1 Modeling Dependencies with Copulae

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1.1 Introduction

The modeling and estimation of multivariate distributions is one of the most critical issues in financial and economic applications. The distributions are usually restricted to the class of multivariate elliptical distributions. This limits the analysis to a very narrow class of candidate distribution and requires the estimation of a large number of parameters. Two further problems are illustrated in Figure 1.1. The scatter plot in the first figure shows realizations of two Gaussian random variables, the points are symmetric and no extreme outliers can be observed. In contrast, the second picture exhibits numerous outliers. The outliers in the first and third quadrants show that extreme values often occur simultaneously for both variables. Such behavior is observed in crisis periods, when strong negative movements on financial markets occur simultaneously. In the third figure we observe that the dependency between negative values is different compared to positive values. This type of non-symmetric dependency cannot be modeled by elliptical distributions, because they impose a very specific radially symmetric dependency structure. Both types of dependencies are often observed in financial applications. The assumption of Gaussian distribution is therefore rarely consistent with the empirical evidence and possibly leads to incorrect inferences from financial models. Moreover, the correlation coefficient is equal for all three samples, despite clear differences in the dependencies. This questions the suitability of the correlation coefficient as the key measure of dependence for financial data.

The seminal result of Sklar (1959) provides a partial solution to these problems. It allows the separation of marginal distributions from the dependency structure between the random variables. Since the theory on modeling and estimation of univariate distributions is well established compared to the multivariate case, the initial problem reduces to modeling the dependency by copulae. This approach has several important advantages. Firstly, it dramatically widens the class of candidate distribution. Secondly, it allows a simple

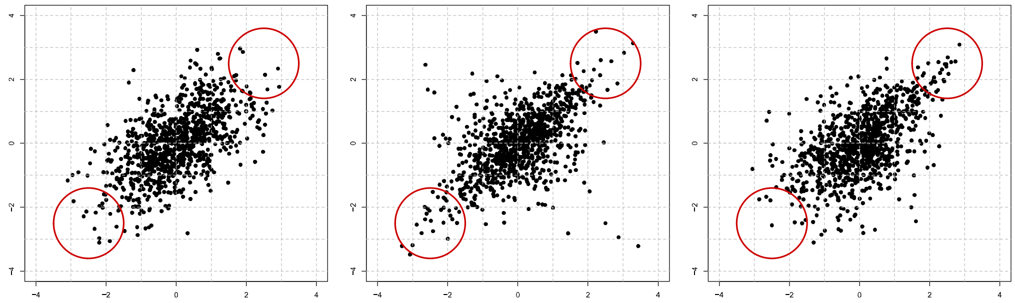

Figure 1.1. Scatter plots of bivariate samples with different dependency structures

construction of distributions with less parameters than imposed by elliptical models. Thirdly, the copula-based models reflect the real-world relationships on financial markets better.

The purpose of this chapter is twofold. Firstly, to provide the theoretical background, dealing with the estimation, simulation and testing of copula-based models, and secondly, to discuss several important applications of copulae to financial problems. The chapter is structured as follows. The next section provides a review of bivariate copulae. Here we also consider different copula families and dependency measures. The third section extends the discussion to a multivariate framework. The fourth and fifth sections provide estimation and simulation techniques. The final section illustrates the use of copulae in financial problems. We omit all proofs and follow the notation used in Joe (1997).

1.2 Bivariate Copulae

Modeling and measuring the dependency between two random variables using copulae is the subject of this section. There are several equivalent definitions of the copula function. We define it as a bivariate distribution function with both marginal distributions being uniform on $[0, 1]$.

DEFINITION 1.1 *The bivariate copula is a function* $C\colon [0,1]^2 \to [0,1]$ *with the following properties:*

1. *For every* $u_1, u_2 \in [0,1]$ $C(u_1, 0) = 0 = C(0, u_2)$.
2. *For every* $u_1, u_2 \in [0,1]$ $C(u_1, 1) = u_1$ *and* $C(1, u_2) = u_2$.
3. *For every* (u_1, u_2), $(u_1', u_2') \in [0,1]^2$ *such that* $u_1 \leq u_2$ *and* $u_1' \leq u_2'$

$$C(u_2, u_2') - C(u_2, u_1') - C(u_1, u_2') + C(u_1, u_1') \geq 0.$$

Copulae gained their popularity due to a seminal paper by Sklar (1959), where this term was first coined. The separation of the bivariate distribution function into the copula function and margins is formalized in the next theorem (Nelsen (2006), Theorem 2.3.3).

PROPOSITION 1.1 *Let F be a bivariate distribution function with margins F_1 and F_2, then there exists a copula C such that*

$$F(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\}, \quad x_1, x_2 \in \overline{\mathbb{R}}. \tag{1.1}$$

If F_1 and F_2 are continuous then C is unique. Otherwise C is uniquely determined on $F_1(\overline{\mathbb{R}}) \times F_2(\overline{\mathbb{R}})$.

Conversely, if C is a copula and F_1 and F_2 are univariate distribution functions, then function F in (1.1) is a bivariate distribution function with margins F_1 and F_2.

The theorem allows us to depart an arbitrary continuous bivariate distribution into its marginal distributions and the dependency structure. The latter is defined by the copula function.

The representation (1.1) also shows how new bivariate distributions can be constructed. We can extend the class of standard elliptical distributions by keeping the same elliptical copula function and varying the marginal distributions or vice versa. Going further we can take elliptical margins and impose some non-symmetric form of dependency by considering non-elliptical copulae. This shows that copulae substantially widen the family of elliptical distributions. To determine the copula function of a given bivariate distribution we use the transformation

$$C(u_1, u_2) = F\{F_1^{-1}(u_1), F_2^{-1}(u_2)\}, \quad u_1, u_2 \in [0, 1], \tag{1.2}$$

where F_i^{-1}, $i = 1, 2$ are generalized inverses of the marginal distribution functions.

Since the copula function is a bivariate distribution with uniform margins, it follows that the copula density could be determined in the usual way

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}, \quad u_1, u_2 \in [0, 1]. \tag{1.3}$$

Being armed with the Theorem 1.1 and (1.3) we could write density function $f(\cdot)$ of the bivariate distribution F in terms of copula as follows

$$f(x_1, x_2) = c\{F_1(x_1), F_2(x_2)\} f_1(x_1) f_2(x_2), \quad x_1, x_2 \in \overline{\mathbb{R}}.$$

A very important property of copulae is given in Theorem 2.4.3 in Nelsen (2006), in it, it is shown that copula is invariant under strictly monotone transformations. This implies that the copulae capture only those features of the joint distribution, which are invariant under increasing transformations.

1.2.1 Copula Families

Naturally there is an infinite number of different copula functions satisfying the assumptions of Definition 1. In this section we discuss in details three important sub-classes of simple, elliptical and Archimedean copulae.

Simplest Copulae

We are often interested in some extreme, special cases, like independence and perfect positive or negative dependence. If two random variables X_1 and X_2 are stochastically independent, from the Theorem 1.1 the structure of such a relationship is given by the product (independence) copula defined as

$$\Pi(u_1, u_2) = u_1 u_2, \quad u_1, u_2 \in [0, 1].$$

The contour diagrams of the bivariate density function with product copula and either Gaussian or t-distributed margins are given in Figure 1.2.

Another two extremes are the lower and upper Fréchet-Hoeffding bounds. They represent the perfect negative and positive dependences respectively

$$W(u_1, u_2) = \max(0, u_1 + u_2 - 1) \text{ and } M(u_1, u_2) = \min(u_1, u_2), \quad u_1, u_2 \in [0, 1].$$

If $C = W$ and $(X_1, X_2) \sim C(F_1, F_2)$ then X_2 is a decreasing function of X_1. Similarly, if $C = M$, then X_2 is an increasing function of X_1. In general we can argue that an arbitrary copula which represents some dependency structure lies between these two bounds, i.e.

$$W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2), \quad u_1, u_2 \in [0, 1].$$

The bounds serve as benchmarks for the evaluation of the dependency magnitude.

Elliptical Family

Due to the popularity of Gaussian and t-distributions in financial applications, the elliptical copulae also play an important role. The construction of this type of copulae is based directly on the Theorem 1.1 and (1.2). By the Theorem 2.3.7 of Nelsen (2006) bivariate copula is elliptical (has reflection symmetry) if and only if

$$C(u_1, u_2, \theta) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2, \theta), \quad u_1, u_2 \in [0, 1].$$

From (1.2) the Gaussian copula and its copula density are given by

$$\begin{aligned} C_N(u_1, u_2, \delta) &= \mathbf{\Phi}_\delta\{\Phi^{-1}(u_1), \Phi^{-1}(u_2)\}, \\ c_N(u_1, u_2, \delta) &= (1 - \delta^2)^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2}(1 - \delta^2)^{-1}(u_1^2 + u_2^2 - 2\delta u_1 u_2)\Big\} \\ &\times \exp\Big\{\frac{1}{2}(u_1^2 + u_2^2)\Big\}, \quad \text{for all } u_1, u_2 \in [0, 1],\ \delta \in [-1, 1] \end{aligned}$$

where Φ is the distribution function of $\mathrm{N}(0, 1)$, Φ^{-1} is the functional inverse of Φ and $\mathbf{\Phi}_\delta$ denotes the bivariate standard normal distribution function with the correlation coefficient δ. The level plots of the respective density are given in Figure 1.2. The t-distributed margins lead to more mass and variability in the tails of the distribution. However, the curves are symmetric, which reflects the ellipticity of the underlying copula.

In the bivariate case the t-copula and its density are given by

$$\begin{aligned} C_t(u_1, u_2, \nu, \delta) &= \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{(1-\delta^2)}} \\ &\times \left(1 + \frac{x_1^2 - 2\delta x_1 x_2 + x_2^2}{(1-\delta^2)\nu}\right)^{-\frac{\nu}{2}-1} dx_1 dx_2, \\ c_t(u_1, u_2, \nu, \delta) &= \frac{f_{\nu\delta}\{t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\}}{f_\nu\{t^{-1}(u_1)\} f_\nu\{t^{-1}(u_2)\}}, \quad u_1, u_2, \delta \in [0, 1], \end{aligned}$$

where δ denotes the correlation coefficient, ν is the number of degrees of freedom. $f_{\nu\delta}$ and f_ν are joint and marginal t-distributions respectively, while t_ν^{-1} denotes the quantile function of the t_ν distribution. In-depth analysis of the t-copula is done in Demarta and McNeil (2004).

