2(u_2,\dots,u_u) > t\right\} \text{ for } (u_1,\dots,u_k) \in [0,1]^k.
\end{aligned}
$$

Further simplification of the previous formula is unfortunately too difficult because of the lack of knowledge of a recursive formula for the HAC density, which is difficult to derive in a general form. □

Proof of Theorem 4.1: Here we prove one special case, from which by analogical thinking follows the statement of the Theorem 4.1. Let us consider the case of two feasible hierarchical Archimedean copulas C^1 and C^2 which differ only by the generator functions on the top level, and which satisfy the condition $\phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$, then $C^1 \prec_c C^2$.

Let $\mathbf{X} \sim C^1$ and $\mathbf{X}' \sim C^2$. To show the concordance property, it is necessary to prove that

$$P\{X_i \leq x_i, i = 1,\dots,k\} \leq P\{X_i' \leq x_i, i = 1,\dots,k\} \tag{A.3}$$

and

$$P\{X_i > x_i, i = 1,\dots,k\} \leq P\{X_i' > x_i, i = 1,\dots,k\} \tag{A.4}$$

for all $\mathbf{x} \in (0,1)^k$. The first inequality is the same as saying $C^1(\mathbf{x}) \leq C^2(\mathbf{x})$ for all $\mathbf{x} \in (0,1)^k$, while the second one is equivalent to $\overline{C}^1(\mathbf{x}) \leq \overline{C}^2(\mathbf{x})$ for all $\mathbf{x} \in (0,1)^k$, where $\overline{C}$ is the survival copula of C.

As mentioned in Chapter 1, Archimedean copulas, from which C^1 and C^2 are composed, arise from the Laplace transform

$$\phi\{\phi^{-1}(u_1) + \dots + \phi^{-1}(u_k)\} = \int_0^\infty \prod_{i=1}^k G_{\phi^{-1}}^\alpha(u)\, dM_\phi(\alpha),$$

where the generator function $\phi(s) = \int_0^\infty e^{-sw}\, dM_\phi(w),\ s \geq 0$ is the Laplace transform of some univariate cumulative distribution function $M_\phi(\cdot)$ of a positive random variable and $G_{\phi^{-1}}(u) = \exp\{-\phi^{-1}(u)\}$. By hypothesis C^1 and C^2 are proper HACs and differ only by a generator function on the highest level such as $\phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$. If we denote the second level copulas by $z_i,\ i = 1,\dots,m$, then

$$\begin{aligned}
C^1(\mathbf{u}) &= C^1\{C_{k_1 r_1}(u_1,\dots,u_{k_1}),\dots,C_{k_m-k_{m-1},r_m}(u_{k_{m-1}+1},\dots,u_{k_m=k})\} \\
&= \phi_1[\phi_1^{-1}\{C_{k_1 r_1}(u_1,\dots,u_{k_1})\} + \cdots \\
&\quad + \phi_1^{-1}\{C_{k_m-k_{m-1},r_m}(u_{k_{m-1}+1},\dots,u_{k_m=k})\}] \\
&= \phi_1\{\phi_1^{-1}(z_1) + \cdots + \phi_1^{-1}(z_m)\}
\end{aligned}$$

$$\begin{aligned}
C^2(\mathbf{u}) &= C^2\{C_{k_1 r_1}(u_1,\dots,u_{k_1}),\dots,C_{k_m-k_{m-1},r_m}(u_{k_{m-1}+1},\dots,u_{k_m=k})\} \\
&= \phi_2[\phi_2^{-1}\{C_{k_1 r_1}(u_1,\dots,u_{k_1})\} + \cdots \\
&\quad + \phi_2^{-1}\{C_{k_m-k_{m-1},r_m}(u_{k_{m-1}+1},\dots,u_{k_m=k})\}] \\
&= \phi_2\{\phi_2^{-1}(z_1) + \cdots + \phi_2^{-1}(z_m)\}
\end{aligned}$$

Let $\nu = \phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$. Then from Theorem A.2 of [11], $\chi_\alpha(u) = \exp\{-\alpha\nu(u)\}$ is the Laplace transform of some $M_\nu(\cdot;\alpha)$. This means that

$$\chi_\alpha(u) = \exp\{\alpha\phi_1^{-1} \circ \phi_2(u)\} = \int_0^\infty e^{-u\xi}\, dM_\nu(\xi,\alpha).$$

