Boundaries of the risk aversion coefficient: Should we invest in the global minimum variance portfolio?

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1. Introduction

The portfolio theory developed in [20] provides a simple and intuitive methodology for wealth allocation. In the next decades numerous generalizations were proposed, extending the classical theory to dynamic portfolio theory, considering predictive factors for asset returns, taking the estimation and model uncertainty into account (see [7] for a review). An alternative approach to the theory of Markowitz is based on the expected utility maximization. As shown in [13], the Makowitz's method is equivalent to maximizing the expected quadratic utility, assuming k-variate Gaussian asset returns with the mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. More formally let $\mathbf{w}$ denote the k-dimensional vector of optimal portfolio weights. We consider a portfolio problem without a risk-free asset, however, the short-sales are allowed. Then the optimal portfolio composition is determined by solving the maximization problem

$$\max_{\mathbf{w}} \mathbf{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \text{ s.t., } \mathbf{w}'\mathbf{1} = 1. \tag{1}$$

The solution to this problem is given by

$$\mathbf{w}_{EU} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-1}\mathbf{R}\boldsymbol{\mu}, \text{ with } \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{2}$$

The factor γ denotes the risk aversion coefficient and it measures the marginal reward for bearing an additional amount of risk. The choice or determining of the value of γ in practice is unclear. There are a few paper dealing with the estimation of the risk aversion coefficient from market data. [14] derives the implied absolute risk-aversion coefficient by estimating the risk-neutral and historical probabilities from option prices. Estimators relying on realized volatility were suggested by Bollerslev et al. [5]. It is important to note, that the corresponding risk aversion characterizes the aggregate and not an individual investor. The usual values of γ considered in empirical applications lie between 1 and 50. If γ tends to infinity the optimal portfolio weights tend to the weights of the GMV portfolio

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$$\mathbf{w}_{GMV} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{3}$$

The GMV portfolio appears to be extremely important and usually used as benchmark for measuring the performance of other portfolio strategies (see [8,9]).

In this paper we contribute to the existing literature by deriving the boundaries of the risk aversion coefficient. The portfolios on the efficient frontier, i.e., the solutions of (1), with higher risk aversions are statistically indistinguishable from the GMV portfolio. We consider the weights, the portfolio return and the variance of the portfolio return as characterizing quantities of the portfolios on the efficient frontier (EF). For each of them we provide a methodology for determining the set of risk aversion coefficients that lead to portfolios that are indistinguishable from the GMV portfolio. In other words, for each characteristic we provide such value of γ^*, that all portfolios on the EF with $\gamma \geqslant \gamma^*$ are not significantly different from the GMV portfolio. We show in the empirical study that the boundaries are relatively small, i.e., a large fractions of the efficient portfolios are statistically equivalent to the GMV portfolio. Thus under some circumstances, particularly under high estimation risk, it is redundant to know the risk aversion precisely.

The main drawback of the portfolio selection problem and the driving force of the paper are the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. The portfolio weights in (2) are nor plausible as long as we have no reliable values for the moments of asset returns. In practice we estimate them from a sample of size n of historical returns. This leads to a substantial estimation risk recognized since [17]. Assuming independent Gaussian returns, [12] obtained the distribution of the estimated variance of the mean-variance portfolio, while [22] derived the distributional properties of the estimated optimal portfolio weights and, thus, quantified the estimation risk. Kan and Zhou [16] analyze analytically the economic loss which arises due the estimation risk, while [6] introduce the estimation uncertainty into the optimization problem. To reduce the estimation risk, several papers suggested shrinkage estimation for the parameters. Particularly, [15] suggest a shrinkage estimator for mean vector and Boyle et al. [18] consider shrinkage estimators for the covariance matrix. Shrinkage of the portfolio weights was elaborated in [11,10,19]. Alternatively, Bayesian methods can be used to account for estimation uncertainty as in [23]. Note, however, that such refinements do not allow for explicit finite sample or even asymptotic statements. This stresses the need for improved estimators of portfolio weights [15,11]. The next section contains a detailed description of the methodology, while Section 3 provides an empirical illustration of the results.

