Dynamical characterization of the Lyapunov form of matrices

Victor Ayala [a], Fritz Colonius [b], Wolfgang Kliemann [c,*]

[a]*Departamento de Matematicas, Universidad Catolica del Norte, Antofagasta, Chile*
[b]*Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany*
[c]*Department of Mathematics, Iowa State University, Ames, IA 50011, USA*

1. Introduction

Spectral properties of matrices can be characterized in various ways: The algebraic approach via the characteristic polynomial yields the eigenvalues and corresponding (generalized) eigenspaces resulting in the Jordan normal form. The linear-algebraic approach using similarity of matrices again results in a characterization via the Jordan form. Furthermore, the dynamical approach via diffeomorphic

* Corresponding author.
E-mail addresses: vayala@ucn.cl (V. Ayala), fritz.colonius@math.uni-augsburg.de (F. Colonius), kliemann@iastate.edu (W. Kliemann).

conjugacy of linear flows $e^{At}x$ and $e^{Bt}x$ again implies similarity of the matrices A and B. If one weakens 'diffeomorphic conjugacy' to 'homeomorphic conjugacy' (or homeomorphic equivalence), homeomorphic conjugacy of $e^{At}x$ and $e^{Bt}x$ is equivalent (in case there are no eigenvalues on the imaginary axis) to the dimensions of the stable (or unstable) subspaces of A and B being equal.

In applications, such as nonlinear differential equations, one is often interested in matrix normal forms that are 'rougher' than the Jordan form, and finer than the characterization via stable subspaces: typical examples are the idea of invariant manifolds in dynamical systems theory, or stability and stabilizability of control systems. These approaches work with the exponential growth behavior of a flow $e^{At}x$ and are thus interested in the real parts of the eigenvalues and the corresponding subspace decomposition (Lyapunov normal form). While this form can, of course, be derived from the Jordan form, there is no obvious dynamical characterization of the Lyapunov normal form in $\mathbb{R}^d$.

In this paper we derive dynamical characterizations of the Lyapunov form for matrices. In Section 2 we start with a short review of known results on dynamical characterizations of spectral properties of matrices via their linear flows in $\mathbb{R}^d$. Then we introduce the Lyapunov normal form in Sections 3 and 4 recalls some general facts on conjugacies and equivalences. Section 5 characterizes the Lyapunov form by looking at the induced flows on projective space and the existence of homeomorphisms which respect the finest Morse decomposition, i.e., map Morse sets onto Morse sets and respect their order. Section 6 studies the induced flows on the flag manifold. It turns out that the Morse sets and their order on the full flag do not characterize the Lyapunov form. Instead, also the projections to the Grassmannians and the order of the corresponding Morse sets has to be taken into account. This results in a constructive characterization via an order graph which we call the Grassmann graph associated to a matrix.

2. Conjugacy and equivalence for linear flows in $\mathbb{R}^d$

In this section we review some of the known concepts and results for the dynamical characterization of matrices. We denote the set of $d \times d$ matrices with real entries by $gl(d, \mathbb{R})$, and the set of invertible matrices by $Gl(d, \mathbb{R})$. The space of vector fields on a manifold M is denoted by $\mathscr{X}(M)$.

Recall the following definitions of conjugacy and equivalence for vector fields, compare, e.g., [5,7,8].

Definition 2.1. Two vector fields $X, Y \in \mathscr{X}(M)$ are:

(i) C^k-*equivalent* ($k \geqslant 1$) if there exists a (local) C^k diffeomorphism $h : M \to M$ such that h takes orbits of $\varphi(t, x)$ (of X) onto orbits of $\psi(t, y)$ (of Y), preserving the orientation (but not necessarily parametrization by time), i.e.

a. for each $x \in M$ there is a strictly increasing and continuous parametrization map $\tau_x : \mathbb{R} \to \mathbb{R}$ such that $h(\varphi(t, x)) = \psi(\tau_x(t), h(x))$ or, equivalently,
b. for all $x \in M$ and $\delta > 0$ there exists $\varepsilon > 0$ such that for all $t \in (0, \delta)$ we have $h(\varphi(t, x)) = \psi(t', h(x))$ for some $t' \in (0, \varepsilon)$.