Similarly to the case of Archimedean copulas, C^1 and C^2 can be then transformed as follows

$$\begin{aligned}
C^1 &= \phi_1\{\phi_1^{-1}(z_1) + \cdots + \phi_1^{-1}(z_m)\} = \phi_1\{\nu \circ \phi_2^{-1}(z_1) + \cdots + \nu \circ \phi_2^{-1}(z_m)\} \\
&= \int_0^\infty e^{-\alpha \sum_{i=1}^m \nu\{\phi_2^{-1}(z_i)\}}\, dM_{\phi_1}(\alpha) = \int_0^\infty \prod_{i=1}^m \left[e^{-\alpha\nu\{\phi_2^{-1}(z_i)\}}\right] dM_{\phi_1}(\alpha) \\
&= \int_0^\infty \prod_{i=1}^m \left[\chi_\alpha\{\phi_2^{-1}(z_i)\}\right] dM_{\phi_1}(\alpha) \\
&= \int_0^\infty \prod_{i=1}^m \left[\int_0^\infty e^{-\gamma\phi_2^{-1}(z_i)}\, dM_\nu(\gamma,\alpha)\right] dM_{\phi_1}(\alpha) \\
&= \int_0^\infty \prod_{i=1}^m \left[\int_0^\infty G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha)\right] dM_{\phi_1}(\alpha) \\
&= \phi_1\left\{-\frac{1}{\alpha}\log \prod_{i=1}^m \int_0^\infty G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha)\right\}
\end{aligned}$$

$$\begin{aligned}
C^2 &= \phi_2\{\phi_2^{-1}(z_1)+\cdots+\phi_2^{-1}(z_m)\} = \phi_1\circ\nu\{\phi_2^{-1}(z_1)+\cdots+\phi_2^{-1}(z_m)\}\\
&= \int_0^\infty \exp[-\alpha\nu\{\phi_2^{-1}(z_i)+\cdots+\phi_2^{-1}(z_m)\}]dM_{\phi_1}(\alpha)\\
&= \int_0^\infty \chi_\alpha\left\{\phi_2^{-1}(z_i)+\cdots+\phi_2^{-1}(z_m)\right\}dM_{\phi_1}(\alpha)\\
&= \int_0^\infty\int_0^\infty e^{-\gamma\sum_{i=1}^m \phi_2^{-1}(z_i)}\,dM_\nu(\gamma,\alpha)dM_{\phi_1}(\alpha)\\
&= \int_0^\infty\int_0^\infty \prod_{i=1}^m e^{-\gamma\phi_2^{-1}(z_i)}\,dM_\nu(\gamma,\alpha)dM_{\phi_1}(\alpha)\\
&= \int_0^\infty\int_0^\infty \prod_{i=1}^m G^\gamma_{\phi_2^{-1}}(z_i)\,dM_\nu(\gamma,\alpha)dM_{\phi_1}(\alpha)\\
&= \phi_1\left[-\frac{1}{\alpha}\log\int_0^\infty\prod_{i=1}^m G^\gamma_{\phi_2^{-1}}(z_i)\,dM_\nu(\gamma,\alpha)\right].
\end{aligned}$$

Note that $\phi_1\{-a\log(\cdot)\}$ is decreasing since it is the composition of continuous monotone decreasing functions. Since the concordance order is invariant under monotone transformations, to prove (A.3) it suffices to show that

$$\prod_{i=1}^n\int_0^\infty G^\gamma_{\phi_2^{-1}}(z_i)\,dM_\nu(\gamma,\alpha) \le \int_0^\infty\prod_{i=1}^n G^\gamma_{\phi_2^{-1}}(z_i)\,dM_\nu(\gamma,\alpha).$$

