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Taras Bodnar, Yarema Okhrin

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TARAS BODNAR

Department of Statistics, European University Viadrina

YAREMA OKHRIN

Department of Statistics, University of Augsburg

ABSTRACT. In this article we analyse the product of the inverse Wishart matrix and a normal vector. We derive the explicit joint distribution of the components of the product. Furthermore, we suggest several exact tests of general linear hypothesis about the elements of the product. We illustrate the developed techniques on examples from discriminant analysis and from portfolio theory.

1. Introduction

The multivariate normal distribution is a standard assumption in many statistical applications. It results in the fact that many expressions and test statistics depend on the estimated mean vector $\hat{\mu}$ and the estimated covariance matrix $\hat{\Sigma}$ of a k-dimensional sample $X_1, \ldots, X_n$ drawn from $\mathcal{N}_k(\mu, \Sigma)$ (k-dimensional normal distribution with mean μ and covariance matrix Σ). We consider the classical unbiased estimators

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad \hat{\Sigma} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\mu})(X_i - \hat{\mu})'.$$

Under the assumption of normality it follows that $\hat{\mu} \sim \mathcal{N}(\mu, \Sigma/n)$ and $(n-1)\hat{\Sigma} \sim \mathcal{W}_k(\Sigma; n-1)$, where $\mathcal{W}_k(\cdot; n-1)$ denotes a k-dimensional Wishart distribution with $n-1$ degrees of freedom (Muirhead, 1982, section 3). There are numerous publications dealing either with the properties of $\hat{\mu}$ and or the properties of $\hat{\Sigma}$. Stein (1956) and Jorion (1986) discuss improvement techniques for the mean estimator. A minimum volume confidence region for the mean vector of the multivariate normal distribution is derived by Efron (2006). The distributional properties of the Wishart matrices, the inverse Wishart matrices and related quantities were discussed in detail by Bodnar & Okhrin (2008), Drton *et al.* (2008), Díaz-García *et al.* (1997), von Rosen (1988), Styan (1989) and others. Improved estimation techniques for covariance matrices and inverse covariance matrices were elaborated by Ledoit & Wolf (2004), Bodnar & Gupta (2010) and others. Despite this little has been done on the expressions involving both $\hat{\mu}$ and $\hat{\Sigma}$. Besides the classical result for the joint distribution as in Muirhead (1982), only results involving quadratic forms in $\hat{\mu}$ and $\hat{\Sigma}$ were considered, for example corollary 3.2.9 of Muirhead (1982) and Mathai & Provost (1992).

In this article we consider the expressions which depend on $A^{-1}z$ where A is a Wishart matrix and z is a Gaussian vector, which is independent on A. To our knowledge, this kind of expression has not been considered in the literature. First we derive the density function of $LA^{-1}z$ for an arbitrary matrix L and the characteristic function of $A^{-1}z$. If L is a vector,

then the density function has an integral representation in terms of the densities of χ^2, normal and non-central F distributions. Second, we develop tests of a general linear hypothesis $H_0 : \mathbf{L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \mathbf{r}$. Special attention is paid to the case when $\mathbf{L}$ is a vector. The tests developed for this case allow more insights into the structure of the elements of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$.

It appears that the vector $\mathbf{A}^{-1}\mathbf{z}$ plays a central role in many problems. Here we concentrate on two interesting applications. In the discriminant analysis we determine a linear combination of variables (discriminant function) which maximizes the discrepancy between two datasets. The coefficients of the linear combination are expressed as a product of inverse Wishart and normal distributions. Exact distribution and tests for the discriminant function allow us to assess the statistical significance of the individual variables for discrimination between two samples. The second application arises in the portfolio theory. Consider an investor who aims to allocate his wealth among k risky assets. The optimal vector of the portfolio weights, which maximizes the quadratic utility of the investor, is proportional to $\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$. Applying the results of this article we can test the significance of the investment in any particular asset and to construct confidence intervals for several characteristics of optimal portfolios. To our knowledge this is the first article which develops exact tests for the discriminant coefficients and an exact confidence interval for the weights of the optimal portfolio in the sense of maximizing the Sharpe ratio assuming Gaussian observations.

The article is structured as follows. We derive the density of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ and the characteristic function of $\mathbf{A}^{-1}\mathbf{z}$ in section 2. This section also contains the results on the exact multivariate test for $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$. The applications to the discriminant analysis and portfolio theory are discussed in section 3. The proofs are given in the Appendix.

2. Main results

Let $\mathbf{A} \sim W_k(\boldsymbol{\Sigma}; n)$, that is, $\mathbf{A}$ follows a k-dimensional Wishart distribution with n degrees of freedom and the parameter matrix $\boldsymbol{\Sigma}$. In the whole article we assume, that $n > k$, that is, the distribution of $\mathbf{A}$ is non-singular. Furthermore, let $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$. We assume that $\lambda > 0$ and $\boldsymbol{\Sigma}$ is positive definite. In theorem 1 we present the joint density of p linear combinations of the elements of the random vector $\mathbf{A}^{-1}\mathbf{z}$, that is, $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$, where $\mathbf{L}$ is a $p \times k$ matrix of constants. The proof of the theorem is non-trivial and exploits dimension-reduction techniques for integrals of quadratic and linear forms. Note that the direct integration is carried out over $\mathbb{R}^k$ and over the set of positive definite k-dimensional matrices. However, the theorem shows that this integral can be reduced to a four-dimensional integral for an arbitrary value of k. This integration can be carried out numerically with high precision and reasonable computational expanses.

Theorem 1. *Let $\mathbf{A} \sim W_k(\boldsymbol{\Sigma}; n)$ and $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$ with $\lambda > 0$ and $\boldsymbol{\Sigma}$ positive definite. Furthermore, let $\mathbf{A}$ and $\mathbf{z}$ be independent and $\mathbf{L}$ be a $p \times k$ matrix of constants.*

(a) *Let $n > k > p \geq 3$ and $k - p \geq 2$. Then the density of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ is given by*

$$f_{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}(\mathbf{y}) = \frac{\lambda^{-5/2}\pi^{-(p+1)/2}}{2^{(n-k+6)/2}\Gamma((n-k+1)/2)} \frac{\exp(-(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})/2\lambda)\det(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}}{\left(\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \dfrac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}\right)^{(p-3)/4} (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}})^{(k-p-2)/4}}$$

$$\times \frac{\Gamma((p+n-k+2)/2)}{\Gamma((n-k+2)/2)} \int_0^\infty \int_0^\infty \int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}} \frac{(w_{11} - w_{12}^2)^{(p-3)/4} w_{21}^{(k-p-4)/4}}{(w_{11} + w_{21})^{(p-1)/2}}$$

$$\times \exp\left(-\frac{w_{11} + w_{21}}{2\lambda} + \lambda^{-1}\frac{\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}}}{\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}w_{12}\right)$$

$$\times I_{(p-3)/2}\left(\lambda^{-1}\sqrt{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}-\frac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}\sqrt{w_{11}-w_{12}^2}\right)I_{(k-p-2)/2}\left(\lambda^{-1}\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}}\sqrt{w_{21}}\right)$$

$$\times\int_0^\infty s^{\frac{(n-k+1+2p)}{2}-1}\left(1+s^2\frac{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}(w_{21}+w_{12}^2)}{(w_{11}+w_{21})w_{21}}+\frac{w_{11}-2s\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}w_{12}}{w_{21}}\right)^{-(p+n-k+2)/2}$$

$$\times\exp\left(\frac{-s}{2}\right)\mathrm{d}s\,\mathrm{d}w_{12}\,\mathrm{d}w_{11}\,\mathrm{d}w_{21},$$

where $\tilde{\mathbf{y}}=(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}\mathbf{y}$ *and* $\tilde{\boldsymbol{\mu}}=(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}\mathbf{L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$. $I_q(\cdot)$ *stands for the modified Bessel function of the first kind as in Abramowitz & Stegun* (1972).

(b) *Let* $n>k>p=1$. *Then the density of* $\mathbf{l}'\mathbf{A}^{-1}\mathbf{z}$ *is given by*

$$f_{\hat{w}_l}(x)=\frac{n-k+2}{\lambda(k-1)}\int_0^\infty z f_{\chi^2_{n-k+1}}(z)\int_0^\infty f_{N(\mathbf{l}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu},\,(\lambda+y)\mathbf{l}'\boldsymbol{\Sigma}^{-1}\mathbf{l})}(xz)f_{F_{k-1,n-k+2,\frac{s}{\lambda}}}$$

$$\times\left(\frac{n-k+2}{\lambda(k-1)}y\right)\mathrm{d}y\,\mathrm{d}z,$$

where $s=\boldsymbol{\mu}'\mathbf{R}_1\boldsymbol{\mu}$ *and* $\mathbf{R}_1=\boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\mathbf{l}\mathbf{l}'\boldsymbol{\Sigma}^{-1}/\mathbf{l}'\boldsymbol{\Sigma}^{-1}\mathbf{l}$. $f_{\chi^2}, f_{\mathcal{N}_{(\cdot,\cdot)}}$ *and* $f_{F_{\cdot,\cdot,\cdot}}$ *denote the densities of* χ^2, *normal and non-central F distributions.*

The density for $p>1$ is a complicated expression with four integrals and generalized function. However, the univariate counterpart is simple. In the case $p=1$ the density can be easily evaluated using common mathematical software packages. The same refers to the characteristic function of $\mathbf{A}^{-1}\mathbf{z}$ provided in the next theorem. Here the expression appears to be even simpler.

Theorem 2. *Let* $\mathbf{A}\sim W_k(\boldsymbol{\Sigma};n)$, $n>k$ *and* $\mathbf{z}\sim\mathcal{N}_k(\boldsymbol{\mu},\lambda\boldsymbol{\Sigma})$. *We assumed that* $\lambda>0$ *and* $\boldsymbol{\Sigma}$ *is positive definite. Then the characteristic function of* $\mathbf{A}^{-1}\mathbf{z}$ *is given by*

$$\varphi_{\mathbf{A}^{-1}\mathbf{z}}(\mathbf{t})=\int_0^\infty\exp\left(i\frac{\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{z}-\frac{\lambda\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2z^2}\right)f_{\chi^2_{n-k+1}}(z)\varphi_{F_{k-1,\,n-k+2,\,\frac{s}{\lambda}}}\left(\frac{i\lambda(k-1)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2(n-k+2)z^2}\right)\mathrm{d}z,$$

where $\varphi_{F_{k-1,n-k+2,\frac{s}{\lambda}}}(\cdot)$ *denotes the characteristic function of the non-central F-distribution with* $k-1$ *and* $n-k+2$ *degrees of freedom and the non-centrality parameter* s/λ.

From Johnson *et al.* (1995, p. 483) the characteristic function of the non-central F-distribution can be presented as an infinite series. We obtain

$$\varphi_{F_{k-1,\,n-k+2,\,\frac{s}{\lambda}}}\left(\frac{i\lambda(k-1)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2(n-k+2)z^2}\right)=\exp\left(-\frac{s}{2\lambda}\right)\sum_{j=0}^\infty\frac{(s/2\lambda)^j}{j!}$$

$$\times{}_1F_1\left(\frac{k-1}{2}+j,\,-\frac{n-k+2}{2};\frac{\lambda\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2z^2}\right).$$

2.1. Multivariate tests with arbitrary p

The techniques developed in the proof of theorem 1 allows us also to construct tests for the components of the the vector $\mathbf{w}=\mathbf{L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$. We consider the test problem given by the hypotheses

$$H_0:\mathbf{w}=\mathbf{r}\quad\text{against}\quad H_1:\mathbf{w}\neq\mathbf{r},\tag{1}$$

where $\mathbf{r}\in\mathbb{R}^p$ is a vector of constants. If $\mathbf{A}$ and $\mathbf{z}$ are estimators for $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$, respectively, then a test statistic of the linear hypothesis can be given by

$$T = \frac{(\mathbf{L}\mathbf{A}^{-1}\mathbf{z} - \mathbf{r})'(\mathbf{L}\mathbf{R}_z\mathbf{L}')^{-1}(\mathbf{L}\mathbf{A}^{-1}\mathbf{z} - \mathbf{r})}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}, \tag{2}$$

where $\mathbf{R}_z = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{z}\mathbf{z}'\mathbf{A}^{-1}/\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}$. From the definition of $\mathbf{R}_z$ it holds that it is a $k \times k$ positive semidefinite matrix of rank $k-1$ with $\mathbf{R}_z\mathbf{z} = \mathbf{0}$. Hence, the matrix $\mathbf{L}\mathbf{R}_z\mathbf{L}'$ is positive definite iff the matrix $(\mathbf{L}, \mathbf{z})$ is of full rank $p+1$. The last statement holds with probability 1 because $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$ and, consequently, it is equal to a linear combination of the columns of $\mathbf{L}$ with probability 0. The test statistic (2) is a generalization of the multivariate test for the mean vector (Muirhead, 1982).

