Controllability and stabilization of one-dimensional systems near bifurcation points

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Abstract

Bifurcation theory deals with the change of qualitative behavior in a parameter dependent family of differential equations. For one-dimensional equations the possible bifurcation scenarios are well understood. If the family of differential equations can be controlled by admissible controls with compact range, the question arises, whether the systems are controllable near a bifurcation point and whether stabilization around unstable bifurcation branches via bounded feedback is possible. In this paper we show that controllability for parametrized families of one-dimensional control systems can be characterized in terms of two parameters, the original bifurcation parameter and the size of the control range. These results are used to construct (nonsmooth) stabilizing feedbacks and to describe the set of initial values, from which stabilization is possible. Furthermore, robustness properties of the stabilizing feedback are discussed.

Keywords: Control sets; Bifurcation; Discontinuous feedback

1. Introduction

Recently, much attention has been devoted to two areas in nonlinear control theory: the construction of smooth and nonsmooth stabilizing feedbacks, and the control of systems around bifurcation points of the underlying, uncontrolled dynamics. This paper is an attempt to characterize controllability in parametrized control systems of dimension one for systems with compact control range, and to use the results for the design of (not necessarily smooth) static feedback laws.

For one-dimensional control systems of the form $\dot{x} = f(x, u(t))$, with $u \in \mathcal{U} = \{u : \mathbb{R} \to U, \text{measurable}\}$ and $U \subset \mathbb{R}$ compact, connected, $0 \in \operatorname{int} U$, the control sets (i.e. regions of complete controllability) are easily computed, see e.g. [2]. For a parametrized family of control systems $\dot{x} = f_\alpha(x, u(t))$ with $\alpha \in I$, an open interval in $\mathbb{R}$, the control sets need not depend continuously on α, even if f_α is analytic in α. The bifurcation behavior of the control sets for these systems is actually characterized by two parameters: the original bifurcation parameter α, and a parameter $\rho \geqslant 0$, which determines the size of the control range via $U^\rho = \rho \cdot U$. For common types of bifurcation diagrams the bifurcation structure of associated control systems is presented in Section 4. Some background material together with a discussion of the domains of attraction for control sets of a single control system can be found in Section 2.

The basic question when dealing with static feedback stabilization of nonlinear systems is: when does (asymptotic) controllability imply stabilizability? The answer to this question depends heavily on the smoothness that one requires for the feedback

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law. Even for one-dimensional systems there may exist piecewise constant feedbacks, when no smooth feedback is available, see [6] and the survey [5]. In this paper, we do not require smoothness of the feedbacks (they will be piecewise constant). This leads to the characterization that a one-dimensional control system is (asymptotically) stabilizable via static state feedback at a point x^0 iff x^0 is contained in some control set, see Theorems 4 and 6. If dynamic feedback is considered, almost-smooth stabilization can be guaranteed, see [3]. Since nonlinear systems, even under the Lie algebra rank condition, need not be completely controllable nor asymptotically controllable to a point x^0 from all initial values, the problem of characterizing the domain for a feedback law has to be addressed. We show that this set coincides with the domain of attraction $\mathscr{A}(D)$ of the control set D with $x^0 \in D$. The existence of smooth 'semiglobal' laws as compared to global ones was analyzed in [8].

Section 5 is devoted to feedback stabilization near bifurcation points: does there exist a feedback law that will delay or advance the original bifurcation point α_0? The answer here depends on whether the value of the bifurcation parameter is known (or estimated, which leads to concepts of adaptive control), or a common feedback law for a range of α-values is desired (robustness with respect to α). For points x^0, where the Lie algebra rank condition is satisfied (regular systems), robust feedback laws that stabilize at x^0 for a range of α-values do not exist in general. If, however, x^0 is a common fixed point for all $u \in U$ (singular systems), one obtains, for fixed control range, a precise estimate for the α-range where robust stabilization is possible, i.e. for advance or delay of the original bifurcation point α_0. Ref. [9] uses washout filters, i.e. dynamic feedback to preserve the original bifurcation diagram even for regular systems.

2. Global structure of one-dimensional control systems

In this paper we consider single-input one-dimensional systems of the form

$$\dot{x} = f(x, u(t)) \quad \text{in } \mathbb{R}, \tag{1}$$

where $u \in \mathscr{U} = \{u : \mathbb{R} \to U, \text{ measurable}\}$ and $U \subset \mathbb{R}$ compact, connected with $0 \in \operatorname{int} U$. We assume that f is a continuous function in both components, such that for every $u \in U$ there are at most finitely many zeros of $f(\cdot, u)$. The solutions of (1) are supposed to exist for all times $t \geqslant 0$. All results are also valid for systems on compact, one-dimensional manifolds, see [2].

The basic concept for understanding stabilization of control systems of the form (1) is that of control sets: A system can be stabilized via feedback exactly at those points that are contained in control sets (with one exception, see Theorem 6, below). Denote by $\varphi(t, x, u)$ the solution of (1) at time $t \in \mathbb{R}$ with initial value $x = \varphi(0, x, u)$ under the control action $u \in \mathscr{U}$. The positive (forward in time) orbit of a point $x \in \mathbb{R}$ is defined as $\mathcal{O}^+(x) := \{y \in \mathbb{R};$ there are $u \in \mathscr{U}$ and $t \geqslant 0$ with $\varphi(t, x, u) = y\}$, similarly for the negative orbit $\mathcal{O}^-(x)$. A control set $D \subset \mathbb{R}$ satisfies (a) $\mathit{cl}\,\mathcal{O}^+(x) \supset D$ for all $x \in D$, (b) for all $x \in D$ there exists $u \in \mathscr{U}$ with $\varphi(t, x, u) \in D$ for all $t \geqslant 0$, and (c) maximality with respect to set inclusion. ('cl' denotes the closure of a set.)

