Covering space for monotonic homotopy of trajectories of control systems☆

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1. Introduction

The subject matter of this article is monotonic (or causal) homotopy between trajectories of control systems. This is a variant of the usual homotopy where two trajectories are considered to be homotopic if they can be deformed to each other continuously

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through trajectories. Equivalently, monotonic homotopy holds when the trajectories belong to the same path component of the space of all trajectories of the control system.

The study of this sort of homotopy is motivated by different sources. First in the control theoretic setting one is interested in understanding such complex objects like accessible sets, control sets, local control sets, etc. Of course, it is to be expected that topological invariants, adapted to the dynamics of the system, can be extremely helpful in getting at least rough descriptions of these sets. This was done by Colonius–Spadini [3], where monotonic fundamental semigroups of local control sets are defined and used to detect the existence of local control sets within control sets.

Also, in semigroup theory monotonic homotopy was considered by Lawson [7,8] (in a slight different setting than ours). The objective in these papers is to extend to Lie semigroups the classical construction of the universal covering groups.

Our objective in this paper is to construct, for monotonic homotopy, the analogue of the simply connected covering space of a topological space. In this regard our main result reads as follows: let Σ be a control system on the state space M (a finite-dimensional manifold). Fixing an initial point x_0 in M, we select a subset of "regular" trajectories and denote by $\Gamma(\Sigma, x_0)$ the set of monotonic homotopy classes of regular trajectories starting at x_0. Then we show that there exists a finite-dimensional manifold structure on $\Gamma(\Sigma, x_0)$ such that the end point mapping $\varepsilon : \Gamma(\Sigma, x_0) \to M$ is a local diffeomorphism in the sense that its differential is an isomorphism at every point of $\Gamma(\Sigma, x_0)$. The image of ε is contained in the interior $\operatorname{int}\mathcal{A}(x_0)$ of the accessible set from x_0, and is in fact $\operatorname{int}\mathcal{A}(x_0)$ if the Lie algebra rank condition holds.

In this case the mapping $\varepsilon : \Gamma(\Sigma, x_0) \to \operatorname{int}\mathcal{A}(x_0)$ is a close analogue of the classical simply connected covering space. In fact, since ε is a local diffeomorphism, we can lift Σ to a control system, say $\widehat{\Sigma}$, on $\Gamma(\Sigma, x_0)$. The trajectories of $\widehat{\Sigma}$ are mapped by ε onto the trajectories of Σ. Conversely, modulo some technical questions related to the fact that x_0 may not belong to $\operatorname{int}\mathcal{A}(x_0)$, we can lift trajectories of Σ to trajectories of $\widehat{\Sigma}$. Then, roughly speaking, we get the following results: (1) two trajectories of Σ are monotonically homotopic if and only if their liftings have the same end point. (2) If N is a manifold endowed with a control system $\widetilde{\Sigma}$ and $p : N \to \operatorname{int}\mathcal{A}(x_0)$ is a local diffeomorphism mapping $\widetilde{\Sigma}$ to Σ, then there exists a lifting mapping $f : \Gamma(\Sigma, x_0) \to N$ which relates $\widehat{\Sigma}$ and $\widetilde{\Sigma}$. The last property shows that $\Gamma(\Sigma, x_0)$ is universal in the same sense as the simply connected covering spaces.

Despite of these properties we stress that, in general, $\Gamma(\Sigma, x_0)$ is not the simply connected covering of $\operatorname{int}\mathcal{A}(x_0)$. Actually, it is not even true that $\varepsilon : \Gamma(\Sigma, x_0) \to \operatorname{int}\mathcal{A}(x_0)$ is a covering mapping. In fact, if ε is a covering then two trajectories which are homotopic (in the usual sense of the word) are also monotonically homotopic. However we exhibit an example of a system Σ having homotopic trajectories which are not monotonically homotopic.

The paper is organized as follows. After describing the set-up in Section 2, Section 3 discusses regular controls on which our constructions are based. In particular it is shown that normal controls are regular. Section 4 introduces the basic object of this paper, monotonic homotopies. As a preparation for their analysis, Section 5 proves basic properties of local diffeomorphisms for which we could not find an adequate reference.

Section 6 proves the manifold structure of the space of monotonic equivalence classes. The control system is lifted in Section 7 to this manifold, and in Section 8 a universality property is shown. Section 9 discusses local control sets and the fundamental semigroup; also the relation to coverings is noted. The final Section 10 presents an example where monotonic homotopy is not implied by homotopy.

2. Set-up

Let M be an n-dimensional connected smooth ($\mathcal{C}^{\infty}$) manifold. For topological purposes we assume that M is given with a Riemannian metric which induces a distance function d_R. We consider a finite-dimensional vector subspace E of the vector space (over $\mathbb{R}$) of smooth vector fields on M. In order to have a topology on E and on corresponding function spaces we assume that E is endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $\Sigma \subset E$ be a convex cone, which is assumed to be generating in E, that is, Σ is not contained in a proper subspace of E and hence with a norm in such a way that the inclusion of E into the space of vector fields is smooth. Although not essential to some of our results we will assume, once and for all, that the vector fields in Σ are forward complete. Also, we assume throughout the paper that Σ satisfies the Lie algebra rank condition, that is, $\mathcal{L}(x) = T_x M$ for all $x \in M$, where $\mathcal{L}$ denotes the smallest Lie algebra of vector fields containing Σ (or E).

Given $x \in M$, denote by $E(x)$ the subspace of the tangent space $T_x M$ obtained by evaluating at x the vector fields in E. The same way the evaluation map yields a convex cone $\Sigma(x) \in T_x M$.

By a trajectory of Σ we understand an absolutely continuous curve α in M with $\alpha'(t) \in \Sigma(\alpha(t))$. In principle a trajectory can be defined in an arbitrary interval $[0, T]$. We are mainly concerned with the geometrical properties of the trajectories, that is, with their traces. Hence we use the fact that Σ is a cone to reparametrize the trajectories and define them in $[0, 1]$. In fact, if $\alpha : [0, T] \to M$, $T > 0$, is a trajectory then the curve $\beta : [0, 1] \to M$, $\beta(t) = \alpha(Tt)$, satisfies $\beta'(t) = T\alpha(Tt) \in \Sigma(\beta(t))$ and thus is also a trajectory.

Denote by $\mathcal{E}$ the Banach space of bounded measurable functions $u : [0, 1] \to E$ endowed with the ess sup-norm $||\cdot||_{\infty}$, where the norm on E comes from the inner product. Let $\mathcal{U}$ be the convex cone formed by those functions $u \in \mathcal{E}$ which assume values in Σ. The assumption that Σ is a generating cone in E implies that $\mathcal{U}$ has non-empty interior in $\mathcal{E}$ (w.r.t. the sup-norm). We call the elements in $\mathcal{U}$ control functions of Σ. For a control function $u : [0, 1] \to \Sigma$ and an initial condition $x \in M$ the corresponding trajectory $\mathrm{trj}_x(u) : [0, 1] \to M$ is the solution of the differential equation $\dot{x} = u(t)(x)$ starting at x.

Apart from the norm (strong) topology it is sometimes convenient to endow $\mathcal{E}$ with the weak* topology, which is the weakest topology such that for all $y \in L_1([0, 1], E)$ the linear functional $u \mapsto \int_0^1 \langle y(t), u(t) \rangle \, dt$ is continuous (cf. Colonius–Kliemann [2]).

Let $T(\Sigma)$ denote the set of trajectories of Σ, $T(\Sigma, x)$ the set of trajectories starting at x and $T(\Sigma, x, y)$ the trajectories starting at x and ending at y. Also write $\mathcal{A}_{\Sigma}(x)$ or simply $\mathcal{A}(x)$ for the accessible set from x, that is, the set of end points of the trajectories $\mathrm{trj}_x(u)$, $u \in \mathcal{U}$. Equivalently, $\mathcal{A}(x)$ is the image of the map $e_x : \mathcal{U} \to M$ which associates to u the end point $\mathrm{trj}_x(u)(1)$ of its trajectory.

We denote the flow defined by the control u by ϕ_t^u (or simply ϕ_t if u is understood). Explicitly, $\phi_t^u(x) = \mathrm{trj}_x(u)(t)$. By the existence and uniqueness theory ϕ_t^u is a diffeomorphism between open subsets of M.

The set of trajectories is topologized with the $\mathcal{C}^1$-topology which is a metric space given by the distance

$$d_1(\alpha, \beta) = \sup_{t\in[0,1]} d_R(\alpha(t), \beta(t)) + \operatorname{ess}\sup_{t\in[0,1]} \left|\alpha'(t) - \beta'(t)\right|.$$

It is a well-known consequence of the continuous dependence of solutions on parameters that for any x the map

$$\mathrm{trj}_x : \mathcal{U} \longrightarrow T(\Sigma, x)$$

is continuous. Furthermore, with respect to the $\mathcal{C}^1$-topology on the set of trajectories the mapping trj_x is also an open mapping. Hence a subset $A \subset T(\Sigma, x)$ is open if and only if its pre-image $\mathrm{trj}_x^{-1}(A)$ is open in $\mathcal{U}$.

3. Regular controls

Given a fixed $x \in M$ we defined above the map trj_x which associates to a control $u \in \mathcal{U}$ the trajectory starting at x. We denote the end point of this trajectory by $e_x(u) = \mathrm{trj}_x(u)(1)$, so that we have the well defined evaluation map $e_x : \mathcal{U} \to M$. Note that this map can be defined in the whole Banach space $\mathcal{E}$ (in case the system is complete). From the usual theorems on dependence of solutions on parameters we have that e_x is differentiable.

Definition 3.1. A control function u is said to be *regular* at $x \in M$ if $u \in \operatorname{int}\mathcal{U}$ and the differential $d(e_x)_u$ of e_x at u is surjective. The set of regular controls at x is denoted by $\mathcal{R}_\Sigma(x)$. A trajectory α is regular at x if $\alpha = \mathrm{trj}_x(u)$ for some $u \in \mathcal{R}_\Sigma(x)$. The set of regular trajectories at x is denoted by $R(\Sigma, x)$, while the set of regular trajectories from x to $y \in M$ is denoted by $R(\Sigma, x, y)$.

We denote by $\mathcal{A}_R(\Sigma, x)$ the set of points attainable from x by regular controls. An application of the implicit function theorem (see e.g. Lang [6]) ensures that both $\mathcal{R}_\Sigma(x)$ and $\mathcal{A}_R(\Sigma, x)$ are open subsets. It will be proved below that these sets are not empty if the Lie algebra rank condition holds.

Given two controls $u, v : [0, 1] \to \Sigma$ in $\mathcal{U}$, their concatenation is the control $v * u$ defined by

$$(v * u)(t) = \begin{cases} u(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ v(2t-1), & \frac{1}{2} < t \leqslant 1. \end{cases}$$

Proposition 3.2. *Let u and v be controls in* $\operatorname{int} \mathcal{U}$.

(1) *Suppose that u is regular at x_0. Then $v * u$ is regular at x_0.*
(2) *If v is regular at the end point of* $\operatorname{trj}_{x_0}(u)$, *then $v * u$ is regular at x_0.*

Proof. Define the controls

$$u_1(t) = \begin{cases} u(2t), & 0 \leqslant t \leqslant 1/2 \\ 0, & 1/2 \leqslant t \leqslant 1 \end{cases} \quad \text{and} \quad v_1(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1/2, \\ v(2t-1), & 1/2 \leqslant t \leqslant 1 \end{cases}$$

and denote the flows of u_1, v_1 and $v * u$ by φ, ψ and Φ, respectively. For any $w \in \mathcal{E}$, the variation of parameter formula gives

$$(de)_{v*u}(w) = (d\Phi_1)_{x_0} \int_0^1 (d\Phi_t)_{x_0}^{-1} w(t)\, dt. \tag{1}$$

In order to have this formula in terms of u_1 and v_1 write $w \in \mathcal{E}$ as $w = w_1 + w_2$ where

$$w_1(t) = \begin{cases} w(2t), 0 \leqslant t \leqslant 1/2 \\ 0, 1/2 \leqslant t \leqslant 1 \end{cases} \quad \text{and} \quad w_2(t) = \begin{cases} 0, 0 \leqslant t \leqslant 1/2, \\ w(2t-1), 1/2 \leqslant t \leqslant 1. \end{cases}$$

If we write the integral in (1) as $\int_0^1 = \int_0^{1/2} + \int_{1/2}^1$ then a simple computation yields

$$\begin{aligned} &(de)_{v*u}(w) \\ &\quad = d(\psi_1)_{\varphi_1(x_0)}\, d(e_{x_0})_u (w_1) + d\left(\psi_1 \circ \varphi_1 \circ \psi_1^{-1}\right)_{\psi_1 \circ \varphi_1(x_0)} d\left(e_{\varphi_1(x_0)}\right)_v (w_2). \end{aligned}$$

Now suppose that u is regular. Then by choosing w so that $w_2 = 0$ we see that $(de)_{v*u}$ is surjective, proving the first part of the proposition. Analogously, the second part follows by choosing w such that $w_1 = 0$, concluding the proof. □

Now, we check that regularity is preserved under time reversal. Given a curve $\alpha : [0, 1] \to M$, we write $\alpha^-(t) = \alpha(1-t)$. If α is a trajectory of Σ then α^- is a trajectory of $-\Sigma$. In fact, if $\alpha = \operatorname{trj}_x(u)$ then $\alpha^- = \operatorname{trj}_y(-u)$ where y is the end point of α and $-u$ is a control of $-\Sigma$.

