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Angaben zur Veröffentlichung / Publication details:

Spadini, Marco, and Fritz Colonius. 2006. "Fundamental semigroups for dynamical systems." *Discrete and Continuous Dynamical Systems* 14 (3): 447–63. https://doi.org/10.3934/dcds.2006.14.447.

FUNDAMENTAL SEMIGROUPS FOR DYNAMICAL SYSTEMS

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Abstract. Algebraic semigroups describing the dynamic behavior are associated to compact, locally maximal chain transitive subsets. The construction is based on perturbations and associated local control sets. The dependence on the perturbation structure is analyzed.

1. Introduction

This paper introduces a classification of the dynamic behavior of autonomous ordinary differential equations. We restrict attention to the dynamic behavior within a maximal chain transitive set, i.e., a chain recurrent component. In order to find a characterization which is robust with respect to certain perturbations, we allow all time-dependent perturbations taking values in a small set $U \subset \mathbb{R}^m$. This perturbed system may be viewed as a control system, and the chain transitive set blows up to a control set D^U. Using earlier constructions for control systems (see Colonius/San Martin/Spadini [2]), we associate to the control set D^U a semigroup describing the behavior of trajectories. Then, taking the inverse limit of the semigroups as $U \to 0$, we obtain a semigroup for the original differential equation. A number of properties of this procedure are derived. In particular, we study if the semigroup is independent of the perturbation structure and we show that the semigroup remains invariant under conjugacies of the differential equation, if the perturbation structure is respected.

We remark that the spirit of this paper is contrary to other contributions that emphasize the study of the global behavior between Morse sets (hence, outside of the chain recurrent set); see, e.g., Mischaikow [8]. On the other hand, topological properties inside the chain recurrent set have recently also found interest (see Farber et al. [4]).

The contents of the paper are as follows. In Section 2 we collect some properties of algebraic semigroups and their inverse limits. Section 3 recalls results on the relation between maximal chain transitive sets and control sets (and slightly generalizes them by considering local versions). Furthermore, the notion of a fundamental semigroup for local control sets of control

Date: June 29, 2005.

systems is recalled from [2]. Then, for a given perturbation structure, the fundamental semigroup of a locally maximal chain transitive set is defined as the inverse limit of such semigroups. Section 4 gives conditions implying that this inverse limit is independent of the perturbation structure. It is shown that these conditions are met in a natural setting for higher order differential equations. In Section 5 it is shown that the semigroup remains invariant under smooth conjugacies of the given differential equation. Furthermore, a number of simple examples is presented.

Notation. The set of compact and convex subsets of $\mathbb{R}^m$ containing the origin in the interior is denoted by $\mathrm{Co}_0 = \mathrm{Co}_0(\mathbb{R}^m)$.

2. Preliminary Facts on Semigroups and Inverse Limits

This section collects some basic facts on (algebraic) semigroups and their inverse limits. Main references are the classical book by Eilenberg/Steenrod [3] and the book by Howie [7].

A *semigroup* Λ is given by an associative operation $\circ : \Lambda \times \Lambda \to \Lambda$ on a nonvoid set Λ. A semigroup does not necessarily have a unity, i.e., an element e with $e \circ g = g \circ e = g$ for all $g \in \Lambda$. If a unity exists, however, one easily sees that it is unique.

In order to define inverse limits of semigroups, some preliminaries are needed. A *quasi-order* in a set A is a relation $\leq$ that is reflexive and transitive. The set A along with the quasi-order $\leq$ is a *directed set* if for each pair $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A subset A' of A is *cofinal* in A if for each $\alpha \in A$ there exists $\beta \in A'$ such that $\alpha \leq \beta$.

In the present paper the following quasi-ordered set will be relevant.

Example 2.1. *Let* $\mathrm{Co}_0 = \mathrm{Co}_0(\mathbb{R}^m)$ *denote the family of all compact convex subsets of* $\mathbb{R}^m$ *that contain the origin in their interior. For* $U, V \in \mathrm{Co}_0$*, define* $U \leq V$ *if* $V \subset U$*. With this quasi-order* Co_0 *is a directed set; in fact, for any* $U, V \in \mathrm{Co}_0$*, one has* $U \leq U \cap V$ *and* $V \leq U \cap V$*.*

Fix $U \in \mathrm{Co}_0$ *and, for* $\rho > 0$*, let* U^ρ *be the set given by* $\rho \cdot U$*. Then, the family of all the sets of the form* U^ρ *for some* $\rho > 0$ *is cofinal in* Co_0*. In fact, for any* $V \in \mathrm{Co}_0$ *there exists* $\rho_0 > 0$ *such that* $U^{\rho_0} \subset V$*, i.e.,* $V \leq U^{\rho_0}$*.*

We define the notion of inverse limit of a family of semigroups (compare [3], Chapter VIII, for such constructions with general inverse systems).

Let $\{\Lambda_\alpha\}_{\alpha \in A}$ be a family of semigroups, where A is a directed set with ordering '$\leq$'. Assume that for all $\alpha \leq \alpha'$ in A, there exist (semigroup) homomorphisms $\lambda_\alpha^{\alpha'} : \Lambda_{\alpha'} \to \Lambda_\alpha$ such that λ_α^α is the identity of Λ_α and $\lambda_\alpha^{\alpha'} \circ \lambda_{\alpha'}^{\alpha''} = \lambda_\alpha^{\alpha''}$ for $\alpha \leq \alpha' \leq \alpha''$. In this case the family of semigroups $\{\Lambda_\alpha\}_{\alpha \in A}$ along with the homomorphisms $\{\lambda_\alpha^{\alpha'}\}_{\alpha, \alpha' \in A}$ forms an *inverse system* $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha, \alpha' \in A}$ over the directed set A.

Denote by $\prod_{\alpha \in A} \Lambda_\alpha = \left\{ (g_\alpha)_{\alpha \in A}, g_\alpha \in \Lambda_\alpha \text{ for all } \alpha \in A \right\}$ the product of the semigroups Λ_α. When no confusion is possible, we shall often use the compact notation (g_α) for the more cumbersome $(g_\alpha)_{\alpha \in A}$.

The inverse limit of $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha'\in A}$ is the subset of $\prod_{\alpha\in A}\Lambda_\alpha$ given by

$$\overleftarrow{\lim_{\alpha\in A}}\Lambda_\alpha = \{(g_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A}\Lambda_\alpha,\ \lambda_\alpha^{\alpha'}(g_{\alpha'}) = g_\alpha \text{ for all } \alpha \le \alpha'\}.$$

Define the 'projection'

$$\lambda_{\alpha'} : \overleftarrow{\lim_{\alpha\in A}}\Lambda_\alpha \to \Lambda_{\alpha'}, \quad \lambda_{\alpha'} : (g_\alpha)_{\alpha\in A} \mapsto g_{\alpha'}.$$

In $\overleftarrow{\lim_{\alpha\in A}}\Lambda_\alpha$ define an operation of composition as follows: Given $(g_\alpha)_{\alpha\in A}$ and $(h_\alpha)_{\alpha\in A}$, let $(g_\alpha)(h_\alpha) = (g_\alpha h_\alpha)$. This is well defined, since for $\alpha \le \alpha'$ one has

$$\lambda_\alpha^{\alpha'}(g_{\alpha'}h_{\alpha'}) = \lambda_\alpha^{\alpha'}(g_{\alpha'})\lambda_\alpha^{\alpha'}(h_{\alpha'}) = g_\alpha h_\alpha.$$