2. Methodology

We consider the portfolio problem from the perspective of an individual investor. The aim is to maximize the quadratic utility given by (1) with respect to the optimal portfolio weights $\mathbf{w}$. Besides the weights, the resulting portfolio is characterized by the expected portfolio return μ_p and the variance of portfolio return V_p. For the given weights $\mathbf{w}_{EU}$ the latter two quantities we define as

$$\mu_{EU} = \mathbf{w}'_{EU}\boldsymbol{\mu} = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-1}\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}, \tag{4}$$

$$V_{EU} = \mathbf{w}'_{EU}\boldsymbol{\Sigma}\mathbf{w}_{EU} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-2}\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}. \tag{5}$$

Letting the risk aversion coefficient go to infinity we obtain the mean and variance of the return on the GMV portfolio and denote them by μ_{GMV} and V_{GMV}. The equation for the efficient frontier is given by

$$(\mu_p - \mu_{GMV})^2 = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}(V_p - V_{GMV}).$$

We observe that the whole efficient frontier can be uniquely characterized by its vertex with the coordinates μ_{GMV} and V_{GMV} and with the slope of the parabola given by $\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}$. The latter quantity is particularly important. If $\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}$ is small then the efficient frontier is very flat and collapses to a straight line if $\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}$ tends to zero. In this case, the only efficient portfolio is the GMV portfolio.

Practical implementation of the above solution requires feasible values for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and the risk aversion coefficient which uniquely characterizes the risk preferences of the investor. We estimate the moments of the Gaussian asset returns using a historical sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ by[1]

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j \text{ and } \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{X}_j - \hat{\boldsymbol{\mu}})(\mathbf{X}_j - \hat{\boldsymbol{\mu}})'. \tag{6}$$

We obtain the estimated portfolio weights and portfolio characteristics by replacing the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ with $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ respectively and mark them with hats.

Setting a reasonable risk aversion coefficient is, however, not straightforward. One possible solution is to rewrite the portfolio problem as a variance minimization problem with a given required return. Unfortunately, this framework does not reflect the risk attitude of the investor and stresses the interest in the portfolio return. We argue, however, that under some

[1] Alternatively, we can consider improved estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as in [15,18]. In this case, however, it is not possible to derive the required exact finite sample results.

circumstances due to the estimation risk in $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$, there is no need to provide the exact value for the risk aversion coefficient. The aim is to find such bound for the risk aversion coefficient, that all portfolios with higher risk aversion can be seen as equivalent to the GMV portfolio. To construct such bounds we need tests for equality of two portfolios on the efficient frontier. It is a difficult task in general. We consider four different approaches: (a) is the variance of the GMV portfolio different from the variance of the investor's portfolio? (b) is the expected return on the GMV portfolio different from the return of the investor's portfolio? (c) are both the portfolio return and the variance of the GMV portfolio different from the return and variance of the investor's portfolio? (d) are the weights of GMV portfolio different to the weights of the investor's portfolio?

To formalize it let $\boldsymbol{\ell}_i \in \mathbb{R}^k,\ i = 1,\dots,p,\ 1 \leqslant p \leqslant k-1$ and let $\mathbf{L}' = (\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_p)$. Let $\mathbf{w}_{L;p} = \mathbf{L}\mathbf{w}_{GMV}$ and $\hat{\mathbf{w}}_{L;p} = \mathbf{L}\hat{\mathbf{w}}_{GMV}$ correspondingly. We consider the tests

$$H_0:\ V_{GMV} = V_0 \text{ vs. } H_1:\ V_{GMV} \neq V_0, \tag{7a}$$
$$H_0:\ \mu_{GMV} = \mu_0 \text{ vs. } H_1:\ \mu_{GMV} \neq \mu_0, \tag{7b}$$
$$H_0:\ \mu_{GMV} = \mu_0,\quad V_{GMV} = V_0 \text{ vs. } H_1:\ \mu_{GMV} \neq \mu_0 \vee V_{GMV} \neq V_0, \tag{7c}$$
$$H_0:\ \mathbf{L}\mathbf{w}_{GMV} = \mathbf{r}_0 \text{ vs. } H_1:\ \mathbf{L}\mathbf{w}_{GMV} \neq \mathbf{r}_0. \tag{7d}$$

For each of the tests we can establish the corresponding confidence intervals for the characteristics of the GMV portfolio. The value of γ which corresponds to the portfolio on the boundary of the confidence interval we denote by γ^*. Thus, it is the smallest value of the risk aversion coefficient, such that the corresponding EU portfolio (a solution of (1)) is still equivalent to the GMV portfolio in terms of the measure in H_0. Only for risk aversion coefficients lower than γ^* we can speak about significant difference between EU and GMV portfolios. The distributions of the considered test statistics are derived in [1,3]. Note that we can also consider the one-sided variants of the tests in (7b 7c). This requires a minor modification of the methods discussed below.

(a) Method based on the portfolio variance.