Using (1.2) we can derive the copula function for an arbitrary elliptical distribution. The problem is, however, that such copulae depend on the inverse distribution functions and these are rarely available in an explicit form. Therefore, the next class of copulae and their generalizations provide an important flexible and rich family of alternatives to the elliptical copulae.

Archimedean Family

Opposite to elliptical copulae, the Archimedean copulae are not constructed using (1.2), but are related to Laplace transforms of bivariate distribution functions (see Section 1.6.2). Let $\mathbb{L}$ denote the class of Laplace transforms which consists of strictly decreasing differentiable functions Joe (1997), i.e.

$$\mathbb{L} = \{\phi : [0; \infty) \to [0,1] \,|\, \phi(0) = 1,\ \phi(\infty) = 0;\ (-1)^j \phi^{(j)} \geq 0;\ j = 1, \ldots, \infty\}.$$

The function $C : [0,1]^2 \to [0,1]$ defined as

$$C(u_1, u_2) = \phi\{\phi^{-1}(u_1) + \phi^{-1}(u_2)\}, \quad u_1, u_2 \in [0,1]$$

is a 2-dimensional Archimedean copula, where $\phi \in \mathbb{L}$ and is called the generator of the copula. It is straightforward to show that $C(u_1, u_2)$ satisfies the conditions of Definition 1. The generator usually depends on some parameters, however, generators with a single parameter θ are mainly considered. Joe (1997) and Nelsen (2006) provide a thoroughly classified list of popular generators for Archimedean copulae and discuss their properties. The most useful in financial applications (see Patton (2004)) appears to be the Gumbel copula with the generator function

$$\phi(x, \theta) = \exp\left(-x^{1/\theta}\right), \quad 1 \leq \theta < \infty,\ x \in [0, \infty].$$

It leads to the copula function

$$\begin{aligned} C(u_1, u_2, \theta) &= \exp\left[-\left\{(-\log u_1)^\theta + (-\log u_2)^\theta\right\}^{1/\theta}\right], \\ & \quad 1 \leq \theta < \infty,\ u_1, u_2 \in [0,1]. \end{aligned}$$

Consider a bivariate distribution based on the Gumbel copula with univariate extreme valued marginal distributions. Genest and Rivest (1989) show that this distribution is the only bivariate extreme value distribution based on an Archimedean copula. Moreover, all distributions based on Archimedean copulae belong to its domain of attraction under common regularity conditions.

In contrary to the elliptical copulae, the Gumbel copula leads to asymmetric contour diagrams in Figure 1.2. The Gumbel copula shows stronger linkage between positive values, however, more variability and more mass in the negative tail. The opposite is observed for the Clayton copula with the generator and copula functions

$$\begin{aligned} \phi(x, \theta) &= (\theta x + 1)^{-\frac{1}{\theta}}, \quad 1 \leq \theta < \infty, \theta \neq 0, x \in [0, \infty], \\ C(u_1, u_2, \theta) &= (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad 1 \leq \theta < \infty, \theta \neq 0, u_1, u_2 \in [0,1]. \end{aligned}$$

Another popular copula generator is the Frank generator given by

$$\phi(x, \theta) = \theta^{-1} \log\{1 - (1 - e^{-\theta})e^{-x}\}, \quad 0 \leq \theta < \infty,\ x \in [0, \infty].$$

The respective Frank copula is the only elliptical Archimedean copula, with the copula function

$$\begin{aligned} C(u_1, u_2, \theta) &= -\theta^{-1} \log \left\{ \frac{1 - e^{-\theta} - (1 - e^{-\theta u_1})(1 - e^{-\theta u_2})}{1 - e^{-\theta}} \right\}, \\ & \quad 0 \leq \theta < \infty, \; u_1, u_2 \in [0, 1]. \end{aligned}$$

1.2.2 Dependence Measures

Since copulae define the dependency structure between random variables, there is a relationship between the copulae and different dependency measures. The classical measures for continuous random variables are Kendall's τ and Spearman's ρ. Similarly as copula functions, these measures are invariant under strictly increasing transformations. They are equal to 1 or -1 under perfect positive or negative dependence respectively. In contrast to τ and ρ, the Pearson correlation coefficient measures the linear dependence and, therefore, is unsuitable for measuring nonlinear relationships. Next we discuss the relationship between τ, ρ and the underlying copula function.

DEFINITION 1.2 *Let F be a continuous bivariate cumulative distribution function with the copula C. Moreover, let $(X_1, X_2) \sim F$ and $(X_1', X_2') \sim F$ be independent random pairs. Then Kendall's τ_2 is given by*

$$\begin{aligned} \tau_2 &= \mathrm{P}\{(X_1 - X_1')(X_2 - X_2') > 0\} - \mathrm{P}\{(X_1 - X_1')(X_2 - X_2') < 0\} \\ &= 2\mathrm{P}\{(X_1 - X_1')(X_2 - X_2') > 0\} - 1 = 4 \iint_{[0,1]^2} C(u_1, u_2)\, dC(u_1, u_2) - 1. \end{aligned}$$

Kendall's τ represents the difference between the probability of two random concordant pairs and the probability of two random discordant pairs.

For most copula functions with a single parameter θ there is a one-to-one relationship between θ and the Kendall's τ_2. For example, it holds that

$$\begin{aligned} \tau_2(\text{Gaussian and } t) &= \frac{2}{\pi} \arcsin \delta, \\ \tau_2(\text{Archimedean}) &= 4 \int_0^1 \frac{\phi^{-1}(t)}{(\phi^{-1}(t))'}\, dt + 1, \quad \text{(Genest and MacKay (1986))}, \\ \tau_2(\Pi) &= 0, \qquad \tau_2(W) = 1, \qquad \tau_2(M) = -1. \end{aligned}$$

This implies, that for Gaussian, t and an arbitrary Archimedean copula we can estimate the unknown copula parameter θ using a type of method of

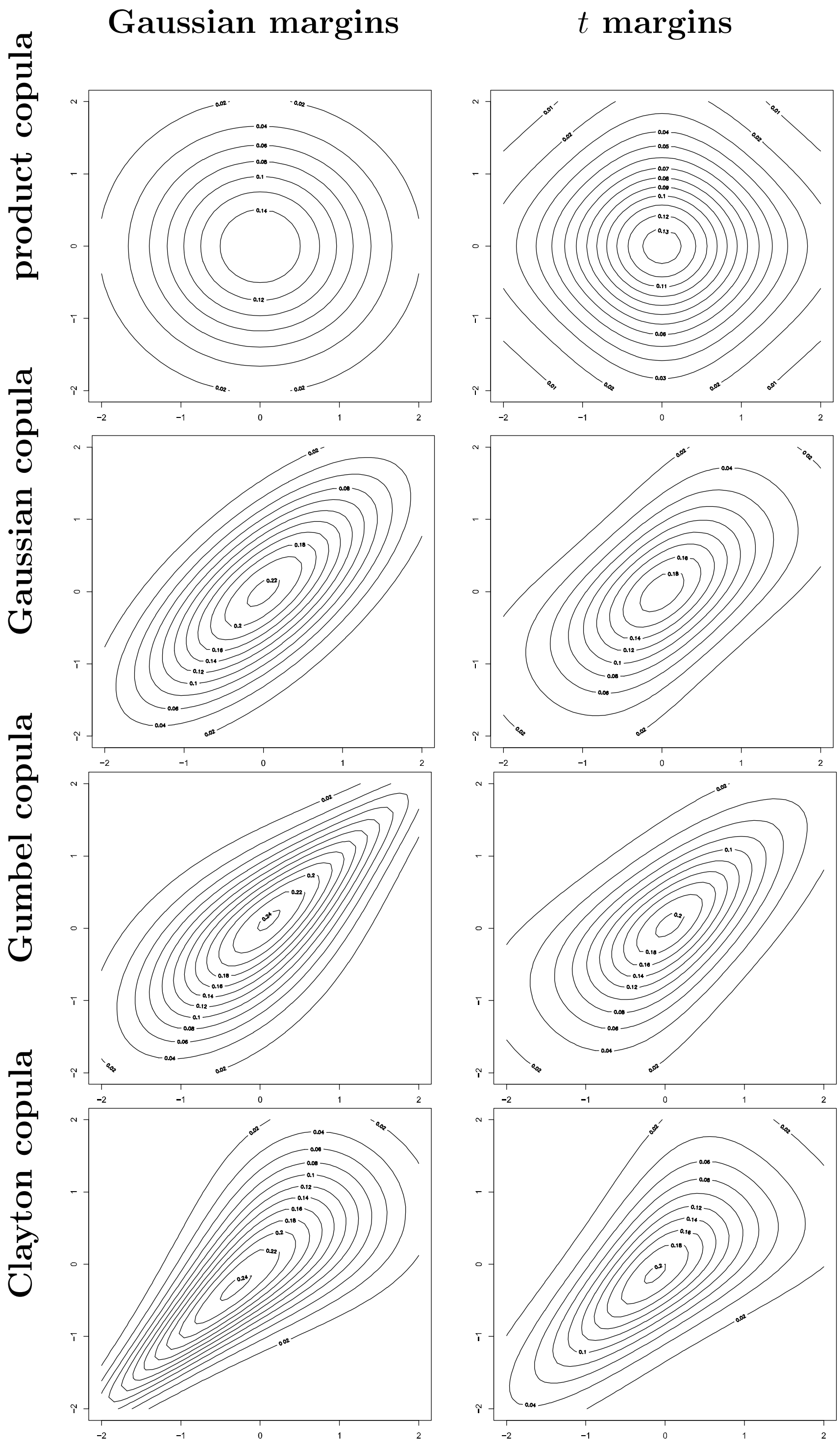

Figure 1.2. Contour diagrams for product, Gaussian, Gumbel and Clayton copulae with Gaussian (left column) and t_3 distributed (right column) margins.

moments procedure with a single moment condition. This requires, however, an estimator of τ_2. Naturally (Kendall (1970)) it is computed by

$$\hat{\tau}_{2n} = \frac{4}{n(n-1)} P_n - 1,$$

where n stands for the sample size and P_n denotes the number of concordant pairs, e.g. such pairs (X_1, X_2) and (X_1', X_2') that $(X_1 - X_1')(X_2 - X_2') > 0$. However, as argued by Genest, Ghoudi and Rivest (1995) the MM estimator of copula parameters is highly inefficient (see Section 1.4). Next we provide the definition and similar results for the Spearman's ρ.