One easily verifies that the product and the inverse limit again are semigroups and that the projections are homomorphisms. Furthermore, if $\alpha \le \alpha'$, then

$$\lambda_\alpha = \lambda_\alpha^{\alpha'} \circ \lambda_{\alpha'},$$

i.e., the following diagram is commutative:

$$\begin{tikzcd} \overleftarrow{\lim_{\alpha\in A}}\Lambda_\alpha \arrow[r, "\lambda_{\alpha'}"] \arrow[dr, "\lambda_\alpha"'] & \Lambda_{\alpha'} \arrow[d, "\lambda_\alpha^{\alpha'}"] \\ & \Lambda_\alpha \end{tikzcd} .$$

Proposition 2.2. *Consider inverse systems of semigroups $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha'\in A}$ and $\{\Gamma_\beta, \gamma_\beta^{\beta'}\}_{\beta,\beta'\in B}$ with directed sets A and B. Assume that there are an order preserving map $i : B \to A$ and homomorphisms $\phi_\beta : \Lambda_{i(\beta)} \to \Gamma_\beta$ for $\beta \in B$, such that for all $\beta \le \beta'$ in B*

$$\begin{tikzcd} \Lambda_{i(\beta')} \arrow[r, "\phi_{\beta'}"] \arrow[d, "\lambda_{i(\beta)}^{i(\beta')}"'] & \Gamma_{\beta'} \arrow[d, "\gamma_\beta^{\beta'}"] \\ \Lambda_{i(\beta)} \arrow[r, "\phi_\beta"] & \Gamma_\beta \end{tikzcd}$$

commutes. Then there is a unique homomorphism ϕ such that for all $\beta \in B$ the following diagram commutes:

$$\begin{tikzcd} \overleftarrow{\lim_{\alpha\in A}}\Lambda_\alpha \arrow[r, "\phi"] \arrow[d, "\lambda_{i(\beta)}"'] & \overleftarrow{\lim_{\beta\in B}}\Gamma_\beta \arrow[d, "\gamma_\beta"] \\ \Lambda_{i(\beta)} \arrow[r, "\phi_\beta"] & \Gamma_\beta \end{tikzcd} .$$

Proof. Define the homomorphism ϕ as follows: If $g \in \overleftarrow{\lim_{\alpha\in A}}\Lambda_\alpha$, set

$$h_\beta = \phi_\beta(g_{i(\beta)}),\ \beta \in B.$$

Then $\phi(g) := (h_\beta)_{\beta\in B} \in \varprojlim_{\beta\in B} \Gamma_\beta$ since, for $\beta \leq \beta'$,

$$\gamma_\beta^{\beta'}(h_{\beta'}) = \gamma_\beta^{\beta'}\big(\phi_{\beta'}(g_{i(\beta')})\big) = \phi_\beta\big(\lambda_{i(\beta)}^{i(\beta')}(g_{i(\beta')})\big) = \phi_\beta(g_{i(\beta)}) = h_\beta.$$

Now, commutativity of the diagram above follows immediately from the definition of ϕ. This, in turn, implies the uniqueness of ϕ. □

Proposition 2.3. *Consider inverse systems of semigroups $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha'\in A}$ and $\{\Gamma_\beta, \gamma_\beta^{\beta'}\}_{\beta,\beta'\in B}$ with directed sets A and B. Assume that there are order preserving maps $i : B \to A$ and $j : A \to B$, and homomorphisms $\phi_\beta : \Lambda_{i(\beta)} \to \Gamma_\beta$ and $\psi_\alpha : \Gamma_{j(\alpha)} \to \Lambda_\alpha$ such that for all $\beta \leq \beta'$ in B and $\alpha \leq \alpha'$ in A, the diagrams*

$$\begin{tikzcd} \Gamma_{j(\alpha')} \arrow[r, "\psi_{\alpha'}"] \arrow[d, "\gamma_{j(\alpha)}^{j(\alpha')}"'] & \Lambda_{\alpha'} \arrow[d, "\lambda_\alpha^{\alpha'}"] \\ \Gamma_{j(\alpha)} \arrow[r, "\psi_\alpha"] & \Lambda_\alpha \end{tikzcd} \qquad \begin{tikzcd} \Lambda_{i(\beta')} \arrow[r, "\phi_{\beta'}"] \arrow[d, "\lambda_{i(\beta)}^{i(\beta')}"'] & \Gamma_{\beta'} \arrow[d, "\gamma_\beta^{\beta'}"] \\ \Lambda_{i(\beta)} \arrow[r, "\phi_\beta"] & \Gamma_\beta \end{tikzcd} \tag{2.1}$$

commute. Assume also that $\beta \leq j(i(\beta))$ and $\alpha \leq i(j(\alpha))$ for every $\alpha \in A$ and $\beta \in B$, and that the diagrams

$$\begin{tikzcd} \Gamma_{j(i(\beta))} \arrow[r, "\psi_{i(\beta)}"] \arrow[d, "\gamma_\beta^{j(i(\beta))}"'] & \Lambda_{i(\beta)} \arrow[dl, "\phi_\beta"] \\ \Gamma_\beta & \end{tikzcd} \qquad \begin{tikzcd} \Lambda_{i(j(\alpha))} \arrow[r, "\phi_{j(\alpha)}"] \arrow[d, "\lambda_\alpha^{i(j(\alpha))}"'] & \Gamma_{j(\alpha)} \arrow[dl, "\psi_\alpha"] \\ \Lambda_\alpha & \end{tikzcd} \tag{2.2}$$

commute. Then the unique homomorphisms

$$\phi : \varprojlim_{\alpha\in A} \Lambda_\alpha \to \varprojlim_{\beta\in B} \Gamma_\beta \quad \textit{and} \quad \psi : \varprojlim_{\beta\in B} \Gamma_\beta \to \varprojlim_{\alpha\in A} \Lambda_\alpha,$$

induced by the diagrams (2.1) as in Proposition 2.2 are inverses of each other.

Proof. We shall refer to the notation of Proposition 2.2. It is enough to prove that for all $h = (h_\alpha) \in \varprojlim_{\alpha\in A} \Lambda_\alpha$ and $g = (g_\beta) \in \varprojlim_{\beta\in B} \Gamma_\beta$, one has

$$\lambda_\alpha\Big(\psi\big(\phi(h)\big)\Big) = h_\alpha \text{ and } \gamma_\beta\Big(\phi\big(\psi(g)\big)\Big) = g_\beta \text{ for all } \alpha \in A, \beta \in B.$$

Let us prove the first one, the second will follow from a similar argument.

Take $h = (h_\alpha) \in \varprojlim_{\alpha\in A} \Lambda_\alpha$. Using Proposition 2.2 twice and (2.2),

$$\begin{aligned} \lambda_\alpha\Big(\psi\big(\phi(h)\big)\Big) &= \psi_\alpha\Big(\gamma_{j(\alpha)}\big(\phi(h)\big)\Big) = \psi_\alpha\big(\phi_{j(\alpha)}(\lambda_{i(j(\alpha))}h)\big) \\ &= \psi_\alpha\big(\phi_{j(\alpha)}h_{i(j(\alpha))}\big) = \lambda_\alpha^{i(j(\alpha))}h_{i(j(\alpha))} = h_\alpha. \end{aligned}$$

This concludes the proof. □

The following lemma, roughly speaking, shows that the inverse limit of an inverse system $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha' \in A}$ is influenced only by Λ_α for "large" (of course in the sense of the quasi-order in A) values of α.