The test of the hypothesis (7a) relies on the test statistic

$$T_V = (n-1)\frac{\hat{V}_{GMV}}{V_0}. \tag{8}$$

Under the H_0 hypothesis the test statistic T_V is χ^2-distributed with $n-k$ degrees of freedom. Substituting $V_0 = V_{EU}$ we get that the null hypothesis is rejected if $T_V \notin [\chi^2_{n-k;\alpha/2}; \chi^2_{n-k;1-\alpha/2}]$. This implies the following confidence interval for the portfolio variance

$$\left[\frac{(n-1)\hat{V}_{GMV}}{\chi^2_{n-k;1-\alpha/2}},\quad \frac{(n-1)\hat{V}_{GMV}}{\chi^2_{n-k;\alpha/2}}\right]. \tag{9}$$

Thus every portfolio with the variance from this interval is statistically indistinguishable from the GMV portfolio.
Taking the nature of our hypothesis into account, we observe that the variance of the EU portfolio can be only higher than the variance of the GMV portfolio. Thus, in our case only the lower bound of the rejection region is of relevance. This implies that

$$\gamma_V^{*2} = \frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}\chi^2_{n-k;\alpha/2}}{\hat{V}_{GMV}(n-1-\chi^2_{n-k;\alpha/2})}. \tag{10}$$

Thus for every investor with the risk aversion coefficient larger than γ^*, the expected utility portfolio is statistically indistinguishable from the GMV portfolio, if we take the portfolio variance as a proximity measure between portfolios.

(b) Method based on the expected portfolio return.

To test the null hypothesis that the expected return on the GMV portfolio μ_{GMV} equals the expected return on the EU portfolio $\mu_0 = \mu_{EU}$ we use the test statistic

$$T_\mu = \sqrt{n\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{\hat{\mu}_{GMV}-\mu_0}{\sqrt{1+\frac{n}{n-1}\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}}. \tag{11}$$

The test statistic follows t-distribution with $n-k$ degrees of freedom. Thus, the null hypothesis is rejected if $|T_\mu| > t_{n-k;1-\alpha/2}$. The corresponding confidence interval for the portfolio return is given by

$$\left[\hat{\mu}_{GMV} - \frac{\sqrt{n-1}}{\sqrt{n-k}}\sqrt{1+\frac{n}{n-1}\hat{s}}\frac{t_{n-k;1-\alpha/2}}{\sqrt{n\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}},\quad \hat{\mu}_{GMV} + \frac{\sqrt{n-1}}{\sqrt{n-k}}\sqrt{1+\frac{n}{n-1}\hat{s}}\frac{t_{n-k;1-\alpha/2}}{\sqrt{n\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}}\right].$$

Taking the upper part we obtain the boundary for the risk aversion coefficient

$$\gamma_\mu^* = \frac{\sqrt{n\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}}{t_{n-k;1-\alpha/2}}\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}{\sqrt{1+\frac{n}{n-1}\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}}. \tag{12}$$

As above, this implies that an arbitrary EU portfolio with the risk-aversion coefficient $\gamma > \gamma_\mu^*$ is statistically indistinguishable in terms of the expected portfolio return from the GMV portfolio.

(c) Method based both on the portfolio return and on the portfolio variance.

Next we consider the joint test for the expected return and the variance of the GMV portfolio. The test is based on the bivariate test statistic $T_{RV} = (T_V, T_R^*)'$ with T_V given in (8) and

$$T_R^* = \sqrt{n}\frac{\hat{\mu}_{GMV} - R_0}{\sqrt{V_0}\sqrt{1 + \frac{n}{n-1}\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}}. \tag{13}$$

It is easier to work with the statistics T_V and T_R^*, since they are independent, in contrary to T_V and T_R, which are dependent. This is the reason why we use T_R^* instead of T_R. It holds that T_R^* is standard normally distributed under H_0 hypothesis.
Let z_α be the α quantile of the standard normal distribution. The joint two-sided confidence interval for the expected return and the variance of the GMV portfolio with the boundaries is defined by (cf. [3])

$$(\mu_p - \hat{\mu}_{GMV})^2 \leqslant z^2_{1-\alpha^*/2}\left(\frac{1}{n} + \frac{\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}{n-1}\right)V_p, \tag{14}$$

$$V_p \in \left[\frac{(n-1)\hat{V}_{GMV}}{\chi^2_{n-k;1-\alpha^*/2}},\quad \frac{(n-1)\hat{V}_{GMV}}{\chi^2_{n-k;\alpha^*/2}}\right] = [\hat{V}_l,\ \hat{V}_u], \tag{15}$$

where $1 - \alpha = (1 - \alpha^*)^2$.
The computation of the investor's coefficient of risk aversion is based on the following idea. First, we find the coordinates of the intersection point between the confidence set (14, 15) and the sample efficient frontier

$$(\mu_p - \hat{\mu}_{GMV})^2 = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}(V_p - \hat{V}_{GMV}). \tag{16}$$

