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# Distributional properties of portfolio weights

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## 1. Introduction

Due to its practical importance, the asset allocation problem has been widely discussed since the development of the mean–variance principle for portfolio selection by Markowitz (1952). This principle defines the behavior of an investor and is internally related to the maximization of the investor's utility function, which is another modeling method for the decision process in asset management. It is

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commonly assumed that asset returns follow multivariate normal distribution. This assumption implies myopic behavior, and thus the investor's problem reduces to a one-period problem and the maximization of the derived quadratic utility function. This subsequently implies the choice of mean–variance optimal portfolios. Samuelson (1970), and Constandinides and Malliaris (1995) provide further discussion on this topic.

As a solution to the utility maximization problem, we obtain the proportions of wealth that must be allocated to particular assets. Despite the key role of portfolio weights in asset allocation, their distributional properties have rarely been studied. In a myopic setup there are at least four important applications of the distribution of weights. First, tests can be applied to determine whether a current portfolio weight differs significantly from a given value. In general, this test allows comparison between portfolios. As a concrete example, we can consider the test of efficiency of a given portfolio. Jobson and Korkie (1989) derive a test for the mean–variance efficiency and introduce a potential performance measure related to the Sharpe ratio. This kind of test is also useful for market and volatility timing and may lead to an adjustment of the portfolio in real time (Barberis, 2000; Fleming et al., 2001). Second, the significance of the allocation to a particular asset can be tested. This can be used, for example, in testing international diversification. As reported in French and Poterba (1991), the fraction of wealth allocated to local assets for the five largest stock markets varies from 0.79 (Germany) to 0.922 (USA), which contradicts the theoretical advantages of diversification. This brought into question the significance of foreign ownership and resulted in much new research trying to explain the reasons for this puzzle. Recent developments in information technology made this problem again pressing (see Stulz, 1995; Britten-Jones, 1999; Ang and Beakert, 2002). Third, one can apply sequential trading strategy based on controlling optimal weights. Golosnoy and Schmid (2005) develop and compare this strategy with other common strategies. This methodology is based on the idea that asset reallocation is performed only in case of significant changes to the estimated portfolio weights. It appears that controlling optimal portfolio weights outperforms the buy-and-hold strategy, as well as the strategies with systematic reallocations. Fourth, since expected portfolio returns play a crucial role in most financial theories, their evaluation is of importance. Therefore, the fact that the optimal weights are random should be incorporated into the derivation of the expected portfolio return as well as into the portfolio variance. Moreover, it is sensible to treat the portfolio return as a function of random asset returns and random portfolio weights and to use its exact distribution.

Excepting Britten-Jones (1999) and Golosnoy and Schmid (2005), the decisions in such problems are usually based on asymptotic results or on direct comparison of the weights, without taking into account the stochastic nature of the returns and the weights, respectively. Jobson and Korkie (1980) derive approximations for the mean and variance of the estimated weights of the Sharpe ratio optimal portfolio, together with the asymptotic covariance matrix. However, asymptotic methods are not suitable for active portfolio management with frequent asset reallocations and short estimation periods. Due to these problems, finite sample properties of the portfolio weights are of interest.

Another important problem that relies on the exact distribution of the weights is the sensitivity of weights to changes in the assets means or variances (Best and Grauer, 1991; Chopra and Ziemba, 1993). Chopra and Ziemba (1993) consider an investor with a risk tolerance of 50 and positively weighted mean–variance efficient portfolios. They conclude for a concrete data set that the cash equivalent values are 11 times more sensitive to changes in the mean than to changes in the variance, and 21 times more sensitive to changes in the mean than to changes in the covariance. Results on the exact distribution of portfolio weights will help analytical analysis of the impact of changes in the mean and the covariance matrix of the returns on the mean and the covariance matrix of the optimal weights.

In this paper we discuss the exact distribution of four types of weights related to the mean–variance optimal portfolios. Let $\boldsymbol{\mu}$ denote the expected return of a portfolio and $\boldsymbol{\Sigma}$ its covariance matrix. $\boldsymbol{\Sigma}$ is assumed to be positive definite. For the expected quadratic utility the portfolio weights are chosen to maximize $\mathbf{w}'\boldsymbol{\mu} - \alpha\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}/2$ subject to $\mathbf{1}'\mathbf{w} = 1$. $\mathbf{1}$ denotes the vector whose components are all equal to 1. $\alpha > 0$ describes the risk aversion of an investor. This leads to the weights of the expected quadratic utility (EU) optimal portfolio

$$\mathbf{w}_{\mathrm{EU}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \alpha^{-1}\mathbf{R}\boldsymbol{\mu} \quad \text{with} \quad \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{1}$$

Taking into account reported evidence that the means of asset returns are difficult to estimate precisely (Merton, 1980) and, therefore, are difficult to distinguish, the global minimum variance portfolio (GMV) should be retained. The weights of the GMV portfolio are given by

$$\mathbf{w}_{\mathrm{GMV}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

Special portfolios arise from Tobin's separation theorem and are referred to as "tangency" portfolios. In terms of the efficient frontier, a tangency portfolio is a portfolio which corresponds to the tangency point between the efficient frontier and a line drawn from the origin or from the point which stands for a riskless asset, if it is available. Thus the presence of the riskless asset is important. In case of no riskless asset, the same tangency portfolio can be obtained via maximization of the Sharpe ratio of a portfolio. The Sharpe ratio is still one of the most popular measures of portfolio and asset performance (Cochrane, 1999; MacKinley and Pastor, 2000; Jobson and Korkie, 1981). The problem can be presented as maximization of $\mathbf{w}'\boldsymbol{\mu}/\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$ subject to $\mathbf{1}'\mathbf{w} = 1$. The Sharpe ratio (SR) optimal weights are given by

$$\mathbf{w}_{\mathrm{SR}} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} \tag{2}$$

provided that $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \neq 0$. If a riskless asset with return $r_f$ is available, the investor's problem reduces to the maximization of the following derived quadratic utility function: $\mathbf{w}'(\boldsymbol{\mu} - r_f\mathbf{1}) - \alpha\,\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}/2$. The solution of this optimization is given by the weights of the tangency portfolio (TP)

$$\mathbf{w}_{\mathrm{TP}} = \alpha^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}). \tag{3}$$

This expression provides the vector of optimal weights of the risky assets in the portfolio. The weight of the riskless asset is given by $w_{\mathrm{TP},0} = 1 - \mathbf{1}'\mathbf{w}_{\mathrm{TP}}$. A similar expression arises in a continuous time setup as in Merton (1971). He considers the consumption-based investment problem with a riskless asset. Taking a constant relative risk aversion utility function (i.e., $U(x) = x^{\gamma}/\gamma$ for $\gamma > 0$ and $U(x) = \log(x)$ for $\gamma = 0$) and a finite investment horizon, we obtain exactly the same solution as in (3) (see Ingersoll, 1987, p. 271).

The optimal weights depend on the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ which must be estimated in practice. By replacing these quantities with their sample counterparts we obtain estimators for the optimal weights. In this paper, the behavior of the estimated optimal weights as functions of random asset returns is discussed for finite and infinite samples. In Section 2 we deal with finite samples. All results in this section are based on the assumption that the underlying returns are independent and multivariate normally distributed. Nowadays, both assumptions are heavily disputed for describing daily returns (e.g., Osborne, 1959; Fama, 1965; Boness et al., 1974; Mittnik and Rachev, 1993). Nevertheless, the assumption of normality is appropriate here due to positive theoretical features, e.g., the consistency with the mean–variance rule, the equivalence of multiperiod and single period decision rules, the implication of the capital asset pricing model (e.g., Stiglitz, 1989; Markowitz, 1991). Moreover, Fama (1976) finds that monthly returns can be described by a normal approach.

Theorem 1 of this paper derives the exact means and the exact covariance matrix of the estimated optimal weights for a $k$-asset portfolio obtained by the expected utility maximization. Special attention is devoted to two-asset portfolios. In Theorem 2 we derive a formula for the higher moments. The exact multivariate density function is given in Proposition 1 for the global minimum variance portfolio. The univariate density of the tangency portfolio is provided in Proposition 3. We determine in Proposition 2 the conditional density for the Sharpe ratio optimal weights. We prove that all moments of order higher than or equal to 1 do not exist at all. In Section 3 we derive the asymptotic distribution of the estimated optimal weights. Here we distinguish again between the $k$-variate and two-variate case. Contrary to Section 2 it is not assumed that the returns are independent. The proofs of all theorems and propositions are given in the appendix (Section 5).

## 2. Properties for finite sample size

We consider a portfolio consisting of $k$ assets. Let $\mathbf{X}_t = (X_{1t}, \ldots, X_{kt})'$ denote the return of the portfolio assets at time point $t$. In this section it is always assumed that the random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are independent and identically distributed with mean

$\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. In order to estimate the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ we choose the estimators

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{X}_j = \bar{\mathbf{X}} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{j=1}^{n} (\mathbf{X}_j - \bar{\mathbf{X}})(\mathbf{X}_j - \bar{\mathbf{X}})'.$$

These estimators are asymptotically normal if the fourth moments exist (cf. Muirhead, 1982). Replacing $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ with $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ in (1)–(3) we obtain estimators for the optimal weights. The estimator of the optimal weights in the sense of maximizing the expected utility is given by

$$\hat{\mathbf{w}}_{\text{EU}} = \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} + \alpha^{-1}\hat{\mathbf{R}}\hat{\boldsymbol{\mu}} \quad \text{with} \quad \hat{\mathbf{R}} = \hat{\boldsymbol{\Sigma}}^{-1} - \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}. \tag{4}$$

The estimators of the global minimum variance portfolio weights, the optimal weights obtained by the Sharpe ratio and the tangency portfolio are

$$\hat{\mathbf{w}}_{\text{GMV}} = \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}, \quad \hat{\mathbf{w}}_{\text{SR}} = \frac{\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}} \quad \text{and} \quad \hat{\mathbf{w}}_{\text{TP}} = \alpha^{-1}\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_f\mathbf{1}). \tag{5}$$

In this section we calculate the mean, the variances, and the covariances of the elements of $\hat{\mathbf{w}}_{\text{EU}}$. Furthermore, we determine the exact density of $\hat{\mathbf{w}}_{\text{GMV}}$, the conditional densities of the components of $\hat{\mathbf{w}}_{\text{SR}}$ and the univariate density of $\hat{\mathbf{w}}_{\text{TP}}$. This is done under the assumption that the random vectors $\mathbf{X}_t$ have a $k$-variate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. We use the notation $\mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to refer to the $k$-variate normal distribution. In the following $\boldsymbol{\Sigma}$ is always assumed to be positive definite.