Proposition 3.3. *Suppose that $u \in \mathcal{R}_\Sigma(x)$ and put $y = e_x(u)$. Then $-u \in \mathcal{R}_{-\Sigma}(y)$. Equivalently, if $\alpha \in R(\Sigma, x, y)$ then $\alpha^- \in R(-\Sigma, y, x)$.*

Proof. Denote by φ and ψ the flows of u and $-u$, respectively. We have

$$d(e_y)_{-u}(w) = (d\psi_1)_y \int_0^1 \left(d\psi_t^{-1}\right)_{\psi_t(y)} w(t)\, dt \in T_x M.$$

The right-hand side is equal to $\int_0^1 d\left(\psi_1 \circ \psi_t^{-1}\right)_{\psi_t(y)} w(t)\, dt$. But $\psi_1 \circ \psi_t^{-1} = \varphi_{1-t}^{-1}$. Hence,

$$d\left(e_y\right)_{-u}(w) = \int_0^1 d\left(\varphi_{1-t}^{-1}\right)_{\varphi_{1-t}(x)} w(t)\, dt. \tag{2}$$

On the other hand,

$$d\left(e_x\right)_u(w) = d\left(\varphi_1\right)_x \int_0^1 d\left(\varphi_t^{-1}\right)_{\varphi_t(x)} w(t)\, dt. \tag{3}$$

Since the integrals in (2) and (3) are the same if $w(t)$ is replaced by $w(1-t)$, and $d\left(\varphi_1\right)_x$ is an isomorphism, it follows that $d\left(e_y\right)_{-u}$ is surjective if and only if $d\left(e_x\right)_u$ is surjective. □

Since we are assuming the Lie algebra rank condition we can construct a plenty supply of piecewise constant controls which are regular. In fact, it is well known that under the Lie algebra rank condition there are normal controls (in the sense of Sussmann [13]). On the other hand we check below that a normal control is regular, provided it belongs to the interior of $\mathcal{U}$. This shows the existence of regular controls.

In order to recall the notion of normal control let us denote by X_t the flow of the vector field X on M. If $X^1, \ldots, X^k$ are vector fields in Σ, we can form the function

$$\rho_x\left(t_1, \ldots, t_k\right) = X_{t_k}^k \circ \cdots \circ X_{t_1}^1(x)$$

with $x \in M$. Clearly, if $t_1, \ldots, t_k \geqslant 0$ then $\rho\left(t_1, \ldots, t_k\right)$ is the end point of a trajectory starting at x defined by a piecewise constant control. According to Sussmann [13] such a control is said to be normal (at x) if the rank of ρ at $\tau = (t_1, \ldots, t_k)$ is $n = \dim M$.

In order to establish the relation between normal and regular controls, let us fix once and for all the vector fields $X^1, \ldots, X^k$ in Σ. Let $\mathbb{R}_+$ be the set of strictly positive real numbers. Each $\tau = (t_1, \ldots, t_k) \in \mathbb{R}_+^k$ determines a piecewise constant control which assumes the value X^i in the interval $[T_{i-1}, T_i)$, where $T_i = t_1 + \cdots + t_i$ (with $t_0 = 0$). This control is defined in the interval $[0, T_k]$.

We reparametrize these piecewise constant controls through the following mappings:

(1) Put

$$\Delta = \{\sigma = (s_1, \ldots, s_k) \in \mathbb{R}_+^k : s_1 + \cdots + s_k = 1\}$$

for the standard simplex in $\mathbb{R}^k$, and let $\eta : \Delta \times \mathbb{R}_+ \to \mathbb{R}_+^k$ be defined by

$$\eta\left(\left(s_1, \ldots, s_k\right), T\right) = T\left(s_1, \ldots, s_k\right).$$

Then η is a diffeomorphism.

(2) Let $\mu : \Delta \times \mathbb{R}_+ \to \mathcal{U}$ be the mapping which associates to the pair $((s_1, \ldots, s_k), T)$ the piecewise constant control defined on [0, 1], whose value in the interval $[S_{j-1}, S_j]$ is TX^i, where $S_i = s_1 + \cdots + s_i$ and $S_0 = 0$.

From the very definition of these mappings we have the following commutative diagram, which gives a reparametrization of the controls defined by $\tau \in \mathbb{R}^k_+$.

$$\begin{array}{ccc} \Delta \times \mathbb{R}_+ & \stackrel{\eta}{\longrightarrow} & \mathbb{R}^k_+ \\ \mu \downarrow & & \downarrow \rho_x \\ \mathcal{U} & \underset{e_x}{\longrightarrow} & M \end{array} \tag{4}$$

With these notations at hand we can prove the following relation between the differentials of e_x and ρ_x.

Proposition 3.4. *Take $\xi \in \Delta \times \mathbb{R}_+$. Then the differential $d(e_x)_{\mu(\xi)}$ is surjective if ρ_x has rank $n = \dim M$ at $\eta(\xi)$.*

Proof. An easy computation shows that

$$\frac{\partial \rho_x}{\partial t_i}(\tau) = dX^k_{t_k} \circ \cdots \circ dX^{i+1}_{t_{i+1}} \left(X^i(z_i) \right),$$

where $z_i = X^i_{t_i} \circ \cdots \circ X^1_{t_1}(x)$. Clearly, these partial derivatives span the image of $\left(d\rho_x\right)_\tau$. On the other hand, for $\xi = ((s_1, \ldots, s_k), T)$ let φ_t, $t \in [0, 1]$, be the flow induced by the control $\mu(\xi)$.

Explicitly,

$$\varphi_t = TX^i_{t-S_{i-1}} \circ \cdots \circ TX^1_{t_1}$$

if $t \in [S_{i-1}, S_i]$, where $S_i = s_1 + \cdots + s_i$ and $S_0 = 0$. The variation of parameter formula gives

$$d(e_x)_{\eta(\xi)}(w) = \left(d\varphi_1\right)_x \int_0^1 \left(d\varphi_t\right)_x^{-1} w(t)\, dt$$

with $w \in \mathcal{E}$. In particular, take w such that $w(t) = 0$ if $t \notin [S_{i-1}, S_i]$ and

$$w(t) = X^i \qquad t \in [S_{i-1}, S_i].$$

Then using the expression for ϕ_t and the fact that $dX_t(X) = X$, for any vector field X, it follows that

$$d\,(e_x)_{\eta(\xi)}\,(w) = T s_i \frac{\partial \rho_x}{\partial t_i}\,(\tau)\,.$$

Therefore, the partial derivatives of ρ_x appear in the image of $d\,(e_x)_{\eta(\xi)}$ proving the claim. □

Remark. An alternative proof of the above proposition would be to show that μ is differentiable and then apply the chain rule to the commutative diagram (4). This would imply that the image of $d\,(e_x)_{\mu(\xi)}$ contains the image of $\left(d\rho_x\right)_{\eta(\xi)}$.

Proposition 3.5. *Under the Lie algebra rank condition the set of regular controls is not empty.*

Proof. By assumption, the convex cone Σ spans the finite-dimensional space E of vector fields. Thus also $\operatorname{int}\Sigma$ spans E and the Lie algebra spanned by the vector fields in $\operatorname{int}\Sigma$ coincides with the Lie algebra $\mathcal{L}$ spanned by E. Then repeating the usual proof that the Lie algebra rank condition implies accessibility (see e.g. Jurdjevic [5]), it follows that there are $X_1, \ldots, X_k$ in $\operatorname{int}\Sigma$ and $(t_1, \ldots, t_k) \in \mathbb{R}^k_+$, which define a normal control. This control belongs to $\operatorname{int}\mathcal{U}$, with respect to the sup-norm topology and it is regular by Proposition 3.4. □

Remark. In [13] it is proved that accessibility (even without eventually the Lie algebra rank condition) implies normal accessibility. This result combined with the other perturbation results of [13] may imply that there are regular controls under accessibility alone. Although the proof above uses that E is finite dimensional it might be true that this condition is not required.

Proposition 3.6. *Assume the Lie algebra rank condition. Then* $\mathcal{A}_R(\Sigma, x) = \operatorname{int}\mathcal{A}(x)$ *and* $\operatorname{cl}\mathcal{A}(x) = \operatorname{cl}(\operatorname{int}\mathcal{A}(x))$.

Proof. The latter equality is well known. Also, it is well known that any point in $\operatorname{int}\mathcal{A}(x)$ is reachable from x by a normal control. The proof that $\operatorname{int}\mathcal{A}(x) = \mathcal{A}_R(\Sigma, x)$ is analogous. □

4. Monotonic homotopy

Monotonic homotopy between trajectories of Σ is a homotopy linking continuously trajectories of Σ through trajectories. Of course, one can define such homotopies between arbitrary trajectories. However, we restrict our definition to regular trajectories, since much sharper results can be obtained in this framework. Recall that, for

$x, y \in M$ the set $R(\Sigma, x, y)$ of regular trajectories of Σ from x to y was endowed with the C^1-topology.

Definition 4.1. Two regular trajectories α and β are said to be monotonically homotopic ($\alpha \simeq_m \beta$) if their extremal points are equal, that is, for some $x, y \in M$, $\alpha, \beta \in R(\Sigma, x, y)$ and α and β belong to the same path component of $R(\Sigma, x, y)$.

This variant of the concept of homotopy appeared in the literature with different names (see Colonius–Spadini [3] and Lawson [7,8]). In view of that we use interchangeably the terms *monotonic homotopy*, *causal homotopy* or *dynamic homotopy*, in the sense of the above definition. In contrast we say *geometric homotopy* for the usual homotopy between curves.

It is clear that the relation of being monotonically homotopic is an equivalence relation. If we fix an initial condition $x \in M$ the set of equivalence classes of these trajectories in $R(\Sigma, x)$ is denoted by $\Gamma(\Sigma, x)$, that is,

$$\Gamma(\Sigma, x) = R(\Sigma, x) / \simeq_m . \tag{5}$$

Denote by $\pi : R(\Sigma, x) \to \Gamma(\Sigma, x)$ the canonical map which associates to α its monotonic homotopy class $[\alpha]$. Also, we write $\tau = \pi \circ \mathrm{trj}$ for the mapping which associates to a control function the monotonic homotopy class of its trajectory.

For later reference we state the following easy consequences of the definition of monotonic homotopy.

Proposition 4.2. *Let $\alpha_1, \alpha_2 \in R(\Sigma, x, y)$ and $\beta_1, \beta_2 \in R(\Sigma, y, z)$ with $x, y, z \in M$. Suppose that $\alpha_1 \simeq_m \alpha_2$ and $\beta_1 \simeq_m \beta_2$. Then, $\beta_1 * \alpha_1 \simeq_m \beta_2 * \alpha_2$.*

Proof. In fact, concatenating homotopies yields a homotopy between $\beta_1 * \alpha_1$ and $\beta_2 * \alpha_2$. □

Remark. We do not know whether the converse to the above proposition holds. However, we prove a partial converse in Lemma 7.3 below.

In the next proposition we let as before α^- be the curve obtained from α by reverting time. By Proposition 3.3, if $\alpha \in R(\Sigma, x, y)$ then $\alpha^- \in R(-\Sigma, y, x)$.

Proposition 4.3. *Let $\alpha_1, \alpha_2 \in R(\Sigma, x, y)$ be such that $\alpha_1 \simeq_m \alpha_2$. Then $\alpha_1^- \simeq_m \alpha_2^-$, for $-\Sigma$.*

Proof. A homotopy between α_1^- and α_2^- is obtained by reverting time of a homotopy between α_1 and α_2. □

5. Local diffeomorphisms

For convenience we shall recollect in this section known results about local diffeomorphisms between manifolds, which will be used later.