Let $_1F_1(a; b; x)$ be the confluent hypergeometric function of the first kind (Andrews et al., 2000, ch. 4), that is,

$$_1F_1(a; b; x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{\Gamma(b+i)} \frac{x^i}{i!}.$$

The density function of the test statistic (2) under H_0 is presented in theorem 3.

Theorem 3. *Let $n \geq k \geq 2$ and $k - p \geq 2$. Then the density of the T statistic under H_0 is:*

$$f_T(x) = \frac{\lambda^{-(p+2)/2}}{2^{(p+2)/2}\sqrt{\pi}} \frac{\exp(-(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})/2\lambda)}{\Gamma(1/2(p-1))(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\mathbf{r}}'\tilde{\mathbf{r}})^{(k-p-2)/4}} \int_0^{\infty} \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k+1}^2}(\xi)$$

$$\times \int_0^{\infty} \int_0^{\infty} \int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}} (w_{11} - w_{12}^2)^{(p-3)/2} \left(1 + \xi^2 \frac{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}} + \frac{w_{11} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{12}}{w_{21}}\right)^{-\frac{(n-k+p+1)}{2}}$$

$$\times {}_1F_1 \left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2} \frac{\xi^2 \frac{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}} + \frac{w_{11} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{12}}{w_{21}}}{1 + \xi^2 \frac{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}} + \frac{w_{11} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{12}}{w_{21}}} \right)$$

$$\times \exp\left(-\frac{(w_{11} + w_{21})}{2\lambda} + \frac{\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}}{\lambda}w_{12}\right) w_{21}^{(k-p-2)/4}$$

$$\times I_{(k-p-2)/2}(\lambda^{-1}\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\mathbf{r}}'\tilde{\mathbf{r}}}\sqrt{w_{21}}) \, dw_{12} \, dw_{11} \, dw_{21} \, d\xi.$$

The distribution is substantially simplified if $\mathbf{r} = \mathbf{0}$. For testing

$$H_0 : \mathbf{w} = \mathbf{0} \quad \text{against} \quad H_1 : \mathbf{w} \neq \mathbf{0} \tag{3}$$

we consider the test statistic which is a partial case of (2) and it is given by

$$\tilde{T} = \frac{n-k+1}{p} \frac{\mathbf{z}'\mathbf{A}^{-1}\mathbf{L}'(\mathbf{L}\hat{\mathbf{R}}_z\mathbf{L}')^{-1}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}. \tag{4}$$

The density function of the test statistic (4) under H_0 is presented in theorem 4. The formula is based on the Gauss hypergeometric function (cf. Andrews et al., 2000, ch. 4) which is defined as:

$$_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)\Gamma(b+i)}{\Gamma(c+i)} \frac{x^i}{i!}.$$

Theorem 4. *Let $n > k \geq 2$ and $k - p \geq 2$. Then the density of the $\tilde{T}$ statistic under H_0 is*

$$f_{\tilde{T}}(x) = \frac{p}{k-p} f_{F_{p,n-k+1}}(x) \int_0^\infty \left(1 + \frac{1}{z}\right)^{-(n-k+p+1)/2} f_{F_{k-p,p,\lambda^{-1}\mu'\Sigma^{-1}\mu}}\left(\frac{p}{k-p}z\right)$$

$$\times {}_2F_1\left(\frac{n-k+p+1}{2}, \frac{n-k+p+1}{2}; \frac{p}{2}; \frac{px}{n-k+1+px}\frac{1}{1+z}\right) \mathrm{d}z,$$

where $f_{F_{\cdot,\cdot}}$ denotes the density of F-distribution.

Note that in practice we are mainly interested in testing a single linear combination of the components of $\mathbf{w}$. This substantially simplifies the testing procedure. Moreover, for the case $p = 1$ we can construct several alternative test statistics, which allow more insights into the behaviour of the components of $\mathbf{w}$. In the next subsection we discuss these procedures in detail.

2.2. Tests with $p = 1$

In this particular case we consider the test problem of the form

$$H_0 : \mathbf{l}'\mathbf{w} = r_0 \quad \text{against} \quad H_1 : \mathbf{l}'\mathbf{w} = r_1 \neq r_0. \tag{5}$$

By setting $p = 1$ and $r_0 = 0$ in theorem 3 we obtain the distribution of the test statistics. However, the distribution has a complicated structure and, therefore, we suggest a computationally more effective procedure which also allows for interesting insights into the behaviour of $\mathbf{w}$. The next corollary gives a stochastic representation of $\mathbf{l}'\mathbf{A}^{-1}\mathbf{z}$.

Corollary 1. *Let $\mathbf{A} \sim W_k(\mathbf{\Sigma}; n)$ and $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$. Then*

$$\mathbf{l}'\mathbf{A}^{-1}\mathbf{z} \stackrel{d}{=} \frac{1}{u_1}\left(\mathbf{l}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \sqrt{\left(\lambda + \frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}}\, u_2\right),$$

where $u_1 \sim \chi^2_{n-k+1}$, $u_2 \sim \mathcal{N}(0,1)$ and $u_3 \sim F((k-1)/2, (n-k+2)/2, s/\lambda)$ with $s = \boldsymbol{\mu}'\mathbf{R}_1\boldsymbol{\mu}$. The random variables u_1, u_2 and u_3 are mutually independently distributed.

From the corollary it follows that to investigate the distributional properties of $\mathbf{l}'\mathbf{A}^{-1}\mathbf{z}$ in a Monte Carlo study it is sufficient to simulate only three random variables u_1, u_2 and u_3. Together with the parameters $\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}$, $\mathbf{l}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$ and $\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$ they fully specify the distribution of $\mathbf{l}'\mathbf{A}^{-1}\mathbf{z}$. This observation motivates a new test.

The idea behind the construction of the test statistic is based on the fact that under H_0

$$\frac{\mathbf{l}'\hat{\mathbf{w}}}{\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}} = \frac{\mathbf{l}'\mathbf{A}^{-1}\mathbf{z}}{\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}} \,\Big|\, \mathbf{z}'\hat{\mathbf{R}}_1\mathbf{z} = y \sim N\left(\frac{\mathbf{l}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}}, (\lambda + y)\frac{1}{\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}}\right). \tag{6}$$

Given $\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}$ it holds that under H_0

$$T_1^* = \sqrt{\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}}\,\frac{\left(\dfrac{\mathbf{l}'\hat{\mathbf{w}}}{(\mathbf{l}'\mathbf{A}^{-1}\mathbf{l})}\right) - \left(\dfrac{r_0}{(\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l})}\right)}{\sqrt{\lambda + \mathbf{z}'\hat{\mathbf{R}}_1\mathbf{z}}} \sim N(0,1). \tag{7}$$

To obtain an unconditional test, which does not depend on the parameter $\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}$, we consider the additional test given by

$$H_0 : \mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l} = v_0 \quad \text{against} \quad H_1 : \mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l} = v_1 \neq v_0 \tag{8}$$

with the test statistic

$$T_2 = \frac{\mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}}{\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}}. \tag{9}$$

Under H_0 the test statistics T_2 follows χ^2-distributed with $n - k + 1$ degrees of freedom. Combining the two tests into the joint two dimensional test, we get the test statistic $T^{(1)} = (T_1, T_2)'$ with

$$T_1 = \sqrt{v_0} \frac{\left(\frac{\mathbf{l}'\hat{\mathbf{w}}}{\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}}\right) - \left(\frac{r_0}{v_0}\right)}{\sqrt{\lambda + \mathbf{z}'\hat{\mathbf{R}}_{\mathbf{l}}\mathbf{z}}} \tag{10}$$

and with the hypotheses

$$H_0 : \mathbf{l}'\mathbf{w} = r_0, \mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l} = v_0 \quad \text{against} \quad H_1 : \mathbf{l}'\mathbf{w}_{TP} = r_1 \neq r_0 \text{ or } \mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l} = v_1 \neq v_0. \tag{11}$$

Bodnar & Schmid (2009) considered the partial case of $T^{(1)}$ with $\mathbf{l} = \mathbf{1}$ for constructing a joint test for the expected return and the variance of the global minimum variance portfolio. Here, we adopt their results. First, we note that the test statistics T_1 and T_2 are independently distributed. The next theorem provides their exact joint distribution and the distribution of the test statistics $T^{(1)}$ of the joint test.

Theorem 5. *Let* $\mathbf{A} \sim W_k(\mathbf{\Sigma}; n)$ *and* $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$.

(a) *The density of* $T^{(1)}$ *is given by*

$$f_{T^{(1)}}(x,z) = \frac{n-k+2}{\lambda(k-1)} \eta^{-1} f_{\chi^2_{n-k+1}}\left(\frac{z}{\eta}\right) \int_0^\infty f_{N(\sqrt{\eta}\delta(y),\eta)}(x) f_{F_{k-1,n-k+2,s/\lambda}}\left(\frac{n-k+2}{\lambda(k-1)}y\right) dy \tag{12}$$

with $\delta(y) = \lambda_1/\sqrt{\lambda+y}$, $\eta = v_1/v_0$, $\lambda_1 = (r_1/v_1 - r_0/v_0)/\sqrt{v_1}$ *and* $s = \boldsymbol{\mu}'\mathbf{R}_{\mathbf{l}}\boldsymbol{\mu}$.

(b) *Under the null hypothesis it holds that* $T_1 \sim N(0,1)$ *and* $T_2 \sim \chi^2_{n-k+1}$. *The joint density of* T_1 *and* T_2, *that is, the density of* $T^{(1)}$, *under* H_0 *is given by*

$$f_{T^{(1)}}(x,z) = \frac{1}{\sqrt{2\pi}} \frac{2^{-(n-k+1)/2}}{\Gamma((n-k+1)/2)} \exp\left(-\frac{z}{2}\right) z^{\frac{n-k+1}{2}-1} \exp\left(-\frac{x^2}{2}\right). \tag{13}$$

For determining the confidence set for $r = \mathbf{l}'\mathbf{w}$ and $v = \mathbf{l}'\mathbf{\Sigma}^{-1}\mathbf{l}$ we make use of the result of theorem 5b. A joint two-sided confidence interval is given by

$$r \in \left[\frac{\mathbf{l}'\hat{\mathbf{w}}}{\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}}v - z_{1-\tilde{\alpha}/2}\sqrt{\lambda + \mathbf{z}'\hat{\mathbf{R}}_{\mathbf{l}}\mathbf{z}}\sqrt{v}, \frac{\mathbf{l}'\hat{\mathbf{w}}}{\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}}v + z_{1-\tilde{\alpha}/2}\sqrt{\lambda + \mathbf{z}'\hat{\mathbf{R}}_{\mathbf{l}}\mathbf{z}}\sqrt{v}\right], \tag{14}$$

$$v \in [\chi^2_{n-k+1;\tilde{\alpha}/2}\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}, \chi^2_{n-k+1;1-\tilde{\alpha}/2}\mathbf{l}'\mathbf{A}^{-1}\mathbf{l}]. \tag{15}$$

The symbol z_β stands for the β-quantile of the standard normal distribution $N(0,1)$, $\chi^2_{p;\beta}$ is the β-quantile of the χ^2-distribution with p degrees of freedom, and $(1-\tilde{\alpha})^2 = 1 - \alpha$.