This is given by

$$\hat{V}^* = \frac{\hat{V}_{GMV}}{1 - z^2_{1-\alpha^*/2}\left(\frac{1}{n\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}} + \frac{1}{n-1}\right)} \tag{17}$$

provided that $\hat{V}^* > \hat{V}_{GMV}$. If the last inequality does not hold then the upper bound of (15), i.e., $\hat{V}_u$, has to be used. Further, the risk aversion coefficient should satisfy the inequality $\min\{\hat{V}^*, \hat{V}_u\} < \hat{V}_{GMV} + \gamma^{-2}\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}$ if $\hat{V}^* > \hat{V}_{GMV}$ and the inequality $\hat{V}_u < \hat{V}_{GMV} + \gamma^{-2}\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}$ otherwise. This leads to the following boundary value

$$\gamma^*_{R,V} = \begin{cases} \frac{\sqrt{\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}}{\sqrt{\min\{\hat{V}^*, \hat{V}_u\} - \hat{V}_{GMV}}} & \text{if } \hat{V}^* > \hat{V}_{GMV}, \\ \frac{\sqrt{\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}}}{\sqrt{\hat{V}_u - \hat{V}_{GMV}}} & \text{if } \hat{V}^* \leqslant \hat{V}_{GMV}. \end{cases} \tag{18}$$

(d) Method based on the weights

To test (7d) the following test statistic can be applied

$$T_w = \frac{n-k}{p}\left(\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\right)\left(\hat{\mathbf{w}}_{L;p} - \mathbf{r}_0\right)'\left(\mathbf{L}\hat{\mathbf{R}}\mathbf{L}'\right)^{-1}\left(\hat{\mathbf{w}}_{L;p} - \mathbf{r}_0\right). \tag{19}$$

Under H_0 it follows the F-distribution with p and $n-k$ degrees of freedom. To obtain a joint test that the first $k-1$ weights of the GMV portfolio are equal to the first $k-1$ weights of the EU portfolio we set $\mathbf{L}_{k-1} = (\mathbf{I}_{k-1}\mathbf{0})$ and $\mathbf{r}_0 = \mathbf{L}_{k-1}\mathbf{w}_{EU}$. Let $F_{p,q;\alpha}$ denote the α-quantile of the $F_{p,q}$-distribution. Then the hypothesis (7d) is rejected if and only if $T_w > F_{p,n-k;1-\alpha}$.
Estimating the unknown quantities and substituting into rejection criterion we obtain the following boundary value for the risk aversion coefficient.

$$\gamma_w^* = \frac{\sqrt{p}}{\sqrt{n-k}}\frac{\sqrt{F_{p,n-k;1-\alpha}}}{\sqrt{\left(\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\right)\left(\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\mathbf{L}'_{k-1}\left(\mathbf{L}_{k-1}\hat{\mathbf{R}}\mathbf{L}'_{k-1}\right)^{-1}\mathbf{L}_{k-1}\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}\right)}}. \tag{20}$$

This implies that an arbitrary EU portfolio with the risk-aversion coefficient $\gamma > \gamma_w^*$ is statistically indistinguishable in terms of the portfolio weights from the GMV portfolio with $\gamma = \infty$. We can also test each weight individually, however, in this case we obtain an individual lower bound γ^* for each asset. Such results would be, however, difficult to summarize. Moreover, it is difficult to monitor the overall type I error, especially, in the case of large portfolios.

The obtained results allow us to make rather general conclusions about the behavior of the boundaries. We verify these statements in the subsequent empirical example. Note that increasing boundary γ^* implies that a smaller fraction of the efficient frontier is statistically indistinguishable from the GMV portfolio. Therefore, it is important for the investor in this case to know his individual risk aversion coefficient precisely. A small bound implies that the majority of investors shall invest in the GMV portfolio without determining the precise individual risk aversion.

In all cases the boundary depends on the estimated slope $\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}$ of the efficient frontier. The smaller this parameter is, the flatter is the efficient frontier and the smaller is the boundary γ^*. Thus the slope $\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}$ can be seen as a generalized measure of estimation error in portfolio theory. Furthermore, it is a known fact in the literature that $\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}$ increases with the number of assets k. Thus increasing the universe of the assets will increase the boundary for the risk aversion coefficient. This appears to be counterintuitive, but reflects the diversification effects arising in large portfolios. Similarly, the sample size n decreases the estimation error and therefore, leads to an increase in the boundary γ^*.

The results obtained in this section are subject to the assumption of independent and identically distributed Gaussian returns. This assumptions are frequently not fulfilled. For example, the asset returns have heavy tails and exhibit weak, but sometimes significant autocorrelation. Similarly, the structural breaks in the parameters over time and poor data quality can deteriorate the results. We argue, however, that all such misspecifications increase the overall estimation error and thus may lead only to a decrease in the boundary γ^*. This implies that model misspecifications will only stress the results of the paper, which provides an optimistic framework for an investor.