For simplicity and to emphasise the argument with respect to which we integrate, we write

$$\prod_{i=1}^n\int_0^\infty g_i(\gamma)\,dM_\nu(\gamma,\alpha) \le \int_0^\infty\prod_{i=1}^n g_i(\gamma)\,dM_\nu(\gamma,\alpha),$$

where $g_i(\gamma) = G^\gamma_{\phi_2^{-1}}(z_i)$ are bounded and decreasing functions in $\gamma \ge 0$ because of the properties of $\exp\{\cdot\}$, while $G^\gamma_{\phi_2^{-1}}(z_i) = \exp\{-\gamma\phi_2^{-1}(z_i)\}$. To prove the inequalities we can use the same approach as in [11]: each bounded decreasing function can be represented as the limit of an infinite sum of piecewise constant functions $\sum_j c_j I_{[0,b_j]}$ for positive constants c_j and b_j. As both sides of the inequality are linear in each $g_i(\gamma)$, it suffices to prove the inequality for $g_i(\gamma) = I_{[0,y_j]}(\gamma)$, $j = 1,\ldots,k$. Suppose $B_1,\ldots,B_n \sim M_\nu(\cdot,\alpha)$ are iid for some fixed α. By the Fréchet upper bound inequality, $P\{B_j \le y_j,\ j = 1,\ldots,n\} \le P\{B_1 \le \min_j y_j\} = \min_j P\{B_j \le y_j\}$, which proves (A.3). This means that C^1 is less positively lower orphant dependent than C^2. To show the whole concordance order, we have to prove that C^1 is less positively upper orphant dependent than C^2. I.e., we need to prove the same inequality, but for the survival functions.

The usual representation of the survival copula is given by

$$\overline{C}(\mathbf{u}) = 1 + \sum_{s\in\mathcal{S}}(-1)^{|s|}C_s(u_j;\ j\in s).$$

In terms of the Laplace transforms the survival copula differs from the copula by taking $H_{\phi^{-1}}(u) = 1 - G_{\phi^{-1}}(u) = 1 - \exp\{-\phi^{-1}(u)\}$ instead of the function $G_{\phi^{-1}}(u)$. Let us denote $H_{\phi^{-1},\gamma}(u) = 1 - G^{\gamma}_{\phi^{-1}}(u)$. Moreover, all $z_i,\ i = 1,\dots,m$ are replaced by $\overline{z}_i,\ i = 1,\dots,m$ which correspond to the respective components of the survival copula. For example if

$$\begin{aligned}
z_1 &= \psi^{-1}\{C_{31}(u_1,u_2,u_3)\}\\
&= \psi^{-1}[C_{31}\{C_{21}(u_1,u_2),u_3\}]\\
&= \psi^{-1}\circ\phi[\phi^{-1}\circ\xi\{\xi^{-1}(u_1)+\xi^{-1}(u_2)\}+\phi^{-1}(u_3)]\\
G^{\eta}_{\phi^{-1}}(z_1) &= \int_0^\infty\int_0^\infty G^{\gamma}_{\xi^{-1}}(u_1)G^{\gamma}_{\xi^{-1}}(u_2)dM_{\phi^{-1}\circ\xi}(\gamma,\beta)G^{\beta}_{\phi^{-1}}(u_3)dM_{\psi^{-1}\circ\phi}(\beta,\eta),
\end{aligned}$$

and then the corresponding $\overline{z}_1$ is

$$\begin{aligned}
H_{\phi^{-1},\eta}(\overline{z}_1) &= 1 - G^{\eta}_{\phi^{-1}}(\overline{z}_1)\\
&= \int_0^\infty\int_0^\infty \{1-G^{\gamma}_{\xi^{-1}}(u_1)\}\{1-G^{\gamma}_{\xi^{-1}}(u_2)\}\\
&\qquad dM_{\phi^{-1}\circ\xi}(\gamma,\beta)\{1-G^{\beta}_{\phi^{-1}}(u_3)\}dM_{\psi^{-1}\circ\phi}(\beta,\eta)\\
&= \int_0^\infty\int_0^\infty H_{\xi^{-1},\gamma}(u_1)H_{\xi^{-1},\gamma}(u_2)\}\\
&\qquad dM_{\phi^{-1}\circ\xi}(\gamma,\beta)H_{\phi^{-1},\beta}(u_3)\}dM_{\psi^{-1}\circ\phi}(\beta,\eta).
\end{aligned}$$

Using similar transformation as in the case of positive lower orphant concordances, we have to prove the following inequality.

