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STOCHASTIC MODELS WITH MULTISTABILITY AND EXTINCTION LEVELS*

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Abstract. For differential equations perturbed by functions of bounded, stationary Markov diffusion processes, the region of multistable points is characterized via controllability properties of an associated control system. This region is given as the union of the domains of attraction for finitely many relatively invariant control sets. For singular systems, i.e., systems with common limit sets for all perturbations, the concept of extinction levels, where the system is absorbed, is introduced. The qualitative theory of stochastic systems, including multistability, is developed for this class of models. Various examples, including a random Lotka–Volterra model and the Lorenz system, are analyzed.

Key words. stochastic dynamical systems, bistability, control sets, absorption

AMS subject classifications. 60H10, 34F05, 93B05, 93C10

1. Introduction.

Recently, the phenomenon of bistability (or, better, multistability) in stochastic systems has attracted considerable attention. A point is multistable if the system response from this point exhibits different limit behavior, each with positive probability. While in deterministic systems the trajectory from an initial point converges to its unique limit set (or possibly to ∞), in systems with random excitation the limit set may not be unique, leading to multistability. This behavior is typically observed in systems with a combination of stable and unstable limit sets. The Duffing oscillator with its fourth-order potential exhibits bistability under random excitation (see, e.g., [31] for a discussion of relative stability under white noise excitation); the mechanics of ship roll motion in random seas leads to a damped version of the Duffing oscillator with two bistable bands (see, e.g., [30] and [11]). Models of chemical reactors with Arrhenius dynamics (see, e.g., [27]) and the Takens–Bogdanov oscillator (modeling the motion of a thin panel in a flow [16], population dynamics [5], or solar gravity [23]) both exhibit the limit set structure that allows us to predict bistability under certain random excitations. In fact, using the results of this paper, the studies [8] (for chemical reactors) and [11] (for the Takens–Bogdanov oscillator) can be interpreted as multistability studies. In this paper we characterize the set of multistable points and the possible system response from these points.

A second phenomenon in stochastic systems that deserves more attention is the levels of extinction. Often models with continuous state space are used as approximations for discrete events, e.g., in population dynamics or chemical reactions. Extinction (or absorption) in these systems will occur already when the solution of the continuous model is still a finite distance away from the absorption level; in particular, extinction may occur in finite time. Think, for example, of the extinction of species with too few offsprings or the dying out of a chemical reaction if the concentration of one of the reactants is too low. To avoid mathematical artifacts in approximating

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continuous models, we will introduce the concept of "extinction levels," i.e. small sets in the state space, in which the stochastic system is stopped. A theory of global qualitative behavior for these systems will be developed in this paper.

More precisely, we consider noisy dynamical systems of the form

$$\dot{x}(t) = X(x(t), \eta(t)) \quad \text{on } M, \tag{1.1}$$

where M is a finite-dimensional, connected C^∞ manifold and X is a C^∞ vector field on M. The noise process $\eta(t)$ is determined by a stochastic differential equation

$$\begin{aligned} d\xi(t) &= Y_o(\xi(t))dt + \sum_{i=1}^{k} Y_i(\xi(t)) \circ dW_i \quad \text{on } N, \\ \eta(t) &= F(\xi(t)) \end{aligned} \tag{1.2}$$

where N is a finite-dimensional, connected C^∞ manifold; Y_j, $j = 0, \ldots, k$, are C^∞ vector fields on N; "o" denotes the symmetric (Stratonovič) stochastic differential; and $F \colon N \to U \subset \mathbb{R}^m$ is a surjective C^∞ function. We assume throughout that (1.2) has a stationary, ergodic solution and satisfies the nondegeneracy condition

$$\dim \mathcal{LA}\{Y_1, \ldots, Y_k\}(p) = \dim N \quad \text{for all } p \in N. \tag{A}$$

Here $\mathcal{LA}\{\cdot\}$ denotes the Lie algebra of a set of vector fields, and $\dim \mathcal{LA}\{\cdot\}(p)$ is the dimension of the (differential geometric) distribution generated by $\mathcal{LA}\{\cdot\}$ in the tangent space T_pN of N at the point $p \in N$. We refer the reader to [4] for details on manifolds, vector fields, and Lie algebras of vector fields; to [18] for details on stochastic differential equations on manifolds; and to [21] for a more detailed description of this setup.

We chose the Stratonovič version of stochastic differential equations for the background noise $\xi(t)$ in (1.2), because one of our main tools, the support theorem, uses this version. Any Itō stochastic differential equation with smooth coefficients can be transformed into a corresponding Stratonovič equation; see, e.g., [18] for the correction formula. This formula affects only the drift of the process; hence assumption (A) holds for the Itō version iff it holds for the Stratonovič versions. Consequently, all results in this paper concerning regions of the state space M in which the system (1.1) has a specific behavior (like stationarity, transience, multistability, or extinction) are the same for both the Itō and the Stratonovič versions. Differences between the two versions will only show up in the simulated trajectories (see equation (4.3) and Figures 4, 10, and 11) and the statistical results based on the simulations, such as the extinction probabilities in Figure 7. Note furthermore that the noise process $\eta(t)$ in our setup can be bounded or unbounded.

The pair process $(x(t), \xi(t))$ from (1.1), (1.2) is a Markov diffusion process if the initial condition is independent of the σ-algebra generated by the Wiener process W; see, e.g., [18]. Note that the component $x(t)$ itself is, in general, not a Markov process.

We are interested in the qualitative behavior of the stochastic dynamical system (1.1), in particular in stationary solutions (i.e., invariant probability measures), areas of multistability, convergence toward stationary solutions, and change of behavior depending on parameters of the system (stochastic bifurcations). We will analyze this behavior in the context of the Markov process $(x(t), \xi(t))$, i.e., on the level of invariant distributions and convergence in distribution, and then "project" the results onto the $x(t)$-component. This allows us to interpret the results in a physically meaningful way.

When considering stochastic dynamical systems of the form (1.1) as components of a Markov diffusion process, two different cases have to be distinguished.

Regular systems are systems for which the supports of the transition probabilities from each point $(x,p) \in M \times N$ have nonvoid interior in $M \times N$; i.e., they form "thick" sets in the state space of the Markov process. A sufficient condition to guarantee this fact in terms of the vector fields is (compare [21])

(B) $\dim \mathcal{LA}\{(X, Y_o), Y_1, \dots, Y_k\}(x,p) = \dim M + \dim N \quad$ for all $(x,p) \in M \times N$.

Assumption (B) can be relaxed somewhat, if one is willing to assume conditions on the transition semigroup of the Markov process (see, e.g., [24]), but we will stick here to assumptions that can be verified directly via the given vector fields.

Singular systems are systems for which there exist invariant sets $I \subset M$ with empty interior. Typical examples are common fixed points for all noise trajectories, i.e., points $x^* \in M$ with $X(x^*, \eta) = 0$ for all $\eta \in U$, or common lower dimensional manifolds, i.e., $X(x, \eta)$ is tangential to I for all $\eta \in U$. Systems of this type are quite common in physics, biology, chemistry, and engineering; compare the examples in § 4. In many situations, these systems behave in a small neighborhood of the set I as if they were absorbed to I, although in the mathematical model the trajectories will not reach the lower dimensional, invariant set from the outside with probability one. A theory of global qualitative behavior for these systems will be developed using the concept of "extinction levels." In particular, we consider aspects of stationarity, convergence, multistability, and bifurcations in the Markovian context. An example for a model with extinction level was treated in [25].

In § 2 we analyze the regions of multistability for regular systems, after a brief review of some analytical tools needed for our analysis. Section 3 is devoted to singular systems with levels of extinction, and in § 4 we present some examples that show, in particular, how the stochastic bifurcation picture can change when extinction levels are present.

2. Multistability in regular stochastic systems. In this section we will characterize the region of multistability for regular stochastic systems. The basic mathematical tool of our approach is the analysis of associated control systems and their relation to stochastic systems via the celebrated "support theorem"; see, e.g., [29] or [22]. We first introduce some notation.

$W(t)$ is a standard, k-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. For a Markov process $\{z(t), t \geq 0\}$ on Ω with topological state space L the family of transition probabilities is denoted by $P(t, z, A), A \in \mathcal{B}(L)$ the Borel σ-algebra on L. A probability measure μ on $(L, \mathcal{B}(L))$ is said to be invariant for $z(t)$ if $P_t\mu = \mu$ for all $t \geq 0$, where $\{P_t, t \geq 0\}$ is the associated Markov semigroup of operators.

To the stochastic dynamical system (1.1), (1.2) we associate a nonlinear control system of the form

$$\dot{x}(t) = X(x(t), u(t)) \quad \text{on } M, \tag{2.1}$$

with $u \in \mathcal{U}\colon = \{u\colon [0,\infty) \to U; u \text{ piecewise constant }\}$. Since we are interested in the long-term behavior of the stochastic system, we assume that all solutions $\varphi(t,x,u)$ of (2.1) with $(x,u) \in M \times \mathcal{U}, \varphi(0,x,u) = x$ are unique and defined for all $t \geq 0$. The control sets of (2.1) play an important role in the analysis of the global behavior of the stochastic system (1.1); see, e.g., [20] and [21]. We recall some facts for easy reference.

The (positive) orbit of a point $x \in M$ up to time t for the control system (2.1) is $\mathcal{O}^+_{\leq t}(x) = \{y \in M;$ there exist $s \in [0,t]$ and $u \in \mathcal{U}$ with $\varphi(s,x,u) = y\}$, and we set

$$\mathcal{O}^+(x) = \bigcup_{t \geq 0} \mathcal{O}^+_{\leq t}(x).$$

DEFINITION 2.1. *A set $D \subset M$ is a* control set *of* (2.1) *if*

(i) $D \subset \mathrm{cl}\, \mathcal{O}^+(x)$ *for all $x \in D$ ("cl" denotes the closure of a set),*

(ii) *there exist $u \in \mathcal{U}$ and $x \in D$ with $\varphi(t,x,u) \in D$ for all $t \geq 0$,*

(iii) *D is maximal (with respect to set inclusion) with the properties* (i) *and* (ii).

A set $L \subset M$ is invariant *for* (2.1) *if the orbits for all $x \in L$ are contained in the closure of L, i.e. $\mathcal{O}^+(x) \subset \mathrm{cl}\, L$ for all $x \in L$. In particular, a control set $C \subset M$ is an* invariant control set *if* $\mathrm{cl}\, \mathcal{O}^+(x) = \mathrm{cl}\, C$ *for all $x \in C$.*

The domain of attraction $\mathcal{A}(A)$ of a set $A \subset M$ is given by all points $x \in M$ for which there exists a control u that steers x arbitrarily close to A; i.e., for a control set $D \subset M$ we define

$$\mathcal{A}(D) = \{y \in M; \mathrm{cl}\, \mathcal{O}^+(y) \cap D \neq \emptyset\}. \tag{2.2}$$

Let $\hat{D} = \{D \subset M; D$ is a control set of (2.1)$\}$ be the set of control sets for the control system. On $\hat{D}$ a natural order is defined through reachability; i.e., for $D_1, D_2 \in \hat{D}$ we set

$$D_1 \prec D_2 \text{ if there exists } x \in D_1 \text{ with } \mathrm{cl}\, \mathcal{O}^+(x) \cap D_2 \neq \emptyset. \tag{2.3}$$

Because of Definition 2.1(iii) this means that any point $x \in D_1$ can be steered approximately by proper choice of a control $u(x)$ to D_2, but not vice versa if $D_1 \neq D_2$. Note that the invariant control sets are maximal elements in $\hat{D}$ in this order.

For the rest of this section we assume that the stochastic system (1.1), (1.2) is regular, i.e., that assumptions (A) and (B) hold. Then we can characterize the possible limit behavior of the stochastic system, i.e., its stationary solutions and invariant measures as follows (compare [20]). A solution $x^0(t)$ of (1.1) is called a stationary Markov solution if the pair process $(x^o(t), \xi(t))$ is a Markov solution of (1.1), (1.2) that is (strictly) stationary. The stationary Markov solutions of (1.1) are determined through invariant control sets of (2.1) in the following way. Equation (1.1) has a stationary Markov solution iff (1.1), (1.2) admits an invariant probability measure. Any ergodic invariant probability μ has support $\mathrm{supp}\mu = C \times N$, where C is an invariant control set of (2.1), and μ is unique on $C \times N$. Recall that all invariant probability measures are convex combinations of the ergodic ones. An invariant control set C with the property that $C \times N$ is the support of an invariant probability measure is called an ergodic control set. While invariant control sets are not always ergodic, we note that the compact ones always have this property. Hence, if all invariant control sets of (2.1) are compact, then there is a one-to-one correspondence between ergodic, stationary Markov solutions and invariant control sets.