Let L and N be manifolds. By a local diffeomorphism we understand a differentiable mapping $f : L \to N$ such that df_x is bijective for any $x \in L$. Clearly, in this case for every $x \in L$ there are neighborhoods V of x and U of $f(x)$ such that f is a diffeomorphism between V and U. A special class of local diffeomorphisms are the differentiable coverings, which have many properties not shared by general local diffeomorphisms.

For our purposes we are interested in the continuous liftings to L of mappings into N. Although this can be done for coverings it is impossible in general. (For example take L to be the interval $(0, 3/2) \subset \mathbb{R}$ and let N be the circle $\mathbb{R}/\mathbb{Z}$. The natural projection $f : L \to N$ is a local diffeomorphism but not a covering, and the path which rounds the circle twice cannot be lifted continuously to L.)

However continuous liftings are possible locally and are unique over connected spaces whenever they exist.

Lemma 5.1. *Let $f : L \to N$ a surjective local diffeomorphism, and I a topological space. Let $\alpha : I \to N$ be a continuous mapping, and take $t_0 \in I$ and $y \in L$ with $f(y) = \alpha(t_0)$. Then there are a neighborhood U of t_0 and a unique mapping $\widetilde{\alpha} : U \to L$ such that $f \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(t_0) = y$. If I is connected and $\widetilde{\alpha}_1, \widetilde{\alpha}_2 : I \to L$ are such that $f \circ \widetilde{\alpha}_i = \alpha$ with $\widetilde{\alpha}_i(t_0) = y$, $i = 1, 2$, then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.*

Proof. Take a neighborhood V of y such that $f : V \to f(V)$ is a diffeomorphism. Then we can define $\widetilde{\alpha}$ locally around t_0 by $f^{-1} \circ \alpha$, where $f^{-1} : f(V) \to V$ is the local inverse of f. Clearly, this local lifting is defined uniquely. The uniqueness follows by noting that the set where $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$ is closed (by continuity) and open (by local uniqueness). □

In the sequel the above lemma will be used mainly to lift curves from N to L. On the other hand the next lemma is concerned with the lifting of homotopies between curves.

Lemma 5.2. *Let $f : L \to N$ be a surjective local diffeomorphism and take continuous curves $\alpha, \beta : [0, 1] \to N$ with $\alpha(0) = \beta(0)$. Let also $H : [0, 1] \times [0, 1] \to N$ be continuous such that $H(0, t) = \alpha(t)$ and $H(1, t) = \beta(t)$, $H(s, 0) = \alpha(0)$. Take $y \in L$ with $f(y) = \alpha(0)$ and suppose that for all $s \in [0, 1]$ the curve $t \mapsto H(s, t)$ lifts to a curve in L, say $\widetilde{H}(s, t)$ with $\widetilde{H}(s, 0) = y$. Then $(s, t) \mapsto \widetilde{H}(s, t)$ is continuous, and hence a homotopy between the liftings $\widetilde{H}(0, t)$ and $\widetilde{H}(1, t)$ of α and β, respectively.*

Proof. Take a local continuous lift of H around $(0, 0)$ and use uniqueness of the lifting of the paths to see that $\widetilde{H}$ is continuous at $(0, 0)$. Now fix $s \in [0, 1]$ and let m be the supremum of t such that H is continuous on (s, τ), $0 \leqslant \tau \leqslant t$. Let V be a neighborhood of $\widetilde{H}(s,m)$ such that $f : V \to f(V)$ is a diffeomorphism. Then in a neighborhood U of (s,m), H lifts continuously to a mapping $\widetilde{H}_1$, having image in V. But if τ is close enough to m, then $\widetilde{H}(s,\tau)$ belongs to V by the continuity $\tau \mapsto \widetilde{H}(s,\tau)$. Thus using the continuity of $\widetilde{H}$ at (s,τ) we conclude that $\widetilde{H}(\sigma,\tau)$ belongs to V if (σ,τ) is close enough to (s,m). Hence by uniqueness of the liftings of the curves $\tau \to \widetilde{H}(\sigma,\tau)$ we conclude that $\widetilde{H} = \widetilde{H}_1$ on U. This implies that $m = 1$, concluding the proof. □

From the preceding lemma we get that homotopic curves lift to curves with the same end points, if the homotopy also lifts.

Lemma 5.3. *Let α, β satisfy the conditions of the previous lemma, and suppose furthermore that $\alpha(1) = \beta(1)$, and that H is a homotopy fixing end points. Then the liftings $\widetilde{\alpha}$ and $\widetilde{\beta}$ starting at y of α and β, respectively, satisfy $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$.*

Proof. In fact, by continuity $\widetilde{H}(s,1)$ is constant as a function of s. □

Remark. In Lemma 5.2 we assumed the existence of $\widetilde{H}(s,t)$ to show its continuity. In general it is not possible to lift such homotopies. For an example, take $f : \mathbb{C}\setminus\{\pm 1\} \to \mathbb{C}$, $f(z) = z^3 - 3z$. It is easy to check that f is a surjective local diffeomorphism. In $\mathbb{C}$ every curve can be shrunk to a point. But since $\mathbb{C}\setminus\{\pm 1\}$ is not simply connected, there are homotopies in $\mathbb{C}$ which cannot be lifted to $\mathbb{C}\setminus\{\pm 1\}$.

Let $f : L \to N$ be a local diffeomorphism and X a vector field on N. Then we define $\widetilde{X}$ on L by $\widetilde{X}(x) = df^{-1}(X(f(x)))$, where f^{-1} is a local inverse of f around x. It follows that the mapping $X \mapsto \widetilde{X}$ is injective and f maps trajectories of $\widetilde{X}$ into trajectories of X. Conversely, if α is a trajectory of X and $\widetilde{\alpha}$ is a curve in L with $f(\widetilde{\alpha}) = \alpha$ then $\widetilde{\alpha}$ is a trajectory of $\widetilde{X}$. However, it is not true that trajectories of X can be entirely lifted to trajectories of $\widetilde{X}$ (see, for example, the local diffeomorphism $(0, 3/2) \to \mathbb{R}/\mathbb{Z}$, mentioned above).

Given a control system Σ if we lift the vector space E to $\widetilde{E}$ we get a control system $\widetilde{\Sigma}$ on L such that both Σ and E are in bijection with $\widetilde{\Sigma}$ and $\widetilde{E}$, respectively. Because of these bijections, the control functions of Σ, are also control functions of $\widetilde{\Sigma}$. In the sequel we use always the same control space $\mathcal{U}$ for systems related by local diffeomorphisms. Clearly, for $u \in \mathcal{U}$ the corresponding trajectories of $\widetilde{\Sigma}$ are mapped into trajectories of Σ. In other words, if $f(y_0) = x_0$ then $\mathrm{trj}_{x_0} = f \circ \widetilde{\mathrm{trj}}_{y_0}$, with $\widetilde{\mathrm{trj}}_{y_0}(u)$ standing for the trajectory of $\widetilde{\Sigma}$. This equality implies immediately the following statement.

Proposition 5.4. *A control u is regular at $z \in L$ (w.r.t. $\widetilde{\Sigma}$) if and only if it is regular at $f(z)$ (w.r.t. Σ).*

For systems related by local diffeomorphisms we introduce the following convenient terminology.

Definition 5.5. Let Σ_1 and Σ_2 be control systems evolving on M_1 and M_2, respectively. We say that a mapping $f : M_1 \to M_2$ is a *control mapping* between Σ_1 and Σ_2 if

f is a local diffeomorphism and $df(\Sigma_1) = \Sigma_2$. We say that the control mapping f is a *control covering* if it is surjective.

6. Manifold structure of $\Gamma(\Sigma, x_0)$

The purpose of this section is to construct a manifold structure on the space $\Gamma(\Sigma, x_0)$ defined in (5). As mentioned before we assume that Σ satisfies the Lie algebra rank condition at every $x \in M$.

Theorem 6.1. *The space of monotonic homotopy classes $\Gamma(\Sigma, x_0)$ has a smooth manifold structure of dimension $n = \dim M$. The end point mapping*

$$\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0) \subset M, \qquad [\gamma] \mapsto \gamma(1),$$

is a local diffeomorphism.

For the construction of the manifold structure on $\Gamma(\Sigma, x_0)$ we use the following well-known way of constructing a differentiable manifold:

Proposition 6.2. *Let X be a set and $\phi_i : W_i \to X$ a collection of mappings with W_i open subsets of $\mathbb{R}^n$. Suppose that*

(1) *Each ϕ_i is a bijection between W_i and its image.*
(2) $X = \bigcup_i \phi_i(W_i)$.
(3) *If i, j are such that $C_{ij} = \phi_i(W_i) \cap \phi_j(W_j) \neq \emptyset$, then the set $\phi_i^{-1}(C_{ij}) \subset W_i$ is open and the mapping $\phi_j^{-1} \circ \phi_i : \phi_i^{-1}(C_{ij}) \to W_j$ is smooth.*

Then (W_i, ϕ_i) defines an atlas for a unique manifold structure on X. This structure carries implicitly a topology on X.

Proof. This is the definition of manifold in many textbooks, where the topology is not defined in advance. $\square$

We define an atlas for the differentiable structure on $\Gamma(\Sigma, x_0)$ through the map e_{x_0}. Since x_0 is fixed in the discussion to follow we suppress the subscripts and write simply e, trj, etc.

Let $x = e(u)$ for the end point of the regular trajectory defined by u. By definition the rank of e at u equals the dimension of M, so that by the implicit function theorem there are open sets $U \subset \mathcal{E}$, $V \subset \ker(de_u)$ and $W \subset \mathbb{R}^n$ such that U is diffeomorphic to $V \times W$ and e restricted to U is equivalent to the projection $V \times W \to W$ (see e.g. Lang [6]). Before proceeding let us remark that the implicit function theorem applies here, because M is finite dimensional and u is regular so that the closed subspace $\ker(de_u)$ is finite codimensional, and hence splits. In view of the diffeomorphisms $V \times W \to U$, we usually identify neighborhoods U in $\mathcal{E}$ with $V \times W$. For fixing ideas let us suppose that $0 \in V$ and u identifies to a point in the slice $\{0\} \times W$, which in turn identifies with W. Then we shall look at W either as an n-dimensional submanifold of U (identified with $\{0\} \times W$) or as an open subset of $\mathbb{R}^n$. We call such W a cross-section of e at u. Note that since $u \in \operatorname{int}\mathcal{U}$, we can shrink U and suppose that $U \subset \operatorname{int}\mathcal{U}$, and that every $v \in U$ is regular.

Given a cross-section W of e at u, the map $\operatorname{trj} : W \to R(\Sigma, x_0)$ which associates to a control $v \in W$ its corresponding trajectory is continuous. Also, it is injective because the end points of the trajectories defined by $v_1 \neq v_2$ in W are different. For the same reason, if we compose trj with the canonical projection $\pi : R(\Sigma, x_0) \to \Gamma(\Sigma, x_0) = R(\Sigma, x_0) / \simeq$, we obtain a one-to-one mapping

$$\psi = \pi \circ \operatorname{trj} : W \longrightarrow \Gamma(\Sigma, x_0),$$

and hence a bijection onto its image.

Our objective is to prove that the collection of bijective mappings $\psi : W \to \psi(W)$ with W running through the cross-sections at every $u \in \mathcal{R}(x_0)$, define an atlas for a differentiable structure on $\Gamma(\Sigma, x_0)$. This is achieved if we check the other conditions of Proposition 6.2, namely

(1) the images $\psi(W)$ cover $\Gamma(\Sigma, x_0)$ and
(2) the transition mappings $\psi_2^{-1} \circ \psi_1$ are differentiable (and have open domains).

By the very definition of $\Gamma(\Sigma, x_0)$ as equivalence classes of regular trajectories it is immediate that any class in $\Gamma(\Sigma, x_0)$ belongs to some $\psi(W)$, thus the first condition follows.

For the differentiability of the transition mappings let $\left(\psi_i(W_i), \psi_i\right)$, $i = 1, 2$, be local charts with $C = \psi_1(W_1) \bigcap \psi_2(W_2) \neq \emptyset$.

Take a class $\xi \in C$ and let $v_i \in W_i$ be such that $\psi_i(v_i) = \xi$. Viewing W_i as subsets of $\mathcal{U}$ we have by definition that the trajectories $\operatorname{trj}(v_i)$ are equivalent. Hence they have the same end point in M, which we denote by x. Let e_i, $i = 1, 2$, be the restriction of e to W_i. Since $e_i : W_i \to e(W_i) \in M$ is a diffeomorphism, we can shrink both W_i, $i = 1, 2$, and suppose that there exists an open set $N \subset M$ such that $e_i : W_i \to N$ are diffeomorphisms.