It should be noted that the EU portfolio used in the above tests is assumed to be a known vector. This means that the investor compares her current (fixed) EU-optimal portfolio composition with the estimated GMV portfolio. A test with estimated EU portfolio weights is discussed in [2,4].

In the next section we apply the developed methodology to the real market data.

3. Empirical study

For the study we use weekly data on the 29 assets listed in the DJIA (Dow Jones Industrial Average) index and traded during the whole period from 22.02.1990 to 21.06.2012.[2] The number of assets we consider reflects the common setting in portfolio problems. Asset allocation problems with much larger number of assets usually rely on factor models (see [7]).

The estimation risk in the portfolio problem is visualized in Fig. 1. The figures show the simulated finite sample densities for the first two portfolio weights (first row), portfolio return and the standard deviation of the portfolio return. The asset returns are simulated from the multivariate normal distribution with the parameters set equal to the sample moments computed on the basis of the full sample. The sample size is set to $n = 100$ and the number of replications is 10^5. We observe that the standard errors of estimators are large, implying that the investor should be uncertain about the obtained characteristics of the portfolio. The standard errors in the portfolio weights are particularly high. This results in severe difficulties of using the portfolio weights for determining the boundaries for risk aversion. The estimation risk in the portfolio return is lower, followed by relatively precise estimators for the portfolio variance.

The results on the boundaries for the risk aversion coefficient based on full sample are summarized in Table 1 and visualized in Fig. 2. To assess the impact of the sample size we estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ using the whole sample and use these values to compute the boundaries γ^* with different sample sizes n by using (10) for the method based on the portfolio variance, (12) for the method based on the expected portfolio return, (18) for the method based both on the expected portfolio return and on the portfolio variance. For each value of the risk aversion coefficient we compute the corresponding value of the required return μ^*_{EU} and of the portfolio variance V^*_{EU} as given in (4) and (5) by replacing the population values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ with their full sample estimators. This results are summarized in Table 1. Note that the boundaries are low. For example, for $n = 60$ any investor with the risk aversion coefficient higher than 3.8955 can invest in the GMV portfolio, since the GMV portfolio is statistically indistinguishable from the corresponding EU portfolio if the variance is taken as a measure of proximity. In the portfolio return is taken as a measure of proximity between portfolios, then the boundary γ^*_μ is much higher due to higher estimation risk (see Merton [21]). Fig. 2 illustrates the parts of the efficient frontier which contain the portfolios which are indistinguishable from the GMV portfolio. Evidently, these portfolios constitute a large part of the frontier.

Next we analyse the impact of the number of assets k on the boundaries γ^* and the dynamics of the boundaries in time. We consider portfolios consisting of $k = 2, 5, 10$ and 20 risky assets. Since the choice of the assets is not unique, we sample randomly k assets out of 29 and generate 10^4 different portfolios. Note that we recompute the efficient frontier for each portfolio of size k and do not test the k-assets portfolio against the portfolios on the efficient frontier with 29 assets. This is a fair approach to assess the impact of k on the boundaries γ^*.

To shorten the presentation we restrict the discussion only to the method based on the variance of the portfolio (Paragraph (a)). For other methods the results are very similar. For each moment of time we estimate the mean returns and the covariance matrix using the last 100 observations. For every portfolio we compute the corresponding γ^*_V and their

[2] The distribution of monthly returns is closer to the Gaussian compared to the shorter term returns. But taking monthly data over longer periods of time may lead to biased results due to non-constant parameters. The daily data causes opposite problems. For this reason we opt for the weekly frequency, which is a trade-off between the two extremes.