Having identified the stationary Markov solutions of the stochastic system, we turn to the behavior of the individual solutions $x(t, x_o)$ of (1.1) with initial value $x_o \in M$ and their convergence toward the stationary limiting solutions. Here solution means that the pair $(x(t,x_o), \xi(t,\xi_o))$ for $(x_o, \xi_o) \in M \times N$ is a Markov solution of (1.1), (1.2). Since the noise component $\xi(t)$ is independent of the x-component and assumed to be uniquely ergodic, $\xi(t,\xi_o)$ will tend, for any initial value $\xi_o \in N$, in distribution to the unique invariant measure of (1.2). Therefore it suffices to analyze $x(t, x_o)$.

A Markov process $z(t)$ with topological state space L is said to *converge to* ∞ (notation: $z(t) \to \infty$) if $z(t)$ is transient on any compact set $K \subset L$. Note that processes in compact invariant sets cannot converge to ∞; this holds, in particular, if L itself is compact.

The solution $x(t, x_0)$ of (1.1) is said to *converge to a closed set* $A \subset M$ if its first hitting time $\tau_A(x_o) = \inf\{t \geq 0; x(t, x_o, \omega) \in A\}$ is finite with positive probability. Note that if A is an invariant set of (2.1), then the trajectories of (1.1) that enter A will stay in A for all later times. We introduce the following notation.

$\hat{C} = \{C \subset M; C \text{ is an invariant control set }\}$ and $\hat{C}_e$ are those invariant control sets of (2.1) such that for $C \in \hat{C}_e$ we have that $C \times N$ is the support of some ergodic, invariant probability of (1.1), (1.2).

$p_C(x_o) = P\{\tau_C(x_o) < \infty\}$ is the probability that the first hitting time of C for the solution with initial value $x_o \in M$ is finite.

$p_\infty(x_0) = P\{x(t, x_o) \to \infty\}$ is the probability that $x(t, x_o)$ converges to ∞. Note that under assumption (B) the control system (2.1) has at most countably many invariant control sets (see, e.g., [20]). The following theorem summarizes some pertinent results from [20].

THEOREM 2.2. *Consider the stochastic system* (1.1), (1.2) *under assumptions* (A) *and* (B).

(i) *For all* $x_o \in M$ *we have* $p_\infty(x_o) + \sum_{C \in \hat{C}_e} p_C(x_o) = 1$.

(ii) *For* $x_o \in M$ *and* $C \in \hat{C}$ *it holds that* $p_C(x_o) > 0$ *iff* $x_o \in \mathcal{A}(C)$.

(iii) *For* $C \in \hat{C}_e$ *denote by* μ_C *the unique ergodic invariant measure of* (1.1), (1.2) *with* $\operatorname{supp} \mu_C = C \times N$. *If* $\sum_{C \in \hat{C}_e} p_C(x_o) = 1$, *then* $(x(t, x_o), \xi(t)) \underset{\mathcal{D}}{\Longrightarrow} \sum p_C(x_o) \mu_C$ *as* $t \to \infty$, *where* $\underset{\mathcal{D}}{\Longrightarrow}$ *denotes convergence in distribution.*

This result says that for any initial value $x_o \in M$ the solution of the stochastic system (1.1) converges either to ergodic control sets or to ∞; no other limit behavior is possible. Furthermore, convergence toward an invariant control set C occurs with positive probability if and only if the initial value x_o is in the control theoretic domain of attraction of C as defined in (2.2). Therefore, in compact, invariant sets $L \subset M$ it suffices to compute the invariant control sets and their domains of attraction in order to describe the long-term behavior of the solutions of (1.1) completely. Now the obvious question arises. For which $x_o \in M$ does there exist exactly one $C \in \hat{C}_e$ with $p_C(x_o) > 0$? This is the problem of multistability.

DEFINITION 2.3. *A point* $x_o \in M$ *is said to be* multistable, *if there exist different stationary solutions* $(x^1(t), \xi(t))$ *and* $(x^2(t), \xi(t))$ *with invariant probability measures* μ_1 *and* μ_2 *such that* $p_{\operatorname{supp} \mu_i}(x_o) > 0$ *for* $i = 1, 2$. *The set of multistable points is denoted by* MS.

By Theorem 2.2, multistability can occur only when (2.1) has more than one ergodic control set. The following proposition shows that this condition is in fact equivalent to $MS \neq \emptyset$ in compact, invariant sets of (2.1).

PROPOSITION 2.4. *Consider the stochastic system* (1.1), (1.2) *under assumptions* (A) *and* (B), *and let* $L \subset M$ *be a compact, invariant set. Then there exists a multistable point* $x_o \in L$ *iff the system* (2.1) *has at least two ergodic control sets.*

Proof. If $x_o \in M$ is multistable, then (2.1) has at least two ergodic control sets, since ergodic invariant probability measures on control sets are unique.

Assume that (2.1) has two ergodic control sets, say C_1 and C_2, in L. Then $\mathcal{A}(C_1) \neq L$, and there exists $y \in \partial\mathcal{A}(C_1) \cap L$, where $\partial\mathcal{A}(C_1)$ denotes the boundary of $\mathcal{A}(C_1)$. By Theorem 2.2(i) there exists an invariant (and hence ergodic) control set

$C_3 \subset L$ with $p_{C_3}(y) > 0$. Under assumption (B) all invariant control sets have nonvoid interior, and therefore there exist a control $u \in \mathcal{U}$ and a time $t_3 > 0$ with $\varphi(t_3, y, u) \in \operatorname{int} C_3$. By continuous dependence of the solutions of ordinary differential equations on the right-hand side, there exists a continuous function $v\colon [0, t_3 + 1] \to U$ with an L^∞-neighborhood $\mathcal{V}(v)$ and a neighborhood $B(y) \subset L$ such that $\varphi(t_3 + 1, z, w) \in \operatorname{int} C_3$ for all $z \in B(y)$ and all $w \in \mathcal{V}(v)$. Now by [22], $P\{F^{-1}\mathcal{V}(v)\} > 0$, and thus $p_{C_3}(z) > 0$. Pick $z \in B(y) \cap \mathcal{A}(C_1)$; then by Theorem 2.2(ii) z is a multistable point. □

Remark 2.5. The concept of multistability, as defined in Definition 2.3, is the standard one used in the physics literature (there often called "bistability"). To prove results for this concept, we must consider a compact, invariant set $L \subset M$ in Proposition 2.4 and in the results that follow in this section. If one generalizes Definition 2.3 to allow for $p_\infty(x_o) > 0$ as one of the possible limit behaviors from x_o, then the compactness assumption is not necessary. This generalization is straightforward and left to the reader.

Proposition 2.4 gives a criterion for the existence of multistable points. We now proceed to describe the set MS more precisely. To avoid certain degeneracies at the boundary ∂L of the compact, invariant set $L \subset M$, we require that all limit sets of the control system (2.1) be (uniformly) bounded away from ∂L. The following strong invariance condition turns out to be sufficient:

(C) $L = \operatorname{cl}(\operatorname{int} L)$, and for all $x \in MS \cap \operatorname{int} L$ there exists $\epsilon(x) > 0$ such that whenever $\varphi(t, x, u) \in MS$ for some $t \geq 0$, $u \in \mathcal{U}$ then $d(\varphi(t, x, u), \partial L) \geq \epsilon(x)$; there exists $\epsilon_0 > 0$ such that for all $x \in \operatorname{cl} MS$ and $u \in \mathcal{U}$ we have that if $y = \lim_{k\to\infty} \varphi(t_k, x, u) \in MS$ for some sequence $t_k \to \infty$, then $d(y, \partial L) \geq \epsilon_0$.

Here $d(\cdot,\cdot)$ denotes the metric on the state space M.

The set MS of multistable points has some obvious topological properties, which we list next. Note that in a compact invariant set $L \subset M$ there are at least one and at most finitely many invariant control sets. From now on we assume that the set L as above is given and that there are l invariant control sets in L. It follows directly from the proof of Proposition 2.4 that the set $MS \subset L$ is open in L and given by

$$MS = \bigcup_{1 \leq i,j \leq l, i \neq j} (\mathcal{A}(C_i) \cap \mathcal{A}(C_j)). \tag{2.4}$$

Let ∂MS and $\partial_L MS$ denote the boundaries of MS in M and in L, respectively. Define

$$\partial_j MS = \partial_L MS \cap \mathcal{A}(C_j), \quad j = 1, \ldots, l,$$
$$\partial_o MS = MS \cap \partial L.$$

Since each $y \in \partial_L MS$ is in the domain of attraction of exactly one invariant control set, we have

$$\partial_L MS = \bigcup_{j=1}^{l} \partial_j MS \quad \text{and} \quad \partial MS = \bigcup_{j=0}^{l} \partial_j MS.$$

$\partial_j MS$ is isolated and open in $\partial_L MS$, and in the topology of M

$$y \in \operatorname{cl} \partial_j MS \cap \operatorname{cl} \partial_i MS \text{ for some } 1 \leq i, j \leq l, i \neq j, \text{ implies } y \in \partial L. \tag{2.5}$$

Note further that $\partial_L MS$ consists of at least two different $\partial_j M$. Analogous definitions can be given for every connected component of MS, and all the properties stated above remain valid.

We will characterize the set MS of multistable points in L via a finite number of specific variant control sets, which, however, share some properties of the invariant control sets. The idea is to use the order between control sets, as defined in (2.3). Recall that the invariant control sets $C_1, \dots, C_l$ in L are the maximal elements in $(\hat{D}, \prec)$. Multistable points are those from which at least two invariant control sets C_1 and C_2 can be reached in (2.1). Hence we are looking for control sets D such that $D \prec C_1$ and $D \prec C_2$. To characterize all multistable points, we must consider the maximal control sets $D \in (\hat{D}, \prec)$ with this property and their corresponding domains of attraction. As we will see below, this idea does, in fact, work. However, it is easier for the proofs to work with the following, more technical definition. Recall from Theorem 2.2(ii) that $x \in MS \subset L$ iff there exist (at least) two invariant control sets $C_1, C_2 \subset L$ with $x \in \mathcal{A}(C_i)$ for $i = 1, 2$.

DEFINITION 2.6. *A control set $D \subset MS$ is called* relatively invariant, *if $x \in D$ and $\varphi(t, x, u) \notin D$ for some $t > 0, u \in \mathcal{U}$ imply $\varphi(t, x, u) \notin MS$.*

The following two lemmas present basic properties of relatively invariant control sets that will be needed later in the article. Recall first of all that by (B) control sets with nonvoid interior are closed iff they are invariant (see [20]). For relatively invariant control sets we obtain an analogous result.

LEMMA 2.7. *For a control set D the following properties are equivalent.*

(i) *D is relatively invariant.*

(ii) $\operatorname{int} D \neq \emptyset$, *and D is closed in MS; i.e., $D \subset MS$ and $(\partial D \backslash D) \cap MS = \emptyset$.*

Proof. If D is relatively invariant, consider $x \in D, u \in \mathcal{U}$ with $\varphi(t, x, u) \in D \subset MS$ for all $t \geq 0$. By assumption (B), $\varphi(t, x, u) \in \mathcal{O}^+_{\leq t}(x) \subset \operatorname{cl}(\operatorname{int} \mathcal{O}^+_{\leq t}(x))$ for $t > 0$ (see, e.g., [19, p. 69] for this standard fact from geometric control theory). Thus we have $MS \cap \operatorname{cl}(\operatorname{int} \mathcal{O}^+_{\leq t}(x)) \neq \emptyset$ and, since MS is open, also $MS \cap \operatorname{int} \mathcal{O}^+_{\leq t}(x) \neq \emptyset$. By relative invariance of D one obtains that $MS \cap \operatorname{int} \mathcal{O}^+_{\leq t}(x) \subset D$ and hence $\operatorname{int} D \neq \emptyset$.