We claim that $\psi_2^{-1} \circ \psi_1 = e_2^{-1} \circ e_1$. In fact, given $v_1 \in \psi_1^{-1}(C) \subset W_1$, $\psi_1(v_1)$ is the monotonic homotopy class of $\operatorname{trj}(v_1)$, and $\psi_2^{-1} \circ \psi_1(v_1) = v_2$ where $\operatorname{trj}(v_2) \simeq \operatorname{trj}(v_1)$. In particular, the end points of $\operatorname{trj}(v_2)$ and $\operatorname{trj}(v_1)$ coincide, that is, $e_1(v_1) = e_2(v_2)$ and hence $v_2 = e_2^{-1} \circ e_1(v_1)$ showing the claim. From $\psi_2^{-1} \circ \psi_1 = e_2^{-1} \circ e_1$ the differentiability of the transition map follows at once, concluding the construction of the manifold structure in $\Gamma(\Sigma, x_0)$.

It remains to show that ε is a local diffeomorphism. In fact, keeping the notation in the construction let $\psi : W \to \psi(W)$ be a chart for the differentiable structure. We have the composition

$$\psi(W) \subset \Gamma(\Sigma, x_0) \xrightarrow{\psi^{-1}} W \xrightarrow{e_{|W}} N \subset M,$$

with $\varepsilon = \left(e_{|W}\right) \circ \psi^{-1}$. Thus ε is a diffeomorphism between the open sets $\psi(W)$ and N, proving the statement. This concludes the proof of Theorem 6.1. □

The atlas built on $\Gamma(\Sigma, x_0)$ provides this set with the manifold topology for which the charts are homeomorphisms. On the other hand, the set of regular controls $\mathcal{R}_\Sigma(x_0)$ is endowed with both the strong and the weak* topologies. In the next statement we establish the continuity properties of the mapping $\tau = \pi \circ \mathrm{trj} : \mathcal{R}_\Sigma(x_0) \to \Gamma(\Sigma, x_0)$ associating to a control function the monotonic homotopy class of its trajectory.

Proposition 6.3. *The mapping τ is continuous with respect to the weak* topology* (*and hence w.r.t. the strong topology*). *Also*, *τ is an open mapping w.r.t. the strong topology* (*and hence w.r.t. the weak* topology*).

Proof. For the continuity observe that a weak* compact set is bounded. Hence, by finite dimensionality of E, uniform local Lipschitz continuity holds for the values of control functions in a weak* compact subset of $\mathcal{E} = L_\infty(\mathbb{R}, E)$. Then the usual Gronwall inequality argument (see e.g. Sontag [12]) shows that trj is continuous w.r.t. the weak* topology on the controls and the uniform convergence topology on trajectories. In particular the evaluation mapping e is continuous in the weak* topology. Now, let $\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0)$ be the local diffeomorphism of Theorem 6.1. Then $e = \varepsilon \circ \tau$ and continuity of τ follows, because locally $\tau = \varepsilon^{-1} \circ e$ where ε^{-1} stands for a local inverse of ε.

The fact that τ is open is an immediate consequence of the definition of the charts by means of the implicit function theorem, as performed above. □

This same proof applies to the mapping $\pi : R(\Sigma, x_0) \to \Gamma(\Sigma, x_0)$ defined on the regular trajectories.

Proposition 6.4. *The mapping $\pi : R(\Sigma, x_0) \to \Gamma(\Sigma, x_0)$ is continuous with respect to the C^0* (*and hence w.r.t. the C^1 topology*). *Also*, *π is an open mapping in the C^1 topology* (*and hence w.r.t. the C^0 topology*).

Remark. The set $\Gamma(\Sigma, x_0)$ can be naturally endowed with the quotient topology where a subset $A \subset \Gamma(\Sigma, x_0)$ is open if and only if its pre-image $\pi^{-1}(A)$ is open in $R(\Sigma, x_0)$ (w.r.t. the C^1 topology). Since π is both continuous and open with respect to the manifold topology, it follows that this topology coincides with the quotient topology.

Next we derive some properties of the topology of $\Gamma(\Sigma, x_0)$.

Proposition 6.5. *The topology of the manifold $\Gamma(\Sigma, x_0)$ is paracompact and Hausdorff.*

Proof. The unit ball in the space $\mathcal{E}$ endowed with the weak* topology is compact metrizable, hence separable. The same is true for the intersection with $\mathcal{U}$, hence also $\mathcal{U}$ is separable. Furthermore, suppose that the differential $d(e_x)_u$ of the end point map has full rank at u. Thus there are $w_1, \ldots, w_n$ such that

$$d(e_x)_u w_1, \ldots, d(e_x)_u w_n$$

are linearly independent. Then for any sequence (u_k) converging in the weak* sense to u and k large enough, also

$$d(e_x)_{u_k} w_1, \ldots, d(e_x)_{u_k} w_n$$

are linearly independent. Thus for all controls $u \in \mathcal{U}$ in a weak* neighborhood of a regular control, the derivative is surjective. In particular, there is a countable subset of regular controls (u_k) of $\mathcal{U}$ approximating every regular control (clearly, the controls u_k may be taken with values in the interior of $\mathcal{U}$). By Proposition 6.3 we have continuity of τ with respect to the weak* topology. Then the continuous image $\{\tau(u_k)\}_{k\in\mathbb{N}}$ is also dense. Since separable manifolds are paracompact, it follows that $\Gamma(\Sigma, x_0)$ is paracompact. The Hausdorff property follows at once from Theorem 6.1 combined with the following lemma. $\square$

Lemma 6.6. *Let L and N be differentiable manifolds and $f : L \to N$ a local diffeomorphism. Then L is Hausdorff if N is Hausdorff.*

Proof. Take $x \neq y \in L$. If $f(x) \neq f(y)$ choose open sets $f(x) \in U_1$ and $f(y) \in U_2$ with $U_1 \cap U_2 = \emptyset$. Then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ separate x from y. Thus suppose that $f(x) = f(y)$. Since L is locally Euclidean it is enough to show the existence of an open set $y \in V$ that does not contain x in its closure. For this choose $V \subset L$ with $y \in V$ so that $f : V \to f(V)$ is a diffeomorphism and suppose that there exists a sequence $x_k \in V$ with $x_k \to x$. Then $f(x_k) \to f(x) = f(y)$. But the restriction of f to V is a diffeomorphism. Hence $x_k \to y$ contradicting the assumption that $x \neq y$. Hence $x \notin \operatorname{cl} V$, concluding the proof. $\square$

7. Lifting Σ to $\Gamma(\Sigma, x_0)$

By Theorem 6.1 the end point map $\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0)$ is a local diffeomorphism. Hence, the restriction of Σ to $\mathcal{A}_R(\Sigma, x_0)$ can be lifted to $\Gamma(\Sigma, x_0)$. We denote the lifted system by $\widehat{\Sigma}$. Accordingly for a vector field X on $\mathcal{A}_R(\Sigma, x_0)$ we write $\widehat{X}$ for its lifting to $\Gamma(\Sigma, x_0)$. Also, we let $\widehat{\mathrm{trj}}_y(u)$ be the trajectory of $\widehat{\Sigma}$ corresponding to the control u and starting at $y \in \Gamma(\Sigma, x_0)$.

The purpose of this section is to study $\widehat{\Sigma}$ and relate its properties to the monotonic homotopy of trajectories of Σ. Here and in the next section we keep our constructions as close as possible to the classical case. However, we must cope with the fact that in general x_0 is not in $\mathcal{A}_R(\Sigma, x_0)$, so that we must take care with the initial point of a lifted curve. This will cause most of the technical difficulties in our proofs.

The first objective is to prove that $\widehat{\Sigma}$ is forward complete if this happens to Σ. Take $y_0 \in \Gamma(\Sigma, x_0)$, a control $u \in \mathcal{U}$ and put $\widehat{\alpha} = \widehat{\mathrm{trj}}_{y_0}(u)$. We must check that $\widehat{\alpha}$ is defined in the whole interval [0, 1]. For this put $z_0 = \varepsilon(y_0) \in \mathcal{A}_R(\Sigma, x_0)$ and let $\alpha = \mathrm{trj}_{z_0}(u)$ be the trajectory of Σ starting at z_0. By assumption Σ is forward complete, so that α extends to [0, 1]. Also, $\widehat{\alpha}$ is a lifting of α. Thus forward completeness follows from Lemma 5.1 if we check that α lifts completely to $\Gamma(\Sigma, x_0)$.

We construct explicitly the lifting of α as follows: Denote by $\overline{\alpha}$ the path in the space of trajectories which is defined by

$$\overline{\alpha}(s)(t) = \alpha(st), \quad s, t \in [0, 1].$$

Clearly $\overline{\alpha}$ is continuous with respect to C_0 topology and for each s, $\overline{\alpha}(s)$ is the piece of α in $[0, s]$.

Let us choose a representative β of $y_0 \in \Gamma(\Sigma, x_0)$. The end point of β is z_0, so that for each $s \in [0, 1]$ we can perform the concatenation $\overline{\alpha}(s) * \beta$. Proposition 3.2 implies that $\overline{\alpha}(s) * \beta$ belongs to $\mathcal{R}_\Sigma(x_0)$. Hence it makes sense to take its class $[\overline{\alpha}(s) * \beta]$, defining the curve $s \mapsto [\overline{\alpha}(s) * \beta]$ in $\Gamma(\Sigma, x_0)$.

Proposition 7.1. *Keep the above notations. Then* $\widehat{\alpha}(s) = [\overline{\alpha}(s) * \beta]$, $s \in [0, 1]$. *In particular, the end point of* $[\overline{\alpha}(s) * \beta]$ *is the class of* $\alpha * \beta$.

Proof. Note that the end point of $s \mapsto \overline{\alpha}(s) * \beta$ is $\alpha(s)$. Hence, by definition of ε, we have $\varepsilon[\overline{\alpha}(s) * \beta] = \alpha(s)$ for all $s \in [0, 1]$. Now, $\alpha(0) * \beta = \beta$, so that $[\overline{\alpha}(0) * \beta] = y_0$. Hence, $s \mapsto [\overline{\alpha}(s) * \beta]$ is the unique lifting of α starting at y_0, showing the claim. □

By the above discussion this proposition shows immediately that $\widehat{\Sigma}$ is forward complete. For later reference we record this fact.

Proposition 7.2. *If* Σ *is forward complete, the lifted system* $\widehat{\Sigma}$ *is forward complete.*

A well-known fact in the theory of covering spaces states that two curves in a space M with the same initial and end points are homotopic if and only if their liftings to the simply connected covering space $\widetilde{M}$ have the same end point if the initial points coincide.

Next we prove an analogous result in the context of monotonic homotopy. We must take care of the fact that in general trajectories starting at x_0 (even the regular ones) are not entirely contained in $\mathcal{A}_R(\Sigma, x_0)$. For example, consider the system Σ in $\mathbb{R}^2$ spanned by the basic vector fields $\partial/\partial x$ and $\partial/\partial y$. Then $\mathcal{A}_R(\Sigma, 0) = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ and the evaluation map $\varepsilon : \Gamma(\Sigma, 0) \to \mathcal{A}_R(\Sigma, 0)$ is a (global) diffeomorphism. Hence, a piecewise constant normal trajectory which stays for some time in one of the axis is not contained in $\mathcal{A}_R(\Sigma, 0)$.

Thus we do not have in advance liftings to $\Gamma(\Sigma, x_0)$ of trajectories of Σ. In order to avoid this problem we consider the following situation which is enough to relate liftings to monotonic homotopy: fixing x_0, take $z_0 \in \mathcal{A}_R(\Sigma, x_0)$. Then we shall prove that two regular trajectories α_1 and α_2 (starting at z_0 and having the same end point) are monotonically homotopic if and only if the end points of their liftings to $\Gamma(\Sigma, x_0)$ (starting at the same class y_0) are equal. Actually we shall prove a stronger result namely that $\Gamma(\Sigma, z_0)$ is an open submanifold of $\Gamma(\Sigma, x_0)$ which is diffeomorphic to $\mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$.

The proof of this result requires the following partial converse of Proposition 4.2.

Lemma 7.3. *Take trajectories $\beta_i \in \mathcal{R}_\Sigma(x_0)$, $i = 1, 2$, with the same end point $z_0 \in \mathcal{A}_R(\Sigma, x_0)$. Let α be a trajectory of Σ starting at z_0. Then $\beta_1 \simeq_m \beta_2$ if $\alpha * \beta_1 \simeq_m \alpha * \beta_2$.*

Proof. Suppose by contradiction that β_1 is not homotopic to β_2, that is, $[\beta_1] \neq [\beta_2]$. For $I = 1, 2$ denote by γ_i the liftings of α starting at $[\beta_i]$, respectively. Then $\gamma_i = \widehat{\mathrm{trj}}_{[\beta_i]}(u)$, $i = 1, 2$, where u is a control function defining α. Since $[\beta_1] \neq [\beta_2]$, it follows that the end points of γ_1 and γ_2 are different, by the uniqueness of the liftings (see Lemma 5.1). But by Proposition 7.1 the end point of γ_i is the class $[\alpha * \beta_i]$, $i = 1, 2$, showing that $\beta_1 \simeq_m \beta_2$ if $\alpha * \beta_1 \simeq_m \alpha * \beta_2$. □

By reverting time we get an analogous relation between monotonic homotopy and concatenations on the right.