DEFINITION 1.3 *Let F be a continuous bivariate distribution function with the copula C and the univariate margins F_1 and F_2 respectively. Assume that $(X_1, X_2) \sim F$. Then the Spearman's ρ is given by*

$$\rho_2 = 12 \iint_{\overline{\mathbb{R}}^2} F_1(x_1) F_2(x_2)\, dF(x_1, x_2) - 3 = 12 \iint_{[0,1]^2} u_1 u_2\, dC(u_1, u_2) - 3.$$

Similarly as for Kendall's τ, we provide the relationship between Spearman's ρ and copulae.

$$\begin{aligned} \rho_2(\text{Gaussian and } t) &= \frac{6}{\pi} \arcsin \frac{\delta}{2}, \\ \rho_2(\Pi) &= 0, \qquad \rho_2(W) = 1, \qquad \rho_2(M) = -1. \end{aligned}$$

Unfortunately there is no explicit representation of Spearman's ρ_2 for Archimedean in terms of generator functions as by Kendall's τ. The estimator of ρ is easily computed using

$$\hat{\rho}_{2n} = \frac{12}{n(n+1)(n-1)} \sum_{i=1}^{n} R_i S_i - 3\frac{n+1}{n-1},$$

where R_i and S_i denote the ranks of two samples. For a detailed discussion and relationship between these two measures we refer to Fredricks and Nelsen (2004), Chen (2004), etc.

1.3 Multivariate Copulae

In this section we generalize the above theory to the multivariate case. First we define the copula function and state Sklar's theorem.

DEFINITION 1.4 *A d-dimensional copula is a function C: $[0,1]^d \to [0,1]$ with the following properties:*

1. *$C(u_1, \ldots, u_d)$ is increasing in each component $u_i \in [0,1]$, $i = 1, \ldots, d$.*
2. *$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $u_i \in [0,1]$, $i = 1, \ldots, d$.*
3. *For all $(u_1, \ldots, u_d)$, $(u'_1, \ldots, u'_d) \in [0,1]^d$ with $u_i < u'_i$ we have*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(v_{j1}, \ldots, v_{jd}) \geq 0,$$

where $v_{j1} = u_j$ and $v_{j2} = u'_j$, for all $j = 1, \ldots, d$.

Thus a d-dimensional copula is the distribution function on $[0,1]^d$ where all marginal distributions are uniform on [0,1]. In the Sklar's theorem the very importance of copulae in the area of multivariate distributions has been recapitulated in an exquisite way.

PROPOSITION 1.2 (Sklar (1959)) *Let F be a multivariate distribution function with margins $F_1, \ldots, F_d$, then there exists the copula C such that*

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \quad x_1, \ldots, x_d \in \overline{\mathbb{R}}.$$

If F_i are continuous for $i = 1, \ldots, d$ then C is unique. Otherwise C is uniquely determined on $F_1(\overline{\mathbb{R}}) \times \cdots \times F_d(\overline{\mathbb{R}})$.

Conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then function F defined above is a multivariate distribution function with margins $F_1, \ldots, F_d$.

The representation in Sklar's Theorem can be used to construct new multivariate distributions by changing either the copula function or the marginal distributions. For an arbitrary continuous multivariate distribution we can determine its copula from the transformation

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \quad u_1, \ldots, u_d \in [0,1], \tag{1.4}$$

where F_i^{-1} are inverse marginal distribution functions. Copula density and density of the multivariate distribution with respect to copula are

$$\begin{aligned} c(u_1, \ldots, u_d) &= \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}, \quad u_1, \ldots, u_d \in [0,1], \\ f(x_1, \ldots, x_d) &= c\{F_1(x_1), \ldots, F_d(x_d)\} \prod_{i=1}^{d} f_i(x_i), \quad x_1, \ldots, x_d \in \overline{\mathbb{R}}. \end{aligned}$$

For the multivariate case as well as for the bivariate case copula function is invariant under monotone transformations.

1.3.1 Copula Families

It is straightforward to generalize the independence copula and the upper and lower Fréchet-Hoeffdings bounds to the multivariate case. The independence copula is defined by the product

$$\Pi(u_1, \dots, u_d) = \prod_{i=1}^{d} u_i.$$

The upper and lower Fréchet-Hoeffdings bounds are given by

$$\begin{aligned} W(u_1, \dots, u_d) &= \max\Big(0, \sum_{i=1}^{d} u_i + 1 - d\Big), \\ M(u_1, \dots, u_d) &= \min(u_1, \dots, u_d), \quad u_1, \dots, u_d \in [0,1]. \end{aligned}$$

respectively. An arbitrary copula $C(u_1, \dots, u_d)$ lies between the upper and lower Fréchet-Hoeffdings bounds

$$W(u_1, \dots, u_d) \le C(u_1, \dots, u_d) \le M(u_1, \dots, u_d).$$

Note, however, that the lower Fréchet-Hoeffding bound is not a copula function for $d > 2$.

The generalization of elliptical copulae to $d > 2$ is straightforward. In the Gaussian case we have:

$$\begin{aligned} C_N(u_1, \dots, u_d, \mathbf{\Sigma}) &= \mathbf{\Phi}_{\mathbf{\Sigma}}\{\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\}, \\ c_N(u_1, \dots, u_d, \mathbf{\Sigma}) &= |\mathbf{\Sigma}|^{-1/2} \times \\ &\exp\left\{-\frac{[\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)](\mathbf{\Sigma}^{-1} - \boldsymbol{I})[\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_k)]}{2}\right\}^{\top}, \\ &\text{for all } u_1, \dots, u_d \in [0,1], \end{aligned}$$

where $\mathbf{\Phi}_{\mathbf{\Sigma}}$ is a d-dimensional normal distribution with zero mean and the correlation matrix $\mathbf{\Sigma}$. The variances of the variables are imposed by the marginal distributions. Note, that in the multivariate case the implementation of elliptical copulae is very involved due to technical difficulties with multivariate cdf's.

Archimedean and Hierarchical Archimedean copulae

In contrast to the bivariate case, the multivariate setting allows construction methods for copulae. The simplest multivariate generalization of the

Archimedean copulae is $C : [0,1]^d \to [0,1]$ and is defined as

$$C(u_1,\dots,u_d) = \phi\{\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d)\}, \quad u_1,\dots,u_d \in [0,1], \qquad (1.5)$$

where $\phi \in \mathbb{L}$. This definition provides a simple, but rather limited technique for the construction of multivariate copulae. The whole complex multivariate dependency structure is determined by a single copula parameter. Furthermore, the multivariate Archimedean copulae imply that the variables are exchangeable. This means, that the distribution of $(u_1,\dots,u_d)$ is the same as of $(u_{j_1},\dots,u_{j_d})$ for all $j_\ell \neq j_v$. This is certainly not an acceptable assumption in practical applications.

A much more flexible method is provided by hierarchical Archimedean copulae (HAC), discussed by Joe (1997), Whelan (2004), Savu and Trede (2006), Embrechts, Lindskog and McNeil (2003), Okhrin, Okhrin and Schmid (2007). In the most general case of fully nested copulae, the copula function is given by

$$\begin{aligned} C(u_1,\dots,u_d) &= \phi_{d-1}\big\{\phi_{d-1}^{-1} \circ \phi_{d-2}\big(\dots[\phi_2^{-1} \circ \phi_1\{\phi_1^{-1}(u_1) + \phi_1^{-1}(u_2)\} \\ &+ \phi_2^{-1}(u_3)] + \dots + \phi_{d-2}^{-1}(u_{d-1})\big) + \phi_{d-1}^{-1}(u_d)\big\} \\ &= \phi_{d-1}[\phi_{d-1}^{-1} \circ C(\{\phi_1,\dots,\phi_{d-2}\})(u_1,\dots,u_{d-1}) + \phi_{d-1}^{-1}(u_d)] \end{aligned} \qquad (1.6)$$

for $\phi_{d-i}^{-1} \circ \phi_{d-j} \in \mathbb{L}^*$, $i < j$, where

$$\begin{aligned} \mathbb{L}^* = \{\omega : [0;\infty) \to [0,\infty) \,|\, \omega(0) = 0, \\ \omega(\infty) = \infty;\ (-1)^{j-1}\omega^{(j)} \geq 0;\ j = 1,\dots,\infty\}, \end{aligned}$$

and "$\circ$" is the composition operator. In contrast to the usual Archimedean copula (1.5), the HAC defines the whole dependency structure in a recursive way. At the lowest level, the dependency between the first two variables is modeled by a copula function with the generator ϕ_1, i.e. $z_1 = C(u_1,u_2) = \phi_1\{\phi_1^{-1}(u_1) + \phi_1^{-1}(u_2)\}$. At the second level another copula function is used to model the dependency between z_1 and u_3, etc. Note, that the generators ϕ_i can come from the same family and differ only through the parameter or, to introduce more flexibility, come from different generator families. As an alternative to the fully nested model, we can consider copula functions, with arbitrary chosen combinations at each copula level, so-called partially nested copulae. For example the following 4-dimensional copula, where the first and the last two variables are joined by individual copulae with generators ϕ_{12} and ϕ_{34}. Further, the resulted copulae are combined by a copula with the generator ϕ.

$$\begin{aligned} C(u_1,u_2,u_3,u_4) &= \phi\big(\phi^{-1}[\phi_{12}\{\phi_{12}^{-1}(u_1) + \phi_{12}^{-1}(u_2)\}] \\ &+ \phi^{-1}[\phi_{34}\{\phi_{34}^{-1}(u_3) + \phi_{34}^{-1}(u_4)\}]\big). \end{aligned} \qquad (1.7)$$

Whelan (2004) and McNeil (2007) provide tools for generating samples from Archimedean copulae, Savu and Trede (2006) derived the density of such copulae and Joe (1997) proves their positive quadrant dependence (see Theorem 4.4). Okhrin et al. (2007) considered methods for determining the optimal structure of the HAC and provided asymptotic theory for the estimated parameters.