Next suppose that D is not closed in MS, i.e., there exists $y \in (\partial D \backslash D) \cap MS$. But MS is open, and hence (B) implies $\emptyset \neq \operatorname{int} \mathcal{O}^+_{\leq t}(y) \subset MS$ for $t > 0$ sufficiently small. By Definition 2.1(iii), $\operatorname{int} \mathcal{O}^+_{\leq t}(y) \cap D = \emptyset$ (otherwise $y \in D$), and hence there exists $z \in \operatorname{int} \mathcal{O}^+_{\leq t}(y) \backslash D$. Now continuous dependence of the solutions on the initial value yields a contradiction to relative invariance.

Conversely, suppose that $\operatorname{int} D \neq \emptyset$ and there are $t > 0, x \in D$, and $u \in \mathcal{U}$ with $\varphi(t, x, u) \in MS \backslash D$. Let $T := \sup\{\tau > 0; \varphi(\tau, x, u) \in D\}$. Clearly $\varphi(T, x, u) \in MS \cap \partial D$. If $\varphi(T, x, u) \in D$ one can steer this point into $\operatorname{int} D$. Now continuous dependence gives a contradiction to maximality of T. □

LEMMA 2.8. *For every $x \in MS$ there exist $y \in \mathcal{O}^+(x)$ and $J(y) \subset \{1, \dots, l\}$ such that $y \in \bigcap_{j \in J} \mathcal{A}(C_j)$ and J is minimal for y in the following sense. If $\varphi(t, y, v) \in MS$ for some $t > 0, v \in \mathcal{U}$, then $\varphi(t, y, v) \in \bigcap_{j \in J} \mathcal{A}(C_j)$.*

Proof. Since $x \in MS$, there exists $J_1 \subset \{1, \dots, l\}$ with $x \in \bigcap_{j \in J_1} \mathcal{A}(C_j)$. If there are $t_1 > 0$ and $v_1 \in \mathcal{U}$ with $y_i := \varphi(t_1, x, v_1) \in MS \backslash \bigcap_{j \in J_1} \mathcal{A}(C_j)$, then there exists $\emptyset \neq J_2 \subsetneqq J_1$ with $y_1 \in \bigcap_{j \in J_2} \mathcal{A}(C_j)$. Proceeding recursively, one ends up, after finitely many steps, with a point $y \in \mathcal{O}^+(x)$ with a minimal index set J. □

Note that a minimal index set as in Lemma 2.8 has at least two elements. Furthermore, the lemma implies that for each relatively invariant control set D there is $J \subset \{1, \dots, l\}$ such that for each $x \in D$ the index set J is minimal. This follows simply from the fact that D is a control set. The following fundamental proposition shows the existence of relatively invariant control sets and gives some insight into their structure.

PROPOSITION 2.9. *Assume* (C). *Let* $x \in MS \subset L$ *with*

$$x \in \bigcap_{j \in J} \mathcal{A}(C_j) \quad \textit{where } J \subset \{1, \dots, l\} \textit{ is some minimal index set for } x.$$

Then there exists a relatively invariant control set $D \subset \operatorname{cl} \mathcal{O}^+(x)$ *with*

$$\partial D \cap \partial_j MS \neq \emptyset \quad \textit{for all } j \in J. \tag{2.6}$$

Proof. We first introduce some notation. Denote the connected component of MS that contains x by M_x, and let

$$Q_o := \{y \in M_x; d(y, \operatorname{cl} \partial_o M_x) \geq \epsilon(x)\},$$

where $\epsilon(x)$ is chosen according to assumption (C). For $\epsilon > 0$ and $j \in J$ define

$$N_j(\epsilon) := \{y \in M_x; d(y, \operatorname{cl} \partial_j M_x) \leq \epsilon\},$$
$$Q_j(\epsilon) := \{y \in M_x \cap Q_o; d(y, \operatorname{cl} \partial_j M_x) = \epsilon\}.$$

Using (2.5), we can choose $\epsilon > 0$ small enough such that

$$d(\partial_j M_x \cap Q_o, \partial_i M_x \cap Q_o) \geq 5\epsilon \quad \text{for all } i, j \in J, i \neq j.$$

Hence the sets $Q_j(\epsilon)$ are nonvoid, compact, and pairwise disjoint with distance at least 3ϵ. Decreasing, if necessary, ϵ further, we may assume that $x \in M_x \backslash \bigcup_{j \in J} N_j(2\epsilon)$.

Every trajectory $\{\varphi(t, y, u), t \geq 0\}$ with $y \in \operatorname{cl} \mathcal{O}^+(x) \cap (M_x \backslash N_j(2\epsilon))$ that approaches C_j for $t \to \infty$ must exit through $\partial_j M_x \cap Q_o$ and must cross $Q_j(\epsilon)$. For every y in this set there exists a control $u \in \mathcal{U}$ with this property. We find that

$$\emptyset \neq Q_j(x, \epsilon) := \operatorname{cl} \mathcal{O}^+(x) \cap Q_j(\epsilon) \subset \bigcap_{j \in J} \mathcal{A}(C_j)$$

for all $j \in J$ and all $\epsilon > 0$ small enough. Define

$$Q(x, \epsilon) := \bigcup_{j \in J} Q_j(x, \epsilon).$$

Now consider the family of sets

$$Q(y, \epsilon) := \operatorname{cl} \mathcal{O}^+(y) \cap Q(x, \epsilon), \quad y \in Q(x, \epsilon).$$

These sets are nonvoid by the remarks above, since $Q_j(x, \epsilon)$ and $Q_i(x, \epsilon)$ have distance at least 3ϵ for $i \neq j$. Furthermore, these sets are compact and ordered via

$$Q(y, \epsilon) \prec Q(y', \epsilon) \quad \text{if } y' \in \operatorname{cl} \mathcal{O}^+(y).$$

Every linearly ordered set $\{Q(y_i, \varepsilon);\ i \in I\}$ has an upper bound

$$Q(y, \varepsilon) = \bigcap_{i \in I} Q(y_i, \varepsilon) \quad \text{for some } y \in \bigcap_{i \in I} Q(y_i, \varepsilon),$$

since this intersection is nonempty. Thus Zorn's lemma implies that this family of sets has a maximal element $Q(y, \epsilon)$. Now the set

$$D := \operatorname{cl} \{z \in MS; z \in \mathcal{O}^+(y)\},$$

where the closure is taken in MS , is a relatively invariant control set due to the following explanation. Observe first that every $z \in D$ is approximately reachable from y. Conversely, $y \in \operatorname{cl} \mathcal{O}^+(z)$ for all $z \in D$, since otherwise $y \notin Q(z, \epsilon) = \operatorname{cl} \mathcal{O}^+(z) \cap$

$Q(x,\epsilon) \subsetneqq Q(y,\epsilon)$, contradicting the maximality of $Q(y,\epsilon)$. The other properties of a control set are obviously satisfied.

Now suppose that, contrary to relative invariance, there are $t > 0$ and $u \in \mathcal{U}$ with $\varphi(t,y,u) \in MS\backslash D$. Let $T := \sup\{\tau > 0; \varphi(\tau,y,u) \in D\}$. Then $\varphi(T,y,u) \in (MS \cap \partial D)\backslash D$. By minimality of J we have $\varphi(T,y,u) \in \bigcap_{j\in J} \mathcal{A}(C_j)$. Hence one can steer $\varphi(T,y,u)$ into some point $z \in \operatorname{cl} \mathcal{O}^+(y) \cap Q(x,\epsilon)$. By definition of $D, z \in D$. This contradicts $\varphi(T,y,u) \notin D$.

Finally, assertion (2.6) follows because in the construction above ϵ can be made arbitrarily small. □

Proposition 2.9 yields the characterization of relatively invariant control sets via the reachability order (2.3) as mentioned before Definition 2.6.

COROLLARY 2.10. *A control set D is relatively invariant iff there exist invariant control sets $C_1 \neq C_2$ with $D \prec C_i, i = 1,2$, and D is maximal with this property; i.e., $D \prec D'$ and $D' \prec C_i, i = 1,2$, for a control set D' implies $D = D'$.*

Proof. Obviously, every relatively invariant control set satisfies the properties stated in the corollary. Conversely, suppose that $D \subset MS$ is not relatively invariant. Then there exists $x \in D$ with $y := \varphi(t,x,u) \in MS\backslash D$ for some $t > 0, u \in \mathcal{U}$. By Lemma 2.8 we may assume that there is a minimal index set J for y. Now Proposition 2.9 yields a relatively invariant control set $D' \neq D$ with $D \prec D'$ and $D' \prec C_i$ for some invariant control sets $C_1 \neq C_2$. Hence D is not maximal. □

Recall that in L the invariant control sets are the maximal elements of $(\hat{D}, \prec)$, hence they are not comparable with respect to the order $\prec$. By Corollary 2.10 the relatively invariant control sets are also not comparable with respect to $\prec$. But there may exist control sets between the invariant and the relatively invariant ones. These are in the domain of attraction of exactly one invariant control set. All control sets that are smaller (with respect to $\prec$) than the relatively invariant control sets are, together with their respective domains of attraction, contained in the set MS of multistable points.

Finally we show that the number of relatively invariant control sets is finite.

PROPOSITION 2.11. *Assume* (C). *Then there are only finitely many relatively invariant control sets in L. In particular, the set MS of multistable points has only finitely many connected components.*

Proof. The second statement follows from the first, since by the construction in the proof of Proposition 2.9 one can find a relatively invariant control set in every connected component of MS.

Now assume, contrary to the first assertion, that there are countable many relatively invariant control sets $D_n, n \in \mathbb{N}$. Thus $D_i \neq D_j$ for $i \neq j$ and $\operatorname{int} D_n \neq \emptyset$ for all $n \in \mathbb{N}$ by Lemma 2.7. In the Hausdorff metric on the nonvoid, compact subsets of $\operatorname{cl} MS$ there exists a cluster point F of $\operatorname{cl} D_n, n \in \mathbb{N}$. Hence we may assume that $\operatorname{cl} D_n$ converges to the compact set $F \subset \operatorname{cl} MS$. Clearly, $\operatorname{int} F = \emptyset$. Furthermore, for all $y \in F$ the strong invariance condition (C) implies that $d(y, \partial L) \geq \epsilon_o$. Every $x \in \operatorname{int} D_n$ is a limit point for $t \to \infty$ of a periodic trajectory in MS and, hence, has a distance at least ϵ_o from ∂L, and $\operatorname{cl} D_n = \operatorname{cl}(\operatorname{int} D_n)$.

Since the number of invariant control sets in L is finite, we may consider a subsequence of D_n (which we denote again by D_n) such that there exists $J \subset \{1,\ldots,l\}$ with $D_n \subset \bigcap_{j\in J} \mathcal{A}(C_j)$ and J is a minimal index set for all $n \in \mathbb{N}$. Applying Proposition 2.9 to $x \in D_n$, we obtain that $\partial D_n \cap \partial_j MS \neq \emptyset$ for all $j \in J$, all $n \in \mathbb{N}$. Using (2.5) we can choose $\epsilon > 0$ and $x_n \in D_n$ such that $d(x_n, \partial_L MS) \geq \epsilon$ for all $n \in \mathbb{N}$. Hence $F \cap MS \neq \emptyset$. By Proposition 2.9 one finds for $x \in F \cap MS$ a relatively invariant control set D with $x \in \mathcal{A}(D)$. Since $\operatorname{int} D \neq \emptyset$ by Lemma 2.7, we obtain a contradiction to the relative invariance of D_n for n large enough. □

The following theorem is the main result of this section. It characterizes the set of multistable points via relatively invariant control sets and summarizes the findings of this section.

THEOREM 2.12. *Consider the stochastic system* (1.1), (1.2) *under assumptions* (A) *and* (B). *Let $L \subset M$ be a compact invariant set satisfying assumption* (C). *Then the set MS of multistable points is given by*

$$MS = \bigcup_{j=1}^{l_1} \mathcal{A}(D_j),$$

where $D_j, j = 1, \dots, l_1 < \infty$, are the relatively invariant control sets of (2.1).