Corollary 7.4. *Let β be a regular trajectory starting at x_0 and having end point at $z_0 \in \mathcal{A}_R(\Sigma, x_0)$. Then two regular trajectories α_i, $i = 1, 2$, starting at z_0 and having the same end point are monotonically homotopic if and only if $\alpha_1 * \beta \simeq_m \alpha_2 * \beta$.*

Proof. Follows immediately from the previous lemma and the fact that monotonic homotopy and regularity are maintained under time reversal of the trajectories. □

Now we can relate $\Gamma(\Sigma, z_0)$ to $\Gamma(\Sigma, x_0)$, $z_0 \in \mathcal{A}_R(\Sigma, x_0)$. Fix a trajectory β from x_0 to z_0. Then the concatenation $\alpha \mapsto \alpha * \beta$ maps $\mathcal{R}_\Sigma(z_0)$ into $\mathcal{R}_\Sigma(x_0)$. By Corollary 7.4, $\alpha_1 * \beta \simeq \alpha_2 * \beta$ if $\alpha_1 \simeq \alpha_2$. Hence, we have a well-defined map $I_\beta : \Gamma(\Sigma, z_0) \to \Gamma(\Sigma, x_0)$, $I_\beta[\alpha] = [\alpha * \beta]$. Again by Corollary 7.4, $\alpha_1 \simeq \alpha_2$ if $\alpha_1 * \beta \simeq \alpha_2 * \beta$, which means that I_β is injective.

Proposition 7.5. *With the notations as above let β be regular. Then image of I_β is $\mathcal{A}_R\left(\widehat{\Sigma}, [\beta]\right)$ and $I_\beta : \Gamma(\Sigma, z_0) \to \mathcal{A}_R\left(\widehat{\Sigma}, [\beta]\right)$ is a diffeomorphism onto its image. Furthermore, if β_1 and β_2 are regular trajectories from x_0 to z_0 then $I_{\beta_1} = I_{\beta_2}$ if and only if $\beta_1 \simeq_m \beta_2$.*

Proof. It was checked before that the liftings of trajectories of Σ are trajectories of $\widehat{\Sigma}$. Hence, the image of I_β is contained in $\mathcal{A}_R\left(\widehat{\Sigma}, [\beta]\right)$. Conversely, I_β is onto $\mathcal{A}_R\left(\widehat{\Sigma}, [\beta]\right)$ because trajectories of $\widehat{\Sigma}$ are projected into trajectories of Σ. Since I_β is injective, it follows that $I_\beta : \Gamma(\Sigma, z_0) \to \mathcal{A}_R\left(\widehat{\Sigma}, [\beta]\right)$ is a bijection. Now, by Proposition 7.1, $I_\beta[\alpha]$ is the end point of the lifting $\widehat{\alpha}$, so that locally $I_\beta = \varepsilon_{x_0}^{-1} \circ \varepsilon_{z_0}$ where $\varepsilon_{x_0}^{-1}$ is a local inverse of the end point map $\varepsilon_{x_0} : \Gamma(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0)$. Hence I_β is differentiable and its differential is an isomorphism at every point, showing that it is a diffeomorphism. □

From this proposition it follows at once that trajectories of Σ starting at z_0 and having the same end point are monotonically homotopic if and only if their liftings have the same end point. For later reference we state this fact.

Corollary 7.6. *Let $z_0 \in \mathcal{A}_R(\Sigma, x_0)$ and fix $y_0 \in \varepsilon_{x_0}^{-1}\{z_0\}$. Let α_1 and α_2 be regular trajectories of Σ starting at z_0 and having the same end point. Denote by $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$, respectively, their liftings to $\Gamma(\Sigma, x_0)$ starting at y_0. Then $\alpha_1 \simeq_m \alpha_2$ if and only if $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ have the same end point.*

Now we look at monotonic homotopy for trajectories in $\Gamma(\Sigma, x_0)$.

Proposition 7.7. *Let δ_1 and δ_2 be trajectories of $\widehat{\Sigma}$ in $\Gamma(\Sigma, x_0)$ having the same initial point $y_0 \in \Gamma(\Sigma, x_0)$. Then δ_1 and δ_2 are monotonically homotopic in $\Gamma(\Sigma, x_0)$ if and only if they have the same end point.*

Proof. The trajectories of Σ, $\alpha_i = \varepsilon(\delta_i)$ have the same initial and end points in $\mathcal{A}_R(\Sigma, x_0)$, and their liftings are δ_1 and δ_2, respectively. By the previous corollary $\alpha_1 \simeq \alpha_2$, hence there is a homotopy by trajectories h_t linking them. For each t the corresponding trajectory lifts to a trajectory $\widehat{h_t}$ of $\widehat{\Sigma}$ starting at y_0. Note that $\widehat{h_t}$ is continuous by Lemma 5.2. Using again the previous corollary, it follows that the end point of each $\widehat{h_t}$ is the same as of δ_i. Hence, $\widehat{h_t}$ is a homotopy linking δ_1 and δ_2, concluding the proof. □

The above proposition shows that $\Gamma(\Sigma, x_0)$ is simply connected in the sense that trajectories with the same initial and end points are monotonically homotopic. Alternatively, the covering construction for $\widehat{\Sigma}$ does not provide new manifolds:

Corollary 7.8. *For any $y_0 \in \Gamma(\Sigma, x_0)$ the space $\Gamma\left(\widehat{\Sigma}, y_0\right)$ coincides with $\mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$ and hence with $\Gamma(\Sigma, z_0)$ if $z_0 = \varepsilon_{x_0}(y_0)$.*

Proof. In fact, regular trajectories starting at y_0 are monotonically homotopic if and only if they have the same end point. □

We conclude this section with a discussion about the topology used for the monotonic homotopy. According to our definition two trajectories are monotonically homotopic if they belong to the same path component of $R(\Sigma, x, y)$, which was endowed with the C^1-topology. Let us consider instead the C^0 topology. It is clear that two trajectories which are C^1 monotonically homotopic are also C^0 monotonically homotopic, since a C^1 continuous path is also C^0 continuous (the C^0 topology is weaker than the C^1). Next we shall apply the lifting results of this section to see that the C^0 topology yields the same monotonic homotopy classes.

Proposition 7.9. *Let α and β be regular trajectories in $R(\Sigma, x, z)$ and suppose that there exists a path ξ in $R(\Sigma, x, z)$ which links α to β and is C^0 continuous. Then α and β are C^1 monotonically homotopic.*

Proof. Take x_0 with $x \in \mathcal{A}_R(\Sigma, x_0)$, so that $\Gamma(\Sigma, x)$ equals $\mathcal{A}_R\left(\widehat{\Sigma}, y\right)$ for any $y \in \varepsilon_{x_0}^{-1}(x)$. Also, put $H(s,t) = \xi(s)(t)$. By Lemma 5.2, H lifts to a homotopy $\widetilde{H}$ in $\mathcal{A}_R\left(\widehat{\Sigma}, y\right)$, because the curves $t \mapsto H(s,t)$ are trajectories so that they lift to $\mathcal{A}_R\left(\widehat{\Sigma}, y\right)$. The liftings $\widehat{\alpha}$ and $\widehat{\beta}$, starting at y, are given by $\widehat{\alpha}(t) = \widetilde{H}(0,1)$ and $\widehat{\beta}(t) = \widetilde{H}(1,t)$. Since $\widetilde{H}$ is a lifting of H, it follows that $\widehat{\alpha}$ and $\widehat{\beta}$ have the same end point. Therefore, by Corollary 7.6, α and β are C^1 monotonically homotopic. $\square$

8. Universal property

In this section we consider a (surjective) control covering $\pi : N \to \mathcal{A}_R(\Sigma, x_0)$ between a system $\widetilde{\Sigma}$ on N and Σ (or rather its restriction to $\mathcal{A}_R(\Sigma, x_0)$). Our objective is to prove the existence of a control mapping $f : \Gamma(\Sigma, x_0) \to N$ between $\widehat{\Sigma}$ and $\widetilde{\Sigma}$. This construction is the analogue of the classical one which gives the covering spaces from the simply connected covering. We note however that, contrary to the classical case, the mapping f is not in general surjective, that is, it is not a control covering. This is due to the lack of controllability of $\widetilde{\Sigma}$.

Throughout this section we assume that the system satisfies the Lie algebra rank condition.

A natural way to define $f : \Gamma(\Sigma, x_0) \to N$ would be to take a regular trajectory starting at x_0 and lift it to both $\Gamma(\Sigma, x_0)$ and N obtaining f by comparing the two liftings. To perform this construction trajectories must belong to $\mathcal{A}_R(\Sigma, x_0)$, which is in general not true. To overcome this problem we lift trajectories starting at points $z_0 \in \mathcal{A}_R(\Sigma, x_0)$, getting mappings defined on $\Gamma(\Sigma, z_0)$. Then we extend these mappings to the whole $\Gamma(\Sigma, x_0)$.

We assume throughout that the system $\widetilde{\Sigma}$ on N is forward complete. Under this condition any trajectory α of Σ lifts uniquely to a trajectory of $\widetilde{\Sigma}$ as soon as an initial point y_0 is prescribed. In fact, if u is a control defining α then $\widetilde{\alpha} = \widetilde{\mathrm{trj}}_{y_0}(u)$ is such lifting, where $\widetilde{\mathrm{trj}}_{y_0}(u)$ denotes the trajectory of $\widetilde{\Sigma}$ corresponding to u starting at y_0. In the sequel we use freely these liftings.

Our approach requires a curve linking x_0 to z_0 which is entirely contained in $\mathcal{A}_R(\Sigma, x_0)$ except possibly for the initial point x_0. Hence we start by building such curve backwards as follows.

Lemma 8.1. *Assume the system satisfies the Lie algebra rank condition, and take $x_0 \in M$ and $z_0 \in \mathcal{A}_R(\Sigma, x_0)$. Then there exists a sequence z_n in $\mathcal{A}_R(\Sigma, x_0)$ with $\lim z_n = x_0$ and such that $z_m \in \mathcal{A}_R(\Sigma, z_n)$ if $m < n$.*

Proof. We construct the sequence inductively, starting from z_0. First choose a sequence of open neighborhoods U_n, $n \geqslant 1$, of x_0 such that $\{x_0\} = \bigcap_{n \geqslant 1} U_n$. Now define $z_1 \in U_1 \cap \mathcal{A}_R(\Sigma, x_0)$ as follows: Take a control function $u \in \mathcal{R}_\Sigma(x_0)$ such that the end point of the corresponding trajectory $e_{x_0}(u) = z_0$. By reverting time we see that $-u$ is a regular control at z_0 for $-\Sigma$. Thus if we write $e_{z_0}^{-}$ for the map which associates a control in $-\Sigma$ to the end point of the trajectory starting at z_0, it follows that its image $\mathrm{im}\, e^{-}_{z_0}$ covers a neighborhood of x_0. Thus $\mathcal{A}_R(-\Sigma, z_0) \cap U_1 \cap \mathcal{A}_R(\Sigma, x_0)$ is not empty. Any z in this intersection satisfies our requirements because $z_0 \in \mathcal{A}_R(\Sigma, z_1)$ if $z_1 \in \mathcal{A}_R(-\Sigma, z_0)$. Now proceed by induction and define analogously $z_{n+1} \in \mathcal{A}_R(-\Sigma, z_n) \cap U_{n+1} \cap \mathcal{A}_R(\Sigma, x_0)$, using a regular trajectory from x_0 to z_n. At each step we get $z_n \in \mathcal{A}_R(\Sigma, z_{n+1})$ implying that $z_m \in \mathcal{A}_R(\Sigma, z_n)$ if $m < n$. □

Given a sequence z_n as built in this lemma we link z_{n+1} to z_n by a trajectory, say β_n, of Σ. In principle β_n is defined in $[0, 1]$, but we can shift time so that β_n becomes defined in the interval $[-n-1, -n]$. Concatenating successively these trajectories we obtain a continuous curve defined in the interval $(-\infty, 0]$ which is entirely contained in $\mathcal{A}_R(\Sigma, x_0)$.