Two fundamentally different cases have to be considered:

(1) regular systems, where the Lie algebra rank condition is satisfied, i.e. for one-dimensional systems: for all $x \in \mathbb{R}$ there exists $u \in U$ with $f(x, u) > 0$, or $f(x, u) < 0$,

(2) singular systems, which contain common fixed points of all $f(\cdot, u)$, $u \in U$.

2.1. Regular systems

Under our assumptions, we consider all control sets of regular systems that have nonvoid interior. Then precise controllability holds in control sets, i.e. $\mathcal{O}^+(x) \supset \operatorname{int} D$ for all $x \in D$. The order between control sets is defined as

$$D \prec D' \text{ if there exists } x \in D \text{ with } \mathit{cl}\,\mathcal{O}^+(x) \cap D' \neq \emptyset.$$

Closed control sets are maximal sets with respect to this order and they are invariant (i.e. $\mathit{cl}\,\mathcal{O}^+(x) = D$ for all $x \in D$), open sets are minimal, and sets between minimal and maximal sets are neither open nor closed.

In general, control sets of (1) form around the Morse sets (which are the fixed points for one-dimensional systems) of $\dot{x} = f(x, 0)$ under the 'inner pair condition', see [1]. For one-dimensional systems more is true: the control sets are the intervals of fixed points of $f(x, u)$, $u \in U$, where the lower boundary a belongs to D iff there exists $u \in U$ with

$f(a,u) > 0$, and similarly for the upper boundary, see [2] for a precise statement. Note that several stable and/or unstable fixed points for a constant control $u \in U$ may be contained in one control set. Closed control sets contain at least one stable fixed point for each $u \in U$, open control sets contain unstable fixed points.

The set of initial values from which the system can be asymptotically controlled to a point $x \in D$ is given by the domain of attraction of D

$$\mathscr{A}(D) := \{y \in \mathbb{R};\ cl\,\mathcal{O}^+(y) \cap D \neq \emptyset\}$$
$$= \{y \in \mathbb{R};\ y \in \mathcal{O}^-(x) \text{ for some } x \in D\}.$$

For regular systems we have the following characterization of $\mathscr{A}(D)$.

Proposition 1. *Let $D \subset \mathbb{R}$ be a control set with nonvoid interior and bounded domain of attraction and define $A(D) := \{x \in \mathbb{R};\ x \in B$ where B is a control set with $B \prec D\}$. Then $\mathscr{A}(D) = (\inf A(D), \sup A(D))$, in particular, $\mathscr{A}(D)$ is an open interval.*

The proof of Proposition 1 follows directly from the characterization of control sets, see [1].

According to Proposition 1, $\mathscr{A}(D)$ contains D and the stable manifolds of all fixed points of $f(\cdot,u)$, $u \in U$ that are in D. In general, $\mathscr{A}(D)$ can be larger than this set and it can be shown that $\mathscr{A}(D)$ consists exactly of D and of the stable manifolds of all periodic trajectories in D.

2.2. Singular systems

If $x^0 \in \mathbb{R}$ is a point, where the Lie algebra rank condition is violated, then x^0 is a common fixed point of the vector fields $f(\cdot,u)$, and hence of all (time varying) vector fields $f(x,u(t))$, $u \in \mathscr{U}$. In this case, $\{x^0\}$ is a one point invariant control set of (1), which we call a singular control set. We extend the order between control sets by defining for a singular control set $\{x^0\}$ and an arbitrary control set D with $\{x^0\} \cap cl\,D = \emptyset$

$\{x^0\} \prec D$ if for all $\varepsilon > 0$ there exists $x \in \mathbb{R}$ with $|x - x^0| < \varepsilon$ and $cl\,\mathcal{O}^+(x) \cap D \neq \emptyset$.

Analogously, we define $D \prec \{x^0\}$. Note that with this extension closed control sets need not be maximal, e.g. $\{x^0\}$, where x^0 is an isolated unstable common fixed point. The domain of attraction for singular control sets is again defined as $\mathscr{A}(\{x^0\}) = \{y \in \mathbb{R};\ cl\,\mathcal{O}^+(y) \cap D \neq \emptyset\}$. Corresponding to Proposition 1 we obtain for singular systems:

Proposition 2. *Let $\{x^0\} \subset \mathbb{R}$ be a singular control set and with bounded domain of attraction, and let $A(\{x^0\}) := \{x \in \mathbb{R}; x \in B$ where B is a control set with $B \prec \{x^0\}\}$. Define $I = [\inf A(\{x^0\}), \sup A(\{x^0\})]$.*

(i) If $\{x^0\} \cap cl\,D = \emptyset$ for all control sets D of (1) with $\{x^0\} \neq D$, and

(a) if $\{x^0\}$ is maximal with respect to $\prec$, then $\mathscr{A}(\{x^0\}) = \operatorname{int} I$,

(b) if $\{x^0\}$ is minimal with respect to $\prec$, $\mathscr{A}(\{x^0\}) = \{x^0\}$,

(c) otherwise $\mathscr{A}(\{x^0\}) = \operatorname{int} I \cup \{x^0\}$.

(ii) If $\{x^0\} \subset cl\,D$ for some control set D of (1) with $\{x^0\} \neq D$, then $\mathscr{A}(\{x^0\}) = \{x^0\} \cup \bigcup\{\mathscr{A}(D); D$ is a control set with $\{x^0\} \subset cl\,D\}$.