1.3.2 Dependence Measures

MMeasuring dependence in a multivariate framework is a tedious task. This is due to the fact that, the generalizations of bivariate measures are not unique. One of the multivariate extensions of the Kendall's τ and its estimator is proposed in Barbe, Genest, Ghoudi and Rémillard (1996)

$$\tau_d = \frac{2^d}{2^{d-1}-1}\,\mathsf{E}(V) - 1 = \frac{2^d}{2^{d-1}-1}\int t\,dK(t) - 1, \tag{1.8}$$

$$\hat{\tau}_{dn} = \frac{2^d}{2^{d-1}-1}\cdot\frac{1}{n}\sum_{i=1}^{n} V_{in} - 1 = \frac{2^d}{2^{d-1}-1}\int t\,dK_n(t) - 1, \tag{1.9}$$

where $V_{in} = \frac{1}{n-1}\sum_{m=1}^{n}\prod_{j=1}^{d}\mathbf{1}(x_{jm} \le x_{im})$ and $V = C\{F_1(X_1),\dots,F_d(X_d)\} \in [0,1]$. $K_n(t)$ and $K(t)$ are distribution functions of V_{in} and V respectively. The expression in (1.8) implies that τ_d is an affine transformation of the expectation of the value of the copula. Genest and Rivest (1993) and Barbe et al. (1996) provide in-depth investigation and derivation of the distribution K.

A multivariate extension of Spearman's ρ based on multivariate copula was introduced in Wolff (1980):

$$\rho_d = \frac{d+1}{2^d-(d+1)}\left\{2^d\int\cdots\int_{[0,1]^d} C(u_1,\dots,u_d)\,du_1\dots du_d - 1\right\}.$$

Schmid and Schmidt (2006a) and Schmid and Schmidt (2006b) discuss its properties and provide a detailed analysis of its estimator given by

$$\hat{\rho}_{dn} = \frac{d+1}{2^d-d-1}\left[\frac{2^d}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}\{1-\hat{F}(x_{ij})\} - 1\right].$$

A version of the pairwise Spearman's ρ was introduced in Kendall (1970)

$$\rho_r = 2^2\sum_{m<l}\binom{d}{2}^{-1}\iint_{[0,1]^2} C_{ml}(u,v)\,dudv - 1,$$

where C_{ml} denotes the bivariate copula of the variables m and l.

Generalizations

There are numerous techniques which allow for the construction of new types of copulae from simple, elliptical or Archimedean copulae. For example, copula families B11 and B12 (Joe (1997)) arise as a combination of the upper Fréchet-Hoeffding bound and the product copula

$$\begin{aligned} C_{B11}(u_1, u_2, \theta) &= \theta M(u_1, u_3) + (1-\theta)\Pi(u_1, u_2) \\ &= \theta \min\{u_1, u_2\} + (1-\theta) u_1 u_2, \\ C_{B12}(u_1, u_2, \theta) &= M^{\theta}(u_1, u_2)\Pi^{1-\theta}(u_1, u_2) \\ &= (\min\{u_1, u_2\})^{\theta}(u_1 u_2)^{1-\theta}, \quad u_1, u_2, \theta \in [0,1]. \end{aligned}$$

For the family B11 we used the property, that every convex combination of copulae is a copula too. Family B12 is also known as Spearman or Cuadras-Augé copula, which is a weighted geometric mean of the upper Fréchet-Hoeffding bound and the product copula. Further generalization is done by using power mean over the upper Fréchet-Hoeffding bound and the product copula

$$\begin{aligned} C_p(u_1, u_2, \theta_1, \theta_2) &= \{\theta_1 M^{\theta_2}(u_1, u_2) + (1-\theta_1)\Pi^{\theta_2}(u_1, u_2)\}^{1/\theta_2} \\ &= \{\theta_1 \min(u_1, u_2)^{\theta_2} + (1-\theta_1)(u_1 u_2)^{\theta_2}\}^{1/\theta_2}, \\ & \quad\ \theta_1 \in [0,1],\ \theta_2 \in \mathbb{R}. \end{aligned}$$

Nelsen (2006), Chapter 3 provides further methods of constructing multivariate copulae, one of them is based on the Archimedean n-copulae. This family of copulae arises from simple multivariate Archimedean copula from reparametrization $\lambda = e^{-\phi t}$. We get

$$C(u_1, \ldots, u_d) = \lambda^{-1}\{\lambda(u_1)\ldots\lambda(u_d)\} = \lambda^{-1}[\Pi\{\lambda(u_1), \ldots, \lambda(u_d)\}].$$

The function λ is known as a *multiplicative generator* of C. Replacing product copula Π with an arbitrary copula C_1 of dimension d we get a new copula family, investigated in Morillas (2005).

Another popular approach to modeling multivariate distributions is based on *vines*. This class was introduced in Joe (1996) and then discussed by Bedford and Cooke (2001), Bedford and Cooke (2002), Kurowicka and Cooke (2006), Aas, Czado, Frignessi and Bakken (2006) and Berg and Aas (2007). The idea is based on the decomposition of a multivariate density into $d(d-1)/2$ bivariate densities. In the literature we have only come accross two types of such structures D-vines and canonical vines. For the D-vine the density is

$$f(x_1,\dots,x_d) = \\ = \prod_{m=1}^{d} f(x_m) \prod_{j=1}^{d-1}\prod_{i=1}^{d-j} c_{ji}\{F(x_i|x_{i+1},\dots,x_{i+j-1}), F(x_{i+j}|x_{i+1},\dots,x_{i+j-1})\},$$

where the conditional distribution is computed as a derivative with respect to known arguments (in details in Sections 2.5 and 2.6.1). To get the copula function we integrate the density over the d-dimensional hyper cube. As noted by Berg and Aas (2007) there are only $d(d-1)/2$ possible copulae to be described using vines. For $d = 10$ it is only 45 different models, while using HAC as in Okhrin et al. (2007) more than 300 million copulae are available. However the estimation of the parameters of the model and simulation from the copula are faster when vines are used.

1.4 Estimation Methods

The estimation of a copula-based multivariate distribution involves both the estimation of the copula parameters $\boldsymbol{\theta}$ and the estimation of the margins F_j, $j = 1,\dots,d$. The properties and quality of the estimator of $\boldsymbol{\theta}$ heavily depend on the estimators of F_j, $j = 1,\dots,d$. We distinguish between a parametric and a nonparametric specification of the margins. If we are interested only in the dependency structure, the estimator of $\boldsymbol{\theta}$ should be independent of any parametric models for the margins. In practical applications, however, we are interested in a complete distribution model and, therefore, parametric models for margins are preferred (see Joe (1997)).

In the bivariate case a standard method of estimating the univariate parameter θ is based on Kendall's τ statistic by Genest and Rivest (1993). The estimator of τ complemented by the method of moments allows for the estimation of the parameters. However, as shown in Genest et al. (1995) the maximum-likelihood method lead to substantially more efficient and general estimators. For non-parametrically estimated margins, Genest et al. (1995) show the consistency and asymptotic normality of ML estimators and derive the moments of the asymptotic distribution. The maximum-likelihood estimation can be performed simultaneously for the parameters of the margins and of the copula function. Alternatively, a two-stage procedure can be applied, where we estimate the parameters of margins at the first stage and the copula parameters at the second stage (see Joe (1997), Joe (2005)). Fermanian and Scaillet (2003), Chen, Fan and Patton (2004) and Chen, Fan and Tsyrennikov (2006) analyze the case of nonparametrically estimated margins.

Chen and Huang (2007) considered a fully nonparametric estimation of the copula. Next we provide details on both approaches.

Parametric margins

Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1^\top, \dots, \boldsymbol{\alpha}_d^\top)^\top$ denote the vector of parameters of marginal distributions and $\boldsymbol{\theta}$ parameters of the copula. The classical full ML estimator $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\theta}^\top)^\top$ solves the system

$$\begin{aligned} \frac{\partial \mathcal{L}(\boldsymbol{\eta}, \mathbf{X})}{\partial \boldsymbol{\eta}^\top} &= \mathbf{0}, \\ \text{where } \mathcal{L}(\boldsymbol{\eta}, \mathbf{X}) &= \sum_{i=1}^n \log \left[c\{F_1(x_{1i}, \boldsymbol{\alpha}_1), \dots, F_d(x_{di}, \boldsymbol{\alpha}_d), \boldsymbol{\theta}\} \prod_{j=1}^d f_j(x_{ji}, \boldsymbol{\alpha}_j) \right] \\ &= \sum_{i=1}^n \Big[\log c\{F_1(x_{1i}, \boldsymbol{\alpha}_1), \dots, F_d(x_{di}, \boldsymbol{\alpha}_d), \boldsymbol{\theta}\} \\ &+ \sum_{j=1}^d \log f_j(x_{ji}, \boldsymbol{\alpha}_j) \Big]. \end{aligned}$$

Following the standard theory on ML estimation, the estimator is efficient and asymptotically normal, however, it is often computationally demanding to solve the system simultaneously. Alternatively the multistage optimization proposed in Joe (1997), Chapter 10 also known as *inference of margins*, can be applied. First, we estimate separately the parameters of the margins and then use them in the estimation of the copula parameters as known quantities. The above optimization problem is then replaced by

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\alpha}_1^\top}, \dots, \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\alpha}_d^\top}, \frac{\partial \mathcal{L}_{d+1}}{\partial \boldsymbol{\theta}^\top} \right)^\top = \mathbf{0}, \tag{1.10}$$

$$\begin{aligned} \text{where} \quad \mathcal{L}_j &= \sum_{i=1}^n l_j(\mathbf{X}_i), \text{ for } j = 1, \dots, d+1, \\ l_j(\mathbf{X}_i) &= \log f_j(x_{ji}, \boldsymbol{\alpha}_j), \text{ for } j = 1, \dots, d, i = 1, \dots, n, \\ l_{d+1}(\mathbf{X}_i) &= \log \Big[c\big\{F_1(x_{1i}, \boldsymbol{\alpha}_1), \dots, F_d(x_{di}, \boldsymbol{\alpha}_d)\big\}\Big], \text{ for } i = 1, \dots, n. \end{aligned}$$

The first d components in (1.10) correspond to the usual ML estimation of the parameters of the marginal distributions. The last component reflects the estimation of the copula parameters. Detailed discussion on this method could be found in Joe and Xu (1996). Note, that this procedure does not lead to efficient estimators; however, as argued by Joe (1997) the loss in the efficiency is modest. The advantage of the two-stage procedure lies in the dramatic reduction of the numerical complexity. This is especially pronounced in the case of hierarchical Archimedean copulae (see Okhrin et al. (2007)). This method is a special case of the generalized method of moments with an identity weighting matrix (see Cherubini, Luciano and Vecchiato (2004), Section 4.5).