Now let $\hat{w}_{\text{EU},i}$ denote the $i$th component of $\hat{\mathbf{w}}_{\text{EU}}$, and also let us use similar notation for $\hat{\mathbf{w}}_{\text{GMV}}$, $\hat{\mathbf{w}}_{\text{SR}}$ and $\hat{\mathbf{w}}_{\text{TP}}$. Let $\mathbf{e}_i$ be the $k$-dimensional vector with $i$th component equal to 1 and all other components equal to zero. As usual $rank(A)$ stands for the rank of a matrix $A$ and $\text{tr}(A)$ for its trace.

### *2.1. Weights of the expected quadratic utility optimal portfolio*

In Theorem 1 we derive the mean and the covariance matrix of $\hat{\mathbf{w}}_{\text{EU}}$.

**Theorem 1.** *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors and let $\mathbf{X}_i \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $i = 1, \ldots, n$.*

(a) *Assume that either*
  (i) $\alpha = \infty$, $n \geqslant k + 1$, *and* $k \geqslant 2$ *or*
  (ii) $\alpha < \infty$, $n \geqslant k + 2$, *and* $k \geqslant 3$

*is satisfied. Then it follows that*

$$\text{E}(\hat{\mathbf{w}}_{\text{EU}}) = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{n-1}{n-k-1}\alpha^{-1}\mathbf{R}\boldsymbol{\mu} \tag{6}$$

(b) *Assume that either*
 (i) $\alpha = \infty, n \geqslant k+2$, *and* $k \geqslant 2$ *or*
 (ii) $\alpha < \infty$, $n \geqslant k+4$, *and* $k \geqslant 3$ *is satisfied.*

*Then the variance of the estimated expected quadratic utility optimal portfolio weights* $\hat{\mathbf{w}}_{\mathrm{EU}}$ *exists. Moreover if* (ii) *is replaced by* (ii′)
(ii′) $\alpha < \infty$, $n \geqslant k+4$, *and* $k \geqslant 4$
*then it holds that*

$$Var(\hat{\mathbf{w}}_{\mathrm{EU}}) = \frac{1}{n-k-1}\frac{\mathbf{R}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} + \alpha^{-2}c_1\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R} + \alpha^{-2}c_2\,\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}\mathbf{R} + \frac{\alpha^{-2}}{n}\left(c_1 + c_2(k-1) + \frac{(n-1)^2}{(n-k-1)^2}\right)\mathbf{R},$$

*with*

$$c_1 = \frac{(n-1)^2(n-k+1)}{(n-k)(n-k-1)^2(n-k-3)}, \quad c_2 = \frac{(n-1)^2}{(n-k)(n-k-1)(n-k-3)}.$$

Note that the estimator $\hat{\mathbf{w}}_{\mathrm{EU}}$ is biased. Nevertheless, it can be shown that the unbiased estimator

$$\hat{\hat{\mathbf{w}}}_{\mathrm{EU}} = \frac{\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}} + \alpha^{-1}\frac{n-k-1}{n-1}\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}$$

performs worse in terms of the mean square error, i.e.

$$MSE(\hat{\mathbf{w}}_{\mathrm{EU}}) = Var(\hat{\mathbf{w}}_{\mathrm{EU}}) + (\mathrm{E}(\hat{\mathbf{w}}_{\mathrm{EU}}) - \mathbf{w}_{\mathrm{EU}})(\mathrm{E}(\hat{\mathbf{w}}_{\mathrm{EU}}) - \mathbf{w}_{\mathrm{EU}})' \leqslant MSE(\hat{\hat{\mathbf{w}}}_{\mathrm{EU}}),$$

where $\mathbf{A} \leqslant \mathbf{B}$ means that $\mathbf{B} - \mathbf{A}$ is a non-negative-definite matrix.

In Theorem 1 we have different assumptions on $k$. These restrictions are necessary in order to apply a basic tool of our proof, Theorem 3.2.11 of Muirhead (1982). For that reason we consider the case of two assets separately in the following. Let

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}, \quad \hat{\boldsymbol{\mu}} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}, \quad \hat{\mathbf{\Sigma}} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{pmatrix}.$$

Then the optimal weight $\hat{w}_{\mathrm{EU},2}$, here briefly $\hat{w}_{\mathrm{EU}}$, is given by

$$\hat{w}_{\mathrm{EU}} = \frac{(\bar{X}_2 - \bar{X}_1)\alpha^{-1} + \hat{\sigma}_1^2 - \hat{\sigma}_{12}}{\hat{\sigma}_2^2 + \hat{\sigma}_1^2 - 2\hat{\sigma}_{12}} = \frac{\alpha^{*-1}(\sum_{i=1}^n X_{2i} - \sum_{i=1}^n X_{1i}) + \sum_{i=1}^n (X_{1i} - \bar{X}_1)^2 - \sum_{i=1}^n (X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2)}{\sum_{i=1}^n (X_{1i} - \bar{X}_1)^2 + \sum_{i=1}^n (X_{2i} - \bar{X}_2)^2 - 2\sum_{i=1}^n (X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2)} \tag{7}$$

with $\alpha^* = n\alpha/(n-1)$. In Theorem 2 we determine the moments of the estimated weights of a two-asset portfolio. Now let $a = n(\mu_2 - \mu_1)/\alpha^*$, $b = n(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})/(2\alpha^{*2})$,

$$d_k := \frac{1}{k!}\frac{\partial^k}{\partial t_1^k} e^{at_1 + bt_1^2}\bigg|_{t_1=0} = \sum_{j=[(k+1)/2]}^{k} \binom{j}{k-j} \frac{1}{j!} b^{k-j} a^{2j-k} \tag{8}$$

$$= \alpha^{*-k} \sum_{j=[(k+1)/2]}^{k} n^j \binom{j}{k-j} \frac{1}{j!} \left(\frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}{2}\right)^{k-j} \times (\mu_2 - \mu_1)^{2j-k} \tag{9}$$

and

$$\alpha_{j,k} = \binom{j}{k-j} \frac{k!}{j!} r^{k-j} q^{2j-k} \tag{10}$$

with

$$q = 2(\sigma_1^2 - \sigma_{12}), \quad r = \sigma_1^2\sigma_2^2 - \sigma_{12}^2. \tag{11}$$

**Theorem 2.** *Let $k = 2$ and $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors and let $\mathbf{X}_i \sim \mathcal{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $i = 1, \ldots, n$. Then it holds that for $n > 2m + 1$*

$$\mathrm{E}(\hat{w}_{\mathrm{EU}}^m) = (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})^{-m} \sum_{k=0}^{m} \frac{m!}{k!} d_{m-k} \sum_{j=[(k+1)/2]}^{k} \frac{\alpha_{j,k}}{2^j \prod_{l=j+1}^{m} (n + 2j - 1 - 2l)}, \tag{12}$$

*where $d_k$ and $\alpha_{j,k}$ are defined in* (8) *and* (10), *respectively.*

It can easily be shown that the distribution of $\hat{w}_{\mathrm{EU}}$ is skewed. For equal means it is positively skewed if $\sigma_{12} < \sigma_1^2$; otherwise it is negatively skewed. For the proof of Theorem 2 the interested reader is referred to Okhrin (2004).

### 2.2. *Weights of the global minimum variance portfolio*

We consider the global minimum variance portfolio weights given by the formula

$$\mathbf{w}_{\mathrm{GMV}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

In this section, we derive the density of $(m-1)$ arbitrary components of this vector. Without loss of generality we restrict ourselves to the first $(m-1)$ components and denote this subvector by $\tilde{\mathbf{w}}_{\mathrm{GMV}}$. Setting $m = 2$ provides the marginal density of a single weight, and $m = k$ provides the exact density of the first $k-1$ weights. In the latter case the distribution of the last weight is obtained from $\hat{\mathbf{w}}_{\mathrm{GMV},k} = 1 - \mathbf{1}_{k-1}' \hat{\tilde{\mathbf{w}}}_{\mathrm{GMV}}$. Let us consider $\tilde{\mathbf{w}}_{\mathrm{GMV}}$ as a function of a sample of size $n$ of returns

$$\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}} = \left(\frac{\mathbf{e}_1'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}, \ldots, \frac{\mathbf{e}_{m-1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\right).$$

**Proposition 1.** *Let* $\mathbf{X}_t \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $n \geqslant k \geqslant m \geqslant 2$. *Then* $\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}}$ *follows an* $m-1$*-variate elliptical t-distribution with* $n-k+1$ *degrees of freedom and has parameters* $\tilde{\mathbf{w}}_{\mathrm{GMV}}$ *and*

$$\frac{1}{n-k+1}\frac{\tilde{\mathbf{R}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}},$$