Proof. By the proof of Proposition 2.4, $x \in MS$ iff there exist invariant control sets C_1, C_2 in L such that $x \in \mathcal{A}(C_i), i = 1, 2$. Corollary 2.10 shows that each point in a relatively invariant control set D and hence each point in $\mathcal{A}(D)$ satisfies this property because relatively invariant control sets are open according to Lemma 2.7. By Proposition 2.11 we have that $l_1 < \infty$, and hence $\bigcup_{j=1}^{l_1} \mathcal{A}(D_j) \subset MS$. Conversely, let $x \in MS$. By Lemma 2.8 there exists $y \in \mathcal{O}^+(x)$ with minimal index set J. Now Proposition 2.9 implies that y, and hence x, are in the domain of attraction of some relatively invariant control set. □

Theorem 2.12 reduces the computation of multistable points for the stochastic system (1.1) to the computation of the finitely many relatively invariant control sets and their domains of attraction for the control system (2.1). It is easy to see that not all variant control sets are relatively invariant, even if there is more than one invariant control set; see Remark 2.13 below and the examples in § 4.

For systems with small noise, the control structure can be described as a perturbation of the limit structure of the system without noise; see, e.g., [8] and [9]. However, even in this case the control sets and their domains of attraction have to be computed numerically (see [15] for an algorithm, which is effective at least in low dimensions).

Remark 2.13. For one-dimensional systems, the control sets can be computed explicitly; see [7] and [10]. One sees easily that under the assumptions of Theorem 2.12 and for $\dim M = 1$, a control set is relatively invariant iff it is open. Hence the connected components of the set MS of multistable points in L are given exactly by the open control sets in L. A similar result does not, in general, hold for $\dim M \geq 2$; compare the examples in § 4.

Finally, we would like to mention that the result of Theorem 2.12 need not hold in unbounded, invariant sets. There multistable points need not be in the domain of attraction of any control set, as some simple examples show. We refer to Remark 2.5 for the necessary generalizations in this case.

3. Extinction levels in singular stochastic systems. For the analysis of extinction in stochastic systems we use the following setup. Let M be an open set in a C^∞ manifold Λ, whose boundary ∂M consists of a finite union of lower dimensional submanifolds $M_i, i = 1, \dots, r$. The M_i need not be of the same dimension. We assume that the system (1.1), (1.2) is regular on M, i.e., that assumptions (A) and (B) hold and that the M_i are invariant for the stochastic system. Note that if the Lie algebra $\mathcal{LA}\{(X, Y_o), Y_1, \dots, Y_k\}$ has maximal rank at some point $(x, \xi) \in M \times N$, then it has maximal rank in a neighborhood of this point. Therefore, this setup describes the typical situation of singular systems.

We are interested in the behavior of the stochastic system as it approaches the boundary of M, because its behavior on M was described in § 2. Locally the behavior near the boundary ∂M could be analyzed, for example, using linearization techniques and Lyapunov exponents, describing convergence or divergence of the trajectories at ∂M in a random neighborhood of the boundary; compare, e.g., [3], [1], or [21]. Here we are interested in the global behavior of systems, i.e. in a complete description of the system on the closure $\mathrm{cl}\, M$ of M, where $\mathrm{cl}\, M = M \cup \bigcup_{i=1}^{r} M_i$, and, in particular, in the absorption behavior at one or more of the boundary sets M_i. If $M_i \subset \partial M$ is an invariant set of (1.1), then all vector fields $X(\cdot, \eta)$, $\eta \in U$, are tangential to M_i, and hence ∂M cannot be reached from M in finite time. With this in mind we model absorption using extinction levels.

Let $z(t)$ be a Markov process on a state space L with continuous trajectories. For a Borel set $A \subset L$ the first hitting time of A by $z(t)$ is defined as

$$\tau_A = \inf\{t \geq 0; z(t) \in A\}.$$

The corresponding absorption process at A (also called stopped process in the literature; see, e.g., [13]) is given by

$$z_A(t) = \begin{cases} z(t), & \text{for } 0 \leq t \leq \tau_A, \\ z(\tau_A) & \text{for } t > \tau_A. \end{cases}$$

The absorption process $z_A(t)$ has continuous trajectories and is a Markov process if $z(t)$ is strong Markov—a property that holds, for example, for the solutions of (1.1), (1.2); see [13] for these and other properties of absorption processes. With this notation we can define the concept of extinction levels.

DEFINITION 3.1. *Let $I \subset \partial M$ be an invariant set of the stochastic system. An extinction level at I of size $\epsilon > 0$ is a set $E(I, \epsilon) = \{y \in \mathrm{cl}\, M; d(y, I) \leq \epsilon\}$, where $d(y, I)$ denotes the distance of y to the set $\mathrm{cl}\, I$ on the underlying manifold Λ. The process absorbed at $E(I, \epsilon)$ is $x_{E(I,\epsilon)}(t, x_o)$, where $x_o \in M$ is the initial value for* (1.1).

According to this definition, an extinction level consists of a size $\epsilon > 0$ and an extinction set $E(I, \epsilon)$. It has associated with it the corresponding absorption processes $x_{E(I,\epsilon)}(t, x_o)$ for initial values in M, and we have, in particular, that $x(\sigma, x_o) \in E(I, \epsilon)$ for some (random) time σ implies $x_{E(I,\epsilon)}(\tau, x_o) \in E(I, \epsilon)$ for all $\tau \geq \sigma$. As we will see below, the introduction of extinction levels amounts mathematically to a (partial) compactification of M.

First of all, we consider the effect of extinction levels on the behavior of the process on invariant control sets.

PROPOSITION 3.2. *Consider the system* (1.1) *with extinction level $E(I, \epsilon)$.*

(i) *Let $C \subset M$ be an ergodic control set, or a control set such that $C \backslash \mathrm{int}\, E(I, \epsilon)$ is compact. If $E(I, \epsilon) \cap \mathrm{int}\, C \neq \emptyset$, then $\tau_{E(I,\epsilon)} < \infty$ with probability one for all $x_o \in C$. In particular, the absorbed process $x_{E(I,\epsilon)}$ does not admit an invariant measure in $C \backslash E(I, \epsilon)$.*

(ii) *Let $C \subset M$ be an invariant control set with $E(I, \epsilon) \cap \mathrm{int}\, C = \emptyset$; then*
$P\{\tau_{E(I,\epsilon)} = \infty\} = 1$ for initial values $x_o \in C \backslash E(I, \epsilon)$, and
$P\{\tau_{E(I,\epsilon)} = 0\} = 1$ for initial values $x_o \in C \cap E(I, \epsilon)$.
If C is an ergodic control set for (1.1) *with invariant probability measure ν, then the same invariant measure is invariant for the absorption process.*

Proof. (i) We discuss the case where C is an ergodic control set; the other case uses similar arguments. Denote $R = E(I,\epsilon) \cap \operatorname{int} C$ and let $x_o \in C$. Then, by Theorem 4.1 in [20], $\tau_R(x_o) < \infty$ with probability one. Suppose that the absorbed process has an invariant measure in $C \backslash E(I,\epsilon)$, say μ, with $\mu(C\backslash E(I,\epsilon)) = 1$. Denote $Q = C\backslash E(I,\epsilon)$. If $Q = \emptyset$, the result is trivial. If $Q \neq \emptyset$, choose two compact sets $K_1 \subset (\operatorname{int} Q \times N)$ and $K_2 \subset (\operatorname{int} R \times N)$ with nonvoid interior. Then there exists, by Proposition 2.3 in [6], a time $T > 0$ such that $\inf_{u\in\mathcal{U}}\{t \geq 0, \varphi(t, \left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right), u) = y\} \leq T$ for all $\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right) \in K_1, y \in K_2$ for the control system associated with (1.1), (1.2). Since the solutions of the associated control system depend continuously on the right-hand side, the support theorem implies that $P\{(x,T,x_o),\xi(t,\xi_o)) \in K_2\} > 0$ for all $\left(\begin{smallmatrix} x_o \\ \xi_o \end{smallmatrix}\right) \in K_1$ (compare, e.g., [20] or [2] for this standard argument). By definition of the absorbed process, this implies that $P\{(x_{E(I,\epsilon)}(t,x_o),\xi(t,\xi_o)) \in E(I,\epsilon) \times N\} > 0$ for all $t \geq T$, and therefore the joint process $(x(t),\xi(t))$ cannot admit an invariant probability measure in $Q \times N$.

(ii) Note that, under assumption (B), $\operatorname{cl} C = \operatorname{cl}(\operatorname{int} C)$ for invariant control sets in M. If $x_o \in C\backslash E(I,\epsilon)$, then $P\{x(t,x_o) \in \operatorname{int} C \text{ for all } t > 0\} = 1$ (compare [2]) and hence, under the assumption in (ii), $P\{\tau_{E(I,\epsilon)} = \infty\} = 1$ for $x_o \in C\backslash E(I,\epsilon)$. By Corollary 2.2 in [2], $\nu(\operatorname{int} C) = 1$ for the invariant measure on an ergodic control set. Hence, if $E(I,\epsilon) \cap \operatorname{int} C = \emptyset$, then ν is also invariant for the absorption process. □

Proposition 3.2 describes the stationary solutions of the absorption process in M as: These are the stationary solutions of the original process, whose supports do not intersect $\operatorname{int} E(I,\epsilon)$. Next, we turn to the multistability behavior of the absorption process.

We introduce the absorption orbits of the control system (2.1) with respect to $E(I,\epsilon)$ as

$$\mathcal{O}^+_{E(I,\epsilon)}(x) = \{y \in M;\ \text{there exist } t \geq 0 \text{ and } u \in \mathcal{U} \text{ with } \varphi(t,x,u) = y \text{ and } \varphi(s,x,u) \in M\backslash E(I,\epsilon) \text{ for all } 0 \leq s \leq t\}.$$

The absorption orbit of $x \in M$ consists of all points that can be reached from x via trajectories that do not pass through the extinction set $E(I,\epsilon)$. The control sets of the absorption system are defined by using $\mathcal{O}^+_{E(I,\epsilon)}$ instead of $\mathcal{O}^+$ in Definition 2.1. In particular, any control set D of (2.1) with $D \subset M\backslash E(I,\epsilon)$ is also a control set of the absorption system. Furthermore, if D^a is a control set of the absorption system, then there exists a unique control set D of (2.1) with $D^a \subset D$. But there may exist control sets of (2.1) that do not contain a control set of the absorption system. A slight modification of the numerical procedure presented in [15] allows us to compute the control sets of the absorption system numerically.

The following lemma shows that extinction sets $E(I,\epsilon)$ play the role of an invariant control set in the global qualitative theory of singular systems with extinction level.

LEMMA 3.3. *Consider the system* (1.1) *with extinction level* $E(I,\epsilon)$. *Let* $x_o \in M$ *be an initial value. Then*

$$P\{x(t,x_o) \to E(I,\epsilon) \text{ as } t \to \infty\} > 0 \quad \text{iff } x_o \in \mathcal{A}(E(I,\epsilon)).$$

Proof. This result is proved in the same way as Theorem 2.2(ii), using the associated control system and the support theorem. □

Theorem 2.2 and Lemma 3.3 give a complete characterization of the convergence toward stationary solutions in systems with extinction levels. To characterize the multistable points in these systems, we adapt Definitions 2.3 and 2.6 in the following way.

A point $x_o \in M$ is said to be multistable for the absorption system if there exist different sets F_1 and F_2 in M with $P\{x(t,x_o) \to F_i \text{ as } t \to \infty\} = p_i > 0$ for $i = 1, 2$, where F_i is either an ergodic control set or an extinction set.

A control set D^a of the absorption system is called relatively invariant if it satisfies the conditions of Definition 2.6 with respect to the set MS^a of multistable points of the absorption system. Similarly, the strong positive invariance condition (C) for the absorption process is reformulated with respect to the set MS^a.

With these definitions, the results of § 2 remain valid in the following form.

PROPOSITION 3.4. *Consider the system* (1.1) *with extinction level* $E(I,\epsilon)$, *and let* $L \subset M$ *be a compact, positively invariant set of the absorption system.*

(i) *There exists a multistable point* $x_o \in L \backslash E(I,\epsilon)$ *iff the absorption control system has at least two sets that are either invariant control sets or extinction sets.*

(ii) *If* L *satisfies the strong positive invariance condition for the absorption control system, then the set of multistable points is given by*

$$MS^a = \bigcup_{j=1}^{l_2} \mathcal{A}(D_j),$$

where $D_1, \ldots, D_{l_2}$ *are the relatively invariant control sets of the absorption control system with* $1 \le l_2 < \infty$.

COROLLARY 3.5. *If* $M \subset \Lambda$ *is bounded and the extinction set is given as* $E(\partial M, \epsilon)$, *then Proposition* 3.4 *holds for* M *instead of* L.