Canonical Maximum Likelihood

In this section we consider a nonparametric estimation of the marginal distributions. The asymptotic properties of the multistage estimators of $\boldsymbol{\theta}$ do not depend explicitly on the type of the nonparametric estimator, but on its convergence properties. Here we use the rectangular kernel (histogram). The estimator is given by

$$\hat{F}_j(x) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbf{1}(x_{ji} \leq x), \quad j = 1, \ldots, d.$$

The factor $n/(n+1)$ is used to bound the cdf from one. Let $\hat{F}_1, \ldots, \hat{F}_d$ denote the nonparametric estimators of $F_1, \ldots, F_d$. The canonical ML estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ solves the system by maximizing the pseudo log-likelihood with estimated margins $\hat{F}_1, \ldots, \hat{F}_d$, i.e.

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}^\top} = \mathbf{0}, \\ \text{where} \qquad &\mathcal{L} = \sum_{i=1}^{n} l(\mathbf{X}_i), \\ &l(\mathbf{X}_i) = log\left[c\left\{\hat{F}_1(x_{1i}), \ldots, \hat{F}_d(x_{di})\right\}\right], \text{ for } i = 1, \ldots, n. \end{aligned}$$

As in the parametric case, the semiparametric estimator $\hat{\boldsymbol{\theta}}$ is asymptotically normal under suitable regularity conditions. This method was first used in Oakes (2005) and then investigated by Genest et al. (1995) and Shih and Louis (1995). For properties we refer to these papers.

1.5 Goodness-of-Fit Tests for Copulae

In this section we review the goodness-of-fit (GOF) tests for copulae. With the GOF tests we test whether the underlying copula is equal to some target copula function or belongs to some copula family. The test problem could be written as a composite or a simple null hypothesis

$$H_0: \quad C \in \mathcal{C}_0, \quad \text{against} \quad H_1: \quad C \notin \mathcal{C}_0,$$

$$H_0: \quad C = C_0, \quad \text{against} \quad H_1: \quad C \neq C_0,$$

where $\mathcal{C}_0$ is some known parametric family of copulae, C_0 is some known target copula and C is the underlying true copula. The test problem is in general equivalent to the GOF tests for multivariate distributions. However, since the margins are estimated we cannot apply the standard test procedures directly.

Several related tests have been introduced into the literature. As a simple generalization of the standard χ^2 an adopted χ^2-test is proposed in Fermanian (2005) (Section 2) which is based directly on the distance between C and C_0. Genest and Rivest (1993) consider, in a bivariate setup, a test based on the true and empirical distributions of the pseudo-variable $Z = C_0(X, Y)$. As a measure they use the L_2 norm. This approach is extended to the multivariate case and other measures of proximity by Barbe et al. (1996), Wang and Wells (2000), Genest, Quessy and Rémillard (2006). Wang and Wells (2000) propose to compute a Crámer-von-Mises statistic of the form

$$S_{n\xi} = \int_{\xi}^{1} \{K_n(w) - K(w)\}^2 \, dw, \quad \xi \in (0,1),$$

where $K_n(w)$ and $K(w)$ are empirical and theoretical K-distributions from Section 2.3.2. However exact p-values for this statistic cannot be computed explicitly. Savu and Trede (2004) propose a χ^2-test based on the K-distribution. Tests of LR type were proposed in Chen and Fan (2005). Unfortunately in most cases the distribution of the test statistic does not follow a standard distribution and either bootstrap or other computationally intensive methods should be used.

An alternative approach is based on the probability integral transform introduced in Rosenblatt (1952) and applied in Breymann, Dias and Embrechts (2003), Chen et al. (2004). The idea of the transformation is to construct the variables

$$\begin{aligned} Y_1 &= F_1(X_1), \\ Y_j &= C_0\{F_j(X_j)|F_1(X_1), \ldots, F_{j-1}(X_{j-1})\}, \quad \text{for} \quad j = 2, \ldots, d, \end{aligned}$$

where the conditional copula is defined as

$$C_0(u_j|u_1, \ldots, u_{j-1}) = \frac{\frac{\partial^{j-1}}{\partial u_1 \ldots \partial u_{j-1}} C_0(u_1, \ldots, u_j, 1, \ldots, 1)}{\frac{\partial^{j-1}}{\partial u_1 \ldots \partial u_{j-1}} C_0(u_1, \ldots, u_{j-1}, 1, \ldots, 1)}.$$

Under H_0 the variables Y_i, for $i = 1, \ldots, d$ are independently and uniformly distributed on $[0, 1]$. Since the variables Y_i are not directly observable, we compute the pseudo variables $\hat{Y}_{ji}$ defined by

$$\begin{aligned} \hat{Y}_{1i} &= \hat{F}_1(X_{1i}), \\ \hat{Y}_{ji} &= C\{\hat{F}_j(X_{ji})|\hat{F}_1(X_{1i}), \ldots, \hat{F}_{j-1}(X_{j-1,i})\}, \end{aligned} \tag{1.11}$$

for $j = 2, \ldots, d$, $i = 1, \ldots, n$. Chen et al. (2004) proposed two tests based on $\hat{Y}_{ji}$. Both can be used for our purposes, however here we discuss the second test. Consider the variable $W = \sum_{j=1}^{d}[\Phi^{-1}(Y_j)]^2$. Under H_0 it holds that $W \sim \chi^2_d$. Similarly as Y_j's, W is not observed and its pseudo-observations are computed as $\hat{W}_i = \sum_{j=1}^{d}[\Phi^{-1}(Y_{ji})]^2$. Breymann et al. (2003) assume that estimating margins and copula parameters does not significantly affect the distribution of $\hat{W}_i$ and apply a standard χ^2 test directly to the pseudo-observations.

Chen et al. (2004) develop a kernel-based test for the distribution of W and, thus, account for estimation errors. Let $\hat{g}_W(w)$ denote the kernel estimator of the density of W, i.e. $\hat{g}_W(w) = \frac{1}{nh}\sum_{i=1}^{n} K_h\{w, F_{\chi^2_d}(\hat{W}_i)\}$, where K_h is the univariate boundary kernel with the second order kernel function $k(\cdot)$. Under H_0 the density $g_W(w)$ is equal to one. As a measure of divergency we use $\hat{J}_n = \int_0^1 \{\hat{g}_W(w) - 1\}^2 dw$. Assuming non-parametric estimator of the marginal distributions Chen et al. (2004) prove under regularity conditions that

$$T_n = (n\sqrt{h}\hat{J}_n - c_n)/\sigma \rightarrow \mathrm{N}(0, 1),$$

where the parameters are defined in Chen et al. (2004). The proof of this statement does not depend explicitly on the type of the non-parametric estimator of the marginals F_i, but uses the order of $\hat{F}_j(X_{ji}) - F_j(X_{ji})$ as a function of n. It can be shown that if the parametric families of marginal distributions are correctly specified and their parameters are consistently estimated, then the statement holds also if we use parametric estimators for marginal distributions. Since the test is distribution-free it is convenient to use it as a GOF measure for different copulae in different dimensions. Moreover as argued by Chen et al. (2004), the power and size of the test are comparable with other more sophisticated tests.

1.6 Simulation Methods

Monte-Carlo simulations are often a single reliable solution method in many financial problems. Within the simulation study the random variables are generated from some prescribed distributions. There are numerous methods of simulating from copula-based distributions (see Frees and Valdez (1998), Whelan (2004), Marshall and Olkin (1988), McNeil (2007), Embrechts, McNeil and Straumann (1999), Frey and McNeil (2003), Devroye (1986), etc.). Here we focus on two of them, on the conditional inversion method and on the method proposed by Marshall and Olkin (1988) for Archimedean copulae with generalizations to hierarchical Archimedean copulae by McNeil (2007).

1.6.1 Conditional Inverse Method

The conditional inverse method is a general approach aimed of simulating random variables from an arbitrary multivariate distribution. Here we sketch this method on the example of simulating from copulae. The idea is to generate random variables recursively from the conditional distributions. Let $u_1, \ldots, u_d$ be the sample we generate and let $v_1, \ldots, v_d \sim U(0,1)$ be a uniformly distributed random sample. We set $u_1 = v_1$. The rest of the variables we generate using the recursion $u_i = C_i^{-1}(v_i|u_1, \ldots, u_{i-1})$ for $i = 2, \ldots, d$, where $C_i = C(u_1, \ldots, u_i, 1, \ldots, 1)$ and the conditional distribution of U_i is given by

$$C_i(u_i|u_1, \ldots, u_{i-1}) = \mathrm{P}(U_i \le u_i|U_1 = u_1 \ldots U_{i-1} = u_{i-1}) = \frac{\frac{\partial^{i-1} C_i(u_1,\ldots,u_i)}{\partial u_1 \ldots \partial u_{i-1}}}{\frac{\partial^{i-1} C_{i-1}(u_1,\ldots,u_{i-1})}{\partial u_1 \ldots \partial u_{i-1}}}.$$

The method is numerically expensive, since it depends on higher order derivatives of C and the inverse of the conditional distribution function.