*where* $\tilde{\mathbf{R}} = \{\mathbf{e}_i'\mathbf{R}\mathbf{e}_j\}_{i,j=1,\ldots,m-1}$ (*see* Muirhead (1982), p. 48). *In formulaic terms, the density function is given by*

$$f_{\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}}}(\mathbf{x}) = \frac{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^{(m-1)/2}}{\pi^{(m-1)/2}\sqrt{det(\tilde{\mathbf{R}})}}\frac{\Gamma((n-k+m)/2)}{\Gamma((n-k+1)/2)} \times((\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\mathbf{x}-\tilde{\mathbf{w}}_{\mathrm{GMV}})'\tilde{\mathbf{R}}^{-1}(\mathbf{x}-\tilde{\mathbf{w}}_{\mathrm{GMV}})+1)^{-(n-k+m)/2}. \tag{13}$$

*Moreover,*

$$\mathrm{E}(\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}}) = \tilde{\mathbf{w}}_{\mathrm{GMV}} \quad \textit{and} \quad Var(\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}}) = \frac{1}{n-k-1}\frac{\tilde{\mathbf{R}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

**Corollary 1.** *The univariate marginal distribution of* $\hat{\mathbf{w}}_{\mathrm{GMV}}$ *is a scaled t-distribution with* $n-k+1$ *degrees of freedom, more precisely*

$$\frac{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\sqrt{n-k+1}}{\sqrt{\mathbf{e}_i'\mathbf{R}\mathbf{e}_i}}(\hat{w}_{\mathrm{GMV},i} - w_{\mathrm{GMV},i}) \sim t_{n-k+1}.$$

The standard distribution of $\hat{\mathbf{w}}_{\mathrm{GMV}}$ simplifies the application of tests, since the marginal distribution of a multivariate $t$-distribution is again a $t$-distribution (see Muirhead (1982)).

### 2.3. Weights of the sharpe ratio optimal portfolio

Next we consider the estimated weights of the Sharpe ratio optimal portfolio. They are given by

$$\hat{\mathbf{w}}_{\mathrm{SR}} = \frac{\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}. \tag{14}$$

As above we consider $k$ risky assets, but restrict the analysis to the $i$th component of $\hat{\mathbf{w}}_{\mathrm{SR}}$. This leads to $\hat{w}_{\mathrm{SR},i} = \mathbf{e}_i'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}/\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$. It is shown that all moments higher or equal to 1 of $\hat{w}_{\mathrm{SR},i}$ do not exist. Because this estimator is commonly used in practice this should serve as a warning to practitioners. It holds that

$$f_{\hat{w}_{\mathrm{SR},i}}(x) = \int \cdots \int f_{\hat{w}_{\mathrm{SR},i}|\hat{\boldsymbol{\mu}}}(x|\mathbf{y}) n_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)(\mathbf{y})\, \mathrm{d}\mathbf{y},$$

where the conditional density of $\hat{w}_{\mathrm{SR},i}|\hat{\boldsymbol{\mu}}$ is given by $f_{\hat{w}_{\mathrm{SR},i}|\hat{\boldsymbol{\mu}}}(x|\mathbf{y})$ and $n_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)(\cdot)$ denotes the density of a $k$-variate Gaussian vector with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}/n$ and stands for the density of $\hat{\boldsymbol{\mu}}$. Using the fact that $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are independent (see Muirhead (1982), Theorem 3.1.2) the conditional distribution of $\hat{\mathbf{w}}_{\mathrm{SR}}$ given $\hat{\boldsymbol{\mu}}$ is specified only by the distribution of $\hat{\boldsymbol{\Sigma}}$.

Assume that the matrix $\mathbf{M}' = (\mathbf{e}_i, \mathbf{1}, \mathbf{y})$ has rank 3. Let $\mathbf{H} = (\mathbf{M}\boldsymbol{\Sigma}^{-1}\mathbf{M}')^{-1}$. Moreover, let $\mathbf{H}^{-1}$ be split into the submatrices $\mathbf{H}_{ij}^{(-)}$ for $i,j = 1,2$, $\mathbf{H}_{22}^{(-)} = \mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}$ and similarly for $\mathbf{H}$ with components $\mathbf{H}_{ij}$ for $i,j = 1,2$. Furthermore let

$$\xi(x) = \frac{|(x\ 1)\mathbf{H}_{12}|}{\sqrt{det(\mathbf{H})}\sqrt{(1\ \ -x)\mathbf{H}_{11}^{(-)}(1\ \ -x)'}}.$$

**Proposition 2.** *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors and let $\mathbf{X}_i \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $i = 1, \ldots, n$. Suppose that $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{y} \neq 0$ and that $rank(\mathbf{M}) = 3$. Let $n \geqslant k+1$ and $k \geqslant 3$. Then the density function of $\hat{w}_{\mathrm{SR},i}$ conditional on $\hat{\boldsymbol{\mu}}$ is given by*

$$f_{\hat{w}_{\mathrm{SR},i}|\hat{\boldsymbol{\mu}}}(x|\mathbf{y}) = \frac{1}{\pi}\frac{1}{\sqrt{det(\mathbf{H}_{11}^{(-)})}}\left(\frac{det(\mathbf{H}^{-1})}{det(\mathbf{H}_{11}^{(-)})\,\mathbf{H}_{22}^{(-)}}\right)^{(n-k)/2}\frac{1}{(x\ 1)\mathbf{H}_{11}(x\ 1)'}$$
$$+\frac{n-k+1}{\pi}\frac{\sqrt{\mathbf{H}_{22}^{(-)}}|(x\ 1)\mathbf{H}_{12}|}{(det(\mathbf{H}^{-1}))^{(n-k+1)/2}}\frac{((x\ 1)\mathbf{H}_{11}(x\ 1)')^{(n-k-1)/2}}{((1\ \ -x)\mathbf{H}_{11}^{(-)}(1\ \ -x)')^{(n-k+2)/2}}$$
$$\times\int_0^{\xi(x)}(1+v^2)^{-(n-k+3)/2}\,\mathrm{d}v.$$

All moments of $\hat{w}_{\mathrm{SR},i}|\hat{\boldsymbol{\mu}}$ of order $\beta \geqslant 1$ do not exist, but all moments of order $\beta \in [0,1)$ exist. To show this, note that for the first term the existence of the moment of order $\beta$ requires the integral of the form

$$\int_0^{\infty}\frac{x^{\beta}}{(x\ 1)\mathbf{H}_{11}(x\ 1)'}\,\mathrm{d}x$$

to be finite, which is possible only if $\beta < 1$. The second term has a representation in terms of a Gaussian hypergeometric function (see Muirhead (1982)) and finite moments for all values of $\beta$. Because both terms in $f_{\hat{w}_{\mathrm{SR},i}|\hat{\boldsymbol{\mu}}}$ are positive, it follows that the same results hold for the moments of $\hat{w}_{\mathrm{SR},i}$.

### *2.4. Weights of the tangency portfolio*

In this section we restrict ourselves to the case $k = 1$. It seems difficult to generalize the results for a portfolio with many stocks. Nevertheless, the problem of allocation to risky and non-risky assets is in its own right of great importance. The weight of the risky asset reduces in this case to the following expression:

$$w_{\mathrm{TP}} = \alpha^{-1}\frac{\mu - r_f}{\sigma^2}$$

and is estimated with $\hat{w}_{\mathrm{TP}} = \alpha^{-1}(\hat{\mu} - r_f)/\hat{\sigma}^2$, where $\mu$ and $\sigma^2$ are the parameters of the returns distribution and $\hat{\mu}$ and $\hat{\sigma}^2$ the corresponding sample estimates. The next proposition provides the density of $\hat{w}_{\mathrm{TP}}$.

**Proposition 3.** *If $X_i \sim \mathcal{N}_1(\mu, \sigma^2)$ for $i = 1, \ldots, n$ and $n \geqslant 2$ then the density function of $\hat{w}_{\mathrm{TP}}$ is given by*

$$f_{\hat{w}_{\mathrm{TP}}}(x) = \frac{\alpha}{2\sqrt{2\pi}\sigma/\sqrt{n}} \left(\frac{n-1}{2\sigma^2}\right)^{(n-1)/2} \frac{1}{\Gamma(\frac{n-1}{2})} \mathrm{e}^{-\frac{n(\mu-r_f)^2}{2\sigma^2}} a^{-(n+1)/4}$$
$$\times \left[\sqrt{a}\Gamma((n+1)/4)\,{}_1F_1\left(\frac{n+1}{4}, \frac{1}{2}; \frac{b^2}{4a}\right) - b\Gamma((n+3)/4)\,{}_1F_1\left(\frac{n+3}{4}, \frac{3}{2}; \frac{b^2}{4a}\right)\right],$$

*with*

$$a = \frac{\alpha^2 x^2}{2\sigma^2/n}, \quad b = \frac{n-1}{2\sigma^2} - n\alpha^2 w_{\mathrm{TP}}$$

*and ${}_1F_1(\cdot, \cdot; \cdot)$ denotes the confluent hypergeometric function (see Muirhead (1982)). Moreover, for $n \geqslant 4$ it holds that*

$$\mathrm{E}(\hat{w}_{\mathrm{TP}}) = \frac{n-1}{n-3} w_{\mathrm{TP}}, \tag{15}$$

*and for $n \geqslant 6$ that*

$$Var(\hat{w}_{\mathrm{TP}}) = \frac{\alpha^{-2}}{\sigma^2} \frac{(n-1)^2}{n(n-3)(n-5)} + 2w_{\mathrm{TP}}^2 \frac{(n-1)^2}{(n-3)^2(n-5)}. \tag{16}$$

For $x = 0$ the value of $f_{\hat{w}_{\mathrm{TP}}}$ is interpreted as the limit $\lim_{x \to 0} f_{\hat{w}_{\mathrm{TP}}}(x)$, which is given by

$$f_{\hat{w}_{\mathrm{TP}}}(0) = \frac{2\sqrt{n}\alpha\sigma}{\sqrt{2\pi}(n-1)} \frac{\Gamma((n+1)/2)}{\Gamma((n-1)/2)} \mathrm{e}^{-\frac{n(\mu-r_f)^2}{2\sigma^2}}.$$

## 3. Asymptotic behavior of the weights

Here we assume that the portfolio assets are serially correlated. Let $\{\mathbf{X}_t\}$ be a $k$-dimensional weakly stationary process with mean $\boldsymbol{\mu} = \{\mu_i\}$ and cross-covariance matrix

$$\boldsymbol{\Gamma}(h) = \{\gamma_{ij}(h)\} = \{\mathrm{E}(X_{i,t+h} - \mu_i)(X_{j,t} - \mu_j)\}_{i,j=1,\ldots,k}$$

at lag $h$. $\gamma_{ij}(h)$ is called cross-covariance at lag $h$. Note that in general $\gamma_{ij}(h) \neq \gamma_{ji}(h)$ but $\gamma_{ij}(h) = \gamma_{ji}(-h)$.