Lemma 2.4. *Let A and B be directed sets and assume that $B \subset A$ is cofinal in A. Consider a family of semigroups $\{\Lambda_\alpha\}_{\alpha \in A}$. Then $\overleftarrow{\lim\limits_{\alpha \in A}} \Lambda_\alpha$ is isomorphic to $\overleftarrow{\lim\limits_{\alpha \in B}} \Lambda_\alpha$.*

Proof. Corollary 3.16 Chap. VIII in [3] provides the required semigroup isomorphism. □

We now provide some results that can be useful for the computation of the inverse limit of an inverse family of semigroups.

Proposition 2.5. *Let $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha' \in A}$ be an inverse system of semigroups.*
(i) If each Λ_α has a unity e_α, then $(e_\alpha)_{\alpha \in A}$ is the unity of $\overleftarrow{\lim\limits_{\alpha \in A}} \Lambda_\alpha$.
(ii) Assume that B is cofinal in A and that Λ_β has a unity for any $\beta \in B$. Then $\overleftarrow{\lim\limits_{\alpha \in A}} \Lambda_\alpha$ has a unity.

Proof. (i) It is enough to observe that for any $(g_\alpha)_{\alpha \in A} \in \overleftarrow{\lim\limits_{\alpha \in A}} \Lambda_\alpha$ one has

$$(e_\alpha)(g_\alpha) = (g_\alpha) = (g_\alpha)(e_\alpha).$$

(ii) Follows from part (i) and Lemma 2.4. □

Proposition 2.6. *Let $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha' \in A}$ be an inverse system of semigroups.*
(i) Assume that for all $\alpha, \alpha' \in A$, with $\alpha \leq \alpha'$, the map $\lambda_\alpha^{\alpha'}$ is an isomorphism. Then for all $\alpha' \in A$

$$\overleftarrow{\lim_{\alpha \in A}} \Lambda_\alpha \simeq \Lambda_{\alpha'}.$$

(ii) If B is cofinal in A and Λ_β consists only of its unity for $\beta \in B$, then $\overleftarrow{\lim\limits_{\alpha \in A}} \Lambda_\alpha$ consists only of its unity.

Proof. (i) Follows from Theorem 3.4 in [3]. (ii) Follows from part (i) and Lemma 2.4. □

Proposition 2.7. *Suppose that the inverse limit of an inverse system of semigroups $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha' \in A}$ admits a unity and let B be cofinal in A. Then for every $\beta \in B$ the semigroup Λ_β contains an idempotent element.*

Proof. By Lemma 2.4 it follows that $\overleftarrow{\lim\limits_{\beta \in B}} \Lambda_\beta$ contains the unity $e = (e_\beta)_{\beta \in B}$. Since

$$(e_\beta) = e = e^2 = (e_\beta)(e_\beta) = (e_\beta^2),$$

the assertion follows. □

We conclude this section with a characterization of inverse limits using a universal property in the category of semigroups. As constructed, the inverse limit of $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha'\in A}$ is given by a semigroup together with homomorphisms λ_α to Λ_α. Propositions 2.2 and 2.6 (i) imply the following property:

If Γ is a semigroup and $\gamma_\alpha : \Gamma \to \Lambda_\alpha$ are homomorphisms, such that for all $\alpha \leq \alpha'$

$$\gamma_\alpha = \lambda_\alpha^{\alpha'} \circ \gamma_{\alpha'},$$

then there is a unique homomorphism $\gamma : \Gamma \to \overleftarrow{\lim_{\alpha\in A}} \Lambda_\alpha$ with

$$\lambda_\alpha \circ \gamma = \gamma_\alpha.$$

This follows, in fact, by considering the inverse system $\{\Gamma_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha'\in A}$ where $\Gamma_\alpha = \Gamma$ for all $\alpha \in A$ and $\lambda_\alpha^{\alpha'}$ is the identity for all $\alpha \leq \alpha'$.

This property characterizes the inverse limit up to isomorphisms. In fact, using commutative diagrams one can easily show that the following fact holds (inverse limits are a categorical construction).

Proposition 2.8. *Let $\{\Lambda_\alpha, \lambda_\alpha^{\alpha'}\}_{\alpha,\alpha'\in A}$ be an inverse system of semigroups. Consider a semigroup Δ together with homomorphisms $\delta_\alpha : \Delta \to \Gamma_\alpha$ such that for all $\alpha \leq \alpha'$*

$$\delta_\alpha = \lambda_\alpha^{\alpha'} \circ \delta_{\alpha'}.$$

Assume that for every semigroup Γ and homomorphisms $\gamma_\alpha : \Gamma \to \Lambda_\alpha$, such that for all $\alpha \leq \alpha'$

$$\gamma_\alpha = \lambda_\alpha^{\alpha'} \circ \gamma_{\alpha'}$$

there is a unique homomorphism $\bar{\gamma} : \Gamma \to \Delta$ with

$$\delta_\alpha \circ \bar{\gamma} = \gamma_\alpha.$$

Then there exists a unique isomorphism $\delta : \Delta \to \overleftarrow{\lim_{\alpha\in A}} \Lambda_\alpha$ such that for all α

$$\lambda_\alpha \circ \delta = \delta_\alpha.$$

Remark 2.9. *Clearly, in Proposition 2.8, it suffices to require the conditions for a cofinal subset of A.*

3. Locally Maximal Chain Transitive Sets

Consider a differential equation

$$\dot{x} = f_0(x) \tag{3.3}$$

given by a C^1-vector field $f_0 : \mathbb{R}^d \to \mathbb{R}^d$, and assume that global solutions $\varphi(t, x)$, $t \in \mathbb{R}$, exist for all considered initial conditions $\varphi(0, x) = x$.

First recall that a chain transitive set $M \subset \mathbb{R}^d$ is a closed invariant set such that for all points $x, y \in N$ and all $\varepsilon, T > 0$ there is an $(\varepsilon, T)-$chain ζ given by $n \in \mathbb{N}$, points $x_0 = x,\ x_1, ..., x_n = y$ in M and times $T_0, ..., T_{n-1} \geq T$ with

$$\mathrm{d}(\varphi(T_i, x_i), x_{i+1}) < \varepsilon \quad \text{for} \quad i = 0, 1, ..., n-1. \tag{3.4}$$

A chain transitive set M is called *locally maximal,* if it has a neighborhood N such that every chain transitive set M' with $M \subset M' \subset N$ satisfies $M = M'$.

The behavior within M will be analyzed via perturbations. Let C^1-vector fields $f_1, ..., f_m$ on $\mathbb{R}^d$ be given and let $F(x) = [f_1(x), ..., f_m(x)]$. For every $U \in \mathrm{Co}_0(\mathbb{R}^m)$ we consider the perturbed system denoted by $\left(\Sigma^U\right)$

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) = f_0(x) + F(x)u, \tag{3.5}$$
$$(u_i) \in \mathcal{U}(U) = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for almost all } t \in \mathbb{R}\}.$$

Again we assume that (absolutely continuous) global solutions $\varphi(t, x, u)$, $t \in \mathbb{R}$, exist for all considered initial conditions $\varphi(0, x, u) = x$ and all u. Interpreting u as a control function, (3.5) can be viewed as a control system. The sets $U \in \mathrm{Co}_0(\mathbb{R}^m)$ will be called admissible control ranges.

Definition 3.1. *A subset D of $\mathbb{R}^d$ with nonempty interior is a local control set if there exists a neighborhood N of* $\mathrm{cl}D$ *such that for each $x, y \in D$ and every $\varepsilon > 0$ there exist $T > 0$ and $u \in \mathcal{U}$ such that*

$$\varphi(t, x, u) \in N \textit{ for all } t \in [0, T] \textit{ and } \quad \mathrm{d}\big(\varphi(T, x, u), y\big) < \varepsilon$$

and for every D' with $D \subset D' \subset N$ which satisfies this property, one has $D' = D$.