For control sets D of a singular system with $\operatorname{int} D \neq \emptyset$, the characterization of $\mathscr{A}(D)$ from Proposition 1 remains true with the extended order as defined above. The proof of Proposition 2 follows again from the characterization of control sets for regular and for singular systems in [1]. It can be shown for singular systems that $\mathscr{A}(D)$ consists again exactly of D and of the stable manifolds of all periodic trajectories in D.

3. Feedback stabilization of one-dimensional control systems

In this section we will use the results from Section 2 to obtain precise feedback stabilization results. We continue to work under the assumptions from Section 2.

Definition 3. The control system (1) is locally feedback stabilizable at $x_0 \in \mathbb{R}$, if there exists an open neighborhood V of x_0 and a piecewise constant feedback function $F: V \to U$ such that x_0 is a fixed point of $\dot{x} = f(x, F(x))$ and this system, restricted to V, has unique solutions for $t \geqslant 0$ and is asymptotically stable.

3.1. Regular systems

Theorem 4. *The system (1) is locally feedback stabilizable at $x_0 \in \mathbb{R}$ iff there exists a control set $D \subset \mathbb{R}$ with $\operatorname{int} D \neq \emptyset$ and $x_0 \in D$. In this case, the set of initial values from which the system can be stabilized* *at x_0, agrees with $\mathscr{A}(D)$; here the discontinuity points of the stabilizing feedback can cluster only at $\partial\mathscr{A}(D)$.*

Proof. If $x_0 \in \mathbb{R}$ is not contained in a control set with nonvoid interior then there exists $\varepsilon > 0$ such that $f(x,u) \geqslant 0$ (or $\leqslant 0$) for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, and all $u \in U$ (see [1, p. 219]). Hence there exists no admissible feedback F that stabilizes the system (1) at x_0.

Conversely, let $x_0 \in D$. Then there exists $u_0 \in U$ with $f(x_0, u_0) = 0$. If there is $z_1 < x_0$ with $f(x, u_0) > 0$ for all $x \in (z_1, x_0)$, choose $x_1 \in (z_1, x_0)$; otherwise set $x_1 = x_0$. In both cases, $U_1 := \{u \in U; f(x_1, u) > 0\}$ is nonempty. For $u \in U_1$ define $Z_2(u)$ as the largest interval containing x_1 such that $f(y,u) > 0$ for all $y \in Z_2(u)$, and $z_2(u) = \inf Z_2(u)$. (Note that $z_2(u)$ may be $-\infty$.) Let $z_2 = \inf\{z_2(u); u \in U_1\}$.

If $z_2 \in \partial\mathscr{A}(D)$, we stop, because for each $x_2 \in \mathscr{A}(D)$, $x_2 < x_1$ we have found $u_2 \in U_1$ with $[x_2, x_1] \subset Z_2(u_2)$. Otherwise, $z_2 \in \mathscr{A}(D)$, and hence $U_2 := \{u \in U; f(z_2, u) > 0\}$ is nonempty. For $u \in U_2$ define $Z_3(u)$ as the largest interval containing z_2 such that $f(y,u) > 0$ for all $y \in Z_3(u)$, and $z_3(u) = \inf Z_3(u)$. Let $z_3 = \inf\{z_3(u); u \in U_2\}$. If $z_3 \in \partial\mathscr{A}(D)$, we stop, otherwise we proceed recursively. By construction we know that for every $y \in \mathscr{A}(D)$ with $y < x_0$ there exists $n \in \mathbb{N}$ with $z_n < y$, hence this procedure leads to $\partial\mathscr{A}(D)$ (in at most countably many steps). Define $x_n := \frac{1}{2}(z_n + z_{n+1})$ for $n \geqslant 2$. Then there exist $x'_n \geqslant z_n$ and $u_n, u'_n \in U_n$ such that $[x'_n, x_{n-1}] \subset Z_n(u_{n-1})$ and $[x_n, x'_n] \subset Z_{n+1}(u'_n)$ for $n \geqslant 2$. The desired feedback on $\mathscr{A}(D) \cap \{y \in \mathbb{R}; y \leqslant x_0\}$ is defined by

$$F(x) = \begin{cases} u_0 & \text{for } x \in [x_1, x_0], \\ u_{n-1} & \text{for } x \in [x'_n, x_{n-1}),\ n \geqslant 2, \\ u'_n & \text{for } x \in [x_n, x'_n),\ n \geqslant 2. \end{cases}$$

We proceed in an analogous fashion for $\mathscr{A}(D) \cap \{y \in \mathbb{R}; y \geqslant x_0\}$. □

The construction of the feedback F in the proof of Theorem 4 is aimed at achieving stabilization with as few switches as possible. Often it is desirable to have feedbacks that are robust with respect to variations in the control, i.e. $F(x)$ should be as close as possible to the midpoint of U. In this case, the selection of the z_i has to be adjusted accordingly, and optimally robust strategies may not exist, or lead to unacceptably slow convergence towards the point x_0.

The feedback law, constructed in the proof of Theorem 4, is piecewise constant. We obtain the following result on the existence of smooth feedbacks.