We define $\boldsymbol{\Gamma}(0) = \boldsymbol{\Sigma} = \{\sigma_{ij}\}$, $\boldsymbol{\theta}_1 = (\boldsymbol{\mu}', vech(\boldsymbol{\Sigma})')'$, $\boldsymbol{\theta}_2 = vech(\boldsymbol{\Sigma})$ with $vech(\boldsymbol{\Sigma}) = (\sigma_{11}, \ldots, \sigma_{k1}, \sigma_{22}, \ldots, \sigma_{k2}, \ldots, \sigma_{k-1,k-1}, \sigma_{k,k-1}, \sigma_{kk})'$. Then $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ have $k(k+3)/2$

and $k(k+1)/2$ components, respectively. We estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by their empirical counterparts $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ as defined in Section 2. The estimators of $\boldsymbol{\theta}_i$ are denoted by $\hat{\boldsymbol{\theta}}_i$ for $i=1,2$. Several authors have shown under various conditions that $\hat{\boldsymbol{\theta}}_i$ $(i=1,2)$ are asymptotically normal with mean $\boldsymbol{\theta}_i$ and covariance matrix $\boldsymbol{\Omega}_i$. In this case, the process must be strictly stationary, linear or Gaussian (see, e.g., Hannan, 1970, p. 228). If, e.g., $\{\mathbf{X}_t\}$ is a Gaussian process, then it holds that

$$\boldsymbol{\Omega}_1 = \begin{pmatrix} \left\{ \sum_{h=-\infty}^{\infty} \gamma_{ij}(h) \right\}_{i,j=1\ldots k} & \mathbf{0}_{k\times k(k+1)/2} \\ \mathbf{0}_{k(k+1)/2\times k} & \left\{ \left\{ \sum_{h=-\infty}^{\infty} (\gamma_{ip}(h)\gamma_{js}(h) + \gamma_{is}(h)\gamma_{jp}(h)) \right\}_{\substack{j=i,\ldots,k \\ s=p,\ldots,k}} \right\}_{i,p=1,\ldots,k} \end{pmatrix}$$

provided that all infinite sums are convergent. $\boldsymbol{\Omega}_2$ contains only the lower block of $\boldsymbol{\Omega}_1$. For $k=2$ we get that

$$\boldsymbol{\Omega}_2 = \begin{pmatrix} \sum\gamma_{11}(h) & \sum\gamma_{12}(h) & 0 & 0 & 0 \\ \sum\gamma_{21}(h) & \sum\gamma_{22}(h) & 0 & 0 & 0 \\ 0 & 0 & 2\sum\gamma_{11}^2(h) & 2\sum\gamma_{11}(h)\gamma_{12}(h) & 2\sum\gamma_{12}^2(h) \\ 0 & 0 & 2\sum\gamma_{11}(h)\gamma_{21}(h) & \sum(\gamma_{11}(h)\gamma_{22}(h) + \gamma_{12}(h)^2) & 2\sum\gamma_{22}(h)\gamma_{12}(h) \\ 0 & 0 & 2\sum\gamma_{21}(h)^2 & 2\sum\gamma_{22}(h)\gamma_{21}(h) & 2\sum\gamma_{22}(h)^2 \end{pmatrix},$$

where $h$ runs in all sums from $-\infty$ to $\infty$.

The optimal weights discussed in Section 2 can be considered as functions of the estimators $\hat{\boldsymbol{\theta}}_i$ $(i=1,2)$. Let $\hat{\mathbf{w}}'_{\mathrm{EU}} = g_1(\hat{\boldsymbol{\theta}}_1)$, $\hat{\mathbf{w}}'_{\mathrm{GMV}} = g_2(\hat{\boldsymbol{\theta}}_2)$, $\hat{\mathbf{w}}'_{\mathrm{SR}} = g_3(\hat{\boldsymbol{\theta}}_1)$ and $\hat{\mathbf{w}}'_{\mathrm{TP}} = g_4(\hat{\boldsymbol{\theta}}_1)$. The components $g_{j1},\ldots,g_{jk}$ of $g_j$, $j\in\{1,4\}$, are continuously differentiable in a neighborhood of $\boldsymbol{\theta}_1$ and $g_{21},\ldots,g_{2k}$ of $g_2$ are continuously differentiable in a neighborhood of $\boldsymbol{\theta}_2$. $g_3$ possesses the same property provided that $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\neq 0$. Now for $j\in\{1,3,4\}$ let $\mathbf{G}_j$ be a $k\times k(k+3)/2$ matrix with components $\partial g_{jv}/\partial\theta_{1s}$ for $v=1,\ldots,k$, $s=1,\ldots,k(k+3)/2$ and $\mathbf{G}_2$ be a $k\times k(k+1)/2$ matrix with components $\partial g_{2v}/\partial\theta_{2s}$ for $v=1,\ldots,k$, $s=1,\ldots,k(k+1)/2$. If $\hat{\boldsymbol{\theta}}_i$ $(i=1,2)$ is asymptotically normal with mean $\boldsymbol{\theta}_i$ and covariance matrix $\boldsymbol{\Omega}_i$ and if all diagonal elements of $\mathbf{G}'_j\boldsymbol{\Omega}_i\mathbf{G}_j$ with $j=1,3,4$ for $i=1$ and $j=2$ for $i=2$ are non-zero, then $\hat{\mathbf{w}}_{\mathrm{EU}}$, $\hat{\mathbf{w}}_{\mathrm{GMV}}$, $\hat{\mathbf{w}}_{\mathrm{SR}}$ and $\hat{\mathbf{w}}_{\mathrm{TP}}$ are asymptotically normal, too (e.g., Brockwell and Davis, 1991, p. 21)

$$\sqrt{n}(g_j(\hat{\boldsymbol{\theta}}_i) - g_j(\hat{\boldsymbol{\theta}}_i)) \xrightarrow{\mathrm{d}} \mathcal{N}_k(0, \mathbf{G}'_j\boldsymbol{\Omega}_i\mathbf{G}_j),$$

with $j=1,3,4$ for $i=1$ and $j=2$ for $i=2$. The partial derivatives of $\mathbf{G}_j$ can immediately be determined by using the rules for matrix differentiation. Let $\boldsymbol{\Delta}_{v\tau} = \boldsymbol{\Sigma}^{-1}(\mathbf{E}_{v\tau} + \mathbf{E}_{\tau v})\boldsymbol{\Sigma}^{-1}$ for $v\neq\tau$ and $\boldsymbol{\Delta}_{vv} = \boldsymbol{\Sigma}^{-1}\mathbf{E}_{vv}\boldsymbol{\Sigma}^{-1}$, where $\mathbf{E}_{v\tau} = \{\delta_{v\tau}\}$ and $\delta_{v\tau}$ denotes Kronecker's delta. We get that

$$\frac{\partial\mathbf{w}'_{\mathrm{EU}}}{\partial\boldsymbol{\mu}} = \alpha^{-1}\mathbf{R},$$

$$\frac{\partial \mathbf{w}_{EU}}{\partial \sigma_{v\tau}} = -\alpha^{-1}\left(\mathbf{\Delta}_{v\tau}\boldsymbol{\mu} - \frac{\mathbf{1}'\mathbf{\Delta}_{v\tau}\boldsymbol{\mu}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\mathbf{\Sigma}^{-1}\mathbf{1}\right) + \frac{\alpha^{-1}\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - 1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\left(\mathbf{\Delta}_{v\tau}\mathbf{1} - \frac{\mathbf{1}'\mathbf{\Delta}_{v\tau}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\mathbf{\Sigma}^{-1}\mathbf{1}\right).$$

The Sharpe ratio optimal weights are

$$\frac{\partial \mathbf{w}'_{SR}}{\partial \boldsymbol{\mu}} = \frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}}{(\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^2}, \quad \frac{\partial \mathbf{w}_{SR}}{\partial \sigma_{v\tau}} = -\frac{\mathbf{\Delta}_{v\tau}\boldsymbol{\mu}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}} + \frac{\mathbf{1}'\mathbf{\Delta}_{v\tau}\boldsymbol{\mu}}{(\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^2}\mathbf{\Sigma}^{-1}\boldsymbol{\mu},$$

and the tangency portfolio weight is

$$\frac{\partial \mathbf{w}'_{TP}}{\partial \boldsymbol{\mu}} = \alpha^{-1}\mathbf{\Sigma}^{-1}, \quad \frac{\partial \mathbf{w}_{TP}}{\partial \sigma_{v\tau}} = -\alpha^{-1}\mathbf{\Delta}_{v\tau}(\boldsymbol{\mu} - r_f\mathbf{1}).$$

Now assume that returns are serially uncorrelated. In this case the next theorem provides the variances of the asymptotic distribution for all types of weights. Note that the variances of $\hat{\mathbf{w}}_{EU}$, $\hat{\mathbf{w}}_{GMV}$ and $\hat{\mathbf{w}}_{TP}$ are equal to the limits of the finite sample variances. However, for $\hat{\mathbf{w}}_{SR}$ this is not the case, since the finite sample variance does not exist.