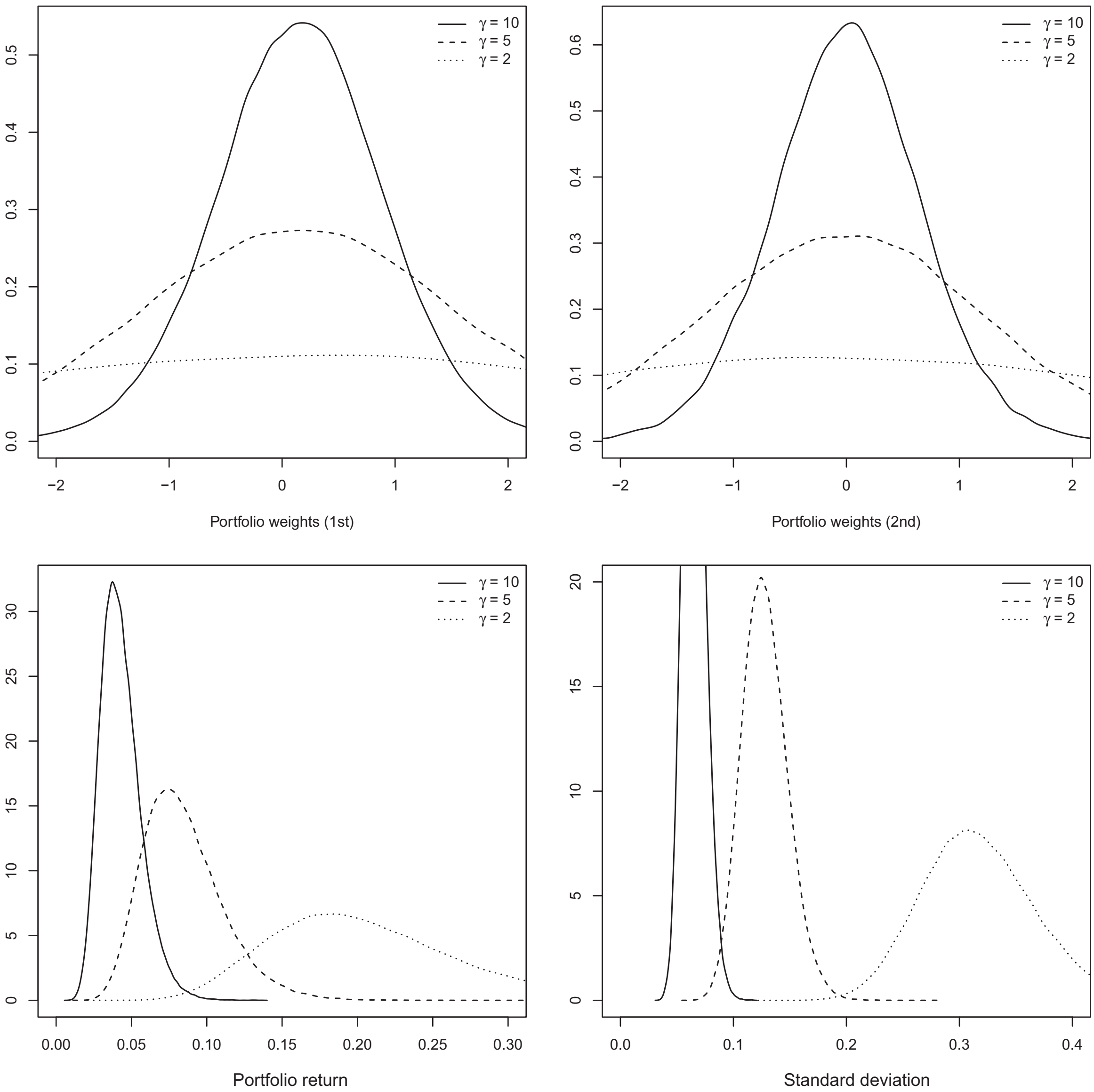

Fig. 1. Simulated finite sample densities for the first two portfolio weights (first row), portfolio return and the standard deviation of the portfolio return (second row) for the EU portfolio with given risk aversion coefficients. The sample size is set to $n = 100$ and the number of replications is 10^5.

Table 1
The boundaries for the risk aversion coefficient with the return and variance of the corresponding portfolios for different sample sizes n. The boundaries for the risk aversion coefficient in each panel are determined using the methods suggested in subsections 2a, 2b and 2c by formulaes (10), (12) and (18), respectively. The targeted values of μ^*_{EU} and V^*_{EU} are computed as in (4) and (5). For the GMV portfolio it holds $\hat{\mu}_{GMV} = 0.001781$ and $\hat{V}_{GMV} = 0.0003265$.

n		60	120	250
γ^*_V		3.8955	6.7410	9.8435
	μ^*_{EU}	0.004787	0.003518	0.002971
	V^*_{EU}	0.001117	0.0005892	0.0004492
γ^*_μ		1.7740	3.1078	4.8711
	μ^*_{EU}	0.008383	0.005550	0.004185
	V^*_{EU}	0.004787	0.003518	0.002971
$\gamma^*_{\mu,V}$		3.6698	6.4096	9.3608
	μ^*_{EU}	0.003518	0.002154	0.001878
	V^*_{EU}	0.001196	0.0006116	0.0004602