Proposition 5. *Consider the system* (1) *and define* $\mathscr{X}^0 = \{(u,x);\ f(x,u) = 0\} \subset U \times \mathbb{R}$, $\mathscr{X}^{\pm} = \{(u,x);\ f(x,u) \gtrless 0\}$. *Let* $\pi_{\mathbb{R}}: U \times \mathbb{R} \to \mathbb{R}$ *be the projection onto the second component. Then the system is stabilizable at* $x_0 \in D$ via continuous feedback in $\mathscr{A}(D)$ *iff there exist*

(i) *a continuous path* $\gamma^+ : [0,1) \to \mathscr{X}^+$ *with*

$$\pi_{\mathbb{R}}\gamma^+(0) = \inf \mathscr{A}(D) \ (\textit{or } \pi_{\mathbb{R}}\gamma^+(s) \to -\infty \textit{ for } s \downarrow 0, \\ \textit{if } \inf \mathscr{A}(D) = -\infty)$$

$$\pi_{\mathbb{R}}\gamma^+(1) = x_0, \pi_{\mathbb{R}}\gamma^+(s_1) < \pi_{\mathbb{R}}\gamma^+(s_2) \textit{ for } s_1 < s_2, \textit{ and}$$

(ii) *a continuous path* $\gamma^- : [0,1) \to \mathscr{X}^-$ *with*

$$\pi_{\mathbb{R}}\gamma^-(0) = \sup \mathscr{A}(D) \ (\textit{or } \pi_{\mathbb{R}}\gamma^-(s) \to \infty \textit{ for } s \downarrow 0, \\ \textit{if } \sup \mathscr{A}(D) = \infty)$$

$$\pi_{\mathbb{R}}\gamma^-(1) = x_0, \pi_{\mathbb{R}}\gamma^-(s_1) > \pi_{\mathbb{R}}\gamma^-(s_2) \textit{ for } s_1 < s_2, \textit{ and}$$

(iii) $\gamma^+(1) = \gamma^-(1) \in \mathscr{X}^0$, *and* γ^+, γ^- *are continuous at* $s = 1$.

Proof. If there exists a continuous stabilizing feedback $F(x)$ in $\mathscr{A}(D)$, then $(F(x), x)$ defines γ^+ and γ^- as desired, after possibly a reparametrization.

Conversely, if γ^+ and γ^- are given as above, define a function $F: \mathscr{A}(D) \to U$ by

$$F(x) = \begin{cases} u \text{ with } (u,x) \in \operatorname{graph} \gamma^+ & \text{for } x \geqslant x_0, \\ v \text{ with } (v,x) \in \operatorname{graph} \gamma^- & \text{for } x \geqslant x_0. \end{cases}$$

$F(x)$ is well defined and continuous, the solutions $\varphi(t,x)$ of $\dot{x} = f(x, F(x))$ exist uniquely for all $t \geqslant 0$ by our general assumptions, and x_0 is a fixed point of $f(x, F(x))$ by construction. Furthermore, if there exists an initial value $\bar{x} \in \mathscr{A}(D)$ with $\lim_{t \to \infty} \varphi(t, \bar{x}) = y \neq x_0$, then $f(y, F(y)) \neq 0$ yields a contradiction, hence the point x^0 is the only limit point in $\mathscr{A}(D)$ and the system is asymptotically stable. □

3.2. Singular systems

Theorem 6. *Let x_0 be a common fixed point of all $f(\cdot, u)$, $u \in U$. The system* (1) *is locally feedback stabilizable at x_0 iff either*

(i) $x_0 \in cl\, D$ *for all control sets* D *of* (1) *with* $\{x_0\} \neq D$, *and* $\{x_0\}$ *is maximal with respect to* $\prec$, *or*
(ii) $x_0 \in cl\, D$ *for some control set* D, *and* $x_0 \in \operatorname{int} \mathscr{A}(\{x_0\})$.

The proof of Theorem 6 uses Proposition 2 and a similar construction as in the proof of Theorem 4. Corollary 5 remains valid in this case. Note that one cannot stabilize the system (1) at points in control sets that are characterized in Proposition 2(i), (b) and (c).

4. Bifurcation of control sets

In order to understand control and stabilization of systems near bifurcation points, we have to generalize the results from the previous sections to control systems depending on a parameter. We will first consider the global bifurcation structure of one-dimensional control systems. In general, these systems are of the form

$$\dot{x} = f_\alpha(x, u) \text{ in } \mathbb{R}^1, \quad \alpha \in I \subset \mathbb{R},$$

with $u \in \mathscr{U}^\rho = \{u : \mathbb{R} \to U^\rho, \text{ measurable}\}$ and $U^\rho = \rho \cdot U$ for $\rho \geqslant 0$, where $U \subset \mathbb{R}$ is compact and connected. For each α and ρ the control sets of the system can be computed as described in Section 2, and their dependence on these parameters can be studied. Here we will only hint at some interesting effects that can occur around bifurcation points. It turns out that the structurally simpler case

$$\dot{x} = X(x) + f(\alpha + u(t))Y(x),$$

$$\alpha \in I \subset \mathbb{R}, \ \rho \geqslant 0, \ u \in \mathscr{U}^\rho \qquad (2_\alpha^\rho)$$

covers all the interesting phenomena, and the results are easy to visualize. Here X, Y, and f are continuous so that the system satisfies the existence and uniqueness conditions stated above. We continue to work under the assumptions from Section 2, and consider only regular systems in this section. The extension to singular systems is discussed in Section 5.

Assume that the Lie algebra rank condition holds for (2_α^ρ) for all $\alpha \in I$, $\rho > 0$. We consider the basic bifurcation types in $\mathbb{R}^1$ (see e.g. [7 or 4]) – results for systems with more complicated bifurcation diagrams are obtained by combining the ideas described below.