**Theorem 3.** *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors and let $\mathbf{X}_i \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ for $i = 1, \ldots, n$. Then the asymptotic variances of $\hat{\mathbf{w}}_{EU}$, $\hat{\mathbf{w}}_{GMV}$, $\hat{\mathbf{w}}_{TP}$ and the variance of the asymptotic distribution of $\hat{\mathbf{w}}_{SR}$ are given by*

(a) $$\lim_{n\longrightarrow\infty} n\, Var(\hat{\mathbf{w}}_{EU}) = \frac{1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\mathbf{R} + \alpha^{-2}\mathbf{R} + \alpha^{-2}\,\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}\mathbf{R} + \alpha^{-2}\,\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R},$$

(b) $$\lim_{n\longrightarrow\infty} n\, Var(\hat{\mathbf{w}}_{GMV}) = \frac{1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\mathbf{R},$$

(c) *If* $\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} \neq 0$ *then*

$$\mathbf{G}'_3\mathbf{\Omega}_1\mathbf{G}_3 = \frac{1 + \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{(\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^4}[(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})\,\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{\Sigma}^{-1} + (\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^2\mathbf{\Sigma}^{-1} - (\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu})\mathbf{\Sigma}^{-1}(\mathbf{1}\boldsymbol{\mu}' + \boldsymbol{\mu}\mathbf{1}')\mathbf{\Sigma}^{-1}],$$

(d) $$\lim_{n\longrightarrow\infty} n\, Var(\hat{\mathbf{w}}_{TP}) = \alpha^{-2}\mathbf{\Sigma}^{-1} + \alpha^{-2}((\boldsymbol{\mu} - r_f\mathbf{1})'\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})\mathbf{\Sigma}^{-1} + \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})(\boldsymbol{\mu} - r_f\mathbf{1})'\mathbf{\Sigma}^{-1}).$$

Let $k = 2$. Then it holds for the weights based on expected quadratic utility that

$$\lim_{n\to\infty} n\, Var(\hat{w}_{EU,2}) = \frac{1}{(\gamma_{11}^2 + \gamma_{22}^2 - 2\gamma_{12})^2}(\alpha^{-2}(\gamma_{11}^2 + \gamma_{22}^2 - 2\gamma_{12}) + 2\alpha^{-2}(\mu_2 - \mu_1)^2 + \gamma_{11}^2\gamma_{22}^2 - \gamma_{12}^2).$$

The asymptotic variance of $\hat{w}_{GMV,2}$ can be determined by letting $\alpha$ increase to infinity. It is of high importance that a similar result does not hold for the weights of the Sharpe ratio optimal portfolio, because the moments of the exact distribution do not exist at all. Estimating the variance by the sample variance obtained from a Monte-Carlo simulation would lead to misleading results.

## 4. Conclusion

This paper discusses the problem of the exact moments and the exact distribution of optimal portfolio weights. We consider four of the most commonly used portfolio allocation strategies. The first two moments are derived in multivariate setup for the expected quadratic utility portfolio weights. The global minimum variance portfolio weights have the most attractive statistical properties due to the standard exact distribution. The density function is derived only in the univariate case for the weights of the tangency portfolio. In the simulation study, which is not provided here but is available in Okhrin (2004), we conclude that the exact moments can deviate significantly from their asymptotic counterparts, and thus, relying on the asymptotic results may be misleading. For example, this deviation can be very large for the covariance matrix of the expected quadratic utility and global minimum variance portfolio weights. This is of special importance for the Sharpe ratio optimal weights, since we show that the first two moments do not exist, and consequently the use of the common estimator leads to untractable results. Moreover, it appears that the moments of the optimal portfolio weights are very sensitive to changes in the moments of stock returns. The sensitivity of the mean appears to diminish for higher risk aversion, but the sensitivity of the covariance matrix is substantial and robust. In general, this paper stresses the drawbacks of the classical estimation procedures for the family of mean–variances efficient portfolios and the need for more sophisticated techniques.

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## Appendix

Here, we denote the $p$-dimensional Wishart distribution with $n$ degrees of freedom and covariance matrix $\mathbf{\Sigma}$ by $W_p(n, \mathbf{\Sigma})$.

**Proof of Theorem 1.** Because $\hat{\boldsymbol{\mu}}$ and $\hat{\mathbf{\Sigma}}$ are independent (see Muirhead, 1982, Theorem 3.1.2) and $\hat{\mathbf{w}}_{\mathrm{EU}} = \mathbf{w}_{\mathrm{EU}}(\hat{\boldsymbol{\mu}}, \hat{\mathbf{\Sigma}})$, the conditional distribution of $\hat{\mathbf{w}}_{\mathrm{EU}}$ given $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}$ is equal to the distribution of the random vector

$$\mathbf{w}_{\mathrm{EU}}(\boldsymbol{\mu}, \hat{\mathbf{\Sigma}}) = \frac{\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}} + \alpha^{-1}\hat{\mathbf{R}}\boldsymbol{\mu}. \tag{17}$$

We consider the $i$th component of this random vector. This leads to

$$w_{\mathrm{EU},i}(\boldsymbol{\mu},\hat{\boldsymbol{\Sigma}}) = \frac{\mathbf{e}_i'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} + \alpha^{-1}\mathbf{e}_i'\hat{\mathbf{R}}\boldsymbol{\mu}.$$

To simplify the notation, the indices are neglected and we briefly write $\hat{w}_i$ instead of $\hat{w}_{\mathrm{EU},i}$.

(a) Hence it follows that $\mathrm{E}(\hat{w}_i) = \mathrm{E}(\mathrm{E}(w_{\mathrm{EU},i}(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}})|\hat{\boldsymbol{\mu}}))$. Let $\mathbf{M}' = (\mathbf{e}_i\ \hat{\boldsymbol{\mu}}\ \mathbf{1})$ with $rank(\mathbf{M}) = 3$ and $\hat{\mathbf{H}}^{(-)} = \mathbf{M}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{M}' = \{\hat{\mathbf{H}}_{ij}^{(-)}\}_{i,j=1,2}$, where $\hat{\mathbf{H}}_{22}^{(-)} = \mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}$. Similarly we define $\hat{\mathbf{H}} = (\hat{\mathbf{H}}^{(-)})^{-1} = \{\hat{\mathbf{H}}_{ij}\}_{i,j=1,2}$. Matrices $\mathbf{H}$ and $\mathbf{H}^{(-)}$ are defined in the same way. Using $\hat{\mathbf{H}}_{22}^{(-)} = (\hat{\mathbf{H}}_{22} - \hat{\mathbf{H}}_{21}\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12})^{-1}$ and $\hat{\mathbf{H}}_{12}^{(-)} = -\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12}(\hat{\mathbf{H}}_{22} - \hat{\mathbf{H}}_{21}\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12})^{-1}$, it can be shown that $\hat{w}_i|\hat{\boldsymbol{\mu}}$ is equal to the first component of the vector $\mathbf{z} = -\hat{\mathbf{H}}_{11}^{-1}(\hat{\mathbf{H}}_{12} - (0\ \alpha^{-1})')$. Note that $(n-1)\hat{\mathbf{H}}\sim W_3(n-k+2,\mathbf{H})$ and, therefore, $(n-1)\hat{\mathbf{H}}_{11}\sim W_2(n-k+2,\mathbf{H}_{11})$. For notational convenience, let $\mathbf{a} = (0\quad \alpha^{-1}(n-1))'$. From Theorem 3.2.10 of Muirhead (1982), it follows that:

$$(n-1)\hat{\mathbf{H}}_{12}|(n-1)\hat{\mathbf{H}}_{11} = \mathbf{X}\sim\mathcal{N}_2(\mathbf{X}\cdot\mathbf{H}_{11}^{-1}\mathbf{H}_{12},(\mathbf{H}_{22}-\mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{X}),$$

$$(n-1)(\hat{\mathbf{H}}_{12} - (0\ \alpha^{-1})')|(n-1)\hat{\mathbf{H}}_{11}$$
$$= \mathbf{X}\sim\mathcal{N}_2(\mathbf{X}\cdot\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - \mathbf{a}',(\mathbf{H}_{22}-\mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{X}),$$

$$\hat{\mathbf{H}}_{11}^{-1}(\hat{\mathbf{H}}_{12} - (0\ \alpha^{-1})')|(n-1)\hat{\mathbf{H}}_{11}$$
$$= \mathbf{X}\sim\mathcal{N}_2(\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - \mathbf{X}^{-1}\mathbf{a}',(\mathbf{H}_{22}-\mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{X}^{-1}).$$

This implies

$$\mathrm{E}(\mathbf{z}) = \mathrm{E}(\mathrm{E}(\mathbf{z}|\hat{\mathbf{H}}_{11})) = -\mathrm{E}(\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - ((n-1)\hat{\mathbf{H}}_{11})^{-1}\mathbf{a}')$$
$$= -\mathbf{H}_{11}^{-1}\left(\mathbf{H}_{12} - \frac{1}{n-k-1}\mathbf{a}'\right).$$

Thus

$$\mathrm{E}(\hat{w}_i|\hat{\boldsymbol{\mu}}) = \frac{\mathbf{e}_i'\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{n-1}{n-k-1}\alpha^{-1}\mathbf{e}_i'\mathbf{R}\hat{\boldsymbol{\mu}}.$$

Taking expectation over $\hat{\boldsymbol{\mu}}$ and an arbitrary component of $\hat{\mathbf{w}}_{\mathrm{EU}}$ proves (a).