Note that the theorem provides the distribution of the test statistics not only under the null hypotheses, but also under the alternative. This allows us to compute the power of the test explicitly. Using (12) it is equal to

$$G_{T^{(1)};\tilde{\alpha}}(\eta, \lambda_1, s) = 1 - \frac{n-k+2}{\lambda(k-1)}(1 - G_{T_2;\tilde{\alpha}}(\eta)) \int_{z_{\tilde{\alpha}/2}}^{z_{1-\tilde{\alpha}/2}} \int_0^\infty f_{N(\sqrt{\eta}\delta(y),\eta)}(x) f_{F_{k-1,n-k+2,s/\lambda}}$$

$$\times \left(\frac{n-k+2}{(k-1)\lambda}y\right) dy\,dx,$$

where $G_{T_2;\alpha}(\eta) = 1 - F_{\chi^2_{n-k+1}}(\chi^2_{n-k+1;1-\alpha/2}/\eta) + F_{\chi^2_{n-k+1}}(\chi^2_{n-k+1;\alpha/2}/\eta)$. The power function depends on the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only via the parameters η, λ_1 and s.

2.3. Tests with $p = 1$ and $\mathbf{r} = \mathbf{0}$.

A particularly simple result can be obtained in the case $r_0 = 0$. The hypotheses of the test are given by

$$H_0 : \mathbf{l'w} = 0 \quad \text{against} \quad H_1 : \mathbf{l'w} = r_1 \neq 0. \tag{16}$$

Similarly as in the previous subsection we consider the test statistics

$$T^{(2)} = \sqrt{n-k+1} \frac{\mathbf{l'A}^{-1}\mathbf{z}}{\sqrt{\mathbf{l'A}^{-1}\mathbf{l}}\sqrt{\lambda + \mathbf{z'}\hat{\mathbf{R}}_1\mathbf{z}}}. \tag{17}$$

Theorem 6 provides the distribution of $T^{(2)}$ both the null and the alternative hypotheses.

Theorem 6. *Let* $\mathbf{A} \sim W_k(\boldsymbol{\Sigma}; n)$ *and* $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$.

(a) *The density of* $T^{(2)}$ *is given by*

$$f_{T^{(2)}}(x) = \frac{n-k+2}{\lambda(k-1)} \int_0^\infty f_{t_{n-k+1,\delta_1(y)}}(x) f_{F_{k-1,n-k+2,s/\lambda}}\left(\frac{n-k+2}{\lambda(k-1)}y\right) \mathrm{d}y \tag{18}$$

with $\delta_1(y) = \lambda_1/\sqrt{\lambda + y}$, $\lambda_1 = \dfrac{\mathbf{l'\Sigma}^{-1}\boldsymbol{\mu}}{\sqrt{\mathbf{l'\Sigma}^{-1}\mathbf{l}}}$, *and* $s = \boldsymbol{\mu'}\mathbf{R}_{l_1}\boldsymbol{\mu}$.

(b) *Under the null hypothesis it holds that* $T^{(2)} \sim t_{n-k+1}$ *and* $T^{(2)}$ *is independent of* $\mathbf{z'}\hat{\mathbf{R}}_1\mathbf{z}$.

The proof follows from the proof of proposition 1 of Bodnar & Schmid (2009). Note that we obtain a standard distribution under the null. This implies that the test is particularly easy to implement.

As a result of the simplicity of the distribution under H_0 it is particularly easy to construct one-sided test using the $T^{(2)}$ statistic. The testing hypotheses are given by

$$H_0 : \mathbf{l'\Sigma}^{-1}\boldsymbol{\mu} \leq 0 \quad \text{against} \quad H_1 : \mathbf{l'\Sigma}^{-1}\boldsymbol{\mu} > 0. \tag{19}$$

For example, if we put $\mathbf{l'} = (1, -1, 0, \ldots, 0)$ the rejection of the null hypothesis means that the first element of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ is significantly larger than the second component of this vector. To run the test we compare $T^{(2)}$ with the quantile of the t-distribution.

3. Applications

3.1. Applications to discriminant analysis

The test (3) can be directly applied in the discriminant analysis to test the significance of the population analogues of the coefficients in the discriminant function. The aim of the discriminant analysis is twofold. First, to model the separation of two or more groups of multivariate observations and to identify the relative contribution of each variable to separation procedure. Second, the discriminant analysis provides tools for predicting the allocation of new observations to the groups. Formally, we look for a linear combination of the variables, such that the standardized distance between the groups of observations, measured using the combination, is maximized.

Let $\mathbf{X}_1^{(1)}, \ldots, \mathbf{X}_{n_1}^{(1)}$ and $\mathbf{X}_1^{(2)}, \ldots, \mathbf{X}_{n_2}^{(2)}$ denote two groups of k-dimensional observations. Let $\mathbf{a}$ be a constant vector and $z_i^{(1)} = \mathbf{a'}\mathbf{X}_i^{(1)}$ for $i = 1, \ldots, n_1$ and similarly $z_i^{(2)} = \mathbf{a'}\mathbf{X}_i^{(2)}$ for $i = 1, \ldots, n_2$.

The respective means we denote by $\bar{z}^{(j)} = \mathbf{a}'\bar{\boldsymbol{\mu}}^{(j)}$ for $j = 1, 2$. We maximize the standardized differences $(\bar{z}^{(1)} - \bar{z}^{(2)})/\sigma_z$. The maximum is achieved at $\hat{\mathbf{a}} = \hat{\boldsymbol{\Sigma}}_{pl}^{-1}(\hat{\boldsymbol{\mu}}^{(1)} - \hat{\boldsymbol{\mu}}^{(2)})$ with the pooled covariance matrix given by $\hat{\boldsymbol{\Sigma}}_{pl} = 1/(n_1 + n_2 - 2)((n_1 - 1)\hat{\boldsymbol{\Sigma}}^{(1)} + (n_2 - 1)\hat{\boldsymbol{\Sigma}}^{(2)})$.

The analysis of the discriminant function given by $\hat{\mathbf{a}}$ is directly linked to the measurement of the relative impact of each variable. Unfortunately, the method available in the literature does not take into account the estimation error in $\hat{\mathbf{a}}$. Rencher (2002) reviews three approaches to the interpretation of the discriminant function. The first method it to compare the standardized coefficients. This, however, is equivalent to direct comparison of realizations of random variables and does not take the stochastic nature of $\hat{\mathbf{a}}$ into account. The second suggests to compare the partial F-statistics, which measure the individual impact of each variable, while keeping the impact of other variables eliminated. The third method is based on the correlation between the variables and the discriminant function. As argued by Rencher (2002) this is equivalent to considering the contribution of each variable in a univariate framework. It is clear that these methods do provide some interesting insights into the discriminant function, but they clearly lack statistical rigidity.

The results of this article allow us to develop arbitrary individual, joint, one- or two-sided tests for the population counterparts of the coefficients $\hat{\mathbf{a}}$. Furthermore the tests are exact, assuming that the underlying sample comes from a normal distribution. Note that $(n_1 + n_2 - 2)\hat{\boldsymbol{\Sigma}}_{pl} \sim W_k(n_1 + n_2 - 2, \boldsymbol{\Sigma})$ and $\hat{\boldsymbol{\mu}}_1 - \hat{\boldsymbol{\mu}}_2 \sim \mathcal{N}_k(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}, (n_1^{-1} + n_2^{-1})\boldsymbol{\Sigma})$. Moreover $\hat{\boldsymbol{\mu}}_1 - \hat{\boldsymbol{\mu}}_2$ and $\hat{\boldsymbol{\Sigma}}_{pl}$ are independent. This implies that the theory developed in section 2 can be directly applied to the components of $\hat{\mathbf{a}}$ by replacing $\mathbf{z}$ with $\hat{\boldsymbol{\mu}}_1 - \hat{\boldsymbol{\mu}}_2$ and taking into account the new constants in the distributions of the estimated parameters. From theorem 1 we obtain the multivariate distribution of $\hat{\mathbf{a}}$ or an arbitrary linear combinations of its components. This allows us to construct exact confidence intervals for individual coefficients or their linear combination. Using theorems 3 and 4 we can run an exact test to verify the hypothesis that two elements of $\hat{\mathbf{a}}$ coincide, that is, if two variables have the same relative impact on the discriminant function. Similarly using theorem 6 we can run an exact test to verify if the impact of one of the variables is higher than that of another variable.

3.2. Applications to portfolio theory

The mean-variance portfolio theory of Markowitz (1952) is a classical approach to asset allocation. The investor allocates his wealth among k risky assets by maximizing the expected return subject to given level of the risk or by minimizing the risk given some predetermined level of expected return. The risk is usually measured by the variance of the portfolio return. Assuming that the asset returns follow normal distribution, $\mathbf{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ the mean-variance portfolio problem is equivalent to maximizing the expected quadratic utility. The investor determines the fractions of wealth allocated to each asset, that is, the portfolio weights $\mathbf{w}$ by maximizing the expected quadratic utility $\mu_p - \frac{\alpha}{2}\sigma_p^2$, where μ_p and σ_p^2 are portfolio return and variance, respectively. α denotes the risk aversion of the investor. If short-selling is allowed and a riskfree asset with return r_f is available then the return on risky assets is given by $\mu_p = \mathbf{w}'(\boldsymbol{\mu} - r_f \mathbf{1}) + r_f$ with the variance $\sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$. Maximizing the utility leads to the tangency portfolio weights

$$\mathbf{w}_{\mathrm{TP}} = \alpha^{-1}\boldsymbol{\Sigma}(\boldsymbol{\mu} - r_f \mathbf{1}). \tag{20}$$

This portfolio lies on the intersection of the mean-variance efficient frontier and the tangency line drawn from the portfolio consisting only of the riskless asset (Ingersoll, 1987).

Another important financial criteria in asset allocation is the Sharpe ratio defined as μ_p/σ_p. It measures the return of the investment per unit of risk given by the standard deviation of

the portfolio return. By maximizing the Sharpe ratio we obtain the Sharpe ratio optimal portfolio weights

$$\mathbf{w}_{\mathrm{SR}} = \frac{\Sigma^{-1}\mu}{\mathbf{1}'\Sigma^{-1}\mu}. \tag{21}$$

In practice the parameters of the distribution of the asset returns are unknown and must be estimated form historical data using $\hat{\mu}$ and $\hat{\Sigma}$. The sample counterparts of the portfolio weights are given by

$$\hat{\mathbf{w}}_{\mathrm{SR}} = \frac{\hat{\Sigma}^{-1}\hat{\mu}}{\mathbf{1}'\hat{\Sigma}^{-1}\mu} \quad \text{and} \quad \hat{\mathbf{w}}_{\mathrm{TP}} = \alpha^{-1}\hat{\Sigma}^{-1}(\hat{\mu} - \mathbf{1}r_f). \tag{22}$$

Furthermore, both types of the portfolio weights are directly linked to the product of inverse Wishart matrix and a Gaussian vector and fall into the framework of the modelling in section 2. For the tangency portfolio this is obvious. For the Sharpe ratio optimal portfolio note that $\mathbf{l}'\hat{\Sigma}^{-1}\hat{\mu}/\mathbf{l}'\hat{\Sigma}^{-1}\mu = r_0$ is equivalent to $(\mathbf{1} - r_0\mathbf{1})'\hat{\Sigma}^{-1}\hat{\mu} = 0$. Okhrin & Schmid (2006) proved that the estimator for the weights of the Sharpe ratio portfolio does not possess the moments of order larger than 1. Thus, the established results provide the explicit distribution of the considered portfolio wights and allow us to construct exact tests for the individual weights or their linear combinations.