Remark 3.6 (on $\epsilon \to 0$).

(i) A multistable point of the stochastic system (1.1) need not be a multistable point of the absorption system. However, for each $x \in MS$ there exists $\epsilon > 0$ such that $x \in MS^a$ for the absorption system with extinction set $E(\partial M, \epsilon)$.

(ii) A similar result need not hold for the stationary solutions of (1.1). Let C be an ergodic control set in M such that $\operatorname{cl} C \cap \operatorname{cl} M \neq \emptyset$. Then, for every $\epsilon > 0$, the absorption system for $E(\partial M, \epsilon)$ does not admit an invariant probability measure with support in C, by Proposition 3.2; compare also the examples in § 4.

Remark 3.7 (on multiple extinction sets and reachability order in singular systems). So far, we have considered singular systems with one extinction set $E(I,\epsilon)$. All results above remain true if we define for invariant sets $I_i \subset \partial M, i \in J$, and different levels $\epsilon_i > 0, i \in J$, the corresponding orbits and control sets of the absorption system for a given family of extinction sets $E(I_i, \epsilon_i)$. With these concepts we can also extend the order defined in (2.3) to singular systems with extinction levels. Now the description of the set of multistable points in Corollary 2.10 extends analogously to the absorption system.

Remark 3.8 (on limit behavior in singular systems). Using Proposition 3.2 and Lemma 3.3 we see that Theorem 2.2(i) remains valid for singular systems with extinction levels if we replace the union over all ergodic control sets $C \in \widehat{C}_e$ by the union over all ergodic control sets and all extinction sets. Similarly, Theorem 2.2(iii) holds when conditioned on the set $\{x(t,x_o) \to \bigcup_{C \in \hat{C}_e} C\}$. Note that unique ergodicity need not hold for the absorption process in extinction sets.

Remark 3.9 (on the system behavior on ∂M). If we take the dynamics of the system on ∂M into account, then some of the results above can be made more detailed. Define F as the union of all invariant control sets of the system on ∂M, and let G be an extinction set containing F, possibly with different extinction levels. Then one obtains that Corollary 3.5 holds with G instead of $E(\partial M, \epsilon)$ and that so does Remark 3.6(ii). Including all invariant control sets in $\mathrm{cl}\, M$, with an obvious extension of the order, allows a description of the multistable points with respect to $\mathrm{cl}\, M$ as in Corollary 2.10.

Remark 3.10 (on one-dimensional systems). If levels of extinction are introduced, the only remaining gap in the discussion of one-dimensional systems (see [10, § 4.1]) can be closed. Suppose $\dim \Lambda = 1$ and there is an invariant control set C (with nonvoid interior) that has a common fixed point x^o of the vector fields $X(\cdot, u), u \in U$, at its boundary. If the other end point x^1 of the interval C belongs to C, then an extinction set $E(\{x^o\}, \epsilon^o)$ such that $C_a = C \backslash E(\{x^o\}, \epsilon^o) \neq \emptyset$ (i.e., $\epsilon^o < |x^1 - x^o|$) has the following effect; C_a is a variant control set of the absorption system and solutions starting in C_a are absorbed at $E(\{x^o\}, \epsilon^o)$ in finite time with probability one. If $x^1 \notin C$, then x^1 is also a common fixed point, and the introduction of a corresponding extinction set $E(\{x^1\}, \epsilon^1)$ with $D_a = C \backslash (E(\{x^o\}, \epsilon^o) \cup E(\{x^1\}, \epsilon^1)) \neq \emptyset$ makes D_a a relatively invariant control set of the absorption system; i.e., D_a is a connected component of the set MS^a of multistable points.

4. Examples. In this section we analyze four examples of stochastic systems that exhibit multistability behavior and/or are subjected to the effects of extinction levels.

4.1. A one-dimensional system with imperfect pitchfork bifurcation. Consider the one-dimensional cubic differential equation

$$\dot{x} = -\frac{1}{2}x^3 + bx^2 + c(q - q_c)x =: X(x, q), \tag{4.1}$$

where $b, c > 0$ and $q_c \in \mathbb{R}$ are constants. If we consider q as a bifurcation parameter, then this model undergoes an (imperfect) pitchfork bifurcation at $q = q_c$. Stochastic versions of (4.1) have been treated, for example, in [17], [10], or [26]. Here we are interested in the case where q is a stochastic process $\eta(t)$ as described in § 1 with values in $U_\alpha^\rho = [\alpha - \rho, \alpha + \rho], \alpha \in \mathbb{R}, \rho > 0$. This model was analyzed in [10, § 4.3] and we continue the discussion here by considering the effect of an extinction level at $x^o = 0$. According to Remark 3.10, the interesting case occurs when x^o belongs to the boundary of an invariant control set of the associated control system.

Define $q_{hc} = q_c - b^2/2c$, and let $\rho < \frac{1}{2}(q_c - q_{hc}) = b^2/4c$. Figure 1 shows the associated control sets for a range of α-values. For each α, the corresponding shaded areas indicate the following items:

- C_1, C_2: *invariant control sets;*
- B, D: *variant control sets;*
- F: *the fixed point* $x^o = 0$.

The control sets are obtained by projecting, for a fixed α-value, the shaded area over this α-value onto the x-axis.

Introducing a level of extinction $E(\{x^o\}, \epsilon)$ with $\epsilon < b$ yields control sets of the absorption system as indicated in Figure 2. According to Proposition 3.2 and Lemma 3.3 we obtain the following results.

For $\alpha \geq \alpha_1, Ca_1 = C_1$, and the system with extinction level $\epsilon < b$ has unique stationary Markov solutions in Ca_1 for each $\alpha \geq \alpha_1$. If $\alpha \geq \alpha_2$, then $Ca_2 = C_2$, which again are the supports for unique stationary solutions. For $\alpha < \alpha_2$, the invariant control sets in region Ca_2 do not carry invariant measures, and the process with initial value $x(0) < 0$ gets absorbed at the extinction set in finite time with probability one.

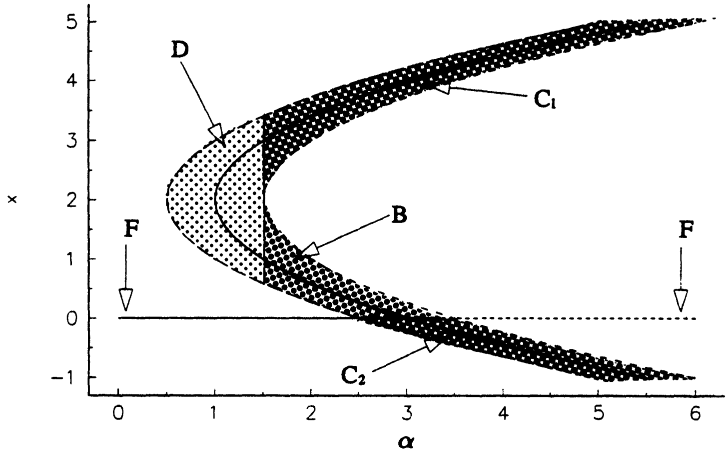


FIG. 1. *Control sets associated to the system* (4.1) *with* $b = 2, c = 1, q_c = 3$, *and* $\rho = 0.5$.

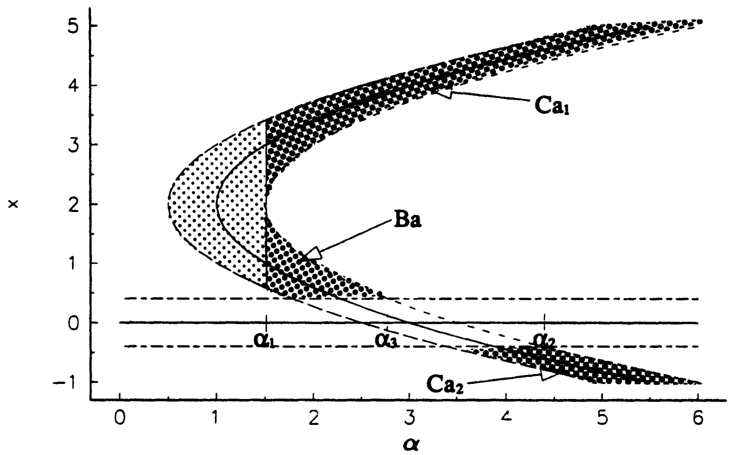


FIG. 2. *Control sets of the absorption control system corresponding to* (4.1) *with* $E(\{0\}, \epsilon)$ *and* $\epsilon = 0.4$.

For $\alpha \in [\alpha_1, \alpha_3)$, the control sets in region Ba consist exactly of the bistability points of the absorption process; here bistability is with respect to Ca_1 and $E(\{0\}, \epsilon)$.

If $\epsilon \geq b$, then the bistability region disappears and some of the invariant control sets in region Ca are no longer ergodic. This occurs iff $\min Ca_1 < b$, according to Proposition 3.2.

4.2. Bacterial respiration process. In [12], Degn and Harrison proposed a model for the existence of a maximal oxygen consumption rate at low oxygen concentration in *Klebsiella aerogenes* cultures. Fairén and Velarde [14] analyzed this model with respect to its limit cycle and bifurcation behavior. They also carried out simulations of a stochastic version with white noise fluctuations. Using the theory developed above, we discuss the following bistability phenomena in the model under bounded, diffusion type excitations.

The Degn–Harrison model reads in dimensionless units as (see [14])

$$\begin{aligned} \dot{x} &= b - x - \frac{xy}{1+qx^2}, \\ \dot{y} &= a - \frac{xy}{1+qx^2}. \end{aligned} \tag{4.2}$$

Here a and q are positive constants and b is the critical parameter, depending on the concentration rates in the underlying chemical reaction scheme. In our analysis, we treat the case where the parameter b is disturbed by an underlying diffusion process as described in § 1. Using the values $a = 11.0, q = 0.5, b(t) \in U = [19.97, 20.03]$, we obtain the control sets of the associated control system as depicted in Figure 3.

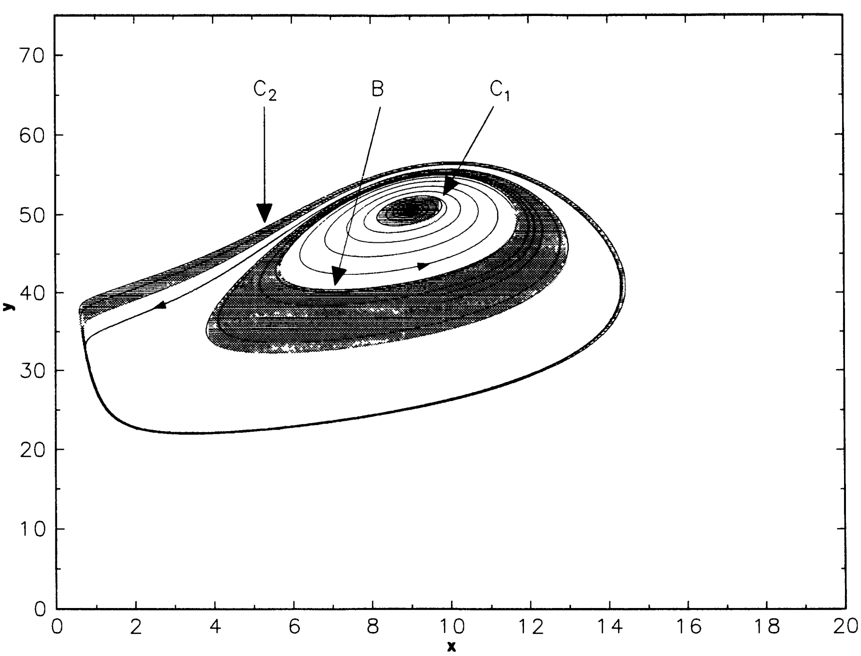


FIG. 3. *Control sets of* (4.2) *with* $a = 11.0$, $q = 0.5$, *and* $b \in [19.97, 20.03]$.