$$\prod_{i=1}^{m}\int_0^\infty H_{\phi_2^{-1},\gamma}(\overline{z}_i)\,dM_\nu(\gamma,\alpha) \le \int_0^\infty \prod_{i=1}^{m} H_{\phi_2^{-1},\gamma}(\overline{z}_i)\,dM_\nu(\gamma,\alpha) \tag{A.5}$$

or

$$\prod_{i=1}^{m}\int_0^\infty h_i(\gamma)\,dM_\nu(\gamma,\alpha) \le \int_0^\infty \prod_{i=1}^{m} h_i(\gamma)\,dM_\nu(\gamma,\alpha),$$

where $h_i(\gamma) = H_{\phi_2^{-1},\gamma}(\overline{z}_i) = 1 - G_{\phi_2^{-1},\gamma}(\overline{z}_i)$ are bounded increasing functions of $\gamma \ge 0$. Similarly to the case presented above, any increasing and bounded function can be approximated by a series $\sum_j c_j I_{[b_j,\infty)}$. It is sufficient to consider only one component of the sum. Similarly, taking $B_1,\dots,B_m \sim M_\nu(\cdot,\alpha)$ for some fixed α, the Fréchet upper bound inequality implies that $P\{B_j > y_j,\ j = 1,\dots,m\} \le P\{B_1 > \max_j y_j\} = \min_j P\{B_j > y_j\}$. This proves (A.4) and completes the proof of $C^1 \prec C^2$. □

Proof of Theorem 5.2: Consider $\boldsymbol{X}_1 = (X_{11},\dots,X_{1k})^\top,\dots,\boldsymbol{X}_n = (x_{n1},\dots,x_{nk})^\top$ as a random sample from $C(u_1,\dots,u_k)$, and then the $M_1 = \max\{X_{i1}\},\dots,M_k = \max\{X_{ik}\}$, follow the distribution

$$P\{M_1 \le x_1,\dots,M_k \le x_k\} = C_n(x_1,\dots,x_k),$$

where the copulae C_n have the same structure as C but are based on the generator functions $\phi_{\ell n} = [\phi_\ell(t)]^n$, $\ell = 1,\dots,r$, and inverse generator functions $\phi_{\ell n}^{-1} = \phi_\ell^{-1}(t^{1/n})$, $\ell = 1,\dots,r$. For example, in a three dimensional example with the copula function

$$\begin{aligned} C(u_1,u_2,u_3) &= C_1\{C_2(u_1,u_2),u_3\} \\ &= \phi_1[\phi_1^{-1}\circ\phi_2\{\phi_2^{-1}(u_1)+\phi_2^{-1}(u_2)\}+\phi_1^{-1}(u_3)], \end{aligned}$$

the extreme value copula is given by

$$\begin{aligned} &P\{M_1 \le x_1,\, M_2 \le x_2,\, M_3 \le x_3\} \\ &\quad = \big(\phi_1[\phi_1^{-1}\circ\phi_2\{\phi_2^{-1}(x_1)+\phi_2^{-1}(x_2)\}+\phi_1^{-1}(x_3)]\big)^n \\ &\quad = \phi_{1n}\{\phi_{1n}^{-1}\big([\phi_2\{\phi_2^{-1}(x_1)+\phi_2^{-1}(x_2)\}]^n\big)+\phi_{1n}^{-1}(x_3^n)\} \\ &\quad = \phi_{1n}[\phi_{1n}^{-1}\circ\phi_{2n}\{\phi_{2n}^{-1}(x_1^n)+\phi_{2n}^{-1}(x_2^n)\}+\phi_{1n}^{-1}(x_3^n)] = C_n(x_1,x_2,x_3). \end{aligned}$$

For the next step, we have to prove the existence of the limit of $C_n(x_1,\dots,x_k)$ when n tends to infinity. By mimicking [6], this limit exists if and only if there exists

$$\left.\frac{\partial[\phi_\ell^{-1}(t)/(\phi_\ell^{-1})'(t)]}{\partial t}\right|_{t=1}, \text{ where } \ell = 1,\dots,k.$$

Taking into account that the extreme-value distribution belong to its own domain of attraction, we have

$$C_n(x_1,\dots,x_k) = C(x_1,\dots,x_k), \text{ for } 0 < x_1,\dots,x_k < 1. \tag{A.6}$$