Next we apply the theory from the previous section to real data. We consider 200 monthly observations on Morgan Stanley Capital International (MSCI) index returns for United Kingdom, Germany, France, the Netherlands, United States, Canada, Japan, Italy, Spain, Switzerland for the period ending in April 2004. The risk aversion coefficient is set to 10. The estimators of the mean vector and covariance matrix are denoted by $\hat{\mu}$ and $\hat{\Sigma}$, respectively. The estimated tangency and Sharpe ratio portfolio weights are given by

$$\hat{\mathbf{w}}_{\mathrm{TP}} = (-0.14, -0.239, 0.143, 0.053, 0.376, -0.156, 0.016, 0.001, -0.079, 0.333),$$
$$\hat{\mathbf{w}}_{\mathrm{SR}} = (-0.454, -0.776, 0.464, 0.171, 1.223, -0.506, 0.051, 0.003, -0.257, 1.081).$$

Since many portfolio weights for both portfolios are close to zero or negative, we wish to run individual tests to verify if these weights are equal to zero ($r_0 = 0$). In practice this would reduce the size of the portfolio and the transaction costs. We restrict the discussion only to the Sharpe ratio optimal portfolios.

The test hypotheses are given by

$$H_0 : \mathbf{l}'\mathbf{w}_{\mathrm{SR}} = r_0 \quad \text{against} \quad H_1 : \mathbf{l}'\mathbf{w}_{\mathrm{SR}} = r_1 \neq r_0.$$

The hypotheses can be rewritten as:

$$H_0 : \mathbf{l}_1'\Sigma^{-1}\mu = (\mathbf{1} - r_0\mathbf{1})'\Sigma^{-1}\mu = \mathbf{0} \quad \text{against} \quad \mathbf{H}_1 : \mathbf{l}_1'\Sigma^{-1}\mu \neq \mathbf{0}. \tag{23}$$

Note that the bivariate test can be applied in the most general framework for non-zero values of the statistic of interest under zero hypothesis. In our particular example we can rely on the simplified version of the test given in section 2.3. The test statistic adopted to the example at hand is given by

$$T_{\mathrm{SR}} = \sqrt{n}\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{(\mathbf{1}-r_0\mathbf{1})'\hat{\Sigma}^{-1}\hat{\mu}}{\sqrt{(\mathbf{1}-r_0\mathbf{1})'\hat{\Sigma}^{-1}(\mathbf{1}-r_0\mathbf{1})}\sqrt{1+(n/(n-1))\hat{\mu}'\hat{\mathbf{R}}_{\mathbf{l}_1}\hat{\mu}}}, \tag{24}$$

which is t-distribution with $n-k$ degrees of freedom under H_0. For the real data we obtain that among 10 assets the largest test statistics is equal to 1.4819 (Swiss market) and the

smallest is -1.198 (Canadian market). Thus, we are unable to prove the statistical significance of the portfolio weights for all assets. This situation is common and implies that the classical approach to the estimation of the portfolio weights leads to very unprecise results, despite its popularity. Next we consider the confidence intervals arising from the test and provide insights into the reasons of the weak performance of the test.

Let $t_{p;\beta}$ denote the β-quantile of the central t-distribution with p degrees of freedom. Then the two-sided $(1 - \alpha)$-confidence interval for $r = \mathbf{l}'\mathbf{w}_{SR}$ is evaluated as:

$$\sqrt{n}\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{(\mathbf{l}-r\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\sqrt{(\mathbf{l}-r\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\mathbf{l}-r\mathbf{1})}\sqrt{1+(n/(n-1))\hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}_{\mathbf{l}_1}\hat{\boldsymbol{\mu}}}} \leq t_{n-k;1-\alpha/2}. \tag{25}$$

Let $K = t_{n-k;1-\alpha/2}\sqrt{n-1}/\sqrt{n}\sqrt{n-k}$ and

$$\hat{\mathbf{H}} = (\hat{h}_{ij})_{ij=1,2,3} = \begin{pmatrix} \mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1} & \mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{l} & \mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} \\ \mathbf{l}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1} & \mathbf{l}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{l} & \mathbf{l}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} \\ \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1} & \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{l} & \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} \end{pmatrix}.$$

Then (25) is equivalent to $Ar^2 - 2Br + C \leq 0$ with $A = \hat{h}_{13}^2 - K(\hat{h}_{11} + (n/(n-1))(\hat{h}_{11}\hat{h}_{33} - \hat{h}_{13}^2))$, $B = \hat{h}_{13}\hat{h}_{23} - K(\hat{h}_{12} + (n/(n-1))(\hat{h}_{12}\hat{h}_{33} - \hat{h}_{13}\hat{h}_{23}))$ and $C = \hat{h}_{23}^2 - K(\hat{h}_{22} + (n/(n-1))(\hat{h}_{22}\hat{h}_{33} - \hat{h}_{23}^2))$.

Note that if $A < 0$ we obtain an unbounded confidence interval. In this case the true value of the portfolio weights can be infinitely away from the estimated weights. This implies very low precision of the estimated weights. It holds that $A = \hat{R}_{GMV}^2/\hat{V}_{GMV}^2 - K(1 + n\hat{s}/(n-1))/\hat{V}_{GMV}$, where $\hat{R}_{GMV} = \mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}/\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}$ and $\hat{V}_{GMV} = 1/\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}$ are the estimated expected return and variance of the global minimum variance portfolio. $\hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}_1\hat{\boldsymbol{\mu}}$ is the slope coefficient of the efficient frontier. The distribution theory of $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ is developed in Bodnar & Schmid (2009, lemma 1) who showed that the estimators are independently distributed, $(n-1)\hat{V}_{GMV}/V_{GMV} \sim \chi_{n-k}^2$, $\frac{n(n-k+1)}{(n-1)(k-1)}\hat{s} \sim F_{k-1,n-k+1,ns}$ and $\hat{R}_{GMV}\,|\,\hat{s}=y \sim \mathcal{N}\left(R_{GMV}, \frac{1+(n/(n-1))y}{n}V_{GMV}\right)$. Then $A < 0$ iff

$$|T_R| < t_{n-k;1-\alpha/2}, \quad \text{where} \quad T_R = \sqrt{n}\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{\hat{R}_{GMV}}{\sqrt{\hat{V}_{GMV}}\sqrt{1+(n/(n-1))\hat{s}}}$$

is the test statistic for testing the null hypothesis that the expected return of the global minimum variance portfolio is equal to 0, that is,

$$H_0^R : \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} = 0 \quad \text{versus} \quad H_1^R : \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \neq 0.$$

Because

$$w_{SR} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

the non-rejecting of the null hypothesis means that we are also unable to define the weights of the Sharpe ratio optimal portfolio. In the considered example it holds that $T_R = 1.6890$ with the corresponding p-value of 0.093. Hence, we are unable to reject the null hypothesis with the probability of 9.3 per cent. Our results are in line with the findings of Britten-Jones (1999), who was unable to reject the null hypothesis that the US weight in the Sharpe ratio portfolio is equal to 1 which makes the international diversification questionable. The author also presented a high value of the variance of its estimator. Our results provide a further explanation of this phenomena. The main reason is that we are not able to reject the null

hypothesis of the test for the expected return of the global minimum variance portfolio, that is, the weights of the Sharpe ratio optimal portfolio are not identifiable.

4. Summary

In this article we analyse distribution of the product of the inverse Wishart matrix and a Gaussian vector. We derive the exact density and the characteristic function. Moreover, we develop a series of exact multivariate tests for the components of the product and their linear combinations. The developed results can be successfully applied to the coefficient of the discriminant function and to the portfolio weights of several popular types of portfolios. To our knowledge this is the first article where these types of tests were proposed.

Acknowledgements

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Taras Bodnar, Department of Statistics, European University Viadrina, PO Box 1786, D-15207 Frankfurt (Oder), Germany.
E-mail: bodnar@euv-frankfurt-o.de

Appendix

We start with an important lemma, which we use to derive the unconditional density of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$. The first part of this lemma is lemma 1 of Bodnar & Schmid (2008b), while the second part is lemma 3.4.1 of Fang & Zhang (1990).

Lemma 1. *Let* $k \geq 3$, $\mathbf{a}_1 \neq \mathbf{0}$ *and* $\mathbf{a}_2 \neq \lambda\mathbf{a}_1$ *for some* $\lambda \in \mathbb{R}$. *Then it holds for every non-negative Borel function* f *that*

$$(a) \int_{R^k} f(\mathbf{a}_1'\mathbf{x}, \mathbf{a}_2'\mathbf{x}, \mathbf{x}'\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{\pi^{(k-2)/2}}{\Gamma((k-2)/2)}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}y_1 \mathrm{d}y_2 \int_0^{\infty} u^{(k-4)/2} f\left(\|\mathbf{a}_1\|y_1, \frac{\mathbf{a}_1'\mathbf{a}_1}{\|\mathbf{a}_1\|}y_1 + \sqrt{\|\mathbf{a}_2\| - \frac{(\mathbf{a}_1'\mathbf{a}_2)^2}{\|\mathbf{a}_1\|^2}}\,y_2, y_1^2 + y_2^2 + u\right)\mathrm{d}u.$$

$$(b) \int_{R^n} f(\alpha'\mathbf{x}, \mathbf{x}'\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{\pi^{1/2(n-1)}}{\Gamma(1/2(n-1))}\int_{-\infty}^{\infty}\mathrm{d}y\int_0^{\infty} u^{1/2(n-1)-1} f(y\|\alpha\|, y^2 + u)\,\mathrm{d}u.$$

Proof of theorem 1. (a) Since $\mathbf{A}$ and $\mathbf{z}$ are independently distributed it follows that the conditional distribution of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}\,|\,(\mathbf{z}=\mathbf{z}^*)$ is equal to the distribution of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*$. The latter vector can be rewritten as:

$$\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^* = \mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*}\frac{\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}. \tag{26}$$

From Muirhead (1982, theorem 3.2.12) we get that $\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^* \sim \chi^2_{n-k+1}$ and is independent of $\mathbf{z}^*$. Moreover from Bodnar & Okhrin (2008, theorem 3) it holds that $\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*$ is independent of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*$ for given $\mathbf{z}^*$. Therefore, it is also independent of $\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*$ and correspondingly of $\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}$. From the proof of theorem 1 of Bodnar & Schmid (2008a) it follows that $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*$ has a p-dimensional t-distribution with $n-k+2$ degrees of freedom, the mean vector $\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*$ and the covariance matrix $(1/(n-k+2))\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}'/\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*$, where $\mathbf{R}_{\mathbf{z}^*} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}/\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*$. This implies that

$$\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^* \sim t_{n-k+2}\left(\frac{1}{n-k+2}\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}', \frac{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}{n-k+2}\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}'\right).$$

Hence, the unconditional density of $\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}$ is obtained by integrating out $\mathbf{z}^*$ which is Gaussian.

$$f_{\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) = \int_{\mathbb{R}^k} f_{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*}}(\mathbf{y}\,|\,\mathbf{z}=\mathbf{z}^*) f_{\mathbf{z}}(\mathbf{z}^*)\,\mathrm{d}\mathbf{z}^*$$

$$= \lambda^{-k/2}\frac{\det(\boldsymbol{\Sigma})^{-1/2}}{(2\pi)^{k/2}}\frac{\Gamma((p+n-k+2)/2)}{\pi^{p/2}\Gamma((n-k+2)/2)}\int_{\mathbb{R}^k}\exp\left(-\frac{(\mathbf{z}^*-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{z}^*-\boldsymbol{\mu})}{2\lambda}\right)$$

$$\times\frac{\det(\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}')^{-1/2}}{(\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)^{p/2}}$$

$$\times\left(1+\frac{(\mathbf{y}-\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)'(\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}')^{-1}(\mathbf{y}-\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)}{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}\right)^{-(p+n-k+2)/2}\mathrm{d}\mathbf{z}^*.$$

To use dimension reduction techniques we introduce the following notation. Let $\tilde{\mathbf{y}}=(\mathbf{L}\boldsymbol{\Sigma}^{-1}\times\mathbf{L}')^{-1/2}\mathbf{y}$, $\mathbf{Q}=\boldsymbol{\Sigma}^{-1/2}\mathbf{L}'(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1/2}$ and $\mathbf{P}=\boldsymbol{\Sigma}^{-1/2}\mathbf{L}'(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}$. Note that the $k\times k$ matrix $\mathbf{Q}$ is a singular. It follows also that $\mathbf{Q}$ is a projection matrix with $\mathbf{Q}=\mathbf{P}\mathbf{P}'$. Using Harville (1997, theorem 18.2.8), we obtain

$$(\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}')^{-1}=(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1}+\frac{(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{L}'(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1}}{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*-\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{L}'(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*},$$

$$\det(\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}')=\det(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')\frac{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*-\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{L}'(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}.$$