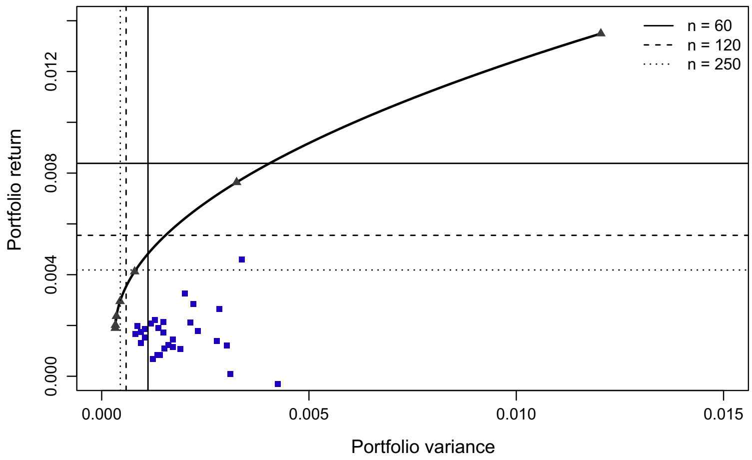

Fig. 2. The efficient frontier with portfolios obtained from boundary risk aversion coefficient γ^*_μ and γ^*_V for different sample sizes. The squares mark the 29 assets in the portfolio. The triangles depict the portfolios with the risk aversion coefficients 1, 2, 5, 10, 20, 50, 100. The intersections of the vertical lines with the efficient frontier correspond to the portfolios with the risk aversion γ^*_V; the intersections of horizontal lines correspond to the portfolios with the risk aversion γ^*_μ.

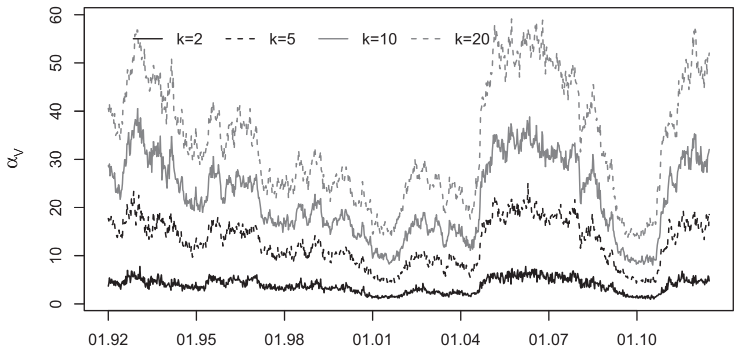

Fig. 3. Historical behavior of the medians of γ^*_V for 10^4 portfolios with $k = 2, 5, 10$ and 20 assets randomly chosen from the assets listed in DJIA.

median over all 10^4 portfolios. This procedure is repeated in each period of time and for each size k. The estimated medians are plotted in Fig. 3.

First, note that the bounds on the risk aversion coefficient are not constant over time. Starting in 1990 we observe a permanent downward trend which ends in 2001. The medians of the risk aversion coefficients lost around a half of their initial values. This implies that in 2000 a much larger part of the efficient frontier was indistinguishable from the GMV portfolio compared to 1993. A similar situation is observed in 2010 with a clear upward trend in the resent time. The periods of low boundaries closely coincide with periods of volatility bursts due to crises. In these periods the high market volatility leads to an increased estimation error, which pushes the boundary risk aversion downwards.

The impact of the portfolio size k on the value of γ^*_V is very strong. Large portfolios are much better diversified and, therefore, allow to construct a GMV portfolio with much lower variance as small portfolios. This implies that the bound on γ which separates significant and not significant EU portfolios should be higher for large portfolios. This intuition is also supported by the empirical results. For portfolios with two assets the average historical median is γ^*_V is 3.729. This value is lower than the usual values for the risk aversion coefficient considered in the literature. This implies that for small k's we cannot distinguish between the GMV portfolio and almost any EU portfolio. For $k = 5$ the situation is still unsatisfactory (the average median is 12.145). Only for larger portfolios with $k = 10$ and 20 we obtain the average medians equal to 20.606 and 31.598, respectively. This implies that only in the case of very large k the commonly taken values of the risk aversion coefficients would lead to portfolios, that are significantly different from the GMV portfolio in terms of the portfolio variance. By small portfolios minor diversification effects have stronger impact than the small estimation risk.