Let us fix some numbers $1 \le j_1 < j_2 \le k$. Then for $x_j = 1,\ j \in \{1,\dots,k\}\setminus\{j_1,j_2\}$, the copula $C(1,\dots,1,x_{j_1},1,\dots,1,x_{j_2},1,\dots,1)$ is a simple bivariate Archimedean copula with some generator function $\phi_{\ell_{12}}$, where $\ell_{12} \in \{1,\dots,k\}$

$$\begin{aligned} C(1,\dots,1,x_{j_1},1,\dots,1,x_{j_2},1,\dots,1) &= C_1(x_{j_1},x_{j_2}) \\ &= \phi_{\ell_{12}}^{-1}\{\phi_{\ell_{12}}(x_{j_1})+\phi_{\ell_{12}}(x_{j_2})\} \end{aligned}$$

such that

$$\begin{aligned} &C_n(1,\dots,1,x_{j_1},1,\dots,1,x_{j_2},1,\dots,1) \\ &\qquad = C(1,\dots,1,x_{j_1},1,\dots,1,x_{j_2},1,\dots,1). \end{aligned}$$

With [6] we get that ϕ_1 is the Gumbel generator. Similarly by taking all other possible pairs $1 \le j_1 < j_2 \le k$, it can be proved that the $\phi_\ell,\ \ell \in \{1,\dots,k\}$ are Gumbel generators. As given in the statement of the theorem that the extreme-value copula also belongs to the family of HAC, then using Proposition 1 allows us to finish the proof, because all the bivariate margins are uniquely determined by the previous steps of the proof. □

Proof of Theorem 5.3:

a) Let $0 < \lambda_0 < \infty$ and $0 < \lambda_{L,i} < \infty$ for all $i = 1,\dots,m$. Using the form of the HAC we obtain

$$\begin{aligned}\lambda_{L,0} &= \lim_{u\to 0^+} u^{-1}\phi_0\Big[\sum_{j=1}^{m}\phi_0^{-1}\big(C_j(uu_{k_{j-1}+1},\dots,uu_{k_j})\big) + \sum_{j=k_m+1}^{k}\phi_0^{-1}(u_j u)\Big]\\ &= \lim_{u\to 0^+} u^{-1}\phi_0\Big[\sum_{j=1}^{m}\phi_0^{-1}\left(u\frac{C_j(uu_{k_{j-1}+1},\dots,uu_{k_j})}{u}\right) + \sum_{j=k_m+1}^{k}\phi_0^{-1}(u_j u)\Big].\end{aligned}$$

We get that

$$\begin{aligned}\lambda_{L,0} &= \lim_{u\to 0^+} u^{-1}\phi_0\Big[\sum_{j=1}^{m}\phi_0^{-1}(\lambda_{L,j}u) + \sum_{j=k_m+1}^{k}\phi_0^{-1}(u_j u)\Big]\\ &= \lim_{u\to 0^+} u^{-1}\phi_0\Big[\phi_0^{-1}(u)\sum_{j=1}^{m}\phi_0^{-1}(\lambda_{L,j}u)/\phi_0^{-1}(u)\\ &\qquad + \phi_0^{-1}(u)\sum_{j=k_m+1}^{k}\phi_0^{-1}(u_j u)/\phi_0^{-1}(u)\Big]\\ &= \lim_{u\to 0^+} u^{-1}\phi_0\Big[\phi_0^{-1}(u)\Big(\sum_{j=1}^{m}\lambda_{L,j}^{-\lambda_0} + \sum_{j=k_m+1}^{k}u_j^{-\lambda_0}\Big)\Big]\\ &= \lim_{v\to\infty}\frac{\phi_0\Big[v\Big(\sum_{j=1}^{m}\lambda_{L,j}^{-\lambda_0} + \sum_{j=k_m+1}^{k}u_j^{-\lambda_0}\Big)\Big]}{\phi_0(v)}\\ &= \Big(\sum_{j=1}^{m}\lambda_{L,j}^{-\lambda_0} + \sum_{j=k_m+1}^{k}u_j^{-\lambda_0}\Big)^{-1/\lambda_0},\end{aligned}$$

where the last two equalities follow from the uniform convergence theorem.