(b) Using the notation from (a) we can compute the second moment of $\mathbf{z}$ as

$$Var(\mathbf{z}) = \mathrm{E}(Var(\mathbf{z}|\hat{\mathbf{H}}_{11})) + Var(\mathrm{E}(\mathbf{z}|\hat{\mathbf{H}}_{11})),$$

with

$$\mathrm{E}(Var(\mathbf{z}|\hat{\mathbf{H}}_{11})) = \mathrm{E}((\mathbf{H}_{22}-\mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})((n-1)\hat{\mathbf{H}}_{11})^{-1})$$
$$= \frac{1}{n-k-1}(\mathbf{H}_{22}-\mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{H}_{11}^{-1}$$
$$= \frac{1}{n-k-1}(\mathbf{H}_{22}^{(-)})^{-1}(\mathbf{H}_{11}^{(-)} - \mathbf{H}_{12}^{(-)}(\mathbf{H}_{22}^{(-)})^{-1}\mathbf{H}_{21}^{(-)}),$$

$$Var(\mathrm{E}(\mathbf{z}|\hat{\mathbf{H}}_{11})) = Var(\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - ((n-1)\hat{\mathbf{H}}_{11})^{-1}\mathbf{a}')$$
$$= \mathrm{E}(((n-1)\hat{\mathbf{H}}_{11})^{-1}\mathbf{aa}'((n-1)\hat{\mathbf{H}}_{11})^{-1}) - \frac{1}{(n-k-1)^2}\mathbf{H}_{11}^{-1}\mathbf{aa}'\mathbf{H}_{11}^{-1}.$$

Using Corollary 14 of Styan (1989), it holds that

$$\mathrm{E}(((n-1)\hat{\mathbf{H}}_{11})^{-1}\mathbf{aa}'((n-1)\hat{\mathbf{H}}_{11})^{-1}) = (d_1 + d_2)\mathbf{H}_{11}^{-1}\mathbf{aa}'\mathbf{H}_{11}^{-1} + d_2\, tr(\mathbf{a}'\mathbf{a}\mathbf{H}_{11}^{-1})\mathbf{H}_{11}^{-1},$$

where $d_2 = 1/(n-k)(n-k-1)(n-k-3)$ and $d_1 = (n-k-2)d_2$. Finally,

$$Var(\mathbf{z}) = \frac{1}{n-k-1}(\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{H}_{11}^{-1}$$
$$+ (d_1 + d_2 - 1/(n-k-1)^2)\mathbf{H}_{11}^{-1}\mathbf{aa}'\mathbf{H}_{11}^{-1} + d_2\, tr(\mathbf{a}'\mathbf{a}\mathbf{H}_{11}^{-1})\mathbf{H}_{11}^{-1}.$$

Taking the first element of $Var(\mathbf{z})$ results in

$$Var(\hat{w}_i|\hat{\boldsymbol{\mu}}) = \frac{1}{n-k-1}\frac{\mathbf{e}_i'\mathbf{R}\mathbf{e}_i}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \alpha^{-2}c_1(\mathbf{e}_i\mathbf{R}\hat{\boldsymbol{\mu}})^2 + \alpha^{-2}c_2\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}\mathbf{e}_i'\mathbf{R}\mathbf{e}_i.$$

We again make use of the fact that $Var(\hat{\mathbf{w}}_{\mathrm{EU}}) = \mathrm{E}(Var(\hat{\mathbf{w}}_{\mathrm{EU}}|\hat{\boldsymbol{\mu}})) + Var(\mathrm{E}(\hat{\mathbf{w}}_{\mathrm{EU}}|\hat{\boldsymbol{\mu}}))$. Because $\hat{\boldsymbol{\mu}} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)$, $\mathbf{R}\boldsymbol{\Sigma}\mathbf{R} = \mathbf{R}$, and $tr(\mathbf{R}\boldsymbol{\Sigma}) = k-1$, it follows that (see Theorem 3.2b.3 of Mathai and Provost, 1992)

$$\mathrm{E}(\mathbf{e}_i'\mathbf{R}\hat{\boldsymbol{\mu}})^2 = (\mathbf{e}_i'\mathbf{R}\boldsymbol{\mu})^2 + \mathbf{e}_i'\mathbf{R}\mathbf{e}_i/n \quad \text{and} \quad \mathrm{E}(\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}) = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu} + (k-1)/n.$$

Substitution proves part b (ii) of the theorem.

Consider now the two arbitrary components of $\hat{\mathbf{w}}_{\mathrm{EU}}$. Let $\mathbf{M}' = (\mathbf{e}_i\ \mathbf{e}_j\ \hat{\boldsymbol{\mu}}\ \mathbf{1})$ with $rank(\mathbf{M}) = 4$. Other notation introduced in (a) is changed correspondingly. Similarly we can show that $(\hat{w}_i, \hat{w}_j)|\hat{\boldsymbol{\mu}}$ is equal to the first two components of the vector $\mathbf{z} = -\hat{\mathbf{H}}_{11}^{-1}(\hat{\mathbf{H}}_{12} - (0\ 0\ \alpha^{-1})')$.

Note that $(n-1)\hat{\mathbf{H}} \sim W_4(n-k+3, \mathbf{H})$ and, therefore, $(n-1)\hat{\mathbf{H}}_{11} \sim W_3(n-k+3, \mathbf{H}_{11})$. Let $\mathbf{a} = (0\ 0\ \alpha^{-1}(n-1))'$. Then from Theorem 3.2.10 of Muirhead (1982), it follows that

$$\hat{\mathbf{H}}_{11}^{-1}(\hat{\mathbf{H}}_{12} - (0\ 0\ \alpha^{-1})')|(n-1)\hat{\mathbf{H}}_{11}$$
$$= \mathbf{X} \sim \mathcal{N}_3(\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - \mathbf{X}^{-1}\mathbf{a}', (\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{X}^{-1}).$$

As we showed in (b)

$$Var(\mathbf{z}) = \frac{1}{n-k-1}(\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{H}_{11}^{-1}$$
$$+ (d_1 + d_2 - 1/(n-k-1)^2)\mathbf{H}_{11}^{-1}\mathbf{aa}'\mathbf{H}_{11}^{-1} + d_2\, tr(\mathbf{a}'\mathbf{a}\mathbf{H}_{11}^{-1})\mathbf{H}_{11}^{-1}.$$

Note that $Cov(\hat{w}_i, \hat{w}_j)$ is equal to the second element in the first row of $Var(\mathbf{z})$. Extracting this element from the last equation shows that

$$Cov(\hat{w}_i, \hat{w}_j|\hat{\boldsymbol{\mu}}) = \frac{1}{n-k-1}\frac{\mathbf{e}_i'\mathbf{R}\mathbf{e}_j}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \alpha^{-2}c_1\mathbf{e}_i\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}\mathbf{e}_j + \alpha^{-2}c_2\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}\mathbf{e}_i'\mathbf{R}\mathbf{e}_j.$$

The unconditional variance is computed using

$$\mathrm{E}(\mathbf{e}_i'\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}\mathbf{e}_j) = (\mathbf{e}_i'\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}\mathbf{e}_j) + \mathbf{e}_i'\mathbf{R}\mathbf{e}_j/n \quad \text{and} \quad \mathrm{E}(\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}) = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu} + (k-1)/n.$$

Substitution completes the proof of the theorem. □

**Proof of Proposition 1.** Let $\mathbf{M}' = (\mathbf{e}_1\ \mathbf{e}_2 \ldots \mathbf{e}_{m-1}\ \mathbf{1})$, $\hat{\mathbf{H}} = (\mathbf{M}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{M}')^{-1} = \{\hat{\mathbf{H}}_{ij}\}_{i,j=1,2}$ and $\hat{\mathbf{H}}^{-1} = \mathbf{M}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{M}' = \{\hat{\mathbf{H}}_{ij}^{(-)}\}_{i,j=1,2}$, with $\hat{\mathbf{H}}_{22}^{(-)} = \mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}$, etc. The matrices $\mathbf{H}^{-1}$ and $\mathbf{H}$ are partitioned in the same way. Since $(n-1)\hat{\boldsymbol{\Sigma}} \sim W_k(n-1, \boldsymbol{\Sigma})$, it follows that $(n-1)\hat{H} \sim W_m(n-k+m-1, (\mathbf{M}\boldsymbol{\Sigma}^{-1}\mathbf{M}')^{-1})$. Then

$$\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}} = \frac{\hat{\mathbf{H}}_{12}^{(-)}}{\hat{\mathbf{H}}_{22}^{(-)}} = \frac{-\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12}(\hat{\mathbf{H}}_{22} - \hat{\mathbf{H}}_{21}\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12})^{-1}}{(\hat{\mathbf{H}}_{22} - \hat{\mathbf{H}}_{21}\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12})^{-1}} = -\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12}.$$