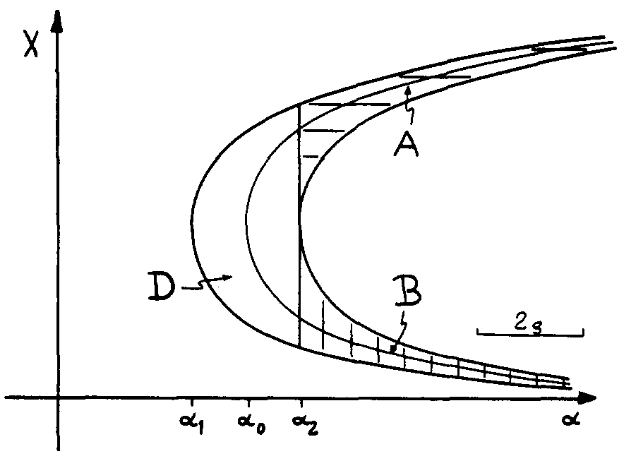

Fig. 1. Control sets around a saddle-node bifurcation point (x^0, α_0). The original bifurcation branches are contained in control sets which are indicated, for each α, by the region drawn around the bifurcation branches: For $\alpha < \alpha_1$, there are no control sets; for $\alpha_1 < \alpha < \alpha_2$ one variant control set exists in region D containing both of the bifurcation branches; for $\alpha \geqslant \alpha_2$ the control set has split up into two sets, an invariant control set in region A and a variant control set in region B.

4.1. Saddle-node bifurcation

The typical example is $\dot{x} = -x^2 + \alpha$, $\alpha \in I$. We assume the following behavior for a saddle-node bifurcation:

There exists an open interval $I \subset \mathbb{R}$ and $\alpha_0 \in I$ such that the uncontrolled system $\dot{x} = X(x) + f(\alpha)Y(x)$ shows

for $\alpha < \alpha_0$: no fixed point,
for $\alpha = \alpha_0$: one fixed point x^0,
for $\alpha > \alpha_0$: two branches of fixed points $x^+(\alpha)$, $x^-(\alpha)$, depending continuously on $\alpha \geqslant \alpha_0$, $x^+(\alpha)$ is (strictly) monotone increasing, $x^-(\alpha)$ is strictly monotone decreasing for $\alpha \geqslant \alpha_0$.

Without loss of generality, we assume that the points $x^+(\alpha)$ are stable and the $x^-(\alpha)$ are unstable. In this situation one obtains for regular systems of the form (2_α^ρ):

Theorem 7. *The bifurcation structure of the control sets of* (2_α^ρ) *around* (x^0, α_0) (*see Fig.* 1) *is as follows*:
– *For* $\alpha < \alpha_0$ *there is* $\rho(\alpha) = \alpha_0 - \alpha$ *such that one has for*

$\rho < \rho(\alpha)$: *no control set of* (2_α^ρ),
$\rho = \rho(\alpha)$: *one control set with void interior, namely* $\{x^0\}$,
$\rho > \rho(\alpha)$: *one control set* $(x^-(\alpha + \rho), x^+(\alpha + \rho)]$.

For $\alpha = \alpha_0$ *there is one control set* $(x^-(\alpha_0 + \rho), x^+(\alpha_0 + \rho)]$.

– *For* $\alpha > \alpha_0$ *there exists* $\rho(\alpha) = \alpha - \alpha_0$ *such that there are for*

$\rho \leqslant \rho(\alpha)$: *two control sets*, $[x^+(\alpha - \rho), x^+(\alpha + \rho)]$ *and* $(x^-(\alpha + \rho), x^-(\alpha - \rho))$,

$\rho > \rho(\alpha)$: *one control set*, $(x^-(\alpha + \rho), x^+(\alpha + \rho)]$.

The proof of Theorem 7 simply uses the characterization of control sets in [1]. Note that in the case of two control sets we have $(x^-(\alpha + \rho), x^-(\alpha - \rho)) \prec [x^+(\alpha - \rho), x^+(\alpha + \rho)]$.

4.2. *Transcritical bifurcation*

The typical example in this case is $\dot{x} = -x^2 + \alpha x$, $\alpha \in I$, but this leads to a singular system, because x^0 is a common fixed point. The differential equations $\dot{x} = (x - a\alpha)(\alpha - x) = -x^2 + \alpha(-ax + x + ax)$, $\alpha \in I$, can serve as an example for systems with transcritical bifurcations that lead to regular control systems. Note that for $a = 0$ we obtain the singular system above. It turns out that the control sets of the corresponding control system are quite different, depending on whether $a > 0$ or $a < 0$ holds. We consider the following two cases of transcritical bifurcations:

Case 1. There exists an open interval $I \subset \mathbb{R}$ and $\alpha_0 \in I$ such that the uncontrolled system has two fixed points $x^+(\alpha), x^-(\alpha)$ for all $\alpha \in I$, $\alpha \neq \alpha_0$, and one fixed point x^0 at $\alpha = a_0$. Assume that both branches $x^+ = \{x^+(\alpha), \alpha \in I\}$ and $x^- = \{x^-(\alpha), \alpha \in I\}$ are differentiable in α with, both derivatives are positive (or negative) in I.