Lemma 8.2. *Let the notations and assumptions be as above. Then there exists a continuous curve $\beta : (-\infty, 0] \to \mathcal{A}_R(\Sigma, x_0)$ such that $\beta(0) = z_0$, each piece $\beta_{|[a,b]}$, $a < b \leqslant 0$, of β is a (reparametrization of a) trajectory of Σ, and there exists a sequence $t_n \to -\infty$ such that $\beta(t_n) \to x_0$.*

Next we lift the curve β in the lemma to a curve $\widehat{\beta}$ in $\Gamma(\Sigma, x_0)$. To avoid the problem of existence of such liftings we construct $\widehat{\beta}$ by embedding $\mathcal{A}_R(\Sigma, x_0)$ into $\mathcal{A}_R(\Sigma, x_1)$ for some x_1 with $x_0 \in \mathcal{A}_R(\Sigma, x_1)$. By Proposition 7.5, $\Gamma(\Sigma, x_0)$ becomes diffeomorphic to $\mathcal{A}_R\left(\widehat{\Sigma}, \widehat{x}\right)$ if $\widehat{x} \in \Gamma(\Sigma, x_1)$ is chosen so that it projects down to x_0. Thus the above lemma applies, giving a curve in $\mathcal{A}_R\left(\widehat{\Sigma}, \widehat{x}\right)$ instead of $\mathcal{A}_R(\Sigma, x)$.

Lemma 8.3. *Let $\widehat{x} \in \Gamma(\Sigma, x_1)$ be chosen so that it projects down to x_0 and fix $y_0 \in \Gamma(\Sigma, x_0) = \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{x}\right)$. Then there exists a continuous curve $\widehat{\beta} : (-\infty, 0] \to \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{x}\right)$ such that $\widehat{\beta}(0) = y_0$, each piece $\widehat{\beta}_{|[a,b]}$, $a < b \leqslant 0$, of $\widehat{\beta}$ is a (reparametrization of a) trajectory of $\widehat{\Sigma}$, and there exists a sequence $t_n \to -\infty$ such that $\widehat{\beta}(t_n) \to \widehat{x}$.*

Now we can start the construction of the desired map $\Gamma(\Sigma, x_0) \to N$. Let $\pi : N \to \mathcal{A}_R(\Sigma, x_0)$ be a control covering and fix $z_0 \in \mathcal{A}_R(\Sigma, x_0)$, $y_0 \in \varepsilon^{-1}\{z_0\}$ and $w_0 \in \pi^{-1}\{z_0\}$. Given these points we define a mapping

$$f_{z_0,y_0,w_0} : \Gamma(\Sigma, z_0) \to N$$

as follows: By Proposition 7.5, $\Gamma(\Sigma, z_0)$ is diffeomorphic to $\mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$. Now, take $y \in \mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$. Then there exists a regular trajectory α of Σ starting at z_0 such that its lifting $\widehat{\alpha}$ with initial point y_0 has end point y. Denote by $\widetilde{\alpha}$ the lifting of α to N with initial point w_0. Then we declare $f_{z_0,y_0,w_0}(y)$ to be the end point of $\widetilde{\alpha}$.

Lemma 8.4. *The mapping f_{z_0,y_0,w_0} is independent of the trajectory α used in the definition.*

Proof. Let α_1 be another trajectory whose lifting $\widehat{\alpha}_1$ also has end point y. Denote by $\widetilde{\alpha}_1$ the lifting of α_1 to N with initial point w_0. Since the end points of $\widehat{\alpha}$ and $\widehat{\alpha}_1$ coincide, it follows by Corollary 7.6 that α_1 is monotonically homotopic to α. Hence the liftings $\widetilde{\alpha}$ and $\widetilde{\alpha}_1$ to N have the same end point, since the initial point w_0 is the same (see Lemma 5.3). Thus f_{z_0,y_0,w_0} is well defined. □

Lemma 8.5. *The mapping f_{z_0,y_0,w_0} is a local diffeomorphism.*

Proof. In fact, since f_{z_0,y_0,w_0} is defined by the end point of a lifting it satisfies $\pi \circ f_{z_0,y_0,w_0} = \varepsilon$. But π is a local diffeomorphism. Hence, locally f_{z_0,y_0,w_0} is given by $\pi^{-1} \circ \varepsilon$, where π^{-1} is a local inverse of π. It follows that f_{z_0,y_0,w_0} is also a local diffeomorphism. □

Having obtained a map $\mathcal{A}_R\left(\widehat{\Sigma}, y_0\right) \to N$, we repeat the same construction along the path $\widehat{\beta} : (-\infty, 0] \to \Gamma(\Sigma, x_0) = \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{x}\right)$ of Lemma 8.3.

With z_0, y_0 and w_0 as above, let $\widehat{\beta}$ be the curve of Lemma 8.3 and put $\beta = \varepsilon\left(\widehat{\beta}\right)$. Also, let $\widetilde{\beta}$ be the lifting of β to N with $\widetilde{\beta}(0) = w_0$. For each $t \in (-\infty, 0]$ we have a local diffeomorphism

$$f_{\beta(t),\widehat{\beta}(t),\widetilde{\beta}(t)} : \Gamma\left(\Sigma, \beta(t)\right) \to N.$$

For simplicity we write f_t instead of $f_{\beta(t),\widehat{\beta}(t),\widetilde{\beta}(t)}$. The next lemmas show that these maps are combined together to form a local diffeomorphism $\Gamma(\Sigma, x_0) \to N$.

Lemma 8.6. $\Gamma(\Sigma, x_0) = \bigcup_{t\in(-\infty,0]} \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{\beta}(t)\right)$.

Proof. As in Lemma 8.3 we view $\Gamma(\Sigma, x_0)$ as the accessible set $\mathcal{A}_R\left(\widehat{\Sigma}, \widehat{x}\right) \subset \Gamma(\Sigma, x_1)$ with $x_0 \in \mathcal{A}_R(\Sigma, x_1)$. Take $y \in \Gamma(\Sigma, x_0)$ and let $\widehat{\alpha}$ be a trajectory of $\widehat{\Sigma}$ regular at $\widehat{x}$ ending at y. Reverting time we see that $\widehat{\alpha}^-$ is regular at y for $-\widehat{\Sigma}$. Therefore there exists a neighborhood U of $\widehat{x}$ in $\Gamma(\Sigma, x_1)$ such that for every $y' \in U$, $y \in \mathcal{A}_R\left(\widehat{\Sigma}, y'\right)$. Since $\widehat{\beta}(t) \to \widehat{x}$, it follows that $y \in \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{\beta}(t)\right)$ for some t, concluding the proof. □

Lemma 8.7. *Given $t_1, t_2 \in (-\infty, 0]$, suppose that $y \in \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{\beta}(t_1)\right) \cap \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{\beta}(t_2)\right)$. Then $f_{t_1}(y) = f_{t_2}(y)$.*

Proof. To fix ideas suppose that $t_1 < t_2$ and denote by $\widehat{\beta}_{t_1,t_2}$ the restriction of $\widehat{\beta}$ to $[t_1, t_2]$, which is a trajectory of $\widehat{\Sigma}$. Analogously let β_{t_1,t_2} be the projection of $\widehat{\beta}_{t_1,t_2}$ to $\mathcal{A}_R(\Sigma, x_0)$.

Now take regular trajectories $\widehat{\alpha}_i$ of $\widehat{\Sigma}$ starting at $\widehat{\beta}(t_i)$, $i = 1, 2$, and having y as end point. Denote by α_i their projections to $\mathcal{A}_R(\Sigma, x_0)$, and let $\widetilde{\alpha}_1$ be the lifting of α_1 to N starting at $\widetilde{\beta}(t_1)$ and $\widetilde{\alpha}_2$ the lifting of α_2 starting at $\widetilde{\beta}(t_2)$. By definition $f_{t_i}(y)$ is the end point of $\widetilde{\alpha}_i$, $i = 1, 2$.

Note that the end point of $\widehat{\beta}_{t_1,t_2}$ is $\widehat{\beta}(t_2)$, so that it makes sense to take the concatenation $\widehat{\alpha}_2 * \widehat{\beta}_{t_1,t_2}$, obtaining a trajectory of $\widehat{\Sigma}$ starting at $\widehat{\beta}(t_1)$ and ending at y. Thus both $\widehat{\alpha}_1$ and $\widehat{\alpha}_2 * \widehat{\beta}_{t_1,t_2}$ have the same end point y, implying that their projections α_1 and $\alpha_2 * \beta_{t_1,t_2}$, respectively, are monotonically homotopic. Hence, their liftings $\widetilde{\alpha}_1$ and $\widetilde{\left(\alpha_2 * \beta_{t_1,t_2}\right)}$ starting at $\widetilde{\beta}(t_1)$ have the same end point. Therefore, to conclude the proof it is enough to observe that the end points of $\widetilde{\alpha}_2$ and $\widetilde{\left(\alpha_2 * \beta_{t_1,t_2}\right)}$ coincide. But this follows from the fact that $\widetilde{\alpha}_2$ starts at $\widetilde{\beta}(t_2)$, which implies that $\widetilde{\left(\alpha_2 * \beta_{t_1,t_2}\right)} = \widetilde{\alpha}_2 * \widetilde{\beta}_{t_1,t_2}$, so that the end points are indeed the same, showing that $f_{t_1}(y) = f_{t_2}(y)$. □

From these two lemmas we have a well-defined mapping

$$f : \Gamma(\Sigma, x_0) \longrightarrow N \tag{6}$$

given by $f(y) = f_t(y)$ where $t \in (-\infty, 0]$ is any value such that $y \in \mathcal{A}_R\left(\widehat{\Sigma}, \widehat{\beta}(t)\right)$. Summarizing, we have

Theorem 8.8. *Assume that the system Σ on M satisfies the Lie algebra rank condition and consider $x_0 \in M$. Let $\pi : N \to \mathcal{A}_R(\Sigma, x_0)$ be a control covering for a system $\widetilde{\Sigma}$ on N, and assume Σ and $\widetilde{\Sigma}$ are forward complete. Then there exists a control mapping $f : \Gamma(\Sigma, x_0) \to N$ such that $\pi \circ f = \varepsilon$.*

Proof. By construction f is equal to f_t on the open set $\mathcal{A}_R\left(\widehat{\Sigma}, \widehat{\beta}(t)\right)$. Hence the properties of f_t are inherited by f, showing that it is a local diffeomorphism which maps $\widehat{\Sigma}$ into $\widetilde{\Sigma}$. □

We note that in general the mapping $f : \Gamma(\Sigma, x_0) \to N$ is not surjective. In fact, the image of f_t is the accessible set $\mathcal{A}_R\left(\widetilde{\Sigma}, \widetilde{\beta}(t)\right)$, so that the image of f is

$$\operatorname{im} f = \bigcup_{t \in (-\infty, 0]} \mathcal{A}_R\left(\widetilde{\Sigma}, \widetilde{\beta}(t)\right), \tag{7}$$

which may be a proper subset of N, since $\widetilde{\Sigma}$ may not be controllable. In Section 10, below we give an example with $N = \widetilde{\mathcal{A}_R(\Sigma, x_0)}$, the universal covering of $\mathcal{A}_R(\Sigma, x_0)$, where the lifted system $\widetilde{\Sigma}$ is not controllable.

9. Control sets

In this section we specialize the previous results to forward orbits starting at $x_0 \in M$ such that $x_0 \in \mathcal{A}_R(\Sigma, x_0)$, or equivalently $x_0 \in \operatorname{int} \mathcal{A}(x_0)$. As is well known this condition holds if and only if x_0 belongs to the interior of a control set of Σ. In this case our previous constructions become more transparent and closer to the classical situation, since in any covering of $\mathcal{A}_R(\Sigma, x_0)$ we can always take a reference point above x_0. Also, the existence of periodic regular trajectories through x_0 allows the introduction of a fundamental semigroup based at x_0, analogous to the fundamental group of a topological space (cf. [3]).

Before proceeding we note that the condition $x_0 \in \operatorname{int} \mathcal{A}(x_0)$ implies that $\mathcal{A}(x_0)$ is open, and hence (under the Lie algebra rank condition) coincides with $\mathcal{A}_R(\Sigma, x_0)$, that is, every point attainable from x_0 is actually regularly attainable.

As before, let $\widehat{\Sigma}$ be the system lifted to $\Gamma(\Sigma, x_0)$. Recall that by Proposition 7.5, $\Gamma(\Sigma, x_0)$ is diffeomorphic to its subset $\mathcal{A}_R\left(\widehat{\Sigma}, z_0\right)$ for any $z_0 \in \varepsilon^{-1}(x_0)$. Thus we can take $\mathcal{A}_R\left(\widehat{\Sigma}, z_0\right)$ as a realization of $\Gamma(\Sigma, x_0)$, and get an easier construction of the covering mapping given by Theorem 8.8.