b) Let $\lambda_0 = 0$, $0 < \lambda_{L,i} < 1$ for $i \in J \subset \{1,\dots,m\}$, and $\lambda_{L,i} = 0$ for $i \notin J$. Since ϕ is nonincreasing, for $a > 0$ the function $\phi_0\Big[\phi_0^{-1}(au)+b\Big]$ is nondecreasing in $u \in (0,1)$. Furthermore, for all $0 < \varepsilon < 1$ there exists a u^* such that $\max_{i\notin J} C_i(uu_{k_{i-1}+1},\dots,uu_{k_i})/u \le \varepsilon$ for all $0 < u < u^*$. Now let $K = \max\{\varepsilon, \lambda_{L,i}, i \in J, u_{k_m+1},\dots,u_k\}$. Because $K < 1$ and $\phi_0^{-1}(Ku) \ge \phi_0^{-1}(u)$, we get that

$$\begin{aligned}\lambda_{L,0} &\le \lim_{u\to 0^+} u^{-1}\phi_0\Big[\phi_0^{-1}(Ku)(m+k-k_m)\Big]\\ &= \lim_{v\to\infty}\phi_0\Big[v(m+k-k_m)\Big]/\phi_0(v) = 0\end{aligned}$$

since $k - k_m + m \ge 2$.

c) Let $\lambda_0 = \infty$ and $0 < \lambda_{L,i} < 1$. Then

$$\max_i C_i(uu_{k_{i-1}+1}, \dots, uu_{k_i})/u \le \lambda_{L,i} - \varepsilon$$

for all $u < u^*$ with $\min_i(\lambda_{L,i} - \varepsilon) > 0$. Let $m = \min\{u_{k_m+1}, \dots, u_k, \lambda_{L,1} - \varepsilon, \dots, \lambda_{L,m} - \varepsilon\}$. Then it follows that for u sufficiently small,

$$\frac{C(uu_1, \dots, uu_k)}{u} \ge \frac{\phi_0\Big[\phi_0^{-1}(um)(m + k - k_m)\Big]}{u} \ge \lambda m,$$

where $0 < \lambda < 1$ since $\phi \in RV_0(\infty)$. Moreover, we get with $M = \min\{u_{k_m+1}, \dots, u_k, \lambda_{L,1} + \varepsilon, \dots, \lambda_{L,m} + \varepsilon\}$ and $\max_i(\lambda_{L,i} + \varepsilon) < 1$ that for u sufficiently small,

$$\frac{C(uu_1, \dots, uu_k)}{u} \le \frac{\phi_0\Big[\sum_{j=1}^m \phi_0^{-1}(u(\lambda_{L,i} + \varepsilon)) + \sum_{j=k_m+1}^k \phi_0^{-1}(u_j u)\Big]}{u} \le M. \tag{A.7}$$

For λ increasing to 1 and ε decreasing to 0 the result follows.

d) Let $\lambda_0 = \infty$, $0 < \lambda_{L,i} < 1$ for $i \in J \subset \{1, \dots, m\}$, and $\lambda_{L,i} = 0$ for $i \notin J$. Note that (A.7) still holds for $M \le \varepsilon$ and thus the result follows.

e) Let $0 < \lambda_0 < \infty$, $0 < \lambda_{L,i} < 1$ for $i \in J \subset \{1, \dots, m\}$, and $\lambda_{L,i} = 0$ for $i \notin J$. Following the proof in part (a) and using the fact that $\lim_{a \to 0+}(a^{-1/p} + 1)^{-p} = 0$ for $p > 0$, we obtain $\lambda_{L,0} = 0$. □

Proof of Theorem 5.4: Relying on the structure of C, we perform the transformation

$$\lim_{u \to 0+} \frac{C(uu_1, \dots, uu_k)}{C^*(u)} = \lim_{u \to 0+} \frac{\phi_0\big[Z(u) + y(u)\psi[Z(u)]\big]}{\phi_0[Z(u)]}$$

with $Z(u) = \sum_{j=1}^m \phi_0^{-1}[C_j^*(u)] + \sum_{j=k_m+1}^k \phi_0^{-1}(u)$ and

$$y(u) = \sum_{j=1}^m \frac{\phi_0^{-1}\big[C_j(u_{k_{j-1}+1}u, \dots, u_{k_j}u)\big] - \phi_0^{-1}\big[C_j^*(u)\big]}{\psi[\phi_0^{-1}\{C_j^*(u)\}]} \cdot \frac{\psi[\phi_0^{-1}\{C_j^*(u)\}]}{\psi[Z(u)]} + \sum_{j=k_m+1}^k \frac{\phi_0^{-1}(u_j u) - \phi_0^{-1}(u)}{\psi[\phi_0^{-1}(u)]} \cdot \frac{\psi[\phi_0^{-1}(u)]}{\psi[Z(u)]}.$$

Next we establish the limits needed to evaluate the above expression.