Without loss of generality we assume that the points $\{x^+(\alpha), \alpha < \alpha_0\}$ and $\{x^-(\alpha), \alpha > \alpha_0\}$ are stable, the other fixed points are unstable, and that $(d/d\alpha)x^+(\alpha) < (d/d\alpha)x^-(\alpha)$ for $\alpha \in I$. In this situation we obtain for regular control systems:

Theorem 8. *In Case* 1 *the bifurcation structure of the control sets of* (2^ϱ_α) *around* (x^0, α_0) *(see Fig. 2) is as follows: For* $\alpha \in I$ *define* $\rho^-(\alpha)$ *to be the (unique)* ρ *such that* $x^+(\alpha - \rho) = x^-(\alpha + \rho)$, *and* $\rho^+(\alpha)$ *to be the (unique)* ρ *with* $x^-(\alpha - \rho) = x^+(\alpha + \rho)$. *Then we have*

– *for* $\alpha < \alpha_0$ *and*

$\rho \leqslant \rho^-(\alpha)$: *two control sets* $(x^-(\alpha - \rho), x^-(\alpha + \rho)) \prec [x^+(\alpha - \rho), x^+(\alpha + \rho)]$,

$\rho > \rho^-(\alpha)$: *one control set* $(x^-(\alpha - \rho), x^+(\alpha + \rho)]$,

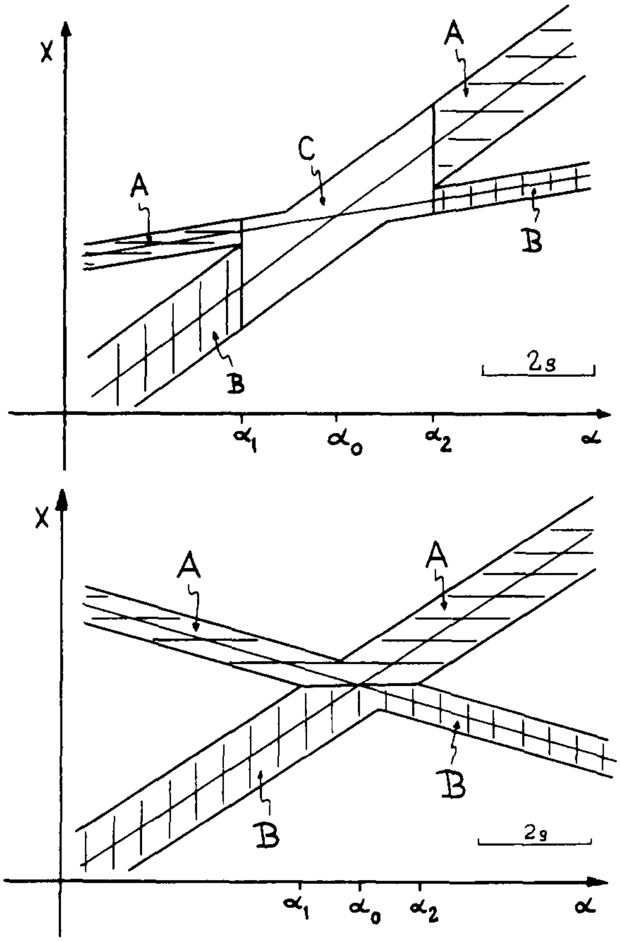

Fig. 2. Control sets around transcritical bifurcation points (x^0, α_0). The top figure illustrates Theorem 8, where for $\alpha_1 < \alpha < \alpha_2$ the invariant (region A) and the variant (region B) control set merge into one variant control set in region C. In the bottom figure, where the slopes of the original bifurcation branches have opposite sign, no region C occurs. For $\alpha_1 < \alpha < \alpha_2$ the regions A and B touch.

– *for* $\alpha = \alpha_0$: *one control set* $(x^-(\alpha_0 - \rho), x^+(\alpha_0 + \rho)]$,

– *for* $\alpha > \alpha_0$ *and*

$\rho \leqslant \rho^+(\alpha)$: *two control sets* $(x^+(\alpha - \rho), x^+(\alpha + \rho)) \prec [x^-(\alpha - \rho), x^-(\alpha + \rho)]$,

$\rho > \rho^+(\alpha)$: *one control set* $(x^+(\alpha - \rho), x^-(\alpha + \rho)]$.

Case 2. The derivative $(d/d\alpha)x^+(\alpha)$ is negative and $(d/d\alpha)x^-(\alpha)$ is positive on I, all other assumptions listed under Case 1 remain valid.

Theorem 9. *In Case* 2 *the bifurcation structure of the control sets of* (2^ϱ_α) *around* (x^0, α_0) *is as follows: For* $\alpha \in I$ *define* $\rho^-(\alpha)$ *to be the (unique)* ρ *such that* $x^+(\alpha + \rho) = x^-(\alpha + \rho)$, *and* $\rho^+(\alpha)$ *to be the (unique)*

ρ with $x^+(\alpha - \rho) = x^-(\alpha - \rho)$. Then one has
- *for $\alpha < \alpha_0$ and*
  - $\rho \leqslant \rho^-(\alpha)$: *two control sets* $(x^-(\alpha - \rho), x^-(\alpha + \rho)) \prec [x^+(\alpha + \rho), x^+(\alpha - \rho)]$,
  - $\rho > \rho^-(\alpha)$: *two control sets* $(x^-(\alpha - \rho), x^0) \prec [x^0, x^+(\alpha - \rho)]$,
- *for $\alpha = \alpha_0$: two control sets* $(x^-(\alpha_0 - \rho), x^0) \prec [x^0, x^+(\alpha_0 - \rho)]$,
- *for $\alpha > \alpha_0$ and*
  - $\rho > \rho^+(\alpha)$: *two control sets* $(x^+(\alpha + \rho), x^0) \prec [x^0, x^-(\alpha + \rho)]$,
  - $\rho \leqslant \rho^+(\alpha)$: *two control sets* $(x^+(\alpha + \rho), x^+(\alpha - \rho)) \prec [x^-(\alpha - \rho), x^-(\alpha + \rho)]$.

4.3. Pitchfork bifurcation

The bifurcation structure of control sets around bifurcation points for a pitchfork bifurcation can be analyzed similarly to B for regular systems. However, bistabilities can occur, i.e. there may exist control sets that, with respect to the order $\prec$, are smaller than two different maximal control sets. We have indicated the situation in Fig. 3(a), where for $\alpha_4 < \alpha < \alpha_5$ the minimal control sets in region B are smaller than the maximal (invariant) control sets in either region A.