The transformation $\mathbf{t}=\boldsymbol{\Sigma}^{-1/2}\mathbf{z}^*$ in the last integral leads to

$$f_{\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y})=\lambda^{-k/2}\frac{\det(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}}{(2\pi)^{k/2}}\frac{\Gamma((p+n-k+2)/2)}{\pi^{p/2}\Gamma((n-k+2)/2)}\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right)$$

$$\times\int_{\mathbb{R}^k}\left(1+\frac{1}{\mathbf{t}'\mathbf{t}}\left(\tilde{\mathbf{y}}'\tilde{\mathbf{y}}-2\tilde{\mathbf{y}}'\mathbf{P}'\mathbf{t}+\mathbf{t}'\mathbf{Q}\mathbf{t}+\frac{(\mathbf{t}'\mathbf{Q}\mathbf{t}-\tilde{\mathbf{y}}'\mathbf{P}'\mathbf{t})^2}{\mathbf{t}'(\mathbf{I}-\mathbf{Q})\mathbf{t}}\right)\right)^{-(p+n-k+2)/2}$$

$$\times\frac{1}{(\mathbf{t}'\mathbf{t})^{(p-1)/2}(\mathbf{t}'(\mathbf{I}-\mathbf{Q})\mathbf{t})^{1/2}}\exp\left(-\frac{\mathbf{t}'\mathbf{t}}{2\lambda}+\lambda^{-1}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1/2}\mathbf{t}\right)\mathrm{d}\mathbf{t}.$$

Since $\mathbf{P}$ is a $k\times p$ matrix with $\mathrm{rank}(\mathbf{P})=p$ and $\mathbf{Q}$ is a projection matrix, then it follows that $\mathrm{rank}(\mathbf{I}-\mathbf{Q})=k-p$ (Muirhead, 1982, theorem 12.3.4). This implies that we can find such a $(k-p)\times k$ matrix $(\mathbf{I}-\mathbf{Q})^{1/2}$ that $(\mathbf{I}-\mathbf{Q})^{1/2\prime}(\mathbf{I}-\mathbf{Q})^{1/2}=(\mathbf{I}-\mathbf{Q})$ and $\mathrm{rank}((\mathbf{I}-\mathbf{Q})^{1/2})=k-p$. This justifies the transformation $\mathbf{v}_1=\mathbf{P}'\mathbf{t}$, $\mathbf{v}_2=(\mathbf{I}-\mathbf{Q})^{1/2\prime}\mathbf{t}$ where $\mathbf{v}_1\in\mathbb{R}^p$ and $\mathbf{v}_2\in\mathbb{R}^{k-p}$. The Jacobian of the transformation is 1. This leads to the integral

$$f_{\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y})=\lambda^{-k/2}\frac{\det(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}}{(2\pi)^{k/2}}\frac{\Gamma((p+n-k+2)/2)}{\pi^{p/2}\Gamma((n-k+2)/2)}\exp\left(-\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right)$$

$$\times\int_{\mathbb{R}^k}\frac{\left(1+\frac{1}{\mathbf{v}_1'\mathbf{v}_1+\mathbf{v}_2'\mathbf{v}_2}\left(\tilde{\mathbf{y}}'\tilde{\mathbf{y}}-2\tilde{\mathbf{y}}'\mathbf{v}_1+\mathbf{v}_1'\mathbf{v}_1+\frac{(\mathbf{v}_1'\mathbf{v}_1-\tilde{\mathbf{y}}'\mathbf{v}_1)^2}{\mathbf{v}_2'\mathbf{v}_2}\right)\right)^{-(p+n-k+2)/2}}{(\mathbf{v}_1'\mathbf{v}_1+\mathbf{v}_2'\mathbf{v}_2)^{(p-1)/2}(\mathbf{v}_2'\mathbf{v}_2)^{1/2}}$$

$$\times\exp\left(-\frac{\mathbf{v}_1'\mathbf{v}_1+\mathbf{v}_2'\mathbf{v}_2}{2\lambda}+\lambda^{-1}(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1/2}\mathbf{P}\mathbf{v}_1+\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1/2}(\mathbf{I}-\mathbf{Q})^{1/2}\mathbf{v}_2)\right)\mathrm{d}\mathbf{v}_1\,\mathrm{d}\mathbf{v}_2.$$

Application of lemma 1 of Bodnar & Schmid (2008b) and lemma 3.4.1 of Fang & Zhang (1990), which we restate in the Appendix, for $\mathbf{v}_1$ with $\mathbf{a}_1=\tilde{\mathbf{y}}$ and $\mathbf{a}_2=\mathbf{P}'\boldsymbol{\Sigma}^{-1/2}$ and for $\mathbf{v}_2$ with $\mathbf{a}_1=(\mathbf{I}-\mathbf{Q})^{1/2\prime}\boldsymbol{\Sigma}^{-1/2}$ yields

$$f_{\mathbf{z}'\mathbf{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) = \lambda^{-k/2} \frac{\pi^{1/2(p-2)}}{\Gamma(1/2(p-2))} \frac{\pi^{1/2(k-p-1)}}{\Gamma(1/2(k-p-1))} \frac{\det(\mathbf{L}\mathbf{\Sigma}^{-1}\mathbf{L}')^{-1/2}}{(2\pi)^{k/2}} \frac{\Gamma((p+n-k+2)/2)}{\pi^{p/2}\Gamma((n-k+2)/2)}$$

$$\times \exp\left(-\frac{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty u_1^{(p-4)/2}$$

$$\times \int_0^\infty \int_{-\infty}^\infty u_2^{(k-p-3)/2} \left(1 + \frac{1}{u_2 + v_{22}^2 + u_1 + v_{11}^2 + v_{12}^2}\right.$$

$$\times \left(\tilde{\mathbf{y}}'\tilde{\mathbf{y}} - 2\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}v_{11} + u_1 + v_{11}^2 + v_{12}^2\right.$$

$$\left.\left. + \frac{(u_1 + v_{11}^2 + v_{12}^2 - \sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}v_{11})^2}{u_2 + v_{21}^2}\right)\right)^{-(p+n-k+2)/2}$$

$$\times \frac{(u_2 + v_{21}^2 + u_1 + v_{11}^2 + v_{12}^2)^{-(p-1)/2}}{(u_2 + v_{21}^2)^{1/2}} \exp\left(-\frac{u_2 + v_{21}^2}{2\lambda} + \lambda^{-1}\sqrt{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}}v_{21}\right)$$

$$\times \exp\left(-\frac{u_1 + v_{11}^2 + v_{12}^2}{2\lambda} + \lambda^{-1}\left(\frac{\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}}}{\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}v_{11}\right.\right.$$

$$\left.\left. + \sqrt{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \frac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}v_{12}\right)\right) \mathrm{d}u_1\,\mathrm{d}v_{11}\,\mathrm{d}v_{12}\,\mathrm{d}u_2\,\mathrm{d}v_{21},$$

where $\tilde{\boldsymbol{\mu}} = (\mathbf{L}\mathbf{\Sigma}^{-1}\mathbf{L}')^{-1/2}\mathbf{L}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$. Transformation of $w_{11} = u_1 + v_{11}^2 + v_{12}^2$, $w_{12} = v_{11}$, $w_{13} = -v_{12}$, $w_{21} = u_2 + v_{21}^2$, $w_{22} = -v_{21}$ with the Jacobian 1 leads to

$$f_{\mathbf{z}'\mathbf{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) = \lambda^{-k/2} \frac{\pi^{1/2(p-2)}}{\Gamma(1/2(p-2))} \frac{\pi^{1/2(k-p-1)}}{\Gamma(1/2(k-p-1))} \frac{\det(\mathbf{L}\mathbf{\Sigma}^{-1}\mathbf{L}')^{-1/2}}{(2\pi)^{k/2}} \frac{\Gamma((p+n-k+2)/2)}{\pi^{p/2}\Gamma((n-k+2)/2)}$$

$$\times \exp\left(-\frac{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right) \int_0^\infty \int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}} \int_0^\infty \frac{1}{(w_{11} + w_{21})^{(p-1)/2}} \frac{1}{\sqrt{w_{21}}}$$

$$\times \exp\left(-\frac{w_{11} + w_{21}}{2\lambda} + \lambda^{-1}\frac{\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}}}{\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}w_{12}\right)$$

$$\times \left(1 + \frac{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}} + \frac{w_{11} - 2\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}w_{12}}{w_{21}}\right)^{-(p+n-k+2)/2}$$

$$\times U_1(w_{11} - w_{12}^2)U_2(w_{21})\,\mathrm{d}w_{11}\,\mathrm{d}w_{12}\,\mathrm{d}w_{21},$$

where

$$U_1(w_{11} - w_{12}^2) = \int_{-\sqrt{w_{11}-w_{12}^2}}^{\sqrt{w_{11}-w_{12}^2}} (w_{11} - w_{12}^2 - w_{13}^2)^{(p-4)/2} \exp\left(-\lambda^{-1}\sqrt{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \frac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}w_{13}\right)\mathrm{d}w_{13}$$

$$= \frac{2^{(p-3)/2}\sqrt{\pi}\Gamma((p-2)/2)}{\left(\lambda^{-1}\sqrt{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \frac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}\right)^{(p-3)/2}} (w_{11} - w_{12}^2)^{(p-3/4)}$$

$$\times I_{(p-3)/2}\left(\lambda^{-1}\sqrt{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \frac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}\sqrt{w_{11} - w_{12}^2}\right),$$

$$U_2(w_{21}) = \int_{-\sqrt{w_{21}}}^{\sqrt{w_{21}}} (w_{21} - w_{22}^2)^{(k-p-3)/2} \exp\left(-\lambda^{-1}\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}}\, w_{22}\right) \mathrm{d}w_{22}$$

$$= \frac{2^{(k-p-2)/2}\sqrt{\pi}\,\Gamma((k-p-1)/2)}{(\lambda^{-1}\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}})^{(k-p-2)/2}} w_{21}^{(k-p-2)/2} I_{(k-p-2)/4}\left(\lambda^{-1}\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}}\sqrt{w_{21}}\right)$$

and $I.(\cdot)$ denotes the modified Bessel function of the first kind. The first two integrals are evaluated based on the results of Andrews *et al.* (2000, p. 235). Putting the pieces together, we obtain

$$f_{\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) = \frac{\lambda^{-5/2}}{2^{5/2}} \frac{\exp\left(-\dfrac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}}{2\lambda}\right)\det(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}')^{-1/2}}{\pi^{(p+1)/2}\left(\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \dfrac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}\right)^{(p-3)/4}(\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}})^{(k-p-2)/4}}$$

$$\times \frac{\Gamma((p+n-k+2)/2)}{\Gamma((n-k+2)/2)} \int_0^\infty \int_0^\infty \int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}} \frac{1}{(w_{11}+w_{21})^{\frac{(p-1)}{2}}}$$

$$\times \exp\left(-\frac{(w_{11}+w_{21})}{2\lambda} + \lambda^{-1}\frac{\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}}}{\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}} w_{12}\right)\left(1 + \frac{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}(w_{21}+w_{12}^2)}{(w_{11}+w_{21})w_{21}}\right.$$

$$\left.+ \frac{w_{11} - 2\sqrt{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}\, w_{12}}{w_{21}}\right)^{-(p+n-k+2)/2}(w_{11} - w_{12}^2)^{(p-3)/4}w_{21}^{(k-p-4)/4}$$

$$\times I_{(p-3)/2}\left(\frac{1}{\lambda}\sqrt{\tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}} - \frac{(\tilde{\mathbf{y}}'\tilde{\boldsymbol{\mu}})^2}{\tilde{\mathbf{y}}'\tilde{\mathbf{y}}}}\sqrt{w_{11} - w_{12}^2}\right) I_{(k-p-2)/2}$$

$$\times \left(\frac{1}{\lambda}\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\mu}}}\sqrt{w_{21}}\right)\mathrm{d}w_{12}\,\mathrm{d}w_{11}\,\mathrm{d}w_{21}.$$

The rest of the proof follows from the fact that $\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}$ and $\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}$ are independent and $\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}/\mathbf{z}'\mathbf{A}^{-1}\mathbf{z} \sim \chi_{n-k+1}^2$. This implies that

$$f_{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}(\mathbf{y}) = \int_0^{+\infty} f_{\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}\,s)f_{\chi_{n-k+1}^2}(s)s^p\,\mathrm{d}s.$$

This completes the proof of the statement of theorem 1a.