Proposition 9.1. *For $x_0 \in \operatorname{int} \mathcal{A}(x_0)$ let $\pi : N \to \mathcal{A}(x_0)$ be a control covering and take $y_0 \in \varepsilon^{-1}(x_0)$ and $w_0 \in \pi^{-1}(x_0)$. Then there exists a unique control mapping $f : \Gamma(\Sigma, x_0) \to N$ such that $\pi \circ f = \varepsilon$ and $f(y_0) = w_0$.*

Proof. See Lemmas 8.4 and 8.5. $\square$

In the context of control sets one can construct the fundamental semigroup related to monotonic homotopy. Fix as above $x_0 \in \operatorname{int} \mathcal{A}(x_0)$ and put $P(\Sigma, x_0) = R(\Sigma, x_0, x_0)$ for the set of regular periodic trajectories through x_0. Clearly, the concatenation of trajectories defines a product in $P(\Sigma, x_0)$. Note that by the way we defined the concatenation, this product is not associative, since $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ do not have the same parametrizations. However, it is a consequence of Corollary 7.6 that the curves $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ in $P(\Sigma, x_0)$ are monotonically homotopic. In fact, the liftings of these curves to $\Gamma(\Sigma, x_0)$ (starting at a prescribed y_0) are the successive concatenations of the liftings of γ, β and α. Although these curves are not equally parametrized, they have the same trace. In particular, the liftings have the same end point. Hence, by Corollary 7.6, $\alpha * (\beta * \gamma) \simeq_m (\alpha * \beta) * \gamma$. Since by Proposition 4.2 monotonic homotopy is well behaved under concatenation, it follows that the quotient space $P(\Sigma, x_0) / \simeq_m$ is an associative semigroup.

Definition 9.2. Suppose $x_0 \in \operatorname{int} \mathcal{A}(x_0)$. Then the *fundamental semigroup* based at x_0 is defined as $\Lambda(\Sigma, x_0) = P(\Sigma, x_0) / \simeq_m$.

Remark. In the above definition we restricted attention to regular trajectories because this is the case which fits to our results. Of course, one can define a semigroup $\Lambda(\Sigma, x_0)$ for arbitrary periodic trajectories. But then the associativity property must be verified directly [3].

By the results of Section 7, it follows that two trajectories in $P(\Sigma, x_0)$ are monotonically homotopic if and only if their liftings to $\Gamma(\Sigma, x_0)$, starting at a given point, have the same end point. Using this fact it is easy to prove that $\Lambda(\Sigma, x_0)$ is given by the fiber of $\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0)$ above x_0. Of course, this result is analogous to the well-known fact that the fundamental group is isomorphic to the group of deck transformations, and hence to the fiber of the simply connected covering.

Proposition 9.3. *Let* $x_0 \in \operatorname{int} \mathcal{A}(x_0)$ *and take* $y_0 \in \varepsilon^{-1}(x_0)$. *Then* $\Lambda(\Sigma, x_0)$ *is in bijection with* $\varepsilon^{-1}(x_0) \cap \mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$.

Proof. Clearly, the periodic trajectories in $P(\Sigma, x_0)$ are the trajectories whose liftings to $\Gamma(\Sigma, x_0)$ starting at y_0 have end point in the fiber $\varepsilon^{-1}(x_0) \cap \mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$. The result is then an immediate consequence of Proposition 7.5, which ensures that $\Gamma(\Sigma, x_0)$ is diffeomorphic to $\mathcal{A}_R\left(\widehat{\Sigma}, y_0\right)$. $\square$

Remark. Proposition 9.3 implies that the topology of the fundamental semigroup is discrete. In fact, as the fiber over a point it is a discrete set because of the local diffeomorphism property of the end point mapping ε.

Regarding the structure of $\Lambda(\Sigma, x_0)$ we note the following useful algebraic property.

Proposition 9.4. *The semigroup* $\Lambda(\Sigma, x_0)$ *is cancellative, that is,* $y = z$ *if either* $xy = xz$ *or* $yx = zx$.

Proof. Note that the cancellative property to the left is exactly the statement of Lemma 7.3. To see the cancellation to the right, take trajectories $\alpha, \beta, \gamma \in P(\Sigma, x_0)$. Then $\beta * \alpha \simeq_m \gamma * \alpha$ means that the liftings to $\Gamma(\Sigma, x_0)$ of these curves have the same end point, say $w \in \Gamma(\Sigma, x_0)$. But then the liftings of β and γ have the same end point because by Lemma 5.1 the lifting of α having w as end point is unique. $\square$

Now we pose the problem of relating monotonic homotopy to plain homotopy between trajectories. Of course, monotonic homotopy implies homotopy between the trajectories. In general the converse is not true, as the example of Section 10 shows. Thus it is required to understand when geometric homotopy implies dynamic homotopy.

In order to state this question precisely, note that it is relevant to specify the set where the homotopies take place. Since two monotonically homotopic trajectories (with the same end points) are homotopic inside the interior of the accessible set of the common starting point, the right question to be posed is whether geometric homotopy inside $\mathcal{A}(x_0)$ entails monotonic homotopy. Having this in mind we write $\alpha \simeq_{\mathcal{A}} \beta$ if α and β are homotopic (with fixed end points) inside $\mathcal{A}(x_0)$, where x_0 is the common initial point.

Definition 9.5. We say that a system Σ is *geometric* at x_0 if monotonic homotopy is equivalent to geometric homotopy. More precisely, this means that for regular trajectories α and β starting at x_0 and having the same end point, the property $\alpha \simeq_{\mathcal{A}} \beta$ implies $\alpha \simeq_m \beta$.

Remark. In the context of topological semigroups Lawson [8] uses the term *compatible homotopy structure* when geometric homotopy coincides with monotonic homotopy.

Regarding the geometric property of Σ our main result is the example of next section. Here we shall prove only the following simple result, which shows that this problem is related to the possibility of lifting homotopies to $\Gamma(\Sigma, x_0)$. Recall that a mapping $p : E \to B$ is said to satisfy the covering homotopy property (CHP) for a space X if for every mapping $f^* : X \to E$ and every homotopy $H : [0, 1] \times X \to B$ with $f_0 = p \circ f^*$, there exists a lifting $H^* : [0, 1] \times X \to E$ with $H^*(0, \cdot) = f^*$ (see e.g. [4]).

Proposition 9.6. *Let* $x_0 \in \operatorname{int} \mathcal{A}(x_0)$ *and suppose that* $\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}(x_0)$ *satisfies CHP for* $[0, 1]$. *Then* Σ *is geometric at* x_0.

Proof. Take $\alpha, \beta \in R(\Sigma, x, z)$ which are homotopic in $\mathcal{A}(x)$. Fix $y \in \varepsilon_x^{-1}(x)$ and let $\widehat{\alpha}$ be the lifting starting at y. Also, let H be a homotopy (with fixed end points) between α and β. By CHP, H lifts to $\widehat{H}$ with $\widehat{H}(0, \cdot) = \widehat{\alpha}$. Since $\widehat{H}$ is a lifting of H, it follows that $\widehat{H}(1, \cdot) = \widehat{\beta}$. Therefore, $\widehat{\alpha}$ and $\widehat{\beta}$ have the same end points, because H fixes end points. This shows that $\alpha \simeq_m \beta$, concluding the proof. □

Corollary 9.7. *Let* $x_0 \in \operatorname{int} \mathcal{A}(x_0)$ *and suppose that* $\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}(x_0)$ *is a covering. Then* Σ *is geometric at* x_0. *In particular, the fundamental semigroup* $\Lambda(\Sigma, x_0)$ *coincides with the fundamental group.*

Proof. In fact, any covering satisfies CHP for $[0, 1]$. □

Remark. These results should be true without the assumption that $x_0 \in \operatorname{int} \mathcal{A}(x_0)$. We note, however, that without this assumption a technical difficulty arises due to the fact that trajectories in $\mathcal{A}(x_0)$ may not lift to $\Gamma(\Sigma, x_0)$.

10. An example

We shall exhibit here an example of a system Σ admitting trajectories which are homotopic but not monotonically homotopic. The idea is to take Σ evolving on M which is controllable from some $x_0 \in M$ but in such a way that the system $\widetilde{\Sigma}$ lifted to the simply connected covering $\widetilde{M}$ of M is not controllable from $w_0 \in \widetilde{M}$ sitting above x_0. Then we search for trajectories $\widetilde{\alpha}$ and $\widetilde{\beta}$ of $\widetilde{\Sigma}$ starting at w_0, having the same end points, but which are not homotopic inside the accessible set from w_0. The projections to M of these trajectories, say α and β, are homotopic (because their liftings have the same end points), but not monotonically homotopic (otherwise $\widetilde{\alpha}$ and $\widetilde{\beta}$ would be homotopic within the accessible set of w_0).

We take M to be the flag manifold $\mathbb{F} = \mathbb{F}^3(1,2)$ whose elements are flags $(V_1 \subset V_2)$ where $V_l \subset \mathbb{R}^3$ is a subspace with $\dim V_l = l$, $l = 1, 2$. Let us recall some properties of $\mathbb{F}$. First the group $\mathrm{Sl}(3,\mathbb{R})$ of 3×3 unimodular matrices acts transitively on $\mathbb{F}$ by $g(V_1 \subset V_2) = (gV_1 \subset gV_2)$. This action restricts to an action of $\mathrm{SO}(3,\mathbb{R})$ which is transitive as well. By these actions there are identifications of $\mathbb{F}$ either with $\mathrm{Sl}(3,\mathbb{R})/P$ or with $\mathrm{SO}(3,\mathbb{R})/Z_0$, where $P \subset \mathrm{Sl}(3,\mathbb{R})$ is the subgroup of upper triangular matrices and $Z_0 \subset \mathrm{SO}(3,\mathbb{R})$ is the subgroup of diagonal matrices with ± 1 entries.

As usual we denote by $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{so}(3,\mathbb{R})$ the Lie algebras of $\mathrm{Sl}(3,\mathbb{R})$ and $\mathrm{SO}(3,\mathbb{R})$, respectively, viewed as the set of right invariant vector fields.

The sphere $\mathbb{S}^3$ is the simply connected covering of both $\mathbb{F}$ and $\mathrm{SO}(3,\mathbb{R})$. The covering maps $p : \mathbb{S}^3 \to \mathbb{F}$ and $\theta : \mathbb{S}^3 \to \mathrm{SO}(3,\mathbb{R})$ are described via Lie group actions as follows: denote by $\{i, j, k\}$ the standard basis of $\mathbb{R}^3$, viewed as the imaginary part of the space $\mathbb{H}$ of quaternions $a_1 + a_2 i + a_3 j + a_4 k$ with real coefficients.

The unit sphere $\mathbb{S}^3 \subset \mathbb{H}$ is a compact group with quaternionic multiplication, having $\mathfrak{so}(3,\mathbb{R})$ as Lie algebra. It represents in the three-dimensional space of imaginary quaternions via the onto homomorphism $\theta : \mathbb{S}^3 \to \mathrm{SO}(3,\mathbb{R})$, by $\theta(z)w = zw\bar{z}$, having $\ker\theta = \{\pm 1\}$. Thus $\mathbb{S}^3$ is a two-fold covering of $\mathrm{SO}(3,\mathbb{R})$. By composing the action of $\mathrm{SO}(3,\mathbb{R})$ with θ we obtain an action of $\mathbb{S}^3$ on $\mathbb{F}$. An easy computation yields $\theta(i) = \mathrm{diag}\{1,-1,-1\}$, $\theta(j) = \mathrm{diag}\{-1,1,-1\}$ and $\theta(k) = \mathrm{diag}\{-1,-1,1\}$. This implies that through the action of $\mathbb{S}^3$ on $\mathbb{F}$, we can identify $\mathbb{F}$ with $\mathbb{S}^3/Z$, where

$$Z = \theta^{-1}(Z_0) = \{\pm 1, \pm i, \pm j, \pm k\}.$$

Hence, the canonical map $p : \mathbb{S}^3 \to \mathbb{F} = \mathbb{S}^3/Z$ is an eight-fold covering of $\mathbb{F}$. Furthermore, Z acts on the right on $\mathbb{S}^3$ and its orbits are the fibers of p. Explicitly, p is given by

$$p : z \in \mathbb{S}^3 \mapsto (\mathrm{span}\{zi\bar{z}\} \subset \mathrm{span}\{zi\bar{z}, zj\bar{z}\}) \in \mathbb{F}. \tag{8}$$

Denote by G the simply connected covering of $\mathrm{Sl}(3,\mathbb{R})$, which is a Lie group with Lie algebra $\mathfrak{sl}(3,\mathbb{R})$, and contains a copy of $\mathbb{S}^3$. We denote also by $\theta : G \to \mathrm{Sl}(3,\mathbb{R})$ the covering homomorphism, since it extends $\theta : \mathbb{S}^3 \to \mathrm{SO}(3,\mathbb{R})$. The group G has Iwasawa decomposition $G = \mathbb{S}^3 T$, with $\mathbb{S}^3$ being the maximal compact subgroup and T a subgroup isomorphic to P. Thus G acts on $\mathbb{S}^3$ by identifying it with G/T. Any action of $\mathrm{Sl}(3,\mathbb{R})$ can be turned into an action of G by composing with θ.