For all three types of bifurcations the domains of attraction are computed according to the results in Section 2. The discussions above show that the bifurcation behavior of control sets for systems with bounded control range depends on two parameters, namely on α and on ρ, where ρ indicates, how the global dynamics of the system affect controllability. The fact is illustrated again when comparing Figs. 3(a) and 3(b): For increasing ρ the two regions A (of maximal control sets) and B (of minimal control sets) in Fig. 3(a) have merged into one region C (of minimal control sets) in Fig. 3(b). In particular, the control sets do not vary continuously with α nor ρ. It can be shown, however, that for bifurcation diagrams with only finitely many bifurcation branches, there are at most finitely many points of discontinuity, and these points can be computed similarly to the determination of ρ^- and ρ^+ in Theorems 8 and 9.

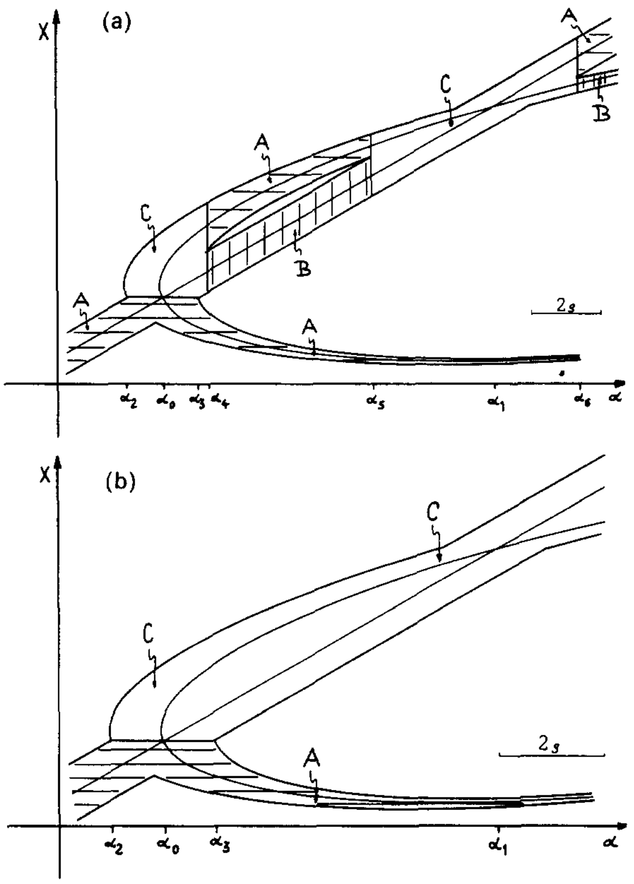

Fig. 3. Control sets for a system with pitchfork bifurcation (x^0, α_0) and transcritical bifurcation (x^1, α_1) for two different control ranges ρ. Comparison between Fig. 3(a) and Fig. 3(b) shows how controllability around bifurcation points is affected by the size ρ of the control range: While the lower region of invariant control sets varies continuously with ρ, the entire upper region in Fig. 3(a) has merged into one variant control set in region C for Fig. 3(b). This is due to the fact that with increasing ρ more of the global dynamics determines the controllability behavior, while the original bifurcation parameter α describes the situation locally.

5. Feedback stabilization near bifurcation points

In a family of one-dimensional differential equations $\dot{x} = X_\alpha(x)$, $\alpha \in I$, bifurcation means change of stability behavior for a branch of fixed points. If the differential equations are controlled (via bounded inputs), the question arises, whether there exist admissible feedbacks that stabilize the system around a bifurcation point. This question can be made more precise in various ways:

(a) The stabilizing feedback can depend on the bifurcation parameter α, i.e. α is known, and

(b) the stabilizing feedback should work for a wide range of the parameter α, i.e. one looks for robust stabilization with respect to α, while the stabilizing feedback does not depend on α.

The answer to (a) is obtained through a combination of the techniques presented in Sections 3 and 4.

In the regular case, let (x^0, α_0) be a bifurcation point of the uncontrolled equation such that the bifurcation branches emanating from x^0 are continuously increasing and decreasing, respectively, around x^0. Then x^0 lies in the interior of a control set of $(2^\rho_{\alpha_0})$ for all $\rho > 0$, and hence the system can be stabilized at x^0 via bounded feedback with values in U^ρ. For $\alpha \neq \alpha_0$ there exists $\rho^*(\alpha)$ (which may be ∞), such that for $\rho > \rho^*(\alpha)$ the corresponding control system is stabilizable at x^0, while for $\rho < \rho^*(\alpha)$ this may not be possible. Here ρ^* is defined by $\rho^*(\alpha) = \inf\{\rho > 0; x^0 \notin \operatorname{int} D^\rho_\alpha\}$, where D^ρ_α is the control set of (2^ρ_α) containing x^0. (If no such control set exists for some $\alpha \in I$ and all $\rho > 0$, then $\rho^*(\alpha) = 0$.) Note that $\rho^*(\alpha)$ need not be equal to the bifurcation point $\rho(\alpha)$ of control sets as introduced in Section 4. Consider, for example, the bifurcation point (x^1, α_1) in Fig. 3(a). For the given ρ, the bifurcation points of the control sets are α_5 and α_6, while the system is stabilizable at x^1 only for a smaller α-range. For singular systems, however, the quantities $\rho(\alpha)$ and $\rho^*(\alpha)$ agree for bifurcations from the common fixed point x^0. The set of initial values, from which the system can be stabilized at the fixed point x^0 is given by the domain of attraction as described in Section 2.