From the properties of the Wishart distribution we get

$$\begin{aligned} &-(n-1)\hat{\mathbf{H}}_{12}|(n-1)\hat{\mathbf{H}}_{11} \\ &\quad \sim \mathcal{N}_{m-1}(-(n-1)\hat{\mathbf{H}}_{11}\mathbf{H}_{11}^{-1}\mathbf{H}_{12}, (\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})(n-1)\hat{\mathbf{H}}_{11}), \end{aligned}$$

$$(n-1)\hat{\mathbf{H}}_{11} \sim W_{m-1}(n-k+m-1, \mathbf{H}_{11}).$$

Consequently it holds that

$$-\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12}|(n-1)\hat{\mathbf{H}}_{11} \sim \mathcal{N}_{m-1}(-\mathbf{H}_{11}^{-1}\mathbf{H}_{12}, (\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})((n-1)\hat{\mathbf{H}}_{11})^{-1}).$$

$$\begin{aligned} f_{\hat{\hat{\mathbf{w}}}_{\mathrm{GMV}}}(\mathbf{x}) &= \int_{\mathbf{Y}>0} f_{\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12}|(n-1)\hat{\mathbf{H}}_{11}}(\mathbf{x}|\mathbf{Y}) f_{(n-1)\hat{\mathbf{H}}_{11}}(\mathbf{Y})\,\mathrm{d}\mathbf{Y} \\ &= \frac{2^{-(n-k+m)(m-1)/2}(det(\mathbf{H}_{11}))^{-(n-k+m-1)/2}}{\pi^{(m-1)/2}(\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})^{(m-1)/2}\Gamma_{m-1}((n-k+m-1)/2)} \\ &\quad \times \int_{\mathbf{Y}>0} (det(\mathbf{Y}))^{(n-k)/2} \\ &\quad \times etr\left[-\frac{1}{2}\left(\frac{(\mathbf{x} + \mathbf{H}_{11}^{-1}\mathbf{H}_{12})(\mathbf{x} + \mathbf{H}_{11}^{-1}\mathbf{H}_{12})'}{\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12}} + \mathbf{H}_{11}^{-1}\right)\mathbf{Y}\right]\mathrm{d}\mathbf{Y}, \end{aligned}$$

where $etr(\mathbf{A})$ denotes $\mathrm{e}^{tr(\mathbf{A})}$. Using the definition of the multivariate gamma function

$$\Gamma_p(\alpha) = (det(\mathbf{Q}))^{\alpha}\int_{\mathbf{Y}>0}(det(\mathbf{Y}))^{\alpha-(p+1)/2}etr(-\mathbf{Q}\mathbf{Y})\,\mathrm{d}\mathbf{Y},$$

where $p$ denotes the dimension of $\mathbf{Y}$, and applying the properties of the inverse of a partitioned matrix, the statement of the theorem naturally follows. □

**Proof of Proposition 2.** Let $\mathbf{H} = (\mathbf{M}\boldsymbol{\Sigma}^{-1}\mathbf{M}')^{-1} = \{h_{ij}\}_{i,j=1,\ldots,3}$ and $\hat{\mathbf{H}} = (\mathbf{M}\hat{\boldsymbol{\Sigma}}\mathbf{M}')^{-1} = \{\hat{h}_{ij}\}_{i,j=1,\ldots,3}$. We observe that

$$\hat{w}_{\mathrm{SR},i}|(\hat{\boldsymbol{\mu}} = \mathbf{y}) = \frac{\hat{h}_{31}\hat{h}_{22} - \hat{h}_{21}\hat{h}_{32}}{\hat{h}_{11}\hat{h}_{32} - \hat{h}_{31}\hat{h}_{12}} = \frac{(\hat{h}_{31}\hat{h}_{22} - \hat{h}_{21}\hat{h}_{32})/(\hat{h}_{11}\hat{h}_{22} - \hat{h}_{12}^2)}{(\hat{h}_{11}\hat{h}_{32} - \hat{h}_{31}\hat{h}_{12})/(\hat{h}_{11}\hat{h}_{22} - \hat{h}_{12}^2)} = \frac{Z}{N}.$$

Now we split $\mathbf{H}$ and $\hat{\mathbf{H}}$ into block matrices. Let $\hat{\mathbf{H}}_{11} = \{\hat{h}_{ij}\}_{i,j=1,2}$, $\hat{\mathbf{H}}_{12} = (\hat{h}_{13}, \hat{h}_{23})'$ and $\hat{\mathbf{H}}_{22} = \hat{h}_{33}$. The matrix $\mathbf{H}$ is split in the same way. Then $(Z, N)' = \hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12}$. Applying Theorem 3.2.10 (ii) of Muirhead (1982) and using the fact that $(n-1)\hat{\mathbf{H}} \sim W_3(n-k+2, \mathbf{H})$, it follows that

$$(n-1)\hat{\mathbf{H}}_{12} \,|\, (n-1)\hat{\mathbf{H}}_{11} = \mathbf{X} \sim \mathcal{N}_2((\mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{X})', (\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{X}).$$

Consequently,

$$\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12} \,|\, (n-1)\hat{\mathbf{H}}_{11} = \mathbf{X} \sim \mathcal{N}_2(\mathbf{H}_{11}^{-1}\mathbf{H}_{12}, (\mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12})\mathbf{X}^{-1}).$$

We denote $\mathbf{a} = (a_1, a_2)' = \mathbf{H}_{11}^{-1}\mathbf{H}_{12}$ and $b = \mathbf{H}_{22} - \mathbf{H}_{21}\mathbf{H}_{11}^{-1}\mathbf{H}_{12} = 1/h_{33}^{(-)}$. Because $(n-1)\hat{\mathbf{H}}_{11} \sim W_2(n-k+2, \mathbf{H}_{11})$, the joint density of $(n-1)Z$ and $(n-1)N$ is equal to

$$\begin{aligned}
f_{(n-1)Z,(n-1)N}(\mathbf{z}) &= \int \cdots \int f_{\hat{\mathbf{H}}_{11}^{-1}\hat{\mathbf{H}}_{12} \,|\, (n-1)\hat{\mathbf{H}}_{11}}(\mathbf{z}|\mathbf{X}) f_{(n-1)\hat{\mathbf{H}}_{11}}(2, n-k+2, \mathbf{H}_{11})(\mathbf{X})\, \mathrm{d}\mathbf{X} \\
&= \frac{1}{2^{n-k+3}\pi b \Gamma_2((n-k+2)/2)} \frac{1}{(\det H_{11})^{(n-k+2)/2}} \\
&\quad \times \int \cdots \int (\det \mathbf{X})^{(n-k)/2} \exp(-tr((\mathbf{H}_{11}^{-1} + (\mathbf{z}-\boldsymbol{a})(\mathbf{z}-\boldsymbol{a})'/b)\mathbf{X}/2))\, \mathrm{d}\mathbf{X} \\
&= \frac{(n-k+1)\sqrt{\det \mathbf{H}_{11}}}{2\pi b}\left(1 + \frac{(\mathbf{z}-\boldsymbol{a})'\mathbf{H}_{11}(\mathbf{z}-\boldsymbol{a})}{b}\right)^{-(n-k+3)/2}
\end{aligned}$$

using Theorem 2.1.11 of Muirhead (1982). This is the density of the two-dimensional $t$-distribution with mean $\boldsymbol{a}$, covariance matrix $((n-k+1)\mathbf{H}_{11}/b)^{-1}$ and $n-k+1$ degrees of freedom. Then the density function of the ratio $Z/N$ is given by

$$\begin{aligned}
f_{Z/N}(x) &= \int_0^\infty z(f_{Z,N}(xz, z) + f_{Z,N}(-xz, -z))\, \mathrm{d}z \\
&= \frac{(n-k+1)\sqrt{\det \mathbf{H}_{11}}\, b^{(n-k+1)/2}}{2\pi}(I_1 + I_2).
\end{aligned}$$

Now let $w_1 = (x\ 1)\mathbf{H}_{11}(x\ 1)'$, $w_2 = (x\ 1)\mathbf{H}_{12}$, $w_3 = \mathbf{H}_{22}$ and $\xi(x) = |w_2|/\sqrt{w_1 w_3 - w_2^2}$. Straightforward calculations show that

$$\begin{aligned}
I_1 &= \int_0^\infty z(w_1 z^2 - 2w_2 z + w_3)^{-(n-k+3)/2}\, \mathrm{d}z \\
&= \frac{1}{w_1(n-k+1)} w_3^{-(n-k+1)/2} + \frac{w_2}{w_1^{3/2}}\left(w_3 - \frac{w_2^2}{w_1}\right)^{-(n-k+2)/2} \\
&\quad \times \int_{-\xi}^\infty (v^2+1)^{-(n-k+3)/2}\, \mathrm{d}v
\end{aligned}$$

and

$$I_2 = \frac{1}{w_1(n-k+1)} w_3^{-(n-k+1)/2} - \frac{w_2}{w_1^{3/2}} \left( w_3 - \frac{w_2^2}{w_1} \right)^{-(n-k+2)/2} \times \int_{\xi}^{\infty} (v^2+1)^{-(n-k+3)/2} \, \mathrm{d}v.$$

Hence

$$f_{Z/N}(x) = \frac{(n-k+1)\sqrt{det\,\mathbf{H}_{11}}\, b^{(n-k+1)/2}}{2\pi} \times \left( \frac{2w_3^{-(n-k+1)/2}}{w_1(n-k+1)} + 2\,\frac{w_2}{w_1^{3/2}} \left( w_3 - \frac{w_2^2}{w_1} \right)^{-(n-k+2)/2} \times \int_0^{\xi} (v^2+1)^{-(n-k+3)/2} \, \mathrm{d}v \right).$$