Mathematically, the control set picture in Figure 3 can be explained as follows. By a result in [8] and [9], the control sets of a system form around the limit sets of a nominal (uncontrolled) system, and their order represents exactly the Morse order of the limit sets in the nominal system if the control range is small. (A limit set ω_1 is smaller than a limit set ω_2 in the Morse order if there exists a trajectory that converges for $t \to -\infty$ to ω_1 and for $t \to +\infty$ to ω_2; see [9] for details.) If we take the system (4.2) with $b = 20.0$ as the nominal system, then this system has a stable fixed point, surrounded by an unstable periodic solution (limit cycle) which in turn is surrounded by a stable periodic solution. Since the control range ± 0.03 is relatively small, we expect to see two invariant control sets (around the stable fixed point and the stable limit cycle) and one variant control set (around the other limit cycle). Note that for nonlinear systems it is, in general, impossible to compute the control sets explicitly (except for the one-dimensional case; see [7]). Likewise, the size of the control range at which control sets merge so that the one-to-one correspondence between limit sets and control sets does not hold anymore cannot, in general, be computed explicitly. Hence numerical algorithms like the one developed in [15] are needed. (We refer to § 4.3 for an example with a large control range.) Note that for this example the set $L = [0, 20] \times [0, 60]$ is a compact, invariant set that satisfies all requirements in § 2, including the strong invariance condition (C).

In Figure 3 the sets C_1 and C_2 are invariant control sets, hence each of them carries a unique invariant probability measure and, therefore, a unique stationary, ergodic Markov solution of the system. The set B is a relatively invariant control set, hence it consists exactly of the bistable points from which the ergodic control sets C_1 and C_2 are reached with positive probability. Note that the boundaries of C_1 and C_2 belong to these sets, while B is open; i.e., the boundary ∂B is not part of the set B. Typical trajectories emanating from B in Figure 3 show how C_1 and C_2 can be reached. The probabilities $p_{C_i}(x_o)$ of reaching $C_i, i = 1, 2$, from $x_o \in B$ depend, of course, on the dynamics of the underlying diffusion process $\xi(t)$, while the supports of the invariant measures, i.e., the sets C_1, C_2, and the bistability area B depend only on the range U of the perturbation $\eta(t) = F(\xi(t))$.

Figure 3 suggests another interesting phenomenon in this model. In the outer control set C_2 the upper left-hand corner shows a larger diameter than the rest of this set. This suggests greater variability of the stochastic trajectories in this area and a rather "deterministic" behavior elsewhere in C_2. By increasing the range of the noise to $b(t) \in [16, 20]$, one does actually observe randomness in the trajectories; compare Figure 4. But regions of larger variation occur around the minimum and the maximum of the x-component, where the control set C_2 is relatively thin. The reason for this behavior lies in the fact that the deterministic vector field has small norm in these areas, hence random fluctuations around it become visible. In other parts of C_2 the deterministic vector field has large norm and the trajectories move so fast that stochastic variations have no time to influence them. It is noteworthy that the y-component does not show significant fluctuations during any part of its cycle.

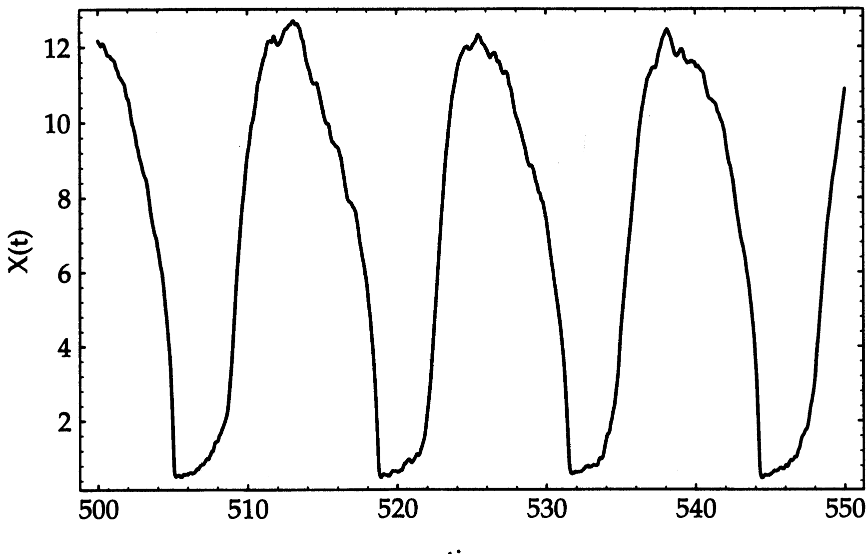
For these figures, the stochastic perturbation process

$$\begin{aligned} &\eta(t) = m + \rho\xi(t), \\ (4.3) \qquad &d\xi = -\frac{1}{\tau}\xi dt + \frac{\gamma}{\sqrt{\tau}}\sqrt{1-\xi^2}\, dW \end{aligned}$$

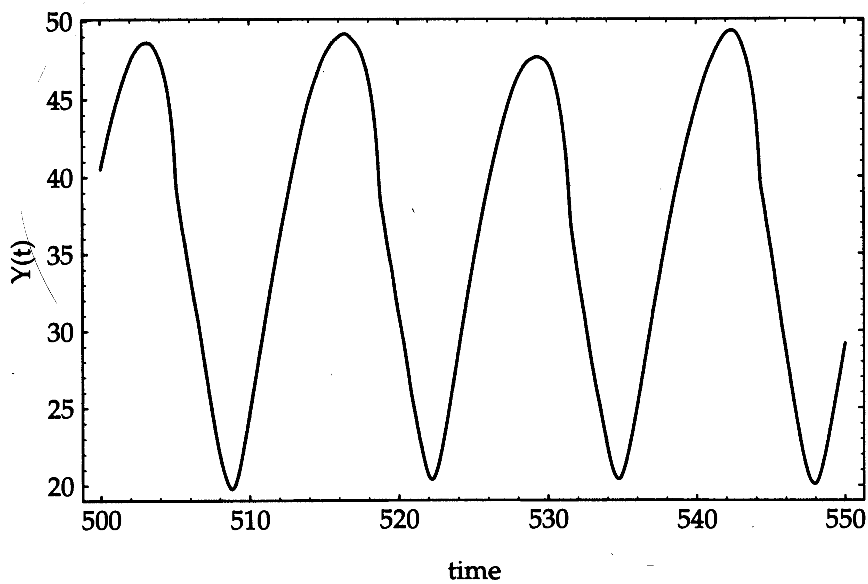
was used, where m and ρ were adapted for the specific noise range. The process ξ has a stationary density of the form (for $\gamma \in [0, 1]$)

$$p(\xi) = \frac{\Gamma(\lambda + 3/2)}{\sqrt{\pi}\Gamma(\lambda + 1)}(1 - \xi^2)^\lambda$$

with $\lambda = (1/\gamma)^2 - 1, \Gamma$ being the gamma function. $p(\xi)$ has expectation 0, is symmetric around $\xi = 0$, and $= 0$ at $\xi = \pm 1$. It converges toward the uniform density on $[-1, 1]$ as $\gamma \to 1$. For our simulations, we used the parameter settings $\tau = 0.1$ and $\gamma = 0.7$.



(a)



(b)

FIG. 4. *Simulated trajectory of* (4.2) *with* $a = 11.0$, $q = 0.5$, *and* $b \in [16, 20]$. (a) *shows the x- and* (b) *the y-component of the trajectory; the horizontal axis is* (*dimensionless*) *time.*

4.3. Lotka–Volterra model with hunting and resting periods. In [17, p. 184ff], Horsthemke and Lefever describe a predator–prey model that includes hunting and resting periods of the predator. They analyze this model under white noise disturbances by approximating it with a one-dimensional stochastic differential equation. We will analyze the two-dimensional model under bounded, diffusion type disturbances of different range with respect to bistability and extinction behavior.

The model is given by the equation

$$\dot{x} = \alpha x\left(1 - \frac{1}{K}x\right) - \beta xy,$$
$$\dot{y} = -\beta xy + \gamma(L - y), \tag{4.4}$$

where $\frac{1}{\beta}$ corresponds to the hunting time of the predator y, $\frac{1}{\gamma}$ corresponds to his resting time, and normalization is done via setting $\frac{1}{\gamma} + \frac{1}{\beta} = 1$. Furthermore, $\alpha, K,$ and L are positive constants.

Note first of all that the rectangle $[0, K] \times [0, L]$ is an invariant set of (4.4) as long as β and γ are positive. The y-axis itself is invariant, and so is the interval $\{0\} \times [0, L]$. In the notation of § 3 we have $M = (0, K) \times (0, L)$, and the crucial part of the boundary ∂M is $M_1 \cup M_2$, with $M_1 = \{0\} \times (0, L), M_2 = \{(0, L)\}$, where $(0, L)$ is a fixed point of the system. We are interested in the coexistence and extinction behavior of the two species, as hunting time (and, as a consequence, resting time) of the predator undergoes random fluctuations.

For our analysis, we choose the following parameter values:

$$K = 0.5, L = 1.0, \alpha = 4.0, \beta(t) \in U^1 = [4.1, 4.2], \text{ and } \beta(t) \in U^2 = [3.0, 5.0].$$

4.3.1. The case $\boldsymbol{\beta}(t) \in U^1 = [4.1, 4.2]$.

Stationary solutions (coexistence). In this β-parameter range the (deterministic) system (4.4) has three fixed points in $\mathrm{cl}\, M$: the stable point $(0, L)$, and an unstable and a stable point in M. By the same reasoning as in § 4.2, we expect to see an invariant control set C around the stable points in M and a variant control set D. Figure 5 shows that this is, indeed, the case. In C there exists a unique stationary Markov solution, indicating coexistence of the two species for random hunting periods in U^1.

Bistability. Considering the system on the closed space $\mathrm{cl}\, M$ yields D as a relatively invariant control set with $D \prec C$ and $D \prec \{(0, L)\}$. (Note that the point $(0, L)$ is a common fixed point for all $\beta \in U^1$.) In this model the y-axis is invariant; i.e., the strong invariance condition (C) is not satisfied at the y-axis. It is, however, satisfied on the sets $(0, K) \times \{0.5\}$ and $(0, K) \times \{1.0\}$, and Theorem 2.12 can be used for this example. Hence D together with its domain of attraction $\mathcal{A}(D)$ consist exactly of the bistable points, as shown in Figure 6. From this figure we see that

- for an initial value to the right of the right boundary of $\mathcal{A}(D)$, the system will converge in distribution to the invariant measure in C, and the species will coexist;
- for an initial value to the left of the left boundary of $\mathcal{A}(D)$, the system will converge toward the point $(0, L)$, indicating extinction of the prey x;
- for initial values in $\mathcal{A}(D)$ the system can go either way with positive probability.

Recall that the set MS of multistable points depends only on the size of the random perturbation but not on the specific dynamics of the perturbation process. The probabilities $p_C\binom{x_0}{y_0}$ and $p_f\binom{x_0}{y_0}$ of reaching C or $f = (0, L)$ from a point $\binom{x_0}{y_0} \in \mathcal{A}(D)$ do, of course, depend on the specific process. Using again the perturbation model (4.3), now with range U^1, we have simulated from each point in a grid on $\mathcal{A}(D)$ 1000 trajectories. The corresponding probabilities $p_f\binom{x_0}{y_0}$ of "reaching" $f = (0, L)$ are shown in Figure 7(a). Figure 7(b) shows the corresponding level curves. As expected, the set of $\binom{x_0}{y_0}$ for which $p_f\binom{x_0}{y_0}$ is numerically different from 0 or 1 is a relatively narrow

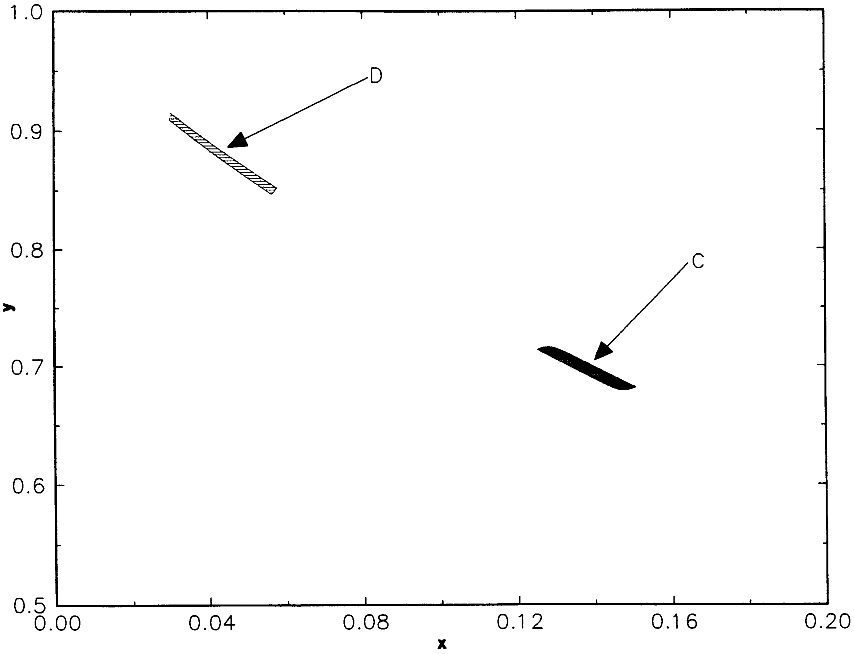


FIG. 5. *Control sets of the system* (4.4) *with* $K = 0.5$, $L = 1.0$, $\alpha = 4.0$, *and* $b \in [4.1, 4.2]$.