(ii) C^k-*conjugate* ($k \geqslant 1$) if there exists a (local) C^k diffeomorphism $h : M \to M$ such that $h(\varphi(t, x)) = \psi(t, h(x))$ for all $x \in M$ and $t \in \mathbb{R}$.

Usually, C^0-*equivalence* is called topological equivalence, and C^0-*conjugacy* is called topological conjugacy or simply conjugacy.

Given two matrices $A, B \in gl(d, \mathbb{R})$ with associated linear flows $\varphi(t, x) = \mathrm{e}^{At}x$ and $\psi(t, x) = \mathrm{e}^{Bt}x$ with $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$, equivalence and conjugacy of the linear flows is summarized in the following facts.

Proposition 2.2. *The linear flows φ and ψ in $\mathbb{R}^d$ are C^k-conjugate for $k \geqslant 1$ iff φ and ψ are linearly conjugate*, *i.e.*, *the conjugacy map h is a linear map in $Gl(d, \mathbb{R})$*, *iff A and B are similar*, *i.e.*, $A = TBT^{-1}$ *for some* $T \in Gl(d, \mathbb{R})$.

Each of these statements implies that A and B have the same eigenvalue structure and (up to a linear transformation) the same (generalized) eigenspace structure. In particular, the C^k-conjugacy classes are exactly the Jordan form equivalence classes in $gl(d, \mathbb{R})$.

Proposition 2.3. *The linear flows φ and ψ in $\mathbb{R}^d$ are C^k-equivalent for $k \geqslant 1$ iff φ and ψ are linearly equivalent*, *i.e.*, *the equivalence map h is a linear map in* $Gl(d, \mathbb{R})$, *iff* $A = \alpha TBT^{-1}$ *for some positive real number α and $T \in Gl(d, \mathbb{R})$.*

Each of these statements implies that A and B have the same (real) Jordan structure and their eigenvalues differ by a positive constant. Hence the C^k- equivalence classes are the Jordan form classes modulo a positive constant.

Proposition 2.4. *If A and B are hyperbolic* (*i.e.*, *there are no eigenvalues on the imaginary axis*), *then the linear flows φ and ψ in $\mathbb{R}^d$ are C^0-equivalent* (*and C^0-conjugate*) *iff the dimensions of the stable subspaces* (*and hence the dimensions of the unstable subspaces*) *of A and B agree*.

Recall that the set of hyperbolic matrices is open and dense in $gl(d, \mathbb{R})$. A matrix A is hyperbolic iff it is structurally stable in $gl(d, \mathbb{R})$, i.e., there exists a neighborhood $U \subset gl(d, \mathbb{R})$ such that all $B \in U$ are topologically equivalent to A.

Remark 2.5. The characterization of C^k-conjugacies in Proposition 2.2 remains true for Lipschitz conjugacies, since by Rademacher's theorem a Lipschitz continuous map is differentiable on a dense subset. Hence Lipschitz conjugacies do not fill the gap between C^0- and C^1-conjugacies.

3. The Lyapunov decomposition of matrices

Each similarity class in $gl(d, \mathbb{R})$ is uniquely determined by its real Jordan form, except for the order of the Jordan blocks. We now define several Lyapunov-type forms for matrices that reflect the real part of the spectrum and the associated subspaces in $\mathbb{R}^d$.

From the Jordan form $J(A)$ we construct the Lyapunov normal form $L(A)$ of A as follows:

Let $\lambda_1 < \cdots < \lambda_m$ be the distinct real parts of the eigenvalues of A, with associated Lyapunov spaces $L_i = \bigoplus_j J_{j,i}$ where the $J_{j,i}$ are the subspaces of $\mathbb{R}^d$ corresponding to the Jordan blocks of $J(A)$ whose eigenvalues have real part λ_i. Note that $\mathbb{R}^d = \bigoplus_{i=1}^m L_i$.