In the sequel we let $\widetilde{\sigma}$ and $\widetilde{\sigma}_1$ be the following circles in $\mathbb{S}^3$:

$$\widetilde{\sigma}(t) = \cos t + i \sin t \qquad \widetilde{\sigma}_1(t) = \widetilde{\sigma}(t)\, j = j\cos t + k \sin t \qquad t \in [0, 2\pi].$$

We define now the system on $\mathbb{F}$. Recall that $X \in \mathfrak{sl}(3,\mathbb{R})$ induces a vector field $\overrightarrow{X}$ on the coset space $\mathbb{F}$ by $\overrightarrow{X}(x) = \frac{d}{dt}(\exp tX) \cdot x_{|t=0}$. Then the set

$$E = \{\overrightarrow{X} : X \in \mathfrak{sl}(3,\mathbb{R})\}$$

is a finite-dimensional vector space of vector fields on $\mathbb{F}$. A similar construction holds in $\mathbb{S}^3$ and since $p : \mathbb{S}^3 \to \mathbb{F}$ intertwines the actions of G on $\mathbb{S}^3$ and $\mathbb{F}$, the vector fields thus obtained on $\mathbb{S}^3$ are the liftings of the vector fields in E under p.

The system Σ will be given by a convex cone in the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. To define it let $W \subset \mathbb{R}^3$ be a pointed cone (i.e., $W \cap -W = \{0\}$) which contains i in its interior and such that $W \cap \operatorname{span}\{j, k\} = \{0\}$ (any such cone will do). Put

$$\Sigma = \{X \in \mathfrak{sl}(3, \mathbb{R}) : \forall t > 0,\ \exp(-tX)\, W \subset W\} \qquad \text{and} \qquad \Sigma_1 = -\Sigma.$$

Denote by S the semigroup in $\mathrm{Sl}(3, \mathbb{R})$ generated by $\exp(\Sigma)$ and let $S_1 = S^{-1}$ be the semigroup generated by $\exp(\Sigma_1)$. It follows that Σ_1 is a cone in $\mathfrak{sl}(3, \mathbb{R})$ which contains in its interior any diagonal matrix $\operatorname{diag}\{2a, -a, -a\}$, $a > 0$ (see [11], Theorem 6.12). This implies that S_1 and S have interior points in $\mathrm{Sl}(3, \mathbb{R})$. The accessible sets of Σ_1 are the orbits of S_1, and the accessible sets of Σ are the orbits of S. Analogously, the cones $\Sigma, \Sigma_1 \subset \mathfrak{sl}(3, \mathbb{R})$ generate semigroups $\widetilde{S}, \widetilde{S}_1 \subset G$ whose orbits on $\mathbb{S}^3 = G/T$ are the accessible sets of the systems $\widetilde{\Sigma}$ and $\widetilde{\Sigma}_1$ lifted to $\mathbb{S}^3$.

The descriptions of the accessible sets on $\mathbb{S}^3$ and $\mathbb{F}$ are given by the corresponding control sets. The control sets of S_1 in $\mathbb{F}$ are known (see [10]):

Proposition 10.1. *The semigroup S_1 has a unique invariant control set in $\mathbb{F}$. It is given by*

$$C = \{(V_1 \subset V_2) \in \mathbb{F} : V_1 \subset W \cup -W\}. \tag{9}$$

This implies that $\operatorname{int} C$ *is a control set of S (and Σ) and by uniqueness Σ is controllable from any $x \in$* $\operatorname{int} C$.

Now we describe the minimal control sets of $\widetilde{S}$ (and hence of $\widetilde{\Sigma}$) in $\mathbb{S}^3$ which is the same as the interior of the invariant control sets of $\widetilde{S}_1$. Any such control set in $\mathbb{S}^3$ projects down to $\operatorname{int} C$ under p. Moreover, since the left action of G on $\mathbb{S}^3$ commutes with the right action of Z, it follows that if $D \subset \mathbb{S}^3$ is a control set and $m \in Z$ then $D \cdot m$ is also a control set.

Proposition 10.2. *There are exactly two invariant control sets D_1 and D_2 of $\widetilde{S}_1$ in $\mathbb{S}^3 = G/T$, namely*

$$D_1 = \{z \in \mathbb{S}^3 : zi\overline{z} \in W\} \qquad D_2 = \{z \in \mathbb{S}^3 : zi\overline{z} \in -W\}.$$

Hence the minimal control sets of $\widetilde{S}$ (and $\widetilde{\Sigma}$) on $\mathbb{S}^3$ are $\operatorname{int} D_1$ *and* $\operatorname{int} D_2$.

Proof. First we note that the invariant control sets of $\widetilde{S}_1$ are the connected components of $p^{-1}(C)$. In fact, the invariant control sets are connected, and hence contained in connected components. On the other hand by [9, Proposition 4.3], $\widetilde{S}_1$ is transitive in the interior of each connected component, thus the connected components are indeed the invariant control sets.

Now, by the expressions of p in (8) and of C in (9), it follows that

$$p^{-1}(C) = \{z \in \mathbb{S}^3 : zi\bar{z} \in W \cup -W\}.$$

Consequently the invariant control sets are D_1 and D_2 as in the statement. □

Corollary 10.3. *The circle* $\widetilde{\sigma}$ *is contained in* $\operatorname{int} D_1$ *while* $\widetilde{\sigma}_1$ *is contained in* $\operatorname{int} D_2$.

Proof. This is a consequence of the proposition and the following straightforward computations:

$$(\cos t + i \sin t)\, i\, (\cos t - i \sin t) = i, \quad (j \cos t + k \sin t)\, i\, (j \cos t + k \sin t) = i. \quad \square$$

We can now look at a pair of trajectories α and β of Σ in $\mathbb{F}$ that have the same end point and are homotopic, but not monotonically homotopic. Both trajectories start at x_0 and end at x_1 where

$$x_0 = (\operatorname{span}\{i\} \subset \operatorname{span}\{i, j\}) \qquad x_1 = (\operatorname{span}\{i\} \subset \operatorname{span}\{i, k\}).$$

In fact, they are projections of the two sides of the circle $\widetilde{\sigma} = \{\cos t + i \sin t : t \in \mathbb{R}\} \in \mathbb{S}^3$ lying between 1 and -1 and passing through $\pm i$, respectively.

Lemma 10.4. *The curves* $\widetilde{\alpha}, \widetilde{\beta} : [0, \pi] \to \mathbb{S}^3$, $\widetilde{\alpha}(t) = \cos t + i \sin t$ *and* $\widetilde{\beta}(t) = \cos t - i \sin t$ *are* (*reparametrizations of*) *trajectories of* $\widetilde{\Sigma}$.

Proof. Put $\sigma = \theta(\widetilde{\sigma})$. Since 1 is mapped into x_0 and

$$\theta(\cos t + i \sin t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2t & -\sin 2t \\ 0 & \sin 2t & \cos 2t \end{pmatrix},$$

it follows that σ is the curve $f_s = \left(V_1 \subset V_2^s\right)$, $s \in \mathbb{R}$, where V_2^s is the subspace spanned by i and $j \cos 2s + k \sin 2s$. Now, take a diagonal matrix $H = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ in $\operatorname{int} \Sigma$, with $\lambda_1 < \lambda_2 < \lambda_3$. The existence of such matrix comes from the fact, mentioned earlier that e.g. $X = \operatorname{diag}\{2, -1, -1\}$ belongs to $\operatorname{int} \Sigma_1 \subset \mathfrak{sl}(3, \mathbb{R})$, so that $-X \in \operatorname{int} \Sigma$. Thus we can choose $H \in \operatorname{int} \Sigma$ close to $-X$.

Looking at the concrete realization of σ we see that $\exp(tH)$, $t \in \mathbb{R}$, leaves this circle invariant. In fact, V_1 is invariant under $\exp(tH)$ as well as the subspace $\operatorname{span}\{j, k\}$. Furthermore, the one-parameter group $\exp(tH)$ has just two fixed-points in σ, since the only lines in $\operatorname{span}\{j, k\}$ invariant under $\exp(tH)$ are those spanned by j and k. The other points of σ are in two trajectories of $\exp(tH)$, running from f_0 to $f_{\pi/4}$.

Consider now the one-parameter group $\exp(tH)$ in G. By equivariance it leaves invariant $\widetilde{\sigma}$, and since $\widetilde{\sigma}$ is a two-fold covering of σ, $\exp(tH)$ has four fixed-points and four trajectories, say τ_1 and τ_2 running from 1 to i and from i to -1, respectively, and τ_3 and τ_4, which go from 1 to $-i$ and from $-i$ to -1, respectively.

Since $H \in \Sigma$, each τ_i may be seen as trajectory of Σ defined on the whole real line.

To conclude the proof we shall link the fixed points between the τ_i's to get trajectories defined in compact intervals. For this take for instance the fixed-point 1 and let us link it to τ_1. Since $H \in \operatorname{int}\Sigma$ there exists $a > 0$ small enough such that

$$X = H + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{pmatrix} \in \operatorname{int}\Sigma.$$

Then the one-parameter group $\exp(tX) \in G$ leaves invariant $\widetilde{\sigma}$ and 1 is not a fixed-point. If we take t small enough we link 1 to τ_1. Proceeding analogously with the other fixed-points we verify that the half-circles are indeed trajectories of Σ, concluding the proof of the lemma. $\square$

We denote by α and β the projections into $\mathbb{F}$ of $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. Since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are trajectories of $\widetilde{\Sigma}$, it follows that α and β are trajectories of Σ. Also, the end points of $\widetilde{\alpha}$ and $\widetilde{\beta}$ coincide, so that α and β are homotopic in $\mathbb{F}$ ($= \mathcal{A}_R(\Sigma, x_0)$).

Finally, we prove that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are not monotonically homotopic. This implies that α and β are not monotonically homotopic, since an eventual monotonic homotopy between α and β could be lifted to a monotonic homotopy between $\widetilde{\alpha}$ and $\widetilde{\beta}$.

For the proof that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are not monotonically homotopic we combine the following facts:

(1) By Corollary 10.3 the circle $\widetilde{\sigma} = \cos t + i \sin t$ is contained in $\operatorname{int} D_1$ and $\widetilde{\sigma}_1 = j \cos t + k \sin t$ is contained in $\operatorname{int} D_2$.
(2) The set $\operatorname{int} D_2$ is a minimal control set of $\widetilde{\Sigma}$ hence invariant under backward trajectories of this system. This means that forward trajectories of Σ starting outside $\operatorname{int} D_2$ never enters this set. In particular, a trajectory starting at $1 \in \operatorname{int} D_1$ does not enter $\operatorname{int} D_2$.
(3) The circle $\widetilde{\sigma}$ is not homotopic to a point in $\mathbb{S}^3 \setminus \widetilde{\sigma}_1$ (that is, the linking number of $\widetilde{\sigma}$ and $\widetilde{\sigma}_1$ is not trivial). To see this make a stereographic projection $\mathbb{S}^3 \setminus \{N\} \to \mathbb{R}^3$ with the north pole N taken in $\widetilde{\sigma}_1$. Then $\widetilde{\sigma}_1$ goes to a straight line l through the origin in $\mathbb{R}^3$, while $\widetilde{\sigma}$ goes to a circle which cannot be shrunk to a point without crossing l (see [1, p. 238]).

Therefore a homotopy between $\widetilde{\alpha}$ and $\widetilde{\beta}$ must cross $\widetilde{\sigma}_1$ and hence is not a monotonic homotopy, for otherwise we would have a trajectory starting at 1 and crossing $\widetilde{\sigma}_1 \subset \operatorname{int} D_2$. This concludes the proof that α and β are homotopic trajectories in $\mathbb{F}$ which are not monotonically homotopic.

Remark. By Corollary 9.7, it follows that in this example the local diffeomorphism $\varepsilon : \Gamma(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0)$ is not a covering.

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