The neighborhood N in the definition above will also be called an *isolating neighborhood* of D. If the neighborhood N can be chosen as $\mathbb{R}^d$, we obtain the usual notion of a control set with nonvoid interior as considered, e.g., in [1]. Thus for local control sets the maximality property of control sets is replaced by a local maximality property, and we refer to the latter also as *global* control sets. It is convenient to introduce the following notation for sets $A \subset \mathbb{R}^d$.

$$\mathcal{O}_A^{+,F}(x, U) = \left\{ y \in \mathbb{R}^d, \begin{array}{l} \text{there are } T \geq 0,\ u \in \mathcal{U}(U) \text{ with } y = \varphi(T, x, u) \\ \text{and } \varphi(t, x, u) \in A \text{ for } 0 \leq t \leq T \end{array} \right\}.$$

If $A = \mathbb{R}^d$ or dependence on F is not relevant, we simply omit the index. In this notation, a local control set for (Σ^U) with isolating neighborhood N is a maximal subset D^U of N with nonvoid interior such that

$$D^U \subset \mathrm{cl}\, \mathcal{O}_N^+(x, U) \text{ for all } x \in D^U.$$

Throughout, we assume that for every admissible control range U the control system (Σ^U) is locally accessible, i.e., for every $T > 0$ the sets

$$\mathcal{O}_{\leq T}^{\pm}(x, U) := \{y \in \mathbb{R}^d,\ y = \varphi(\pm t, x, u) \text{ with } u \in \mathcal{U}(U) \text{ and } t \in (0, T]\}$$

have nonvoid interiors. In this case we also say that the vector fields $f_1, ..., f_m$ specify a *perturbation structure* for (3.3).

The following proposition shows that a locally maximal chain transitive set of the unperturbed equation is contained in local control sets for perturbed systems with admissible control range U.

Proposition 3.2. *Let M be a compact locally maximal chain transitive set with isolating neighborhood N of the unperturbed equation (3.3) and consider for given vector fields $f_1, ..., f_m$ the control systems $\left(\Sigma^U\right)$ depending on admissible control ranges $U \in \mathrm{Co}_0$. Suppose that for all $x \in M$ and all U the so-called* inner pair condition *holds, i.e., there is $T > 0$ such that*

$$\varphi(T, x, 0) \in \mathrm{int}\mathcal{O}_N^+(x, U). \tag{3.6}$$

Then for every U there is a (unique) local control set D^U of $\left(\Sigma^U\right)$ with

$$M \subset \mathrm{int}D^U \text{ and } M = \bigcap_{U \in \mathrm{Co}_0} \mathrm{int}D^U = \bigcap_{U \in \mathrm{Co}_0} \mathrm{cl}D^U.$$

Proof. The statement of the proposition and its proof are minor modifications of [1], Corollary 4.7.2. □

Remark 3.3. *Under the assumptions of Proposition 3.2, it is clear that for $U, U' \in \mathrm{Co}_0$ with $U \subset U'$*

$$M \subset \mathrm{int}D^U \subset \mathrm{int}D^{U'}.$$

Remark 3.4. *Special control ranges are obtained in the form $U^\rho = \rho \cdot U$, where U is compact and convex with $0 \in \mathrm{int}U$. This is a case considered e.g. in* [1]*, and in Gayer* [5]*. The latter reference also shows that the inner pair condition (3.6) is e.g. satisfied for many oscillators; cp. also Lemma 4.5 and its proof. Note that there the restriction to a neighborhood N is not explicitly taken into account. However, all arguments are local, and hence immediately carry over to our situation.*

We recall from [2] the following construction which associates algebraic semigroups to local control sets D reflecting the behavior of the trajectories in D. Fix $p_0 \in \mathrm{int}D$. Define $P(D, p_0)$ as the set of all $(x, T) \in W^{1,\infty}([0, 1], \mathbb{R}^d) \times (0, \infty)$ with the following properties:

$x(0) = x(1) = p_0$, $x(t) \in \mathrm{int}D$ for $t \in [0, 1]$; and there are $0 < \gamma^- \leq \gamma^+ < \infty$ and measurable functions $u : [0, 1] \to U$, $\gamma : [0, 1] \to [\gamma^-, \gamma^+]$ such that

$$\dot{x}(t) = \gamma(t) f\big(x(t), u(t)\big),\ t \in [0, 1], \text{ and } T = \int_0^1 \gamma(t)\, dt.$$

Endow $P(D, p_0)$ with the metric structure given by

$$d((x_1, T_1), (x_2, T_2)) = \max(\|x_1 - x_2\|_\infty, |T_1 - T_2|)$$

for $(x_1, T_1), (x_2, T_2) \in P(D, p_0)$. The set $P(D, p_0)$ consists of the $T-$periodic trajectories in $\mathrm{int}D$ of (3.5) starting at p_0 and reparametrized to $[0, 1]$.

Two elements $(x_0, T_0), (x_1, T_1) \in P(D, p_0)$ are *homotopic*, written $(x_0, T_0) \simeq (x_1, T_1)$, if there exists a continuous map (a homotopy) $H : [0, 1] \to P(D, p_0)$ such that $H(0) = (x_0, T_0)$ and $H(1) = (x_1, T_1)$. One can check that this is an equivalence relation. Denote by $\Lambda(D, p_0)$ the quotient $P(D, p_0)/\simeq$. The operation on $P(D, p_0)$ given by $(x, T) * (y, S) = (x * y, S + T)$ with

$$(x * y)(t) = \begin{cases} x(2t) & t \in [0, 1/2] \\ y(2t - 1) & t \in [1/2, 1] \end{cases}, \tag{3.7}$$

is compatible with homotopy equivalence. Hence, this operation induces a semigroup structure on $\Lambda(D, p_0)$ called the *fundamental semigroup of the pointed local control set* (D, p_0).

Let p_0 be an element of the given locally maximal chain transitive set M of (3.3) and let all assumptions above be satisfied. Then consider the fundamental semigroups $\Lambda^U(D^U, p_0)$ of (Σ^U). Note that by Remark 3.3 one has $D^U \subset D^{U'}$ for $U \subset U'$. Hence $P^U(D^U, p_0)$ can be regarded as a subset of $P^{U'}(D^{U'}, p_0)$. The map $\lambda_U^{U'} : \Lambda^U(D^U, p_0) \to \Lambda^{U'}(D^{U'}, p_0)$ associating $[\alpha] \in \Lambda^U(D^U, p_0)$ to the class of α in $P^{U'}(D^{U'}, p_0)$ is, clearly, a well defined homomorphism. Furthermore, the set Co_0 of admissible control ranges U is a directed system with respect to set inclusion. Thus we can define inverse limits and the following semigroup is well defined.

Definition 3.5. *Let M be a compact locally maximal chain transitive set for (3.3). Then the fundamental semigroup of M with respect to the perturbation structure given by (3.5) is*

$$\Lambda(M, f_1, \ldots, f_m, p_0) := \overleftarrow{\lim_U} \Lambda^U(D^U, p_0).$$

This notation will be shortened to $\Lambda(M, p_0)$ whenever an explicit reference to $f_1, \ldots, f_m$ is not needed.

The fundamental semigroup provides an algebraic description of the behavior of trajectories in perturbed locally maximal chain transitive sets. In the next section we will discuss the dependence of this notion on the perturbation structure.