Definition 3.1. The *Lyapunov normal form* $L(A)$ of a matrix $A \in gl(d, \mathbb{R})$ is the diagonal matrix

$$\begin{bmatrix} \Lambda_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \Lambda_m \end{bmatrix} \quad \text{with } \Lambda_i = \begin{bmatrix} \lambda_i & & 0 \\ & \cdot & \\ 0 & & \lambda_i \end{bmatrix},$$

where λ_i is the real part of an eigenvalue of A and the block size of Λ_i is the dimension $d_i = \dim L_i$ of the Lyapunov space L_i. The blocks are arranged according to the order $\lambda_1 < \ldots < \lambda_m$. Two matrices A and B are called *Lyapunov equivalent* if $L(A) = L(B)$.

The λ_i are called the *Lyapunov exponents* of A. Note that Lyapunov equivalence is an equivalence relation on $gl(d, \mathbb{R})$. Each class has a unique representative given by m real numbers $\lambda_1 < \cdots < \lambda_m$ and m natural numbers $d_i = \dim L_i$, the dimension of the ith Lyapunov space.

Remark 3.2. Alternatively, the Lyapunov normal form of a matrix can be obtained in the following way. For $A \in gl(d, \mathbb{R})$ there exist unique matrices $S, N \in gl(d, \mathbb{R})$ such that $A = S + N$, $SN = NS$, S is semisimple and N is nilpotent (see [4], p. 116, Theorem 1). Note that A and S are Lyapunov equivalent. The complexification of the semisimple part S is diagonalizable. Denote by $S^*(A)$ the matrix formed with the real part of this diagonal matrix, ordered according to the (real) diagonal elements. Then $S^*(A)$ is the Lyapunov normal form $L(A)$ of A.

Remark 3.3. The set of all classes of Lyapunov equivalent matrices in $gl(d, \mathbb{R})$ can be parametrized as follows:

(i) a natural number m with $1 \leqslant m \leqslant d$, denoting the number of different Lyapunov exponents;
(ii) a (continuous) parameter of m variables $\mu \in \mathbb{R} \times (\mathbb{R}^+)^{m-1}$ describing the vector of Lyapunov exponents $(\lambda_1, \ldots, \lambda_m)$, where $\mu_1 = \lambda_1$ and $\mu_i = \lambda_i - \lambda_{i-1}$, for $i = 2, \ldots, m$;
(iii) a discrete index set $I_m \subset \{1, \ldots, d - (m-1)\}^{m-1}$ describing the dimensions of the Lyapunov spaces $(L_1, \ldots, L_m)$, where $I_m(i) = \dim L_i$ for $i = 1, \ldots, m-1$ with $\sum_{i=1}^{m-1} I_m(i) =: n_m \leqslant d - 1$ and $\dim L_m = d - n_m$.

The cardinality of I_m is as follows.

Proposition 3.4. *For dimension $d \geqslant 3$ and for m distinct Lyapunov exponents the cardinality of I_m describing the number of possible Lyapunov classes is determined as follows*:

(i) $m = 1$ *or* d *implies* $\mathrm{card}(I_m) = 1$;
(ii) $m = 2$ *or* $d - 1$ *implies* $\mathrm{card}(I_m) = d - 1$;
(iii) *otherwise* $\mathrm{card}(I_m)$ *is given by the formula with* $m - 2$ *terms*

$$\mathrm{card}(I_m) = \sum_{j_1=1}^{d-m+1} \sum_{j_2=1}^{j_1} \cdots \sum_{j_{i-3}=1}^{j_i-4} \sum_{j_{i-2}=1}^{j_i-3} j_{i-2}.$$

Proof. The proof can be seen by a simple counting argument: Let the dimension $d \in \mathbb{N}$ be given and let m be the number of distinct Lyapunov exponents, ordered according to their natural order in $\mathbb{R}$. We have the following initial values:

d	m	I_m	$\mathrm{card}(I_m)$
1	1	$\{1\}$	1
2	1	$\{2\}$	1
	2	$\{1\}$	1
$\geqslant 3$	1 or d	$\{d\}$ or $\{1\}$	1
	2 or $d-1$	$\{1, 2, \ldots, d-1\}$	$d-1$

In the last case $\mathrm{card}(I_m)$ describes the dimension of the first Lyapunov space.