Substitution of $w_1$, $w_2$ and $w_3$ proves the proposition. □

**Proof of Proposition 3.** Taking into account the independence of $\hat{\mu}$ and $\hat{\sigma}^2$, it holds that

$$f_{\hat{w}_{\mathrm{TP}}}(x) = \int_0^{\infty} f_{\hat{w}_{\mathrm{TP}}|\hat{\sigma}^2}(x|y) f_{\hat{\sigma}^2}(y) \, \mathrm{d}y, \quad \text{with}$$

$$\hat{w}_{\mathrm{TP}}|\hat{\sigma}^2 = y \sim \mathcal{N}_1\left( \alpha^{-1} \frac{\mu - r_f}{y}, \frac{\alpha^{-2}}{n} \frac{\sigma^2}{y^2} \right) \quad \text{and} \quad (n-1)\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Substitution of the densities of the last two distributions into the integral for $f_{\hat{w}_{\mathrm{TP}}}(x)$ results in the expression

$$f_{\hat{w}_{\mathrm{TP}}}(x) = \frac{\sqrt{n}\alpha(\frac{n-1}{2\sigma^2})^{(n-1)/2}}{(2\pi)^{1/2}\Gamma(\frac{n-1}{2})\sigma} \exp\left[ -\frac{n}{2} \frac{(\mu - r_f)^2}{\sigma^2} \right] \times \int_0^{\infty} y^{(n-1)/2} \exp\left[ -\frac{n\alpha^2}{2} \frac{x^2}{\sigma^2} y^2 - \frac{1}{2} \frac{n - 1 - 2n\alpha x(\mu - r_f)}{\sigma^2} y \right] \mathrm{d}y.$$

Note that the integral in the last expression is equal to the Laplace transform of $s^{p-1}\mathrm{e}^{-as^2}$ and $b$ can be of both signs. Let $p = (n+1)/2$, and $a > 0$, i.e. $x \neq 0$, then

$$\int_0^{\infty} s^{p-1} \mathrm{e}^{-as^2 - bs} \, \mathrm{d}s = \int_0^{\infty} s^{p-1} \mathrm{e}^{-as^2} \sum_{i=0}^{\infty} \frac{(-1)^i b^i s^i}{i!} \, \mathrm{d}s$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i b^i}{i!} \int_0^{\infty} s^{p+i-1} \mathrm{e}^{-as^2} \, \mathrm{d}s = \frac{1}{2} a^{-p/2} \sum_{i=0}^{\infty} (-1)^i \frac{(b/\sqrt{a})^i \Gamma((p+i)/2)}{i!}$$
$$= \frac{1}{2} a^{-p/2} \sum_{i=0}^{\infty} \left( \frac{(b/\sqrt{a})^{2i} \Gamma((p+2i)/2)}{(2i)!} - \frac{(b/\sqrt{a})^{2i+1} \Gamma((p+2i+1)/2)}{(2i+1)!} \right)$$
$$= \frac{1}{2} a^{-p/2} \sum_{i=0}^{\infty} \left( \frac{(b^2/4a)^i 2^{2i} \Gamma(p/2+i)}{(2i)!} - \frac{b}{\sqrt{a}} \frac{(b^2/4a)^i 2^{2i} \Gamma((p+1)/2+i)}{(2i+1)!} \right).$$

Noting that $2^{2i}/(2i)! = \Gamma(1/2)/\Gamma(i+1/2)i!$ and using the series representation of confluent hypergeometric function (see Gradshteyn and Ryzhik, 2000), the expression for the integral follows:

$$\int_0^{\infty} s^{p-1} \mathrm{e}^{-as^2-bs} \, \mathrm{d}s = \frac{1}{2} a^{-(p+1)/2} \left[ \sqrt{a} \Gamma(p/2) {}_1F_1 \left( \frac{p}{2}, \frac{1}{2}; \frac{b^2}{4a} \right) - b \Gamma((p+1)/2) {}_1F_1 \times \left( \frac{p+1}{2}, \frac{3}{2}; \frac{b^2}{4a} \right) \right].$$

The interchange of the integral and infinite sum is justified due to absolute convergence of both. Applying this result to $f_{\hat{w}_{TP}}(x)$ proves the proposition. □

**Proof of Theorem 3.** We will give only the proofs of (c) and (d). For (a) and (b) a similar technique can be applied.

(c) Let $\mathbf{D}_k$ denote a $k^2 \times k(k+1)/2$ duplication matrix of (see Rogers, 1980; Magnus and Neudecker, 1999) such that $\mathbf{D}_k vech(\mathbf{X}) = vec(\mathbf{X})$ for an arbitrary $k \times k$-matrix $\mathbf{X}$ and $vec(\mathbf{X}) = (x_{11}, \ldots, x_{1k}, x_{21}, \ldots, x_{2k}, \ldots, x_{kk})'$. Moreover, let $\mathbf{D}_k^+ = (\mathbf{D}_k'\mathbf{D}_k)^{-1}\mathbf{D}_k'$ with the property that $\mathbf{D}_k^+ vec(\mathbf{X}) = vech(\mathbf{X})$. Then the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}_1$ is given by

$$\lim_{n \longrightarrow \infty} n \, Var(\hat{\boldsymbol{\theta}}_1) = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0}_{k \times k^2} \\ \mathbf{0}_{k^2 \times k} & \mathbf{D}_k^+(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime} \end{pmatrix},$$

where $\mathbf{K}_k$ is a commutation matrix (see Magnus and Neudecker, 1999). Consequently the asymptotic covariance matrix of the Sharpe ratio optimal weights can be split as follows

$$\mathbf{G}_3'\boldsymbol{\Omega}_1\mathbf{G}_3 = \left( \frac{\partial \mathbf{w}_{SR}'}{\partial \boldsymbol{\mu}} \right)' \boldsymbol{\Sigma} \frac{\partial \mathbf{w}_{SR}'}{\partial \boldsymbol{\mu}} + \left( \frac{\partial \mathbf{w}_{SR}'}{\partial \, vech(\boldsymbol{\Sigma})} \right)' \mathbf{D}_k^+(\mathbf{I}_{k^2} + \mathbf{K}_k)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})\mathbf{D}_k^{+\prime} \frac{\partial \mathbf{w}_{SR}'}{\partial vech(\boldsymbol{\Sigma})}. \quad (18)$$

Note that

$$\frac{\partial \mathbf{w}_{SR}'}{\partial \boldsymbol{\mu}} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}$$

and

$$\frac{\partial \mathbf{w}'_{\text{SR}}}{\partial\, vech(\boldsymbol{\Sigma})} = \frac{\partial \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}}{\partial\, vech(\boldsymbol{\Sigma})}\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} - \frac{\partial \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\partial\, vech(\boldsymbol{\Sigma})}\frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}$$

$$= \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}\frac{\partial(vec(\boldsymbol{\Sigma}^{-1}))'}{\partial\, vech(\boldsymbol{\Sigma})}[(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\boldsymbol{\mu}\otimes\mathbf{I}_k) - (\boldsymbol{\mu}\otimes\mathbf{1})\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}],$$

where

$$\frac{\partial(vec(\boldsymbol{\Sigma}^{-1}))'}{\partial\, vech(\boldsymbol{\Sigma})} = \frac{\partial\,(vech(\boldsymbol{\Sigma}^{-1}))'}{\partial\, vech(\boldsymbol{\Sigma})}\mathbf{D}'_k = -\mathbf{D}'_k(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}^{+\prime}_k\mathbf{D}'_k \tag{19}$$

(see Harville, 1997, p. 368). The following properties of duplication and commutation matrices are important in the remainder of the proof: (i) $\mathbf{D}_k\mathbf{D}^+_k = \frac{1}{2}(\mathbf{I}_{k^2} + \mathbf{K}_k)$; (ii) $\mathbf{K}'_k = \mathbf{K}_k = \mathbf{K}^{-1}_k$; (iii) if $\mathbf{X}$ is a $m\times n$ matrix and $\mathbf{x}$ is a $m\times 1$ vector, then $\mathbf{K}_m(\mathbf{X}\otimes\mathbf{x}) = (\mathbf{x}\otimes\mathbf{X})$. Using these properties, the second term of (18) can be evaluated as

$$\left(\frac{\partial \mathbf{w}'_{\text{SR}}}{\partial\, vech(\boldsymbol{\Sigma})}\right)'\mathbf{D}^+_k(\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}^{+\prime}_k\frac{\partial \mathbf{w}'_{\text{SR}}}{\partial\, vech(\boldsymbol{\Sigma})}$$

$$= \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^4}[(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\boldsymbol{\mu}'\otimes\mathbf{I}_k) - \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}(\boldsymbol{\mu}'\otimes\mathbf{1}')]$$

$$\times(\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})[(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\boldsymbol{\mu}\otimes\mathbf{I}_k) - (\boldsymbol{\mu}\otimes\mathbf{1})\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}]$$

$$= \frac{1}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^4}[(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1} + (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}$$

$$- (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{1}\boldsymbol{\mu}' + \boldsymbol{\mu}\mathbf{1}')\boldsymbol{\Sigma}^{-1}].$$

Note that substitution of $\mathbf{1}$ instead of $\boldsymbol{\mu}$ proves part (b) of the theorem. Adding $(\partial\mathbf{w}'_{\text{SR}}/\partial\boldsymbol{\mu})'\boldsymbol{\Sigma}\frac{\partial\mathbf{w}'_{\text{SR}}}{\partial\boldsymbol{\mu}}$ to the last expression proves (c).

(d) In this case a modification of the equation (18) can be used as well. The derivatives of the vector of weights with respect to $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given by

$$\frac{\partial \mathbf{w}'_{\text{TP}}}{\partial\boldsymbol{\mu}} = \alpha^{-1}\boldsymbol{\Sigma}^{-1},$$

$$\frac{\partial \mathbf{w}'_{\text{TP}}}{\partial\, vech\,(\boldsymbol{\Sigma})} = \alpha^{-1}\frac{\partial\,(vech(\boldsymbol{\Sigma}^{-1}))'}{\partial vech(\boldsymbol{\Sigma})}\mathbf{D}'_k((\boldsymbol{\mu} - r_f\mathbf{1})\otimes\mathbf{I}_k).$$

Substitution into (18) and use of (19) complete the proof. □

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