4. Compatible Perturbation Structures

In this section we will indicate a condition which guarantees that the fundamental semigroup is independent of the perturbing vector fields $f_1, ..., f_m$. It is convenient to write (3.5) in the form

$$\dot{x} = f_0(x) + F(x)u, \ u \in \mathcal{U}(U). \tag{4.8}$$

Along with (4.8) consider the family of perturbed systems

$$\dot{x} = f_0(x) + G(x)v, \ v \in \mathcal{U}(V) \tag{4.9}$$

with $G(x) = [g_1(x), ..., g_l(x)]$ and C^1-vector fields g_i and admissible control ranges $V \in \mathrm{Co}_0(\mathbb{R}^l)$. Denote the corresponding trajectories by $\varphi^F(t, x, u)$ and $\varphi^G(t, x, v)$, respectively, and assume their global existence.

Throughout this section a compact locally maximal chain transitive set M with compact isolating neighborhood N of the system $\dot{x} = f_0(x)$ is given. The following property will be crucial in order to show that the fundamental semigroups for the perturbation structures given by F and G coincide. It requires that the trajectories for F are reproducible by those for G and conversely.

Definition 4.1. *The perturbation structures given by F and G are called compatible, whenever the following two conditions are satisfied:*

For all admissible control ranges $U \in \mathrm{Co}_0(\mathbb{R}^m)$ there exists an admissible control range $V = i(U) \in \mathrm{Co}_0(\mathbb{R}^l)$ such that for all $x \in N$ and $v \in \mathcal{U}(V)$ with $\varphi^G(t,x,v) \in N$ for $t \geq 0$, there is $u \in \mathcal{U}(U)$ with

$$\varphi^F(t,x,u) = \varphi^G(t,x,v) \text{ for all } t \geq 0;$$

conversely, for all $V \in \mathrm{Co}_0(\mathbb{R}^l)$ there exists $U = j(V) \in \mathrm{Co}_0(\mathbb{R}^m)$ such that for all $x \in N$ and $u \in \mathcal{U}(U)$ with $\varphi^F(t,x,u) \in N$ for $t \geq 0$, there is $v \in \mathcal{U}(V)$ with

$$\varphi^F(t,x,u) = \varphi^G(t,x,v) \text{ for all } t \geq 0.$$

Remark 4.2. *Obviously, one may assume that $i(j(V)) \subset V$ and $j(i(U)) \subset U$. Thus in the order on admissible control ranges (cp. Example 2.1) one has $i(j(V)) \geq V$ and $j(i(U)) \geq U$.*

The following result shows that compatible perturbation structures lead to the same fundamental semigroup. Denote the control sets containing M for (4.8) and (4.9) with control ranges U and V by $D^{U,F}$ and $D^{V,G}$, respectively.

Theorem 4.3. *Let M be a compact locally maximal chain transitive set for (3.3) and consider compatible perturbation structures F and G given by (4.8) and (4.9), respectively. Assume that for every $x \in M$ and every $U \in \mathrm{Co}_0(\mathbb{R}^m)$ there is $T > 0$ such that $\varphi(T,x) \in \mathrm{int}\mathcal{O}_N^{+,F}(x,U)$ and, similarly, that for every $x \in M$ and every $V \in \mathrm{Co}_0(\mathbb{R}^l)$ there is $T > 0$ with $\varphi(T,x) \in \mathrm{int}\mathcal{O}_N^{+,G}(x,V)$.*

Then for $p_0 \in M$ the fundamental semigroups

$$\Lambda(M,F,p_0) = \overleftarrow{\lim_{U \in \mathrm{Co}_0(\mathbb{R}^m)}} \Lambda^U(D^{U,F},F,p_0)$$

and

$$\Lambda(M,G,p_0) = \overleftarrow{\lim_{V \in \mathrm{Co}_0(\mathbb{R}^l)}} \Lambda^V(D^{V,G},G,p_0)$$

are isomorphic.

Proof. Let the corresponding loop sets be

$$P^U(D^{U,F},F,p_0) \text{ and } P^V(D^{V,G},G,p_0),$$

respectively. Since the perturbation structures are compatible, there exist natural (injective) maps

$$P^{j(V)}(D^{j(V),F},F,p_0) \to P^V(D^{V,G},G,p_0),$$
$$P^{i(U)}(D^{i(U),G},G;p_0) \to P^U(D^{U,F},F,p_0).$$

These maps are easily seen to pass to the quotient and to induce maps

$$\psi^V : \Lambda^{j(V)}(D^{j(V),F},F,p_0) \longrightarrow \Lambda^V(D^{V,G},G,p_0),$$
$$\phi^U : \Lambda^{i(U)}(D^{i(U),G},G,p_0) \longrightarrow \Lambda^U(D^{U,F},F,p_0).$$

By Remark 4.2, we can assume $i(j(V)) \geq V$ and $j(i(U)) \geq U$. One can also check that the diagrams (2.1) and (2.2) in Proposition 2.3 commute. Thus, passing to the inverse limits, one obtains an isomorphism

$$\overleftarrow{\lim_{U \in \mathrm{Co}_0(\mathbb{R}^m)}} \Lambda^U(D^{U,F}, F, p_0) \simeq \overleftarrow{\lim_{V \in \mathrm{Co}_0(\mathbb{R}^l)}} \Lambda^V(D^{V,G}, G, p_0),$$

as claimed. $\square$

Next we turn to the question, how one can guarantee that two perturbation structures are compatible. We can answer this only in the special case of higher order differential equations in $\mathbb{R}^N$ of the following form:

$$\begin{cases} x_1^{(n_1)} + h_1(x, \ldots, x^{(n_1-1)}) = 0, \\ \quad \vdots \\ x_N^{(n_N)} + h_N(x, \ldots, x^{(n_N-1)}) = 0, \end{cases} \tag{4.10}$$

where $n_1,\ldots,n_N \in \mathbb{N}$ are given, and $x^{(j)} = (x_1^{(j)}, \ldots, x_N^{(j)})$ denotes the j-th derivative of $x = (x_1, \ldots, x_N)$; all $h_i : \mathbb{R}^{n_i N} \to \mathbb{R}^N$ are C^1-functions.

Associate to (4.10) the control system

$$\begin{cases} x_1^{(n_1)} + h_1(x, \ldots, x^{(n_1-1)}) = f_1(x, \ldots, x^{(n_1-1)}) u_1(t), \\ \quad \vdots \\ x_N^{(n_N)} + h_N(x, \ldots, x^{(n_N-1)}) = f_N(x, \ldots, x^{(n_N-1)}) u_N(t), \end{cases} \tag{4.11}$$

where the functions f_i are C^1 and the controls $u = (u_1, \ldots, u_N)$ are measurable and take values in an admissible control range $U \in \mathrm{Co}_0(\mathbb{R}^N)$.

Both systems (4.10) and (4.11) can, in the standard way, be considered as first order systems in $\mathbb{R}^{n_1+\ldots+n_N}$. In particular, for (4.11) one obtains as a special case of (3.5) with $x_{i,j} = x_i^{(j-1)}, i = 1, ..., N, j = 1, ..., n_i$,

$$\begin{cases} \dot{x}_{1,1} = x_{1,2}, \\ \quad \vdots \\ \dot{x}_{1,n_1} + h_1(x_{1,1}, \ldots, x_{N,n_1}) = f_1(x_{1,1}, \ldots, x_{N,n_1}) u_1(t), \\ \qquad \vdots \\ \dot{x}_{N,1} = x_{N,2}, \\ \quad \vdots \\ \dot{x}_{N,n_N} + h_N(x_{1,1}, \ldots, x_{N,n_N}) = f_N(x_{1,1}, \ldots, x_{N,n_N}) u_N(t), \end{cases} \tag{4.12}$$

Fix a compact, locally maximal chain transitive sets $M \subset \mathbb{R}^{n_1+\ldots+n_N}$ of system (4.10). As we shall see, when $f_1, \ldots, f_N$ are bounded away from 0, there are local control sets as in Proposition 3.2 for (4.12); and, as long as this condition holds, the inverse limit of the semigroups associated to these local control sets does not depend on the choice of the functions $f_1, \ldots, f_N$.