Now let $d \geqslant 5$, and $m \in \{3, \ldots, d-2\}$ be given. The dimension d_m of the Lyapunov spaces L_m is always computed as $d - \sum_{i=1}^{m-1} d_i = d_m$.

Let $D := \sum_{i=1}^{m-3} d_i$. Note that $D = d - 3$ implies $d_{m-2} = d_m = 1$, and $D > d - 3$ is not possible. Thus we may suppose that $D \leqslant d - 4$. Then d_{m-2} can range from 1 to $d - 2 - D$, and d_{m-1} can range from 1 to $d - 1 - D - d_{m-2}$. Hence we obtain for

$$d_{m-2} = 1 : d_{m-1} \in \{1, \ldots, d - D - 2\},$$
$$d_{m-2} = i : d_{m-1} \in \{1, \ldots, d - D - 1 - i\},$$
$$d_{m-2} = d - 2 - D : d_{m-1} \in \{1\}.$$

Hence there are $1 + 2 + \cdots + d - D - 2$ possibilities for (d_{m-2}, d_{m-1}). Summing this over all possible values of $(d_1, \ldots, d_{m-3})$ yields the result. □

For the study of linear flows one often needs less information than the Lyapunov normal form of a matrix: the dimension of the Lyapunov spaces, in the natural order of their Lyapunov exponents, may be sufficient. We therefore introduce the short Lyapunov form.

Definition 3.5. The *short Lyapunov form* $SL(A)$ of a matrix $A \in gl(d, \mathbb{R})$ is given by the vector of the dimensions d_i of the Lyapunov spaces (in the natural order of their Lyapunov exponents) : $SL(A) = (m, d_1, \ldots, d_m)$ where $m \leqslant d$ is the number of distinct Lyapunov exponents.

Note that two matrices A and B have the same short Lyapunov form if and only if the blocks of $L(A)$ and $L(B)$ have the same dimensions. The short Lyapunov form defines an equivalence relation on $gl(d, \mathbb{R})$; its classes can be parametrized as in Remark 3.3(i) and (iii).

Definition 3.6. The short zero-Lyapunov form $S^0L(A)$ of a matrix $A \in gl(d, \mathbb{R})$ is given by the vector of dimensions of the Lyapunov spaces and the total multiplicity of the negative Lyapunov exponents:

$$S^0L(A) = (m, m^s, d_1, \ldots, d_m), \quad \text{with } m^s = \sum_{\lambda_i < 0} d_i.$$

We also write d^c for the dimension of the center space corresponding to $\lambda^c = 0$ and $m^u = \sum_{\lambda_i > 0} d_i$.

The short zero-Lyapunov form defines an equivalence relation on $gl(d, \mathbb{R})$. Its parametrization can be constructed as in Remark 3.3(i) and (iii) with an additional parameter $m^s \in \{0, \ldots, m\}$.

Finally, we combine the stable and unstable subspaces to obtain:

Definition 3.7. The *stability Lyapunov form* $L^s(A)$ of a matrix $A \in gl(d, \mathbb{R})$ is given by the dimensions of the stable, center and unstable subspaces $L^s(A) = (m^s, d^c, m^u)$. Alternatively, $L^s(A)$ can be written in matrix form as

$$L^s(A) = \operatorname{diag}(-1, \ldots, -1, 0, \ldots, 0, 1, \ldots, 1),$$

where the diagonal submatrices have the corresponding dimensions.

Again, the stability Lyapunov form defines an equivalence relation on $gl(d, \mathbb{R})$. Its classes, are parametrized by two natural numbers (m^s, d^c) such that $0 \leqslant m^s + d^