1.6.2 Marshal-Olkin Method

For simulating from Archimedean copulae a simpler method is introduced in Marshall and Olkin (1988). The idea of the method is based on the fact that the Archimedean copulae are derived from Laplace transforms. Let M be a univariate cumulative distribution function of a positive random variable (so that $M(0) = 0$) and ϕ is the Laplace transform of M, i.e.

$$\phi(s) = \int_0^\infty \exp\{-sw\}\, dM(w),\ s \ge 0.$$

For any univariate distribution function F, a unique distribution G exists such that

$$F(x) = \int_0^\infty G^\alpha(x)\, dM(\alpha) = \phi\{-\log G(x)\}.$$

Considering d different univariate distributions $F_1, \dots, F_d$, we obtain that

$$C(u_1, \dots, u_d) = \int_0^\infty \prod_{i=1}^d G_i^\alpha \, dM(\alpha) = \phi \left[\sum_{i=1}^d \phi^{-1}\{F_i(u_i)\} \right]$$

is a multivariate distribution function. To add even more generality we replace the product of univariate distributions G_i with an arbitrary copula function R

$$C(u_1, \dots, u_d) = \int_0^\infty \dots \int_0^\infty R(G_1^\alpha, \dots, G_d^\alpha) \, dM(\alpha).$$

Note that for the classical Archimedean copula R is equal to a product copula. Following the paper of Marshall and Olkin (1988) we proceed with the following three steps to make a draw from a distribution described by an Archimedean copula:

1. generate an observation u from M;
2. generate observations $(v_1, \dots, v_d)$ from R;
3. the generated vector is computed by $\mathbf{x} = \{G_1^{-1}(v_1^{1/u}), \dots, G_d^{-1}(v_d^{1/u})\}$.

This method works much faster than the classical conditional inverse technique. The drawback is that the distribution M can be determined explicitly only for a few generator functions ϕ. This can be done, for example, for Frank, Gumbel and Clayton families (see McNeil (2007), Marshall and Olkin (1988)). The same problem arises in the case of hierarchical copulae, where $\phi_i \circ \phi_{i+1}^{-1}$ should satisfy the properties of generator functions. A slightly modified but more simple procedure for simulating from hierarchical Archimedean copulae is considered in McNeil (2007).

1.7 Applications to Finance

The dependency plays a key role in many financial applications. Elliptical distributions, with the correlation coefficient as the main measure of dependency, constitute a well established class of dependency models commonly used in finance. However, the symmetry assumption and the imposed tail behavior do not reflect the empirical evidence on financial time series. This leads to numerous extensions of Gaussian models to copula-based distributions. In this section we discuss three such extensions. Firstly, we consider the asset allocation problem with non-Gaussian asset returns. Secondly, we discuss the peculiarities of the Value-at-Risk estimation in the non-elliptical framework. Thirdly, we consider the time series models with the residuals following a copula-based distribution.

1.7.1 Asset Allocation

In this section we illustrate the extension of the classical asset allocation problem to copula-based models following Patton (2004). Further discussion and application of the impact of copula-based distribution on portfolio selection procedures can be found in Longin and Solnik (2001) and Hennessy and Lapan (2002).

We consider an investor with a CRRA utility function $U(x) = (1-\gamma)^{-1}x^{1-\gamma}$ willing to allocate his wealth to d risky assets. We denote the d-dimensional vector of continuously compounded asset returns at time $t+1$ by $\mathbf{r}_{t+1} = (r_{1,t+1}, \dots, r_{d,t+1})^\top$ and the vector of portfolio weights by $\mathbf{w} = (w_1, \dots, w_d)^\top$. Let F_{t+1} be the d-dimensional distribution function of $\mathbf{r}_{t+1}$ with the mean μ_{t+1} and covariance matrix $\mathbf{\Sigma}_{t+1}$. The aim is to forecast F_{t+1} for the time period $t+1$ using the data up to time t. The estimator is denoted by $\hat{F}_{t+1}$ with the mean $\hat{\mu}_{t+1}$, the covariance matrix $\hat{\mathbf{\Sigma}}_{t+1}$ and the density $\hat{f}_{t+1}$. The objective of the investor is to maximize the expected utility at the time point $t+1$. This leads to the optimization problem

$$\max_{\mathbf{w}\in\mathcal{W}} \mathsf{E}\, U(1+\mathbf{w}^\top \mathbf{r}_{t+1}). \tag{1.12}$$

In the case of no-short-sales constraint we set $\mathcal{W} = \{\mathbf{w} \in [0,1]^d : \mathbf{w}^\top \mathbf{1} = 1\}$ else we set $\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^d : \mathbf{w}^\top \mathbf{1} = 1\}$. The conditional expectation in (1.12) implies that we integrate the utility with respect to the forecasted distribution $\hat{F}_{t+1}$. This reduces the problem (1.12) to the problem

$$\max_{\mathbf{w}\in\mathcal{W}} \int \cdots \int U(1+\mathbf{w}^\top \mathbf{r}_{t+1}) \hat{f}_{t+1}(\mathbf{r}_{t+1}) d\mathbf{r}_{t+1}.$$

There are several alternative parametric approaches to modeling F_{t+1}. Let $\mathbf{\Sigma}_{d,t+1}$ denote the diagonal matrix containing only the main diagonal of $\mathbf{\Sigma}_{t+1}$. Then $\mathbf{\Sigma}_{t+1} = \mathbf{\Sigma}_{d,t+1}^{1/2} \mathbf{R}_{t+1} \mathbf{\Sigma}_{d,t+1}^{1/2}$, where $\mathbf{R}_{t+1}$ denotes the correlation matrix. A standard approach is to define the model of the asset returns in the form

$$\mathbf{\Sigma}_{d,t}^{-1/2}(\mathbf{r}_t - \mu_t) \sim \mathrm{N}_d(\mathbf{0}, \mathbf{R}_t), \tag{1.13}$$

where the conditional moments μ_t and $\mathbf{\Sigma}_t$ are modeled by a GARCH-in-mean type of process (Franke, Härdle and Hafner (2008)). As a simpler alternative we can consider a Bayesian framework where F_{t+1} denotes the predictive distribution of the asset returns as in Barberis (2000). The unknown parameters of the conditional moments are usually estimated numerically using the ML methodology.

To introduce a copula-based distribution into the asset allocation we deviate from the normality assumption and, following the Sklar's theorem, assume that $F = C(F_1, \ldots, F_d)$. Thus the model (1.14) is replaced with the model

$$\mathbf{\Sigma}_{d,t}^{-1/2}(\mathbf{r}_t - \mu_t) \sim C(F_1, \ldots, F_d) \tag{1.14}$$

with some given functional forms of the copula and the marginal distributions. Similarly as above, the parameters of the conditional moments, of the copula and of the marginal distributions are estimated using the ML method.

In Patton (2004) the investor allocates his wealth between small cap and large cap stocks (i.e. $d = 2$). The conditional mean is defined as linear function of the lagged asset returns and additional explanatory variables. The conditional variance is stated in the TARCH(1,1) form. The rotated Gumbel copula with skewed t margins are used to construct the bivariate distribution of the residuals. This model reveals the highest likelihood function and the lowest AIC and BIC criterion. It is concluded that unconstrained portfolios derived from the normality assumption performed worse in 9 of 10 different trading strategies compared to the Gumbel model.

1.7.2 Value-at-Risk

One of the main advantages of copulae is the fact that they allow for flexible modeling of the tail behavior of multivariate distributions. Since the tail behavior explains the simultaneous outliers of asset returns, it is of special interest in risk management. Therefore, in this section we illustrate the use of copulae for computing the Value-at-Risk (VaR) of portfolios following Embrechts et al. (1999) and Junker and May (2005). The VaR of a portfolio at level α is defined as the lower α-quantile of the distribution of the portfolio return $r_p = \mathbf{w}^\top \mathbf{r}$, i.e.

$$VaR(\alpha) = F_{r_p}^{-1}(\alpha).$$

The VaR is a reasonable measure of risk if we assume that the returns are elliptically distributed. This follows from the fact that VaR is a coherent risk measure (see Embrechts et al. (1999)). Moreover, the assumption of ellipticity implies that minimizing the variance in the Markowitz problem also minimizes the VaR, the expected shortfall and any other coherent measure of risk. However, this statement is false in the non-elliptical case. Moreover, regarding the effect of diversification the variance is the smallest (highest) for perfect negative (positive) correlation of the assets. This also holds for the VaR in the elliptical case, however, not for the non-elliptical distributions (see Embrechts et al. (1999), Theorem 5). This implies that for copula-based distribution the VaR should be used with caution and its computation should be awarded more attention.

Consider the probability that the portfolio return r_p does not exceed some predetermined value ξ, i.e. $\mathrm{P}(r_p \leq \xi)$. Our aim is to determine the lower α-quantile of the distribution of r_p, or, equivalently, to determine such ξ that $\mathrm{P}(r_p \leq \xi) = \alpha$. Note that

$$r_p = \mathbf{w}^\top \mathbf{r} = \sum_{i=1}^{d} w_i r_i = \sum_{i=1}^{d} w_i F_i^{-1}(u_i),$$

where F_i denote the marginal distributions of individual asset returns, $u_i = F_i(r_i) \sim U[0,1]$ for all $i = 1, \ldots, d$ and $u_1, \ldots, u_d \sim C$. The copula C defines the dependency structure between the asset returns. This implies that

$$\mathrm{P}(r_p \leq \xi) = \int_{\mathcal{U}} c(u_1, \ldots, u_d) du_1 \ldots du_d, \tag{1.15}$$

with

$$\mathcal{U} = \{[0,1]^{d-1} \times [0, u_d(\xi)]\}, \quad u_d(\xi) \;=\; F_d\Big[\xi/w_d - \sum_{i=1}^{d-1} w_i F_i^{-1}(u_i)/w_d\Big].$$

For fixed α, the VaR is determined by solving (1.15) numerically for ξ. Direct multidimensional numerical integration is a tedious task which can be substantially simplified by using the Monte-Carlo integration. For this purpose we have to generate random samples from C using methods described in Section 1.6.