Let $x = (x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{N,n_N})$ be a point in $\mathbb{R}^{n_1+\ldots+n_N}$. Denote by $\varphi(t, x, u)$ the solution of (4.12) at time t starting from x at time $t = 0$ and driven by the control function u.

Remark 4.4. *Consider the family* $\mathrm{Co}_0(\mathbb{R}^N)$ *of all compact convex neighborhoods of the origin with the partial order defined in Example 2.1. Given any* $\rho_0 > 0$*, the family of '*ρ*-cubes'*

$$[-\rho,\rho]^N := \left\{(\xi_1,\ldots,\xi_N) \in \mathbb{R}^N : \xi_i \in [-\rho,\rho] \text{ for } i = 1,\ldots,N\right\}$$

for $0 < \rho < \rho_0$*, is cofinal. Therefore, by Lemma 2.4, the inverse limit in Theorem 4.6 does not change if we consider only the control sets relative to these control ranges. For simplicity we indicate dependence on such a control range by a superfix* ρ.

Again a compact locally maximal chain transitive set M of system (4.12) is considered.

Lemma 4.5. *If in system (4.12) the functions* $f_1,\ldots,f_N$ *are nonzero on* M*, then there is a compact neighborhood* K *of* M *such that for every* $\rho > 0$ *and every* $x \in \mathbb{R}^{n_1+\ldots+n_N}$ *there exists* $T > 0$ *such that* $\varphi(T,x,0) \in \mathrm{int}\mathcal{O}_K^{\rho,+}(x)$.

Proof. This follows by inspection of the proof of Theorem 19 in Gayer [5]. Observe that by continuity the functions $f_1,\ldots,f_N$ are bounded away from 0 in a compact neighborhood of M. □

By Proposition 3.2 one has that for every U there exists a local control set D^U containing M in its interior with

$$M = \bigcap_{U \in \mathrm{Co}_0} D^U.$$

Fix a point $p_0 \in M$ and consider for (4.12) the family of fundamental semigroups $\{\Lambda^U(D^U, f_1, ..., f_N, p_0)\}_{U \in \mathrm{Co}_0}$. Then the corresponding inverse limit exists. We shall show that it does not depend on the choice of the functions $f_1,\ldots,f_N$.

Theorem 4.6. *Let* $M \subset \mathbb{R}^{n_1+\ldots+n_N}$ *be a compact locally maximal chain transitive set for (4.10). Assume that the functions* $f_1,\ldots,f_N$ *in (4.11) are bounded away from zero in* M*. Then the inverse limit*

$$\overleftarrow{\lim_{U \in \mathrm{Co}_0}} \Lambda^U(D^U, f_1,\ldots,f_N,p_0)$$

is independent (up to an isomorphism) of the choice of $f_1,\ldots,f_N$.

Proof. By Remark 4.4 one can restrict attention to systems having only small cubes centered at the origin as control ranges.

More precisely, consider in addition to (4.11) with $U = [-\rho,\rho]^N$, $\rho > 0$, the following system

$$\begin{cases} x_1^{(n_1)} + h_1(x,\ldots,x^{(n_1-1)}) = g_1(x,\ldots,x^{(n_1-1)})v_1(t), \\ \quad\vdots \\ x_N^{(n_N)} + h_N(x,\ldots,x^{(n_N-1)}) = g_N(x,\ldots,x^{(n_N-1)})v_N(t), \end{cases} \tag{4.13}$$

where for all i, the functions g_i satisfy the same assumptions as the f_i and $v_i(t) \in [-\delta,\delta]$, $\delta > 0$.

Since the f_i and the g_i are nonzero on M, continuity implies existence of a compact neighborhood K of M on which all f_i and all g_i are bounded away from zero. By Lemma 4.5 and Proposition 3.2 there exist families of local control sets $D^{\rho,f}$ and $D^{\delta,g}$ for the control systems (4.11) and (4.13) respectively, such that

$$M = \bigcap_{\rho>0} \mathrm{int} D^{\rho,f} = \bigcap_{\delta>0} \mathrm{int} D^{\delta,g}.$$

Hence one may assume that ρ and δ are so small that $D^{\rho,f} \subset K$ and $D^{\delta,g} \subset K$.

Let $p_0 \in M$ and consider the semigroups $\Lambda^{\rho,f}(D^{\rho,f}, f, p_0)$ and $\Gamma^{\delta,g}(D^{\delta,g}, g, p_0)$ associated to these control sets. We shall verify that the perturbation structures for (4.11) and (4.13) are compatible. Hence Theorem 4.3 shows that the corresponding inverse limits are isomorphic, as claimed.

Denote by $\varphi^f(t,x,u)$ and $\varphi^g(t,x,v)$ the solutions of the first order systems corresponding to (4.11) and (4.13) starting at $x \in \mathbb{R}^{n_1+\dots+n_N}$ for $t = 0$ driven by u and v, respectively. We claim that there exists $\alpha > 0$ with the following property:

Let $x \in \mathbb{R}^{n_1+\dots+n_N}$ and let v be a control with $v(t) \in V = [-\delta, \delta]$ and $\varphi^g(t,x,v) \in K$ for all $t \geq 0$; then there exists a control function u with $u(t) \in U = [-\alpha\delta, \alpha\delta]$ such that $\varphi^f(t,x,u) = \varphi^g(t,x,v)$.

If this claim is verified, the first compatibility property is satisfied with $i([-\rho,\rho]^N) = [-\rho/\alpha, \rho/\alpha]^N$.

To prove the claim, choose

$$\alpha := \frac{\max \max\limits_{i=1,\dots N} |g_i(\bar{x})|}{\min \min\limits_{i=1,\dots,N} |f_i(\bar{x})|},$$

where the minimum and maximum are taken for $\bar{x} \in K$. (Note the slight abuse of notation here: the functions f_i and g_i only depend on certain components of $\bar{x} \in \mathbb{R}^{n_1+\dots+n_N}$.) Define the components of $u(t)$ for $t \geq 0$ by

$$u_i(t) = \frac{g_i\Big(\varphi^g\big(t,x,v\big)\Big)}{f_i\Big(\varphi^g\big(t,x,v\big)\Big)}\, v_i(t).$$

Then $u_i(t) \in [-\alpha\delta, \alpha\delta]$ for all i and, moreover, the function $t \mapsto \phi^f\big(t,x,u\big)$ satisfies equation (4.13) with initial state x. Therefore, by uniqueness, $\phi^f\big(t,x,u\big) = \phi^g\big(t,x,v\big)$. This proves the first compatibility condition. The second one follows by constructing the map j in the same way. □

5. Invariance under conjugacy

In this section we consider a chain recurrent component M for dynamical systems induced by equations of the form (3.3), and prove that the associated fundamental semigroup is invariant under C^1-conjugacies provided that the perturbation structure is preserved.