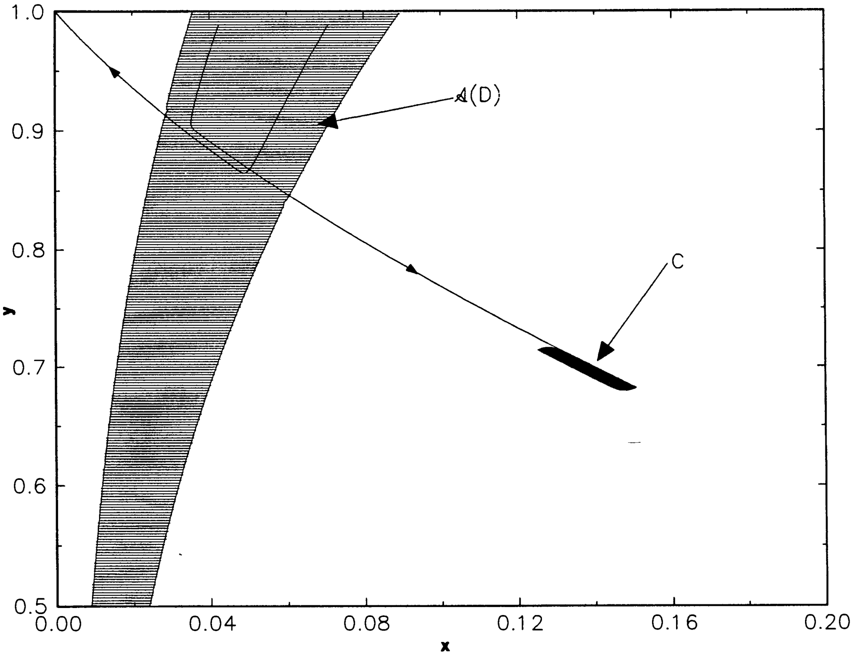


FIG. 6. *The set* $\mathcal{A}(D)$ *of bistable points in system* (4.4) *with* $K = 0.5$, $L = 1.0$, $\alpha = 4.0$, *and* $\beta \in [4.1, 4.2]$.

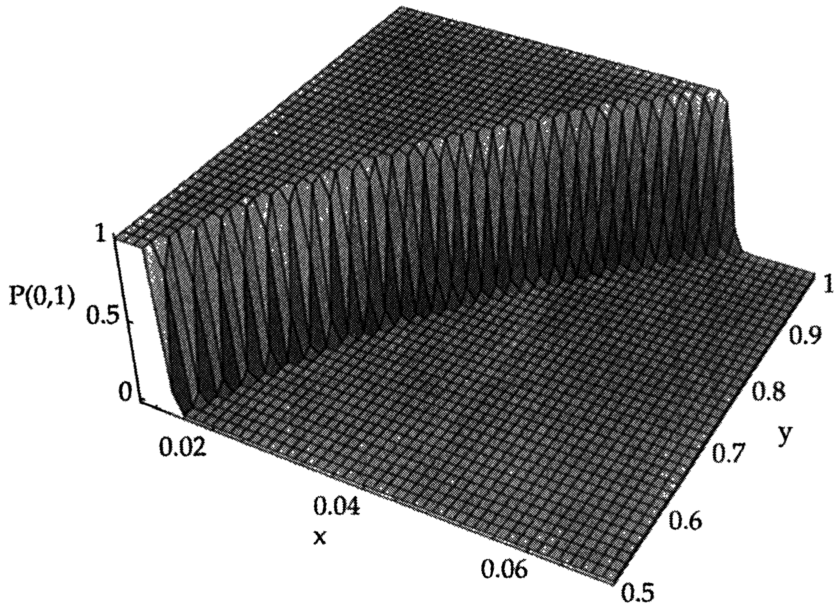


FIG. 7a. *Probabilities of "reaching" the extinction point* $(0,\ L)$ *from the bistability region in system* (4.4) *with* $K = 0.5$, $L = 1.0$, $\alpha = 4.0$, *and* $\beta(t) \in [4.1, 4.2]$.

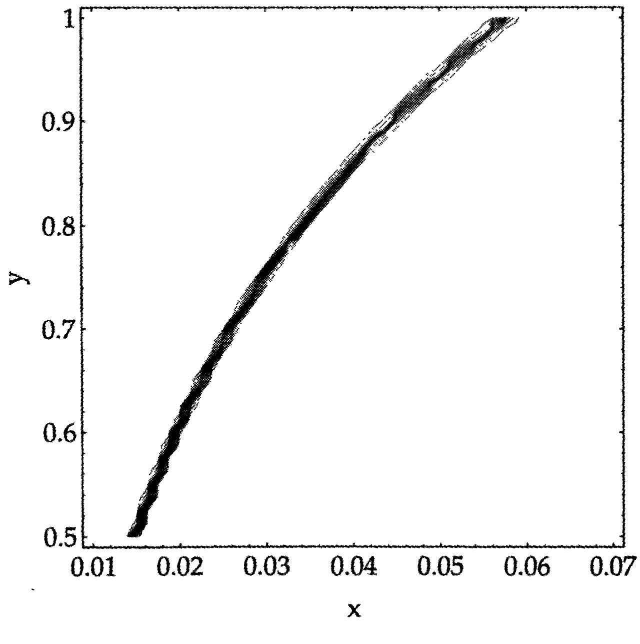


FIG. 7b. *Level curves of the extinction probabilitites in Figure* 7a.

band inside $\mathcal{A}(D)$. Perturbation processes with more skewed invariant distributions will produce a band that is closer to the left (or to the right) boundary of $\mathcal{A}(D)$.

Extinction levels. Equation (4.4) is the continuous space limit of a population model. Therefore, extinction of the prey x should not be considered as the situation where a solution of (4.4) has an x component equal to zero. In fact, since the y-axis is an invariant set of the model, no solution with initial value in M will reach $x = 0$ in finite time. This is the situation that led us to the introduction of extinction levels in § 3. Setting this level to $x = 0.01$ means in the precise language of § 3 that $I = \{0\} \times [0, L], \epsilon = 0.01$, $E(I, \epsilon) = \{(x, y) \in \mathbb{R}^2;\ x \in [0, 0.01],\ y \in [0, L]\}$. Figure 8 shows the effect of introducing this extinction level into the model (4.4) with $U^1 = [4.1, 4.2]$. The lower left corner of the original bistability set, i.e., initial values in $\mathcal{A}(D) \cap E(I, \epsilon)$, are now extinction points. The other areas of the global picture are not affected by the extinction level. The situation changes drastically when we consider a larger range of random perturbations for β.

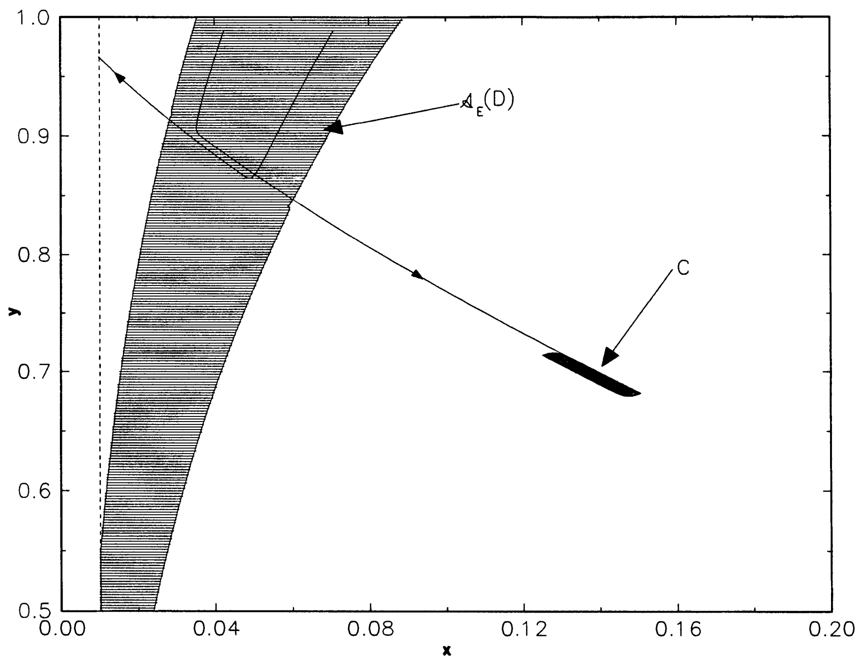


FIG. 8. *Effect of the extinction level* $x = 0.01$ *on the system* (4.4) *with* $K = 0.5$, $L = 1.0$, $\alpha = 4.0$, *and* $\beta(t) \in [4.1, 4.2]$.

4.3.2. The case $\boldsymbol{\beta}(t) \in U^2 = [3.0,\ 5.0]$. For β fixed, the system (4.4) has the following limit behavior in $\operatorname{cl} M$. If $\beta > 4.0$, there exist a stable fixed point $(0, L)$ on the boundary and a stable and an unstable fixed point in M; compare part (I). For $\beta < 4.0$, the fixed point $(0, L)$ is hyperbolic with the stable manifold being the y-axis and the unstable manifold pointing into M; furthermore, there exists one stable fixed point in M. At $\beta = 4.0$ the system undergoes a subcritical bifurcation at (0, L).

For the perturbation range $U^2 = [3, 5]$, we have a mixture of these two limit behaviors. Hence we expect to see one (invariant) control set in M that contains the control sets from (I) and those corresponding to variations around β for $\beta < 4$. With increasing control range, all these control sets have merged into one. Figure 9 shows

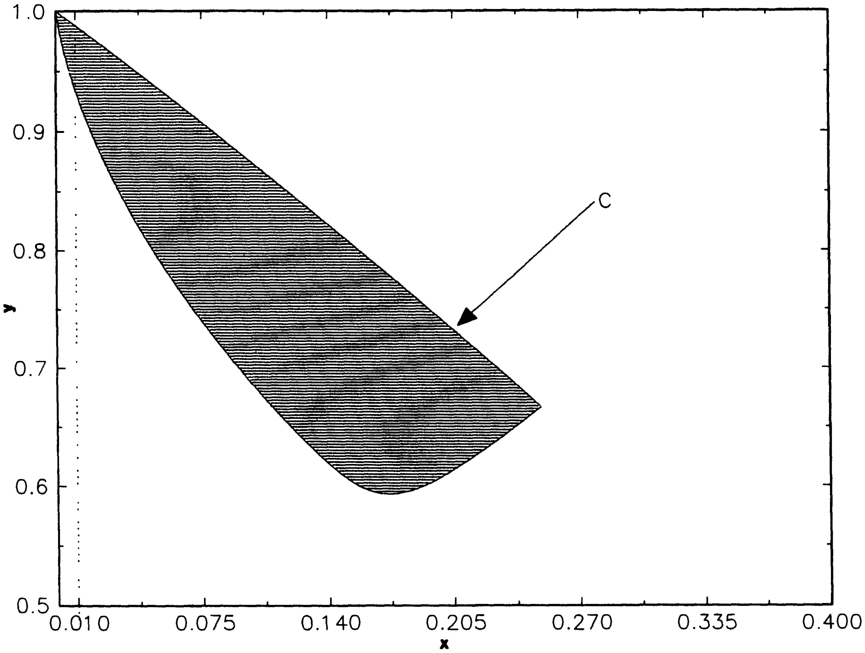


FIG. 9. *The (invariant) control set of the system* (4.4) *with* $K = 0.5$, $L = 1.0$, $\alpha = 4.0$, *and* $\beta \in [3.0, 5.0]$.

the invariant control set for (4.4) with range U^2. Note that according to Proposition 2.4, the set of multistable points is empty.