Junker and May (2005) apply the above methodology to a portfolio consisting of two assets, Hoechts and Volkswagen shares. The returns are standardized by the sample mean and the conditional volatility from the GARCH(1,1) process. The copula function is defined as a convex linear combination of the Frank copula and its survival copula. It is concluded that empirical or t-margins and asymmetric copula-based dependency structures provide the best fit in terms of χ^2 goodness-of-fit test of Diebold, Gunther and Tay (1998). Moreover, the VaR estimator from this model well approximates the empirical estimator. The assumption of Gaussian GARCH(1,1) standardized returns renders the worst results.

1.7.3 Time Series Modeling

Time series models constitute one of the most important tools in dealing with financial data. However, multivariate modeling used up to now does not properly describe financial and economic time series. The reason is that these models are mainly based on the Gaussian or on elliptical distributions. Nowadays there are numerous papers extending the classical time series models to model copula distributed residuals. First we consider the semiparametric copula-based multivariate dynamic model (SCOMDY) of Chen and Fan (2006). Let $\{\mathbf{y}_t^\top, \mathbf{x}_t^\top\}_{t=1}^n$ be stochastic processes, where $\mathbf{x}_t$ contains the exogenous variables and the d-dimensional vector $\mathbf{y}_t$ contains the variables of interest. Let $\mathcal{F}_{t-1}$ denote the information up to the time point t. They specified the model in the following way

$$\begin{aligned}
\mathbf{y}_t &= \mu_t(\boldsymbol{\theta}_1) + \sqrt{\mathbf{H}_t(\boldsymbol{\theta})}\boldsymbol{\varepsilon}_t, \\
\text{where} & \\
\boldsymbol{\theta} &= \{\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top\}^\top \\
\mu_t(\boldsymbol{\theta}_1) &= (\mu_{1,t}(\boldsymbol{\theta}_1), \ldots, \mu_{d,t}(\boldsymbol{\theta}_1))^\top \\
&= \mathsf{E}\{\mathbf{y}_t|\mathcal{F}_{t-1}\} \\
\mathbf{H}_t(\boldsymbol{\theta}) &= \operatorname{diag}\{h_{1,t}(\boldsymbol{\theta}), \ldots, h_{d,t}(\boldsymbol{\theta})\} \\
&= \operatorname{diag}\{h_{1,t}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2), \ldots, h_{d,t}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\} \\
&= \operatorname{diag}\left(\mathsf{E}[\{\mathbf{y}_{1t} - \mu_t(\boldsymbol{\theta}_1)\}^2|\mathcal{F}_{t-1}], \ldots, \mathsf{E}[\{\mathbf{y}_{dt} - \mu_t(\boldsymbol{\theta}_1)\}^2|\mathcal{F}_{t-1}]\right).
\end{aligned}$$

$\mu_t(\cdot)$ is the true conditional mean of the $\mathbf{y}_t$ given $\mathcal{F}_{t-1}$ and $h_{jt}(\cdot)$ is the true conditional variance of the $\mathbf{y}_{jt}$ given $\mathcal{F}_{t-1}$. The residuals are assumed to be serially independent with zero mean and unit variances, i.e. $\mathsf{E}[\boldsymbol{\varepsilon}_{jt}] = 0$ and $\mathsf{E}[\boldsymbol{\varepsilon}_{jt}^2] = 1$ for $j = 1, \ldots, d$. The joint distribution of $\boldsymbol{\varepsilon}$ is assumed to be given by $C\{F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d)\}$, where the margins and the copula function are unknown. This general specification includes the standard processes GARCH, ARCH, VAR as special cases, however, it allows for much more flexibility in the choice of the dependency structure of the residuals. For example considering $\boldsymbol{\theta}_1 = (\boldsymbol{\delta}_1^\top, \ldots, \boldsymbol{\delta}_d^\top)^\top$, $\boldsymbol{\theta}_2 = (\kappa_1, \ldots, \kappa_d; \beta_1, \ldots, \beta_d; \gamma_1, \ldots, \gamma_d)^\top$, $\mu_t = (\mathbf{x}_{1t}^\top\boldsymbol{\delta}_1, \ldots, \mathbf{x}_{dt}^\top\boldsymbol{\delta}_d)^\top$, $\mathbf{H}_t = \operatorname{diag}\{h_{1t}, \ldots, h_{dt}\}$ and the copula for $\boldsymbol{\varepsilon}$ is assumed to be the Gaussian copula, where $\kappa_j > 0$, $\beta_j \geq 0$, $\gamma_j \geq 0$ and $\beta_j + \gamma_j < 1$ for $j = 1, \ldots, d$ we get GARCH(1,1) model with normal innovations

$$\begin{aligned}
y_{jt} &= \mathbf{x}_{jt}^\top\boldsymbol{\delta}_j + \sqrt{h_{jt}}\varepsilon_{jt} \\
h_{jt} &= \kappa_j + \beta_j h_{j,t+1} + \gamma_j(y_{j,t-1} - \mathbf{x}_{j,t-1}^\top\boldsymbol{\delta}_j)^2, \quad j = 1, \ldots, d.
\end{aligned}$$

Chen and Fan (2006) consider maximum likelihood estimators of the parameters in these models and establish large sample properties when the copula is mis-specified. For the choice between two SCOMDY models they introduce a pseudo likelihood ratio test and provide the limiting distribution of the test statistic.

In contrast to the paper by Chen and Fan (2006), Fermanian and Scaillet (2003) consider a nonparametric estimation of copulae for time series and derived asymptotic properties of the kernel estimator of copulae. Further generalization is discussed in Giacomini, Härdle and Spokoiny (2008), where the parameter of the copula is assumed to be time dependent. The aim is to determine the periods with constant dependency structure.

1.8 Simulation Study and Empirical Results

In this section we illustrate the considered algorithms on simulated and real-world data. The next sub-section contains the simulation study, where we show that the aggregated binary structures outperform the alternative strategies of artificial data. In Section 1.8.2 we used Bayes and Akaike information criterion to compare the performance of the HAC-based model with the classical Gaussian and t-models on real data.

1.8.1 Simulation Study

Setup of the study

The aim of this simulation study is the comparison of grouping methods on the example of simulated data. We consider two different true structures $s = (123)(45)$ and $s = (12(34))5$ with the Gumbel generator function given by

$$\phi^{-1} = \{-\log(u)\}^{\theta}, \qquad \phi = \exp(-u^{\frac{1}{\theta}}).$$

This naturally corresponds to the Gumbel copula. The parameters are set for the first structure equal to $\theta_{123} = 4$, $\theta_{45} = 3$ and $\theta_{(123)(45)} = 2$ and for the second structure to $\theta_{34} = 4$, $\theta_{12(34)} = 3$ and $\theta_{(12(34))5} = 2$. Without loss of generality the marginal distributions are taken as uniform on $[0, 1]$. We simulate a sample of 1000 observations. The procedure is repeated 101 times. This number is selected to simplify the interpretation and computation of median structures.

For the simulation we use the conditional inversion method. This method is also used by Frees and Valdez (1998) and Whelan (2004) and we discussed it in Section 1.6. The copula parameters are estimated using the multistage ML method with the nonparametric estimation of margins based on the Epanechnikov kernel. The vector of bandwidths $\mathbf{h} = \{h_i\}_{i=1,\ldots,d}$ in the estimation of the density and in the estimation of the distribution function is based on the Silverman's rule of thumb.

Discussion of the results

The results of the simulation study are summarized in Table 1.1 for the first structure and in Table 1.2 for the second structure. For each simulated data set and for each structure we compare the fit and the structure obtained from the grouping procedures. We consider the simple Archimedean copula (sAC) and groupings based on the Chen et al. (2004) test statistics (Chen), on the θ (θ), binary copulae (θ_{binary}) and aggregated binary copulae ($\theta_{binary\ aggr.}$). As benchmark models we consider the 5-dimensional multivariate normal distribution (N) with $\hat{\mathbf{\Sigma}}$ and $\hat{\mu}$ estimated from the data; the multivariate Gaussian copula with nonparametric margins ($\mathrm{N}_{nonparam.}$); the multivariate t-distribution with eight degrees of freedom and with $\hat{\mathbf{\Sigma}}$ and $\hat{\mu}$ estimated from the data set (t_8); the multivariate t-copula with eight degrees of freedom and nonparametric margins ($t_{nonparam.}$).

For each grouping method and each benchmark we compute the Kullback-Leibler divergence from the empirical distribution function as in Giacomini et al. (2008) and the test statistic of Chen et al. (2004). The Kulback-Leibler functional for the distribution functions estimated using two different methods is

$$\mathcal{K}(\hat{F}_{method\ 1}, \hat{F}_{method\ 2}) = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{\widehat{f}_{method\ 1}(x_{1i}, \ldots, x_{di})}{\widehat{f}_{method\ 2}(x_{1i}, \ldots, x_{di})}\right\}$$

The Kullback-Leibler divergence for the multivariate distribution which is based on copula, can be regarded as a distance between two copula densities.

The first blocks of Table 1.1 and Table 1.2 contain the results for groupings based on the Kullback-Leibler divergence. The columns "$\mathcal{K}$" contain the value of the Kullback-Leibler divergence which is the closest to the median divergence given in parenthesis. The corresponding structure and the test statistic of Chen et al. (2004) are given in the columns "copula structure" and "Chen." respectively. The variance of the Kullback-Leibler divergence is given in the last column. The same holds for the lower blocks of both tables, however, here we find the structure which has the test statistics of Chen et al. (2004) which is the closest to the median of the test statistics. Note that we provide the results for the median performance measures and not for the best replications of the simulation study. This makes the conclusions more robust.