(b) The fact follows from the proof of theorem 1 of Bodnar & Schmid (2008b).

Proof of theorem 2. From the definition of a characteristic function it holds that

$$\varphi_{\mathbf{A}^{-1}\mathbf{z}}(\mathbf{t}) = E(\exp(i\mathbf{t}'\mathbf{A}^{-1}\mathbf{z})) = \int_{-\infty}^\infty \exp(ix)f_{\mathbf{t}'\mathbf{A}^{-1}\mathbf{z}}(x)\,\mathrm{d}x.$$

From theorem 1b, it follows that

$$\varphi_{\mathbf{A}^{-1}\mathbf{z}}(\mathbf{t}) = \frac{n-k+2}{\lambda(k-1)}\int_{-\infty}^\infty \exp(ix)\int_0^\infty z f_{\chi_{n-k+1}^2}(z)\int_0^\infty f_{N(\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu},(\lambda+y)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t})}(xz)f_{F_{k-1,n-k+2,s/\lambda}}$$

$$\times \left(\frac{n-k+2}{\lambda(k-1)}y\right)\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

$$= \frac{n-k+2}{\lambda(k-1)}\int_0^\infty \int_0^\infty f_{\chi_{n-k+1}^2}(z)f_{F_{k-1,n-k+2,s/\lambda}}\left(\frac{n-k+2}{\lambda(k-1)}y\right)$$

$$\times \int_{-\infty}^\infty \exp\left(\frac{i\tilde{x}}{z}\right)f_{N(\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu},(\lambda+y)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t})}(\tilde{x})\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}z,$$

where $\tilde{x} = zx$. Note that the last integral is the characteristic function of the normal distribution with the mean $\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and the variance $(\lambda + y)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}$ evaluated at $1/z$. Hence, it holds

$$\varphi_{\mathbf{A}^{-1}\mathbf{z}}(\mathbf{t}) = \frac{n-k+2}{\lambda(k-1)} \int_0^\infty \int_0^\infty f_{\chi^2_{n-k+1}}(z) f_{F_{k-1,n-k+2,s/\lambda}}\left(\frac{n-k+2}{\lambda(k-1)}y\right)$$
$$\times \exp\left(i\frac{\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{z} - \frac{(\lambda+y)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2z^2}\right) \mathrm{d}y\,\mathrm{d}z$$
$$= \frac{n-k+2}{\lambda(k-1)} \int_0^\infty \exp\left(i\frac{\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{z} - \frac{\lambda\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2z^2}\right) f_{\chi^2_{n-k+1}}(z)$$
$$\times \int_0^\infty \exp\left(-\frac{y\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2z^2}\right) f_{F_{k-1,n-k+2,s/\lambda}}\left(\frac{n-k+2}{\lambda(k-1)}y\right) \mathrm{d}y\,\mathrm{d}z$$
$$= \int_0^\infty \exp\left(i\frac{\mathbf{t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{z} - \frac{\lambda\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2z^2}\right) f_{\chi^2_{n-k+1}}(z)$$
$$\times \int_0^\infty \exp\left(-\frac{\lambda(k-1)\tilde{y}\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t}}{2(n-k+2)z^2}\right) f_{F_{k-1,n-k+2,s/\lambda}}(\tilde{y}) \mathrm{d}\tilde{y}\,\mathrm{d}z,$$

where $\tilde{y} = (n-k+2)y/(\lambda(k-1))$. Again the last integral is the characteristic function of the non-central F-distribution taken at the point $(i\lambda(k-1)\mathbf{t}'\boldsymbol{\Sigma}^{-1}\mathbf{t})/(2(n-k+2)z^2)$. This completes the proof of the theorem.

Proof of theorem 3. Let $\tilde{\mathbf{L}} = (\mathbf{L}', \mathbf{z})$, $\tilde{\mathbf{L}}\mathbf{A}^{-1}\tilde{\mathbf{L}}' = \hat{\mathbf{H}} = \{\hat{\mathbf{H}}_{ij}\}_{i,j=1,2}$ with $\hat{\mathbf{H}}_{22} = \mathbf{z}'\mathbf{A}^{-1}\mathbf{z}$, $\hat{\mathbf{H}}_{12} = \mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ and $\hat{\mathbf{H}}_{11} = \mathbf{L}\mathbf{A}^{-1}\mathbf{L}'$. Because $\mathbf{A}$ and $\mathbf{z}$ are independently distributed it follows that the conditional distribution of $\hat{\mathbf{H}}\,|\,(\mathbf{z} = \mathbf{z}^*)$ is equal to the distribution of $\hat{\mathbf{H}} = \hat{\mathbf{H}}(\mathbf{z}^*)$. Let $\tilde{\mathbf{L}}^* = \tilde{\mathbf{L}}(\mathbf{z}^*)$, $\tilde{\mathbf{L}}^*\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{L}}^{*'} = \{\mathbf{H}_{ij}^*\}_{i,j=1,2}$ with $\mathbf{H}_{22}^* = \mathbf{z}^{*'}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*$, $\mathbf{H}_{12}^* = \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*$, and $\mathbf{H}_{11}^* = \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}'$. Similarly, $\hat{\mathbf{H}}^* = \tilde{\mathbf{L}}^*\mathbf{A}^{-1}\tilde{\mathbf{L}}^{*'} = \{\hat{\mathbf{H}}_{ij}^*\}_{i,j=1,2}$. Thus, $\hat{\mathbf{b}}^* = \mathbf{L}\hat{\mathbf{R}}_{\mathbf{z}^*}\mathbf{L}' = \hat{\mathbf{H}}_{11}^* - \hat{\mathbf{H}}_{12}^*\hat{\mathbf{H}}_{12}^{*'}/\hat{\mathbf{H}}_{22}^*$,

$$\hat{\mathbf{w}}^* = \hat{\mathbf{H}}_{12}^* \quad \text{and} \quad \mathbf{w}^* = \mathbf{H}_{12}^*.$$

Because $\mathbf{A} \sim W_k(\boldsymbol{\Sigma}; n)$ and $\text{rank}(\tilde{\mathbf{L}}^*) = p+1$ we get from theorem 3.2.11 of Muirhead (1982) that

$$(\tilde{\mathbf{L}}^*\mathbf{A}^{-1}\tilde{\mathbf{L}}^{*'})^{-1} \sim W_{p+1}((\tilde{\mathbf{L}}^*\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{L}}^{*'})^{-1}; n-k+p+1).$$

Thus, $\tilde{\mathbf{L}}^*\mathbf{A}^{-1}\tilde{\mathbf{L}}^{*'} \sim W_{p+1}^{-1}(\tilde{\mathbf{L}}^*\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{L}}^{*'}; n-k+2p+3)$. From Bondar & Okhrin (2008), we obtain

$$\hat{\mathbf{H}}_{12}^*\,|\,[\hat{\mathbf{b}}^* \wedge \hat{\mathbf{H}}_{22}^*] \sim \mathcal{N}\left(\mathbf{H}_{12}^*\frac{\hat{\mathbf{H}}_{22}^*}{\mathbf{H}_{22}^*}, \hat{\mathbf{b}}^*\frac{\hat{\mathbf{H}}_{22}^{*2}}{\mathbf{H}_{22}^*}\right).$$

It leads to

$$\hat{\mathbf{w}}^*\,|\,[\hat{\mathbf{b}}^* = \mathbf{c}_{\mathbf{z}^*} \wedge \hat{\mathbf{H}}_{22}^*/\mathbf{H}_{22}^* = 1/\xi] \sim \mathcal{N}\left(\mathbf{H}_{12}^*\frac{1}{\xi}, \mathbf{c}_{\mathbf{z}^*}\mathbf{H}_{22}^*\frac{1}{\xi^2}\right).$$

Then

$$\frac{\mathbf{H}_{22}^*}{\hat{\mathbf{H}}_{22}^*}\frac{\hat{\mathbf{b}}^{*-1/2}(\hat{\mathbf{w}}^* - \mathbf{r})}{\sqrt{\mathbf{H}_{22}^*}}\,|\,[\hat{\mathbf{b}}^* = \mathbf{c}_{\mathbf{z}^*} \wedge \hat{\mathbf{H}}_{22}^*/\mathbf{H}_{22}^* = 1/\xi] \sim \mathcal{N}\left(\mathbf{c}_{\mathbf{z}^*}^{-1/2}\frac{\mathbf{H}_{12}^* - \xi\mathbf{r}}{\sqrt{\mathbf{H}_{22}^*}}, \mathbf{I}\right).$$

The test statistic is then defined by

$$T^* = \left(\frac{\mathbf{H}_{22}^*}{\hat{\mathbf{H}}_{22}^*}\right)^2 \frac{(\hat{\mathbf{w}}^* - \mathbf{r})'\hat{\mathbf{b}}^{*-1}(\hat{\mathbf{w}}^* - \mathbf{r})}{\mathbf{H}_{22}^*} \Bigg/ \left(\frac{\mathbf{H}_{22}^*}{\hat{\mathbf{H}}_{22}^*}\right) = \frac{(\hat{\mathbf{w}}^* - \mathbf{r})'(\mathbf{L}\hat{\mathbf{R}}_{\mathbf{z}^*}\mathbf{L}')^{-1}(\hat{\mathbf{w}}^* - \mathbf{r})}{\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*}.$$

This implies that

$$\left(\frac{\mathbf{H}_{22}^*}{\hat{\mathbf{H}}_{22}^*}\right) T^* \,|\, [\hat{\mathbf{b}}^* = \mathbf{c}_{\mathbf{z}^*} \wedge \hat{\mathbf{H}}_{22}^*/\mathbf{H}_{22}^* = 1/\xi] \sim \chi_p^2(\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)),$$

where

$$\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi) = \frac{(\mathbf{H}_{12}^* - \xi\mathbf{r})'\mathbf{c}_{\mathbf{z}^*}^{-1}(\mathbf{H}_{12}^* - \xi\mathbf{r})}{\mathbf{H}_{22}^*}.$$

From Muirhead (1982, theorem 3.2.12) we know that $\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*/\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^* \sim \chi_{n-k+1}^2$ and is independent of $\mathbf{z}^*$. Hence, $\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z}/\mathbf{z}'\mathbf{A}^{-1}\mathbf{z} \sim \chi_{n-k+1}^2$ and is independent of $\mathbf{z}$. Moreover, applying theorem 3 of Bodnar & Okhrin (2008) it follows that $(\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)/(\mathbf{z}^{*\prime}\mathbf{A}^{-1}\mathbf{z}^*)$ is independent of $\hat{\mathbf{R}}_{\mathbf{z}^*}$. Because $\hat{\mathbf{b}}^{*-1} \sim W_p(n-k+p+1, \mathbf{b}^{-1})$ with $\mathbf{b} = \mathbf{H}_{11}^* - \mathbf{H}_{12}^*\mathbf{H}_{12}^{*\prime}/\mathbf{H}_{22}^*$ we obtain that

$$f_{T^*}(x) = \int_0^\infty \xi f_{\chi_{n-k+1}^2}(\xi) \int_{\mathbf{c}_{\mathbf{z}^*}>0} f_{\chi_{p;\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)}^2}(\xi x) w_p(n-k+p+1, \mathbf{b}^{-1})(\mathbf{c}_{\mathbf{z}^*})\, d\mathbf{c}_{\mathbf{z}^*}\, d\xi.$$