a) Suppose that $\lambda^*_{L,i}(u_{k_{i-1}+1},\dots,u_{k_i})>0$ for all $i=1,\dots,m$.

a$_1$) First note that for any j,

$$\begin{aligned} I &= \lim_{u\to 0+}\frac{\phi_0^{-1}\big[C_j(u_{k_{j-1}+1}u,\dots,u_{k_j}u)\big]-\phi_0^{-1}\big[C_j^*(u)\big]}{\psi[\phi_0^{-1}\{C_j^*(u)\}]}\\ &= \lim_{u\to 0+}\frac{\phi_0^{-1}\big[\lambda^*_{L,j}(u,u_{k_{j-1}+1},\dots,u_{k_j})C_j^*(u)\big]-\phi_0^{-1}\big[C_j^*(u)\big]}{\psi[\phi_0^{-1}\{C_j^*(u)\}]}\\ &= \lim_{u\to 0+}\frac{1}{\psi[\phi_0^{-1}\{C_j^*(u)\}]}\int_{C_j^*(u)}^{\lambda^*_{L,j}(u,u_{k_{j-1}+1},\dots,u_{k_j})C_j^*(u)}[\phi_0^{-1}]'(s)ds.\end{aligned}$$

Since, from the definition of ψ,

$$\psi[\phi_0^{-1}\{C_j^*(u)\}]=-C_j^*(u)[\phi_0^{-1}]'(C_j^*(u)),$$

it follows that

$$\begin{aligned} I &= -\lim_{u\to 0+}\int_{C_j^*(u)}^{\lambda^*_{L,j}(u_{k_{j-1}+1},\dots,u_{k_j})C_j^*(u)}\frac{1}{C_j^*(u)}\frac{[\phi_0^{-1}]'(s)}{[\phi_0^{-1}]'(C_j^*(u))}ds\\ &= -\lim_{u\to 0+}\int_{1}^{\lambda^*_{L,j}(u_{k_{j-1}+1},\dots,u_{k_j})}\frac{1}{v}\frac{\psi[\phi_0^{-1}(vC_j^*(u))]}{\psi[\phi_0^{-1}(C_j^*(u))]}dv.\end{aligned}$$

Since ϕ_0^{-1} is slowly varying at zero and ψ is slowly varying at infinity, the composition $\psi\circ\phi_0^{-1}$ is also slowly varying at zero (see [3, Proposition 1.3.6]). Thus the ratio $\frac{\psi[\phi_0^{-1}(vC_j(u,\dots,u))]}{\psi[\phi_0^{-1}(C_j(u,\dots,u))]}$ tends to one. Therefore the last integral converges to

$$I=-\lim_{u\to 0+}\int_{1}^{\lambda^*_{L,j}(u_{k_{j-1}+1},\dots,u_{k_j})}\frac{1}{v}dv=-\log\{\lambda^*_{L,j}(u_{k_{j-1}+1},\dots,u_{k_j})\}. \quad \text{(A.8)}$$

a$_2$) Moreover, we obtain for the second limit

$$\begin{aligned} II &= \lim_{u\to 0+}\frac{\psi[\phi_0^{-1}\{C_j^*(u)\}]}{\psi[\sum_{j=1}^m\phi_0^{-1}\{C_j^*(u)\}+\sum_{j=k_m+1}^k\phi_0^{-1}(u)]}\\ &= \lim_{u\to 0+}\frac{\psi[\phi_0^{-1}\{u\lambda_j\}]}{\psi[\sum_{j=1}^m\phi_0^{-1}\{u\lambda_j\}+\sum_{j=k_m+1}^k\phi_0^{-1}(u)]}\\ &= (m+k-k_m)^{-\kappa},\end{aligned}$$

since

$$\lim_{u\to 0+}\frac{\phi_0^{-1}(u\lambda_j)/\phi_0^{-1}(u)}{\{\sum_{j=1}^m\phi_0^{-1}\{C_j^*(u)\}+\sum_{j=k_m+1}^k\phi_0^{-1}(u)\}/\phi_0^{-1}(u)}=\frac{1}{m+k-k_m}.$$