In Fig. 4 we provide the histograms of the medians for the last estimation period. For $k = 2$ we observe that a very large part of the portfolios possesses an extremely small bound γ^*_V. This causes the EU portfolios with small k to be very unreliable. With increasing portfolio size the distribution of γ^*_V shifts to the right, however, the difference in the location is obviously significant. The spread of the histograms is high. This implies that in general it is not possible to provide a unique value

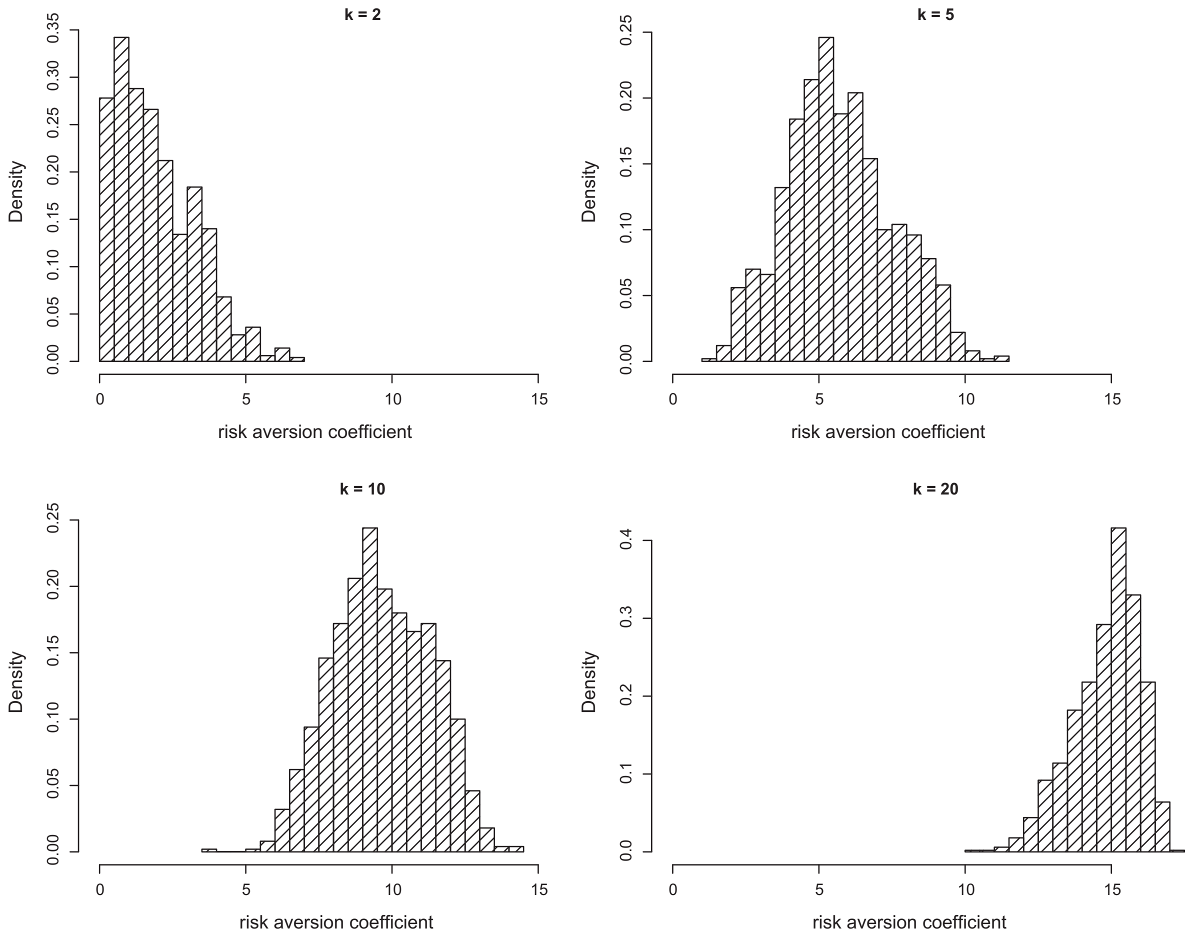

Fig. 4. Histograms of γ_V^* for 10^4 portfolios with $k = 2$, 5, 10 and 20 assets randomly chosen from the assets listed in DJIA.

of γ_V^* which can be used for arbitrary portfolio of given size. This stresses the importance of the methods discussed in the previous section. Nevertheless, there is wide range of the risk aversion coefficients which lead to EU portfolios that are equivalent to the GMV portfolios. This implies that investors with sufficiently large risk aversion can invest directly into the GMV portfolio. This eliminates the need for estimating expected returns and determining the individual risk aversion coefficient.

4. Summary

In this paper we analyze the impact of estimation risk on the portfolio decisions. Particularly, we provide such boundaries γ^*, that all EU portfolios with the risk aversion coefficient $\gamma > \gamma^*$ are statistically equivalent to the GMV portfolio in terms of (a) the expected portfolio return; (b) the portfolio variance; (c) both the portfolio return and variance; (d) the portfolio weights. The empirical study shows that the bounds γ^* increase with the number of assets in the portfolio and uncovers clear historical trends. In general, we conclude that investing in the GMV portfolio is statistically justified for investors with a very wide range of the risk aversion coefficients.

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