Here $f_{\chi_{i;\lambda}^2}$ denotes the density of the non-central χ^2-distribution with degrees i and non-centrality parameter λ and w_p is the density of the Wishart distribution W_p. It holds that (e.g. theorem 1.3.5 of Muirhead, 1982)

$$f_{\chi_{p,\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)}^2}(\xi x) = f_{\chi_p^2}(\xi x) \exp\left(-\frac{\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)}{2}\right) \sum_{i=0}^\infty \frac{1}{(p/2)_i} \frac{\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)^i}{i!} \left(\frac{\xi x}{4}\right)^i,$$

where $(a)_i = a(a+1)\cdots(a+i-1)$. Let us denote

$$k_i(\xi x) = \frac{1}{i!}\frac{1}{(p/2)_i}\left(\frac{\xi x}{4}\right)^i.$$

Then, it follows that

$$f_{T^*}(x) = \sum_{i=0}^\infty \int_0^\infty \xi f_{\chi_p^2}(\xi x) k_i(\xi x) f_{\chi_{n-k+1}^2}(\xi) \int_{\mathbf{c}_{\mathbf{z}^*}>0} \lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)^i \exp\left(-\frac{\lambda(\mathbf{c}_{\mathbf{z}^*}, \xi)}{2}\right)$$

$$\times \frac{1}{2^{p(n-k+p+1)/2}\Gamma_p((n-k+p+1)/2)} \det(\mathbf{b})^{(n-k+p+1)/2} \det(\mathbf{c}_{\mathbf{z}^*})^{(n-k)/2}$$

$$\times \exp(-\mathrm{tr}(\mathbf{b}\mathbf{c}_{\mathbf{z}^*}/2))\, d\mathbf{c}_{\mathbf{z}^*}\, d\xi$$

$$= \int_0^\infty \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k+1}^2}(\xi) \left(1 + \frac{1}{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}(\mathbf{w}^* - \xi\mathbf{r})'\mathbf{b}^{-1}(\mathbf{w}^* - \xi\mathbf{r})\right)^{-(n-k+p+1)/2}$$

$$\times {}_1F_1\left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2} \frac{1/(\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)(\mathbf{w}^* - \xi\mathbf{r})'\mathbf{b}^{-1}(\mathbf{w}^* - \xi\mathbf{r})}{1 + 1/(\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)(\mathbf{w}^* - \xi\mathbf{r})'\mathbf{b}^{-1}(\mathbf{w}^* - \xi\mathbf{r})}\right) d\xi,$$

where the last equality follows from the proof of theorem 2 of Bodnar & Schmid (2008a).

The unconditional density of the T statistic is given by

$$f_T(x) = \int_{\mathbb{R}^k} f_{T^*}(x \mid \mathbf{z} = \mathbf{z}^*) f_{\mathbf{z}}(\mathbf{z}^*) \, d\mathbf{z}^*$$

$$= \lambda^{-k/2} \frac{\det(\Sigma)^{-1/2}}{(2\pi)^{k/2}} \int_0^\infty \int_{\mathbb{R}^k} \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k+1}^2}(\xi) \exp\left(-\frac{(\mathbf{z}^* - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z}^* - \boldsymbol{\mu})}{2\lambda}\right)$$

$$\times \left(1 + \frac{1}{\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*}(\mathbf{w}^* - \xi\mathbf{r})'\mathbf{b}^{-1}(\mathbf{w}^* - \xi\mathbf{r})\right)^{-(n-k+p+1)/2}$$

$$\times {}_1F_1\left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2} \frac{1/(\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*)(\mathbf{w}^* - \xi\mathbf{r})'\mathbf{b}^{-1}(\mathbf{w}^* - \xi\mathbf{r})}{1 + 1/(\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*)(\mathbf{w}^* - \xi\mathbf{r})'\mathbf{b}^{-1}(\mathbf{w}^* - \xi\mathbf{r})}\right) d\mathbf{z}^* \, d\xi.$$

Let $\tilde{\mathbf{r}} = (\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1/2}\mathbf{r}$, $\mathbf{Q} = \Sigma^{-1/2}\mathbf{L}'(\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1}\mathbf{L}\Sigma^{-1/2}$ and $\mathbf{P} = \Sigma^{-1/2}\mathbf{L}'(\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1/2}$. It follows that $\mathbf{Q}$ is a projection matrix with $\mathbf{Q} = \mathbf{PP}'$ (cf. Harville, 1997). Using

$$\mathbf{b}^{-1} = (\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}')^{-1} = \left(\mathbf{L}\Sigma^{-1}\mathbf{L}' - \frac{\mathbf{L}\Sigma^{-1}\mathbf{z}^*\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{L}'}{\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^*}\right)^{-1}$$

$$= (\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1} + \frac{(\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1}\mathbf{L}\Sigma^{-1}\mathbf{z}^*\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{L}'(\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1}}{\mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{z}^* - \mathbf{z}^{*\prime}\Sigma^{-1}\mathbf{L}'(\mathbf{L}\Sigma^{-1}\mathbf{L}')^{-1}\mathbf{L}\Sigma^{-1}\mathbf{z}^*},$$

and making the transformation $\mathbf{t} = \Sigma^{-1/2}\mathbf{z}^*$ we obtain

$$f_T(x) = \frac{\lambda^{-k/2}}{(2\pi)^{k/2}} \exp\left(-\frac{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}{2\lambda}\right) \int_0^\infty \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k+1}^2}(\xi) \int_{\mathbb{R}^k} \exp\left(-\frac{\mathbf{t}'\mathbf{t}}{2\lambda} + \frac{\boldsymbol{\mu}'\Sigma^{-1/2}\mathbf{t}}{\lambda}\right)$$

$$\times \left(1 + \frac{1}{\mathbf{t}'\mathbf{t}}\left(\xi^2\tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\tilde{\mathbf{r}}'\mathbf{P}'\mathbf{t} + \mathbf{t}'\mathbf{Q}\mathbf{t} + \frac{(\mathbf{t}'\mathbf{Q}\mathbf{t} - \xi\tilde{\mathbf{r}}'\mathbf{P}'\mathbf{t})^2}{\mathbf{t}'(\mathbf{I} - \mathbf{Q})\mathbf{t}}\right)\right)^{-(n-k+p+1)/2}$$

$$\times {}_1F_1\left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2}\right.$$

$$\times \left.\frac{\left(\frac{1}{\mathbf{t}'\mathbf{t}}\right)\left(\xi^2\tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\tilde{\mathbf{r}}'\mathbf{P}'\mathbf{t} + \mathbf{t}'\mathbf{Q}\mathbf{t} + \frac{(\mathbf{t}'\mathbf{Q}\mathbf{t} - \xi\tilde{\mathbf{r}}'\mathbf{P}'\mathbf{t})^2}{\mathbf{t}'(\mathbf{I} - \mathbf{Q})\mathbf{t}}\right)}{1 + \left(\frac{1}{\mathbf{t}'\mathbf{t}}\right)\left(\xi^2\tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\tilde{\mathbf{r}}'\mathbf{P}'\mathbf{t} + \mathbf{t}'\mathbf{Q}\mathbf{t} + \frac{(\mathbf{t}'\mathbf{Q}\mathbf{t} - \xi\tilde{\mathbf{r}}'\mathbf{P}'\mathbf{t})^2}{\mathbf{t}'(\mathbf{I} - \mathbf{Q})\mathbf{t}}\right)}\right) d\mathbf{t} \, d\xi.$$

The transformation $\mathbf{v}_1 = \mathbf{P}'\mathbf{t}$, $\mathbf{v}_2 = (\mathbf{I} - \mathbf{Q})^{1/2\prime}\mathbf{t}$ with the Jacobian 1 leads to

$$f_T(x) = \frac{\lambda^{-k/2}}{(2\pi)^{k/2}} \exp\left(-\frac{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}{2\lambda}\right) \int_0^\infty \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k+1}^2}(\xi)$$

$$\times \int_{\mathbb{R}^k} \left(1 + \frac{1}{\mathbf{v}_1'\mathbf{v}_1 + \mathbf{v}_2'\mathbf{v}_2}\left(\xi^2\tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\tilde{\mathbf{r}}'\mathbf{v}_1 + \mathbf{v}_1'\mathbf{v}_1 + \frac{(\mathbf{v}_1'\mathbf{v}_1 - \xi\tilde{\mathbf{r}}'\mathbf{v}_1)^2}{\mathbf{v}_2'\mathbf{v}_2}\right)\right)^{-(n-k+p+1)/2}$$

$$\times {}_1F_1\left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2} \frac{\frac{1}{\mathbf{v}_1'\mathbf{v}_1 + \mathbf{v}_2'\mathbf{v}_2}\left(\xi^2\tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\tilde{\mathbf{r}}'\mathbf{v}_1 + \mathbf{v}_1'\mathbf{v}_1 + \frac{(\mathbf{v}_1'\mathbf{v}_1 - \xi\tilde{\mathbf{r}}'\mathbf{v}_1)^2}{\mathbf{v}_2'\mathbf{v}_2}\right)}{1 + \frac{1}{\mathbf{v}_1'\mathbf{v}_1 + \mathbf{v}_2'\mathbf{v}_2}\left(\xi^2\tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\tilde{\mathbf{r}}'\mathbf{v}_1 + \mathbf{v}_1'\mathbf{v}_1 + \frac{(\mathbf{v}_1'\mathbf{v}_1 - \xi\tilde{\mathbf{r}}'\mathbf{v}_1)^2}{\mathbf{v}_2'\mathbf{v}_2}\right)}\right)$$

$$\times \exp\left(-\frac{(\mathbf{v}_1'\mathbf{v}_1 + \mathbf{v}_2'\mathbf{v}_2)}{2\lambda} + \frac{1}{\lambda}(\boldsymbol{\mu}'\Sigma^{-1/2}\mathbf{P}\mathbf{v}_1 + \boldsymbol{\mu}'\Sigma^{-1/2}(\mathbf{I} - \mathbf{Q})^{1/2}\mathbf{v}_2)\right) d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\xi.$$

Application of lemma 1 to $\mathbf{v}_1$ with $\mathbf{a}_1 = \tilde{\mathbf{r}}$ and to $\mathbf{v}_2$ with $\mathbf{a}_1 = (\mathbf{I} - \mathbf{Q})^{1/2\prime} \mathbf{\Sigma}^{-1/2} \boldsymbol{\mu}$ and the fact $\tilde{\mathbf{r}} = \mathbf{P}' \mathbf{\Sigma}^{-1/2} \boldsymbol{\mu}$ yields

$$
f_T(x) = \frac{\lambda^{-k/2}}{(2\pi)^{k/2}} \frac{\pi^{1/2(p-1)}}{\Gamma(1/2(p-1))} \frac{\pi^{1/2(k-p-1)}}{\Gamma(1/2(k-p-1))} \exp\left(-\frac{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{2\lambda}\right)
$$

$$
\times \int_0^\infty \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k}^2}(\xi) \int_{-\infty}^\infty \int_0^\infty u_1^{(p-3)/2} \int_{-\infty}^\infty \int_0^\infty u_2^{(k-p-3)/2} \left(1 + \frac{1}{u_2 + v_{21}^2 + u_1 + v_{11}^2}\right.
$$

$$
\times \left.\left(\xi^2 \tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}v_{11} + (u_1 + v_{11}^2) + \frac{((u_1 + v_{11}^2) - \xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}v_{11})^2}{u_2 + v_{21}^2}\right)\right)^{-(n-k+p+1)/2}
$$

$$
\times {}_1F_1\left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2}\right.
$$

$$
\times \left.\frac{\dfrac{1}{u_2 + v_{21}^2 + u_1 + v_{11}^2}\left(\xi^2 \tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}v_{11} + (u_1 + v_{11}^2) + \dfrac{((u_1 + v_{11}^2) - \xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}v_{11})^2}{u_2 + v_{21}^2}\right)}{1 + \dfrac{1}{u_2 + v_{21}^2 + u_1 + v_{11}^2}\left(\xi^2 \tilde{\mathbf{r}}'\tilde{\mathbf{r}} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}v_{11} + (u_1 + v_{11}^2) + \dfrac{((u_1 + v_{11}^2) - \xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}v_{11})^2}{u_2 + v_{21}^2}\right)}\right)
$$

$$
\times \exp\left(-\frac{(u_2 + v_{21}^2)}{2\lambda} + \frac{\sqrt{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\mathbf{r}}'\tilde{\mathbf{r}}}}{\lambda}v_{21}\right)
$$

$$
\times \exp\left(-\frac{(u_1 + v_{11}^2)}{2\lambda} + \frac{\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}}{\lambda}v_{11}\right) du_2\, dv_{21}\, du_1\, dv_{11}\, d\xi.
$$