Consider two differential equations on $\mathbb{R}^d$

$$\dot{x} = f(x) \text{ and } \dot{x} = g(x) \tag{5.14}$$

with global flows φ_t^f and $\varphi_t^g, t \in \mathbb{R}$. Assume that they are C^1- conjugate, i.e., there exists a C^1-diffeomorphism H such that the following diagram commutes for all $t \in \mathbb{R}$:

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "\varphi_t"] \arrow[d, "H"'] & \mathbb{R}^d \arrow[d, "H"] \\ \mathbb{R}^d \arrow[r, "\psi_t"] & \mathbb{R}^d \end{tikzcd} .$$

Consider a compact, locally maximal chain transitive set M of $\dot{x} = f(x)$. Then $N := H(M)$ is a compact locally maximal chain transitive set for $\dot{x} = g_0(x)$. For perturbation structures $F(x) = [f_1(x), ..., f_m(x)]$ and $G(x) = [g_1(x), ..., g_l(x)]$, one obtains the control systems

$$\begin{aligned} \dot{x} &= f(x) + F(x)u, \ u \in \mathcal{U}(U), \\ \dot{x} &= g(x) + G(x)v, \ v \in \mathcal{U}(V), \end{aligned} \tag{5.15}$$

with $U \in \mathrm{Co}_0(\mathbb{R}^m), V \in \mathrm{Co}_0(\mathbb{R}^l)$, and corresponding fundamental semigroups

$$\Lambda(M, F, p_0) \text{ and } \Lambda(H(M), G, H(p_0)).$$

We will show that these semigroups are isomorphic if the perturbation structures are compatible under the conjugacy H in an appropriate sense.

Remark 5.1. *Consider a C^1-conjugacy H of φ_t^f and φ_t^g. Then, differentiating the relation $H\big(\varphi_t^f(x)\big) = \varphi_t^g\big(H(x)\big)$ with respect to t at $t = 0$, one gets*

$$H'(x)f(x) = H'(x)(\dot{\varphi}_0^f(x)) = \dot{\varphi}_0^g\big(H(x)\big) = g\big(H(x)\big);$$

here $H'(x)$ denotes the Fréchet derivative of H at x. Thus

$$f(x) = H'(x)^{-1}g\big(H(x)\big),$$

for every $x \in \mathbb{R}^d$.

In order to compare the perturbation structures we transport the vector fields f_i via H and compare them to the g_i.

Lemma 5.2. *Let H be a C^1-conjugacy of the differential equations (5.14) and consider the perturbed equations (5.15) with perturbation structures F and G. Define perturbation structures by*

$$\begin{aligned} H^{-1}G &:= [H'(x)^{-1}g_1\big(H(x)\big), ..., H'(x)^{-1}g_m\big(H(x)\big)], \\ HF &:= [H'(x)f_1\big(H^{-1}(x)\big), ..., H'(x)f_m\big(H^{-1}(x)\big)] \end{aligned} \tag{5.16}$$

Then the perturbation structures HF and G are compatible if and only if F and $H^{-1}G$ are compatible.

Proof. The proof is a straightforward verification that the diffeomorphism H maps solutions $\varphi^F(\cdot, x, u)$ of

$$\dot{x} = f(x) + F(x)u$$

to solutions $\varphi^{HF}(\cdot, x, u)$ of

$$\dot{x} = g(x) + HF(x)u,$$

and that, conversely, H^{-1} maps solutions of

$$\dot{x} = f(x) + G(x)u$$

to solutions of

$$\dot{x} = g(x) + H^{-1}G(x)u.$$

It is, therefore, left to the reader. □

The following theorem shows that the fundamental semigroups remain invariant under C^1-conjugacies H provided that H respects the perturbation structures.

Theorem 5.3. *Assume that H is a C^1-conjugacy of the flows for the differential equations (5.14), and consider a compact, locally maximal chain transitive set M with isolating neighborhood N of $\dot{x} = f(x)$ and perturbation structures F and G given by (5.15). Assume that for every $x \in M$ and every $U \in \mathrm{Co}_0(\mathbb{R}^m)$ there is $T > 0$ such that $\varphi^F(T, x, 0) \in \mathrm{int}\mathcal{O}_N^{+,F}(x, U)$ and, similarly, that for every $x \in M$ and every $V \in \mathrm{Co}_0(\mathbb{R}^l)$ there is $T > 0$ with $\varphi^G(T, H(x), 0) \in \mathrm{int}\mathcal{O}_{H(N)}^{+,G}(H(x), V)$.*

Then for $p_0 \in M$ the fundamental semigroups

$$\Lambda\big(M, F, p_0\big) \text{ and } \Lambda\big(H(M), G, H(p_0)\big)$$

are isomorphic if the perturbation structures G and FH given by (5.16) are compatible.

Proof. By Theorem 4.3, the semigroups

$$\Lambda\big(H(M), HF, H(p_0)\big) \text{ and } \Lambda\big(H(M), G, H(p_0)\big)$$

are isomorphic, since they correspond to compatible perturbation structures. Hence it remains to show that the semigroup remains invariant under the C^1-conjugacy H. As in Lemma 5.2, trajectories of

$$\dot{x} = f(x) + F(x)u$$

are mapped via H onto trajectories of

$$\dot{x} = g(x) + HF(x)u,$$

preserving time, and conversely. Thus for every $U \in \mathrm{Co}_0(\mathbb{R}^m)$ the corresponding local control sets $D^{U,F}$ and $D^{U,HF}$ of these two systems are mapped onto each other and the loop sets for both systems are in a natural bijective correspondence. Hence, H induces an isomorphism between the semigroups

$$\Lambda^U(D^{U,F}, F, p_0) \text{ and } \Lambda^U(D^{U,HF}, HF, H(p_0)),$$

so that their inverse limits are isomorphic. □

As in the preceding section the assumptions of this theorem are satisfied for higher order equations. If the conjugacy respects the form of these equations, then compatibility of the perturbation structures is automatically satisfied. More precisely, in addition to (4.10) with perturbation structure (4.11), consider a higher order equation with perturbation structure of the form

$$\begin{cases} x_1^{(n_1)} + k_1(x, \dots, x^{(n_1-1)}) = g_1(x, \dots, x^{(n_1-1)})\ v_1(t), \\ \vdots \\ x_N^{(n_N)} + k_N(x, \dots, x^{(n_N-1)}) = g_N(x, \dots, x^{(n_N-1)})\ v_N(t) \end{cases} \tag{5.17}$$

Consider C^1-conjugacies $H : \mathbb{R}^{n_1+\dots+n_N} \to \mathbb{R}^{n_1+\dots+n_N}$ of the special form

$$\begin{aligned} &H(x_{1,1}, \dots, x_{N,n_N}) \\ &\quad = \Big(x_{1,1}, \dots, x_{1,n_1-1}, H_1\big(x_{1,1}, \dots, x_{N,n_N}\big), \\ &\qquad\quad \dots, x_{N,1}, \dots, x_{N,n_N-1}, H_N\big(x_{1,1}, \dots, x_{N,n_N}\big)\Big). \end{aligned} \tag{5.18}$$

where the x_{ij} are taken as in (4.12). Note that this kind of conjugacies can be seen as phase-space diffeomorphisms for systems of the form (4.10). The following result shows that for higher order differential equations the fundamental semigroup is essentially independent of the perturbations.

Theorem 5.4. *Let $M \subset \mathbb{R}^{n_1+\dots+n_N}$ be a compact locally maximal chain transitive set for the first order system associated to (4.10), and take $p_0 \in M$. Suppose that H of the form (5.18) is a C^1-conjugacy of the flow associated to this system to the one induced in $\mathbb{R}^{n_1+\dots+n_N}$ by (5.17) when $v_i(t) = 0$ for all t and all i. Suppose that the f_i's in (4.11) do not vanish on M and that the g_i's in (5.17) do not vanish on $H(M)$.*

Then the fundamental semigroups

$$\Lambda(M, f_1, ...f_N, p_0) \text{ and } \Lambda(H(M), g_1, ..., g_N, H(p_0))$$

are isomorphic.