The solution to Problem (b) is treated here in the context of static state feedback, i.e. we do not assume that α can be estimated (which would lead to adaptive control) nor do we introduce feedback dynamics (leading e.g. to washout filters, see [9] for an example). Then feedback stabilization under uncertainty in α takes different forms for regular and for singular systems. For regular systems, one cannot guarantee that one feedback law will stabilize the system at the same point x^* for various values of α. However, if for a given control range $\rho > 0$ one has a connected family D^ρ_α of control sets (i.e. there exists a continuous function $f: I \to \bigcup_{\alpha\in I} D^\rho_\alpha$ with $f(\alpha) \in \operatorname{int} D^\rho_\alpha$) such that $\operatorname{int} \bigcap_{\alpha \in I} D^\rho_\alpha \neq \emptyset$, then there exists a nonempty, open set B of initial values and a common feedback law $u: B \to U^\rho$ such that for each $\alpha \in I$ the system (2^ρ_α) is stabilized at some point in $D = \bigcup_{\alpha\in I} D^\rho_\alpha$. In particular, if $\alpha: \mathbb{R} \to I$ is time varying, then there is $T > 0$ such that the solution of $\dot{x} = X(x) + f(\alpha(t) + u(x)) Y(x)$ lies in D for all $t \geqslant T$. This result follows directly from the feedback construction in the proof of Theorem 4. Note that the set B of initial values can be strictly contained in $\bigcap_{\alpha\in I} \mathscr{A}(D^\rho_\alpha)$. This is for instance the case if for each $\alpha \in I$ the domain of attraction $\mathscr{A}(D^\rho_\alpha)$ contains a (with respect to $\prec$ smallest) control set $\hat{D}^\rho_\alpha$, such that $\hat{D}^\rho_\alpha$ forms a connected family, but $\bigcap_{\alpha\in I} \hat{D}^\rho_\alpha = \emptyset$.

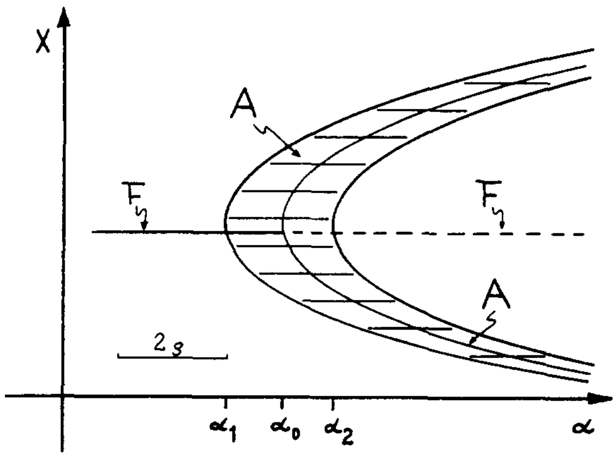

Fig. 4. Control sets around a pitchfork bifurcation (x^0, α_0) in a singular system. In this situation no variant control sets exist, but the fixed point x^0 becomes unstable at α_0. However, the control system is stabilizable at x^0 as long as $\rho \geqslant \alpha - \alpha_0$, see Theorem 10. Hence the bifurcation can be delayed to $\alpha_2 > \alpha_0$.

For singular systems (see Fig. 4) we are interested in a common feedback law $u(x)$ such that the system is stabilized at the common fixed point x^0 for all $\alpha \in I$. The bifurcation behavior of control sets for all singular systems is quite similar, so we will discuss only the case of a pitchfork bifurcation. The typical example in this case is $\dot{x} = -x^3 + \alpha x$, $\alpha \in I \subset \mathbb{R}$. Let us assume the following structure of a pitchfork bifurcation: there exists an open interval $I \subset \mathbb{R}$ and $\alpha_0 \in I$ such that the controlled system $\dot{x} = X(x) + f(\alpha) Y(x)$ has a common fixed point x^0 for all $\alpha \in I$, and for $\alpha > \alpha_0$ there are two branches $x^+(\alpha)$, $x^-(\alpha)$ of fixed points such that $x^+(\alpha)$ and $x^-(\alpha)$ depend continuously on $\alpha \geqslant \alpha_0$, $x^+(\alpha)$ is (strictly) increasing, $x^-(\alpha)$ is decreasing. Without loss of generality we assume that both branches contain stable points. In this situation we obtain:

Theorem 10. *Consider a system with a pitchfork bifurcation as described above. Let $\rho > 0$ be given, and let (x^0, α_0) be the bifurcation point. Then there exists a common feedback $u(x)$ such that the systems (2^ρ_α) are (asymptotically) stable at x^0 for all $\alpha \in I$ with $\alpha_0 \in \operatorname{int} I$, iff $\alpha \leqslant \alpha_0 + \rho$ for all $\alpha \in I$, i.e. iff $\rho \geqslant \alpha - \alpha_0 = \rho(\alpha)$.*

For a proof just notice that if $\rho \geqslant \alpha - \alpha_0$, then there exists $u \in U^\rho$ such that x^0 is asymptotically stable for the system (2_α^ρ) using the constant feedback u.

Conversely, if $\rho < \alpha - \alpha_0$ then we have three control sets, of which $\{x^0\}$ is the smallest one with respect to $\prec$. Hence the system is not stabilizable at x^0 for such an α according to Theorem 6.

Note that the condition on α in Theorem 10 is one-sided because the additional bifurcation branches exist only for $\alpha > \alpha_0$. In case of a transcritical bifurcation an analogous two-sided condition characterizes robust stabilization at x^0.

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