Stationary solutions. The unique invariant control set C of this system is not closed (at the point $(0, L)$), and therefore we cannot use Theorem 2.2 to decide the existence of an invariant measure and, hence, a stationary solution in C. Analyzing the Lyapunov exponents at the crucial point $(0, L)$ we obtain the linearization of (4.4) at $(0, L)$ as

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} \alpha - \beta(t)L & 0 \\ -\beta(t)L & -\dfrac{\beta(t)}{\beta(t)-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

which can be solved explicitly to yield for the initial value (v_1^o, v_2^o)

$$v_1(t) = v_1^o \exp\left(\int_0^t (\alpha - \beta(s)L)\, ds \right),$$

$$v_2(t) = v_2^o \exp\left(\int_0^t -\frac{\beta(s)}{\beta(s)-1}\, ds \right) + \exp\left(\int_0^t -\frac{\beta(s)}{\beta(s)-1}\, ds \right)$$
$$\times \int_0^t \left\{ \exp\left(\int_0^s \frac{\beta(\tau)}{\beta(\tau)-1}\, d\tau \right) \times (-\beta(s) L v_1^0) \exp\left(\int_0^s (\alpha - \beta(\tau)L)\, d\tau \right) \right\} ds.$$

The Lyapunov exponents of the solution are

$$\lambda_1 = \alpha - L\mathbb{E}\beta(t),$$
$$\lambda_2 = -\mathbb{E}(\beta(t)/(\beta(t)-1)),$$

where $\mathbb{E}$ denotes the expectation. The exponent λ_2 corresponds to the invariant manifold being the y-axis and $\lambda_2 < 0$ for this model because $\beta - 1 > 0$. The crucial exponent, therefore, is λ_1, with invariant manifolds pointing into C. For our parameter settings, $\lambda_1 < 0$ iff $\mathbb{E}\beta(t) > 4$ and $\lambda_1 > 0$ iff $\mathbb{E}\beta(t) < 4$. Hence we expect that there exists a (unique) invariant probability measure in C iff $\mathbb{E}\beta(t) < 4$ (compare the arguments in [3]) but we do not have a proof for this conjecture.

Extinction levels. According to Proposition 3.2 introducing an extinction level at $x = \epsilon > 0$ yields (independent of whether there exists an invariant measure in C or not) that the absorption process will hit the extinction set $E(I,\epsilon) = \{(x,y) \in \mathbb{R}^2; x \in [0,\epsilon], y \in [0,L]\}$ from any initial value in $\mathrm{cl}\, M$ in finite time with probability one. This means that the prey population x will become extinct with probability one. This phenomenon occurs, even if the expected hunting time satisfies $1/\mathbb{E}\beta(t) < 1/4$, for which the mean system and the system with small perturbation range as in part (I) show coexistence of the two species. Occasional excursions of the hunting time above 1/4 will eventually lead to the extinction of the prey if even very small extinction levels are included in the model.

The distribution of the extinction time depends, of course, on the dynamics of the noise process $\beta(t)$. Using again the random perturbation model (4.3), we analyzed numerically two cases.

- $U^3 = [2.9, 4.9]$, $\mathbb{E}\beta(t) = 3.9$, $\epsilon = 0.01$, initial value $(x_o, y_o) = (0.03, .91)$. In this case, the simulated trajectories eventually hit the extinction set $E(I,\epsilon)$, but for all simulated trajectories it took over 45,000 time units. Figure 10 shows the x-component of one of the simulated trajectories. Because of this long extinction time, we were not able to simulate sufficiently many trajectories on our equipment to obtain a reliable distribution (average CPU time was about 3 hr per trajectory, with 1000 time steps per time unit).
- $U^4 = [3.1, 5.1]$, $\mathbb{E}\beta(t) = 4.1$, $\epsilon = 0.005$, initial value $(x_o, y_o) = (0.02, 0.95)$. In this case, 1000 simulated trajectories gave a good impression of the distribution of the extinction time. The result is shown in Figure 11.

Our analytical results show that in the model with extinction level $\varepsilon > 0$ extinction of the prey occurs as soon as $\alpha - L\beta > 0$ is possible, independent of the specific dynamics of the random perturbation. Our numerical simulations, however, point out the fact that the extinction time depends drastically on the crucial parameter $\mathbb{E}\beta(t)$, namely on $\alpha - L\mathbb{E}\beta(t)$ being positive or negative. This corresponds to the stochastic bifurcation point at the equilibrium $(0, L)$ as $\lambda_1 = \alpha - L\mathbb{E}\beta(t)$ passes through zero as discussed above. Note that with increasing extinction level ϵ the size of $\mathbb{E}\beta(t)$ will have a less drastic effect.

4.4. The Lorenz model with extinction level. Consider the parametrically excited Lorenz system

$$\begin{aligned} \dot{x}_1 &= p(x_2 - x_1), \\ \dot{x}_2 &= -x_1x_3 + \eta(t)x_1 - x_2, \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{4.5}$$

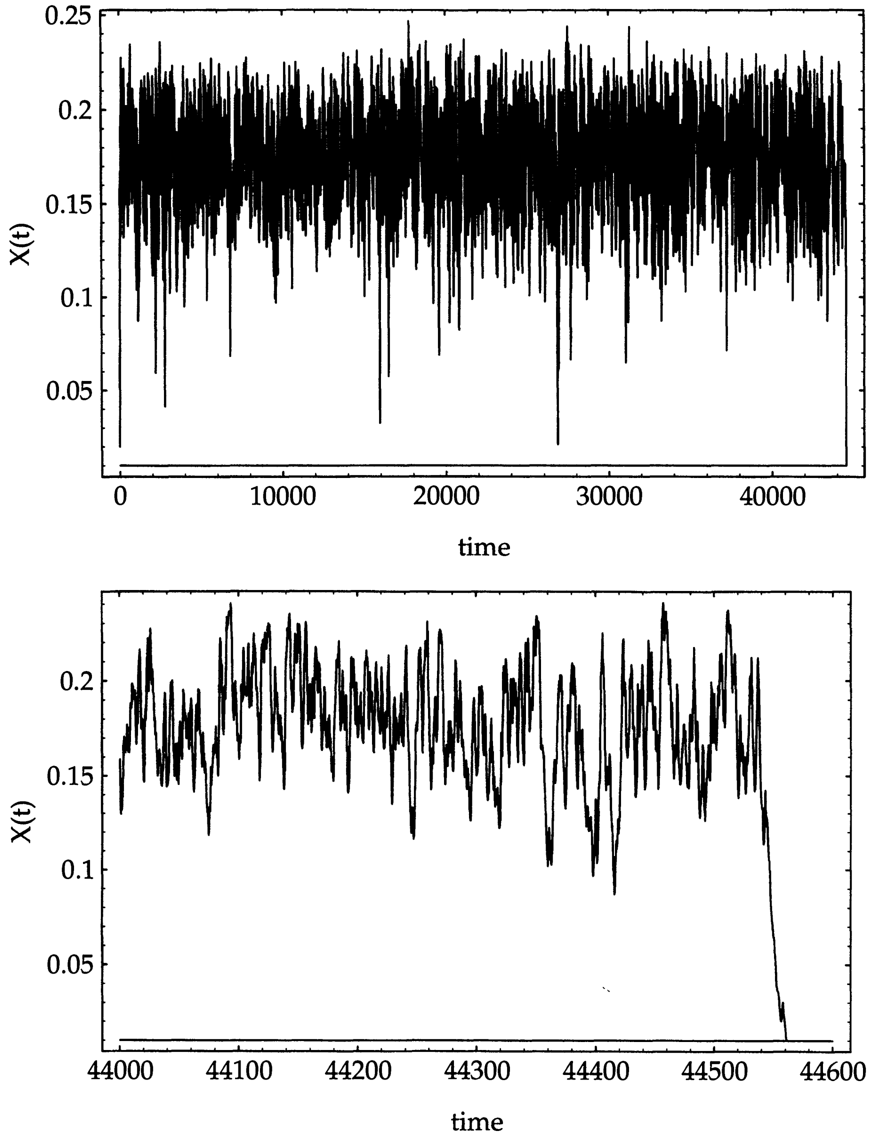


FIG. 10. *Simulated trajectory of the system* (4.4) *with extinction level* $\epsilon = 0.01$, $\beta(t) \in [2.9, 4.9]$, $\mathbb{E}\beta(t) = 3.9$, *and initial value* $(x_o, y_o) = (0.03, 0.91)$.

with $\eta(t) \in U^\rho = [r - \rho, r + \rho], \rho \geq 0$. In the interpretation of the Lorenz equations as a (finite-dimensional) model of the Rayleigh–Bénard convection the random perturbation term corresponds to the Raleigh coefficient, i.e., to the applied temperature difference at the boundary. Note that the x_3-axis is an invariant set of (4.5) for all random excitations. Recall that for $p = 10, b = 8/3, r = 28$ (and $\rho = 0$), this system exhibits a numerically observed "strange attractor" (see, e.g., [28]). We will consider the case where $\rho > 0$ and $r - \rho > 1$ and analyze the effect of extinction levels on the random system (4.5). Note first of all that the set

$$L = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + (x_3 - p - r - \rho)^2 < \alpha^2\}$$

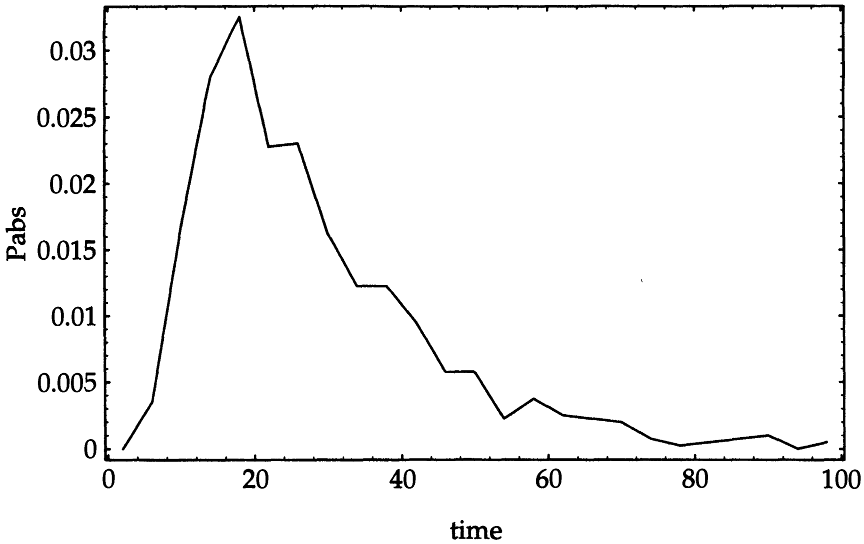


FIG. 11. *Extinction time distribution of the system* (4.4) *with extinction level* $\epsilon = 0.005$, $\beta(t) \in [3.1, 5.1]$, $\mathbb{E}\beta(t) = 4.1$, *and initial value* $(x_o, y_o) = (0.02, 0.95)$.

is strongly positive invariant for α large enough. Hence we can use the results from § 3 for $M = L \backslash Z, \partial M = L \cap Z$, where Z is the x_3-axis, i.e., $Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 = x_2 = 0\}$.

Consider the extinction set $E(I, \epsilon)$ with $\epsilon > 0$ and $I = L \cap Z$. It was shown in [9, Thm. 7.1] that for all $\epsilon > 0$ and all initial values $x^o \in \mathbb{R}^3 \backslash Z$ there exists a control set C of (4.5) with $r = 28, \rho > 0$, such that $E(I, \epsilon) \cap \omega(x^o) \subset \operatorname{int} C$, where $\omega(x^o)$ is the ("chaotic") limit set of the trajectory of (4.5) corresponding to $r = 28, \rho = 0$, and initial value x^o. Since $\operatorname{cl} L$ is a compact set, the control set C is invariant. Hence for any $\epsilon \in (0, \alpha)$ we are in the situation of Proposition 3.2(i). This means that the reaction modeled by the random Lorenz system (4.5) with (arbitrarily small) extinction level $|x_3| = \epsilon > 0$ will become extinct with probability one if fluctuations occur in the Rayleigh coefficient around the chaotic regime.

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