Proof. Due to the special form (5.18) of the conjugacy H, the perturbation structure HF also corresponds to a higher order differential equation of the form (5.17). Clearly, by Remark 5.1, the functions in front of the controls corresponding to the perturbation structure HF are bounded away from zero. The assertion follows from Theorem 4.6 and Theorem 5.3. □

Let us now see some simple examples illustrating the relation between the dynamics and the semigroup.

Example 5.5. *Consider the linear differential equation*

$$\dot{x} = Ax.$$

and additive perturbations leading to the linear control system

$$\dot{x} = Ax + Bu, \quad u(t) \in U,$$

with $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$. *Assume that* (A, B) *is controllable, i.e.,* $\mathrm{rank}[B, AB, ...A^{d-1}B] = d$. *It is known (see* [2, Proposition 4.7]*) that for* U *compact, convex with* $0 \in \mathrm{int}\, U$*, there exists a unique local control set* D^U*, and the semigroup* $\Lambda_U(D^U, 0)$ *consists of just its unity.*

Assuming also that A *is hyperbolic (i.e., its eigenvalues are not purely imaginary), one has that the local chain recurrent component* M *for* $\dot{x} = Ax$ *is compact as it reduces to the origin. Therefore, it makes sense to consider the inverse limit* $\Lambda(M, 0) = \overleftarrow{\lim}_{U\in \mathrm{Co}_0} \Lambda_U(D^U, 0)$. *By Proposition 2.6 (ii),* $\Lambda(M, 0)$ *is just the trivial semigroup.*

Note that Example 5.5 shows that the converse of Theorem 5.3 does not hold. In fact, as shown above, the fundamental semigroup for a linear system, when defined, is trivial regardless of its conjugacy class.

The following two examples show that even for situations where the chain recurrent components exhibit the same topological structure, the fundamental limit semigroup may not be the same.

Example 5.6. *Let* $U \subset \mathbb{R}^m$ *be a compact and convex set containing* 0 *in its interior. Consider control-affine systems of the form*

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x), \ u \in \mathcal{U}^\rho, \tag{5.19}$$

where $\mathcal{U}^\rho$ *denotes the set of measurable functions on* $\mathbb{R}$ *with values in* ρU, $\rho \geq 0$. *Suppose that the uncontrolled system (with* $u \equiv 0$*) has a homoclinic orbit given by*

$$\varphi(t, p_1, 0), \ t \in \mathbb{R}, \ \textit{with} \ \lim_{t\to\pm\infty} \varphi(t, p_1, 0) = p_0,$$

where $p_0 \neq p_1$ *is an equilibrium. Suppose that* $H := \{p_0\} \cup \{\varphi(t, p_1, 0), \ t \in \mathbb{R}\}$ *is a chain recurrent component of the uncontrolled system and that the controllability condition*

$$\mathrm{span}\,\{\mathrm{ad}^k_{f_0} f_i(x), \ i = 1, ..., m, \ k = 0, 1, \ldots\} = \mathbb{R}^d \tag{5.20}$$

holds for all points $x \in H$. *Then for every* $\rho > 0$ *there is a control set* D^ρ *containing* H *in its interior and*

$$\bigcap_{\rho>0} D^\rho = H;$$

see Corollary 4.7.6 in [1] *(the controlled Takens-Bogdanov oscillator is a system where these conditions can be verified; cp. Häckl/Schneider* [6] *or Section 9.4 in* [1]*). It is known (see* [2]*) that the fundamental semigroup* $\Lambda^{U^\rho}(D^{U^\rho}, p_0)$ *contains a unity. Therefore, by Proposition 2.5 (ii) the inverse limit* $\Lambda(H, p_0) = \overleftarrow{\lim}_{U\in \mathrm{Co}_0} \Lambda_U(D^U, 0)$ *contains the unity.*

Example 5.7. *Consider a system (5.19) where the uncontrolled system has a periodic trajectory $\hat{H} = \{\varphi(t, p_0, 0),\ t \in [0, T]\}$. If $\hat{H}$ is a local chain recurrent component of the uncontrolled system and (5.20) holds on $\hat{H}$, then again Corollary 4.7.6 in* [1] *implies the existence of control sets $\hat{D}^{U^\rho}$ containing $\hat{H}$ in the interiors with $\bigcap_{\rho>0} \hat{D}^{U^\rho} = \hat{H}$. It is known (see* [2]*) that the fundamental semigroup $\Lambda^{U^\rho}(D^{U^\rho}, p_0)$ does not contain a unity. We want to show that this is true for the inverse limit as well. For $\rho > 0$ small enough, given any trajectory of (5.19) starting at p_0 and lying in $\hat{D}^{U^\rho}$ one can associate to any $x(t)$ its orthogonal projection $\hat{x}(t)$ onto $\hat{H}$ (this can be proved directly or deduced from the well-known Tubular Neighborhood Theorem). Thus, it makes sense to consider the function that associates the arc length on $\hat{H}$ (measured in the natural trajectory direction) between the point p_0 and $\hat{x}(t)$ to t. Reducing ρ if necessary, we can assume that this function is strictly increasing. Now, topological considerations similar to those used in* [2, Example 4.8] *show that, for $\rho > 0$ small enough, $\Lambda^{U^\rho}(D^{U^\rho}, p_0)$ cannot contain idempotent elements. Consequently, by Proposition 2.7, in this situation $\Lambda(\hat{H}, p_0) = \overleftarrow{\lim_{U \in \mathrm{Co}_0}} \Lambda_U(D^U, 0)$ does not contain a unity.*

Clearly, the periodic solution in Example 5.7 and the homoclinic orbit together with the equilibrium in Example 5.6 are homeomorphic maximal chain transitive sets. However, the corresponding semigroups are not isomorphic, since one contains a unity, while the other does not. These systems are not conjugate illustrating Theorem 5.4.

Example 5.8. *Let $\hat{H}$ and $\Lambda(\hat{H}, p_0)$ be as in Example 5.7. Let γ_ρ be the map that associates to any $n \in \mathbb{N}$ the class of maps in $\Lambda^{U^\rho}(D^{U^\rho}, p_0)$ corresponding to the periodic trajectory $\hat{H}$ gone through n times. Topological considerations show that this map is injective. The discussion preceding Proposition 2.8 shows that there exists a unique homomorphism $\gamma : \mathbb{N} \to \hat{\Lambda}$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\gamma"] \arrow[dr, "\gamma_\rho"'] & \Lambda(\hat{H}, p_0) \arrow[r, "\lambda_{\rho'}"] \arrow[d, "\lambda_{\rho'}"] & \Lambda^{U^{\rho'}}(D^{U^{\rho'}}, p_0) \\
 & \Lambda^{U^\rho}(D^{U^\rho}, p_0) \arrow[ur, "\lambda_\rho^{\rho'}"'] &
\end{tikzcd}$$

An inspection of the above diagram shows that γ is actually an injection of $\mathbb{N}$ into $\Lambda(\hat{H}, p_0)$.

Similar considerations in the case of a homoclinic orbit H described in Example 5.6 show that here an injection of $\mathbb{N} \times \{0\}$ into the limit semigroup $\Lambda(H, p_0)$ exists.

Remark 5.9. *While the examples above show that the constructed semigroup gives some insight into the dynamics on simple, locally maximal chain transitive subsets, they also indicate that its actual computation from a given differential equation is a tremendous task. Notice, however, as shown by the* *discussion following Examples 5.6 and 5.7, that sometimes it is enough to distinguish some of its characteristics. Nevertheless, it remains an open research problem to see wether this construction (or similar ones) can be used to get insight into the behavior of more interesting systems.*

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