a_3) Using a_2) and the proof of Theorem 3.3 of [4], we obtain that

$$\lim_{u\to 0+}\sum_{j=k_m+1}^{k}\frac{\phi_0^{-1}(u_j u)-\phi_0^{-1}(u)}{\psi[\phi_0^{-1}(u)]}\cdot\frac{\psi[\phi_0^{-1}(u)]}{\psi[Z(u)]}$$
$$= -(m+k-k_m)^{-\kappa}\sum_{j=k_m+1}^{k}\log(u_j).$$

a_4) It follows, using (5.4) of [4], that

$$\lim_{u\to 0+}\frac{\phi_0\{Z(u)+y(u)\psi(Z(u))\}}{\phi_0\{Z(u)\}}=e^{-y}$$

if $\lim_{u\to 0+} y(u)=y$. In the present case,

$$\exp(-y)=\prod_{j=1}^{m}[\Lambda_{L,j}^{*}(u_{k_{j-1}+1},\dots,u_{k_j})]^{(m+k-k_m)^{-\kappa}}\prod_{j=k_m+1}^{k}u_j^{(m+k-k_m)^{-\kappa}}.$$

b) Suppose that $\lambda_{L,i}^{*}(u_{k_{i-1}+1},\dots,u_{k_i})=0$. Because it holds for $\varepsilon>0$ that for all u sufficiently small

$$I\ \geq\ \lim_{u\to 0+}\frac{1}{\psi[\phi_0^{-1}\{C_j^{*}(u)\}]}\int_{C_j^{*}(u)}^{\varepsilon C_j^{*}(u)}[\phi_0^{-1}]'(s)ds.$$

Following a_1) we get that $I\geq -log\varepsilon$. Thus $I=\infty$ and the result follows. □

Proof of Theorem 5.5:

a) First assume that $1\leq\gamma_0<\infty$.

$$\lambda_{U,0}\ =\ \lim_{u\to 0+}u^{-1}\Big(1-\phi_0\Big[\sum_{j=1}^{m}\phi_0^{-1}\big(C_j(1-uu_{k_{j-1}+1},\dots,1-uu_{k_j})\big)$$
$$+\sum_{j=k_m+1}^{k}\phi_0^{-1}(1-u_ju)\Big]\Big)$$
$$=\lim_{u\to 0+}u^{-1}\Big(1-\phi_0\Big[\sum_{j=1}^{m}\phi_0^{-1}\Big(1-\frac{1-C_j(1-uu_{k_{j-1}+1},\dots,1-uu_{k_j})}{u}u\Big)$$
$$\sum_{j=k_m+1}^{k}\phi_0^{-1}(1-u_ju)\Big]\Big).$$

We get that

$$\begin{aligned}
\lambda_{U,0} &= \lim_{u\to 0+} u^{-1}\Big(1-\phi_0\Big[\sum_{j=1}^{m}\phi_0^{-1}(1-\lambda_{U,j}u)+\sum_{j=k_m+1}^{k}\phi_0^{-1}(u_j u)\Big]\Big)\\
&= \lim_{u\to 0+} u^{-1}\Big(1-\phi_0\Big[\phi_0^{-1}(1-u)\sum_{j=1}^{m}\phi_0^{-1}(1-\lambda_{U,j}u)/\phi_0^{-1}(1-u)\\
&\qquad\qquad +\phi_0^{-1}(1-u)\sum_{j=k_m+1}^{k}\phi_0^{-1}(1-u_j u)/\phi_0^{-1}(1-u)\Big]\Big)\\
&= \lim_{u\to 0+} u^{-1}\Big(1-\phi_0\Big[\phi_0^{-1}(1-u)\Big(\sum_{j=1}^{m}\lambda_{U,j}^{\gamma_0}+\sum_{j=k_m+1}^{k}u_j^{\gamma_0}\Big)\Big]\Big)\\
&= \lim_{v\to\infty}\frac{1-\phi_0\Big[v\Big(\sum_{j=1}^{m}\lambda_{U,j}^{\gamma_0}+\sum_{j=k_m+1}^{k}u_j^{\gamma_0}\Big)\Big]}{1-\phi_0(v)}\\
&= \Big(\sum_{j=1}^{m}\lambda_{U,j}^{\gamma_0}+\sum_{j=k_m+1}^{k}u_j^{\gamma_0}\Big)^{1/\gamma_0},
\end{aligned}$$

where the last two equalities follow from the uniform convergence theorem.

b) The case $\gamma_0=\infty$ follows along the lines of the proof of Theorem 4.1 in [4]. □

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