Transformation of $w_{11} = u_1 + v_{11}^2$, $w_{12} = v_{11}$, $w_{21} = u_2 + v_{21}^2$, $w_{22} = -v_{21}$ leads to

$$
f_T(x) = \frac{\lambda^{-k/2}}{(2\pi)^{k/2}} \frac{\pi^{1/2(p-1)}}{\Gamma(1/2(p-1))} \frac{\pi^{1/2(k-p-1)}}{\Gamma(1/2(k-p-1))} \exp\left(-\frac{\breve{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{2\lambda}\right) \int_0^\infty \xi f_{\chi_p^2}(\xi x) f_{\chi_{n-k+1}^2}(\xi)
$$

$$
\times \int_0^\infty \int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}} (w_{11} - w_{12}^2)^{(p-3)/2} \int_0^\infty \left(1 + \xi^2 \frac{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}} + \frac{w_{11} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{12}}{w_{21}}\right)^{-(n-k+p+1)/2}
$$

$$
\times {}_1F_1\left(\frac{n-k+p+1}{2}; \frac{p}{2}; \frac{\xi x}{2} \frac{\xi^2\left(\dfrac{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}}\right) + \left(\dfrac{w_{11} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{12}}{w_{21}}\right)}{1 + \xi^2\left(\dfrac{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}(w_{21} + w_{12}^2)}{(w_{11} + w_{21})w_{21}}\right) + \left(\dfrac{w_{11} - 2\xi\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{12}}{w_{21}}\right)}\right)
$$

$$
\times \exp\left(-\frac{(w_{11} + w_{21})}{2\lambda} + \frac{\sqrt{\tilde{\mathbf{r}}'\tilde{\mathbf{r}}}}{\lambda}w_{12}\right) U(w_{21})\, dw_{11}\, dw_{12}\, dw_{21}\, d\xi
$$

with

$$
U(w_{21}) = \int_{-\sqrt{w_{21}}}^{\sqrt{w_{21}}} (w_{21} - w_{22}^2)^{(k-p-3)/2} \exp\left(-\lambda^{-1} n\sqrt{\breve{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\breve{\boldsymbol{\mu}} - \tilde{\mathbf{r}}'\tilde{\mathbf{r}}}w_{22}\right) dw_{22}
$$

$$
= \frac{2^{(k-p-2)/2}\sqrt{\pi}\Gamma((k-p-1)/2)}{(\lambda^{-1}\sqrt{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\mathbf{r}}'\tilde{\mathbf{r}}})^{(k-p-2)/2}} w_{21}^{(k-p-2)/2} I_{(k-p-2)/4}\left(\lambda^{-1}\sqrt{\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \tilde{\mathbf{r}}'\tilde{\mathbf{r}}}\sqrt{w_{21}}\right),
$$

where the last integral is evaluated based on the results of Andrews *et al.* (2000, p. 235). Putting components together leads to the statement of the theorem.

Proof of theorem 4. (a) Because $\tilde{T}=T(n-k+1)/p$, from the proof of theorem 3 it holds that

$$\tilde{T}^*\,|\,[\hat{\mathbf{b}}^*=\mathbf{c}_{\mathbf{z}^*}\wedge\hat{\mathbf{H}}_{22}^*/\mathbf{H}_{22}^*=1/\xi]\sim F_{p,n-k+1}(\lambda(\mathbf{c}_{\mathbf{z}^*})),$$

where $\lambda(\mathbf{c}_{\mathbf{z}^*})=\mathbf{H}_{12}^{*\prime}\mathbf{c}_{\mathbf{z}^*}^{-1}\mathbf{H}_{12}^*/\mathbf{H}_{22}^*$. Because $\tilde{T}^*\,|\,[\hat{\mathbf{b}}^*=\mathbf{c}_{\mathbf{z}^*}\wedge\hat{\mathbf{H}}_{22}^*/\mathbf{H}_{22}^*=1/\xi]$ is independent of ξ, it holds that $\tilde{T}^*\,|\,[\hat{\mathbf{b}}^*=\mathbf{c}_{\mathbf{z}^*}]\sim F_{p,n-k}(\lambda(\mathbf{c}_{\mathbf{z}^*}))$. From the proof of theorem 2 of Bodnar & Schmid (2008a), we obtain

$$f_{\tilde{T}^*}(x)=f_{F_{p,n-k+1}}(x)\left(1+\frac{1}{\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}\mathbf{w}^{*\prime}\mathbf{b}^{-1}\mathbf{w}^*\right)^{-(n-k+p+1)/2}$$

$$\times\,_2F_1\left(\frac{n-k+p+1}{2},\frac{n-k+p+1}{2};\frac{p}{2};\frac{px}{n-k+1+px}\frac{1/(\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)\mathbf{w}^{*\prime}\mathbf{b}^{-1}\mathbf{w}^*}{1+1/(\mathbf{z}^{*\prime}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)\mathbf{w}^{*\prime}\mathbf{b}^{-1}\mathbf{w}^*}\right),$$

where $f_{F_{k_1,k_2}}(\cdot)$ stands for the density function of the central F-distribution with k_1 and k_2 degrees of freedom.

Following the proof of theorem 3, the unconditional density of the $\tilde{T}$ statistic is given by

$$f_{\tilde{T}}(x)=\int_{\mathbb{R}^k}f_{\tilde{T}^*}(x\,|\,\mathbf{z}=\mathbf{z}^*)f_{\mathbf{z}}(\mathbf{z}^*)\,\mathrm{d}\mathbf{z}^*$$

$$=\frac{\lambda^{-(p+2)/2}}{2^{(p+2)/2}\sqrt{\pi}}\frac{\exp(-(\boldsymbol{\mu}^{\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})/2\lambda)}{\Gamma(\frac{1}{2}(p-1))(\boldsymbol{\mu}^{\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}-\tilde{\mathbf{r}}^{\prime}\tilde{\mathbf{r}})^{(k-p-2)/4}}f_{F_{p,n-k+1}}(x)$$

$$\times\int_0^\infty\int_0^\infty\left(\int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}}(w_{11}-w_{12}^2)^{(p-3)/2}\,\mathrm{d}w_{12}\right)\left(1+\frac{w_{11}}{w_{21}}\right)^{-(n-k+p+1)/2}$$

$$\times\,_2F_1\left(\frac{n-k+p+1}{2},\frac{n-k+p+1}{2};\frac{p}{2};\frac{px}{n-k+1+px}\frac{w_{11}}{w_{11}+w_{21}}\right)$$

$$\times\exp\left(-\frac{(w_{11}+w_{21})}{2\lambda}\right)w_{21}^{(k-p-2)/4}I_{(k-p-2)/2}\left(\lambda^{-1}\sqrt{\boldsymbol{\mu}^{\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}\sqrt{w_{21}}\right)\mathrm{d}w_{11}\,\mathrm{d}w_{21}.$$

Using the identity

$$\int_{-\sqrt{w_{11}}}^{\sqrt{w_{11}}}(w_{11}-w_{12}^2)^{(p-3)/2}\,\mathrm{d}w_{12}=w_{11}^{(p-2)/2}\sqrt{\pi}\frac{\Gamma(1/2(p-1))}{\Gamma(1/2p)}$$

and making the transformation $w_{11}=tw_{21}$, we get

$$f_{\tilde{T}}(x)=\frac{\lambda^{-(p+2)/2}}{2^{(p+2)/2}}\frac{\exp(-(\boldsymbol{\mu}^{\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})/2\lambda)}{\Gamma(\frac{1}{2}p)(\boldsymbol{\mu}^{\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{(k-p-2)/4}}f_{F_{p,n-k+1}}(x)\int_0^\infty(1+t)^{-(n-k+p+1)/2}$$

$$\times\,_2F_1\left(\frac{n-k+p+1}{2},\frac{n-k+p+1}{2};\frac{p}{2};\frac{px}{n-k+px}\frac{t}{1+t}\right)t^{(p-2)/2}\tilde{U}(t)\,\mathrm{d}t,$$

where

$$\tilde{U}(t)=\int_0^\infty\exp\left(-\frac{(1+t)w_{21}}{2\lambda}\right)w_{21}^{(k+p-2)/4}I_{(k-p-2)/2}\left(\lambda^{-1}\sqrt{\boldsymbol{\mu}^{\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}\sqrt{w_{21}}\right)\mathrm{d}w_{21}.$$

Making the transformation $y = w_{21}^2$, it follows

$$\tilde{U}(t) = 2 \int_0^\infty \exp\left(-\frac{(1+t)y^2}{2\lambda}\right) y^{(k+p)/2} I_{(k-p-2)/2}\left(\lambda^{-1}\sqrt{\mu'\Sigma^{-1}\mu}\, y\right) dy$$

$$= 2\,{}_1F_1\left(\frac{k}{2}; \frac{k-p}{2}; \frac{\mu'\Sigma^{-1}\mu}{2\lambda}\frac{1}{1+t}\right) \frac{\Gamma(\frac{1}{2}k)(\mu'\Sigma^{-1}\mu)^{(k-p-2)/4}\lambda^{-(k-p-2)/2}}{\Gamma(\frac{1}{2}(k-p))2^{(k-p)/2}((1+t)/2\lambda)^{k/2}},$$

where the last equality follows from the proof of theorem 1 of Bodnar & Schmid (2008b). Hence,

$$f_{\tilde{T}}(x) = \frac{\Gamma(\frac{1}{2}k)}{\Gamma(\frac{1}{2}(k-p))\Gamma(\frac{1}{2}p)} \exp\left(-\frac{\mu'\Sigma^{-1}\mu}{2\lambda}\right) f_{F_{p,n-k+1}}(x) \int_0^\infty (1+t)^{-(n-k+p+1)/2}$$

$$\times {}_2F_1\left(\frac{n-k+p+1}{2}, \frac{n-k+p+1}{2}, \frac{p}{2}; \frac{px}{n-k+1+px}\frac{t}{1+t}\right)$$

$$\times t^{(p-2)/2}(1+t)^{-k/2}\, {}_1F_1\left(\frac{k}{2}, \frac{k-p}{2}; \frac{\mu'\Sigma^{-1}\mu}{2\lambda}\frac{1}{1+t}\right) dt$$

$$= \frac{\Gamma(\frac{1}{2}k)}{\Gamma(\frac{1}{2}(k-p))\Gamma(\frac{1}{2}p)} \exp\left(-\frac{\mu'\Sigma^{-1}\mu}{2\lambda}\right) f_{F_{p,n-k+1}}(x) \int_0^\infty \left(1+\frac{1}{z}\right)^{-(n-k+p+1)/2}$$

$$\times {}_2F_1\left(\frac{n-k+p+1}{2}, \frac{n-k+p+1}{2}, \frac{p}{2}; \frac{px}{n-k+1+px}\frac{1}{1+z}\right)$$

$$\times z^{(k-p-2)/2}(1+z)^{-k/2}\, {}_1F_1\left(\frac{n-k+p+1}{2}, \frac{p}{2}; \frac{\mu'\Sigma^{-1}\mu}{2\lambda}\frac{z}{1+z}\right) dz,$$

where the last equality is obtained by the transformation $t = 1/z$. It completes the proof of theorem 4a.

(b) The statement of theorem 4b is a partial case of theorem 4a.