The results show that the grouping method based on the aggregated binary structure is dominant. It provides for both structures the smallest Kullback-Leibler divergence as well as the lowest test statistics. The simple binary copula also provides good results, however, we see that some of the parameters are very close. This indicates that the variables can be joined together into an aggregated copula. Although, the method based on θ's performs better than the benchmark strategies, it leads, however, to incorrect structures and much higher goodness-of-fit measures compared to the binary copula. The grouping based on the test statistic of Chen et al. (2004) provides very poor results, which indicates a low power of the test against similar structures. Similarly, the ignorance of the hierarchical structure of the distribution imposed by the simple Archimedean copulae leads to the worst results among copula-based methods. The comparison with normal and t-distributions is possible only on the basis of the Kullback-Leibler divergence. We see that, despite of the substantially larger number of parameters, the normal and t-distributions cannot outperform the θ-based grouping methods. Thus we conclude that the proposed grouping methodology based on the aggregated binary structure provides robust and precise results. Moreover, the method is computationally more efficient than the considered alternatives.

method	**copula structure**	**Chen.**	$\mathcal{K}$ ($\hat{\mu}_{\mathcal{K}}$)	$\hat{\sigma}^2_{\mathcal{K}}(10^{-3})$
N			1.074 (1.074)	4.0
$\text{N}_{nonparam.}$			0.282 (0.283)	1.0
t_8			1.104 (1.104)	3.0
$t_{nonparam.}$			0.199 (0.199)	0.0
sAC	(1.2.3.4.5)	85.535463	0.811 (0.809)	3.0
CHEN	$((1.3)_{4.517}.(2.4.5)_{2.341})_{2.34}$	31.432	0.611 (0.613)	78.0
θ	$((1.3.4.5)_{2.288}.2)_{2.286}$	82.510	0.697 (0.560)	142.0
θ_{binary}	$(((1.(2.3)_{4.39})_{4.282}.5)_{2.078}.4)_{2.077}$	3.929	0.132 (0.133)	0.4
$\theta_{binary\ aggr.}$	$(((1.3)_{4.26}.2)_{3.868}.(4.5)_{3.093})_{2.259}$	2.737	0.022 (0.021)	0.0
method	**copula structure**	**Chen.** ($\hat{\mu}_{Chen}$)	$\mathcal{K}$	$\hat{\sigma}^2_{Chen\,stat}$
sAC	(1.2.3.4.5)	88.842 (88.850)	0.704	68.127
CHEN	$((1.2)_{4.316}.(3.4.5)_{2.256})_{2.255}$	31.558 (32.419)	0.585	490.059
θ	$((1.2.4.5)_{2.376}.3)_{2.375}$	56.077 (56.910)	0.769	1407.632
θ_{binary}	$((((1.2)_{4.487}.3)_{4.469}.5)_{2.247}.4)_{2.246}$	4.789 (4.827)	0.112	4.388
$\theta_{binary\ aggr.}$	$(((1.3)_{4.228}.2)_{3.68}.(4.5)_{3.369})_{2.333}$	2.253 (2.248)	0.021	1.914

Table 1.1. Model fit for the true structure (123)(45): Averages of the Kullback-Leibler Divergence and Averages of the Chen Statistics separately

1.8.2 Empirical Example

In this subsection we apply the proposed estimation techniques to financial data. We consider the daily returns of four companies listed in DAX index: Commerzbank (CBK), Merck (MRK), Thyssenkrupp (TKA) and Volkswagen (VOW). The sample period covers more than 2300 observations from

method	copula structure	Chen.	$\mathcal{K}$ ($\hat{\mu}_{\mathcal{K}}$)	$\hat{\sigma}^2_{\mathcal{K}}(10^{-3})$
N			1.088 (1.089)	4
$N_{nonparam.}$			0.289 (0.289)	1
t_8			1.113 (1.114)	3
$t_{nonparam.}$			0.202 (0.202)	1
sAC	(1.2.3.4.5)	78.604	0.502 (0.502)	2
CHEN	$(((1.2)_{3.22}.3)_{3.177}.(4.5)_{2.116})_{2.114}$	8.544	0.305 (0.304)	23
θ	$(((1.2.3)_{3.207}.4)_{3.205}.5)_{2.15}$	5.741	0.079 (0.079)	0
θ_{binary}	$(((1.(3.4)_{4.157})_{3.099}.2)_{3.012}.5)_{2.028}$	2.293	0.003 (0.003)	0
$\theta_{binary\ aggr.}$	$(((3.4)_{4.32}.1.2)_{3.268}.5)_{1.83}$	1.220	0.019 (0.019)	0
method	**copula structure**	**Chen.** ($\hat{\mu}_{Chen.}$)	$\mathcal{K}$	$\hat{\sigma}^2_{Chen\,stat}$
sAC	(1.2.3.4.5)	86.245 (86.278)	0.480	142.714
CHEN	$(((1.3)_{2.835}.5)_{1.987}.(2.4)_{2.898})_{1.986}$	16.263 (16.512)	0.453	281.615
θ	$(((1.2.4)_{3.009}.3)_{3.007}.5)_{1.973}$	4.235 (4.222)	0.083	6.229
θ_{binary}	$(((1.(3.4)_{4.122})_{3.155}.2)_{3.07}.5)_{2.027}$	1.934 (1.955)	0.000	1.520
$\theta_{binary\ aggr.}$	$(((3.4)_{4.195}.1.2)_{3.305}.5)_{1.724}$	2.561 (2.526)	0.014	3.287

Table 1.2. Model fit for the true structure (12(34))5: Averages of the Kullback-Leibler Divergence and Averages of the Chen Statistics separately

13.11.1998 to 18.10.2007. Margins are estimated nonparametrically with Epanechnikov kernel, normal and t-distributed with three degrees of freedom. The results are given in Tables 1.3, 1.4 and 1.5 respectively. For goodness-of-fit measures we choose BIC (Bayes or Schwarz Information Criteria) and AIC (Akaike Information Criteria) and provide the value of the likelihood as intermediate results.

We fit the following multivariate copula functions to the data: HAC with binary and binary aggregated structure, simple Archimedean copula. For comparison purposes we also provide the results for the multivariate normal distribution and multivariate t-distribution with eight degrees of freedom in each table. We also provide the optimal binary and aggregated binary HACs and the simple Archimedean copula for all types of the margins.

We calculate the maximum likelihood value as described in Section 1.4. For the copula-base distributions we use

$$ML = \sum_{i=1}^{n} \log\{c(u_1, \dots, u_d, \boldsymbol{\theta}) f_1(u_1) \dots f_d(u_d)\},$$

where c is the copula density and f_i for $i = 1, \dots, d$ are marginal densities. For the multivariate normal and t-distribution, we computed the likelihood as

$$ML = \sum_{i=1}^{n} \log\{f(u_1, \ldots, u_d, \boldsymbol{\theta})\},$$

where f denotes the joint multivariate density function and $\boldsymbol{\theta}$ is the set of parameters. To penalize the likelihood for large number of parameters we consider the AIC and BIC criterion computed as

$$AIC = -2ML + 2m, \qquad BIC = -2ML + 2\log(m),$$

where m is the number of the parameters to be estimated. The values of ML for the best structure should the the highest, while AIC and BIC should be as small as possible.

We emphasize with bold font the best strategy in each column and with italic the worst strategies. We can conclude that the multivariate t distribution outperforms all other methods and shows the best results for all types of the margins. Nevertheless, note that with the properly selected marginal distributions and copula function, the HAC outperforms the normal distribution. Moreover, note that we considered only HACs based on the Gumbel generator functions. Alternative generator specifications and HACs dependent on several different generators may outperform the t-distribution as well.

	ML	AIC	BIC
HAC	28319.3701	-56632.7402	-56614.9003
HAC_{binary}	28319.3701	-56632.7402	-56614.9003
AC	*28028.5201*	-56055.0403	-56049.0937
N	28027.4098	*-56026.8195*	*-55943.5669*
t_8	**28726.8637**	**-57425.7273**	**-57342.4747**

Table 1.3. Information Criteria: Nonparametric Margins

Optimal binary structure = $(((\text{CBK VOW})_{1.5631}\ \text{TKA})_{1.4855}\ \text{MRK})_{1.1437}$

Optimal structure = $(((\text{CBK VOW})_{1.5631}\ \text{TKA})_{1.4855}\ \text{MRK})_{1.1437}$

Simple Archimedean Copula = $(\text{CBK MRK TKA VOW})_{1.4116}$

	ML	AIC	BIC
HAC	27961.0997	-55918.1995	-55906.3062
HAC_{binary}	27961.2399	-55916.4799	-55898.6400
AC	*27737.7392*	*-55473.4784*	*-55467.5317*
N	28027.4098	-56026.8195	-55943.5669
t_8	**28726.8637**	**-57425.7273**	**-57342.4747**

Table 1.4. Information Criteria: Normal margins

$$\begin{aligned} \text{Optimal binary structure} &= (((\text{CBK VOW})_{1.3756}\ \text{TKA})_{1.3571}\ \text{MRK})_{1.1071} \\ \text{Optimal structure} &= ((\text{CBK TKA VOW})_{1.3756}\ \text{MRK})_{1.1071} \\ \text{Simple Archimedean Copula} &= (\text{CBK MRK TKA VOW})_{1.1944} \end{aligned}$$

	ML	AIC	BIC
HAC	28613.9640	-57223.9280	-57212.0347
HAC_{binary}	28612.2069	-57218.4138	-57200.5740
AC	28404.8899	-56807.7798	-56801.0347
N	*28027.4098*	*-56026.8195*	*-55943.5669*
t_8	**28726.8637**	**-57425.7273**	**-57342.4747**

Table 1.5. Information Criteria: t margins

$$\begin{aligned} \text{Optimal binary structure} &= (((\text{CBK VOW})_{1.3416}\ \text{TKA})_{1.3285}\ \text{MRK})_{1.1007} \\ \text{Optimal structure} &= ((\text{CBK TKA VOW})_{1.3416}\ \text{MRK})_{1.1007} \\ \text{Simple Archimedean Copula} &= (\text{CBK MRK TKA VOW})_{1.1987} \end{aligned}$$

1.9 Summary

In this chapter we provide a detailed review of the copula models in discrete time. We review the construction and simulation of bivariate and multivariate copula models. For practical applications we discuss the alternative estimation procedures and goodness-of-fit tests. Special attention is paid to the hierarchical Archimedean copulae. The chapter is complemented with an extensive simulation study and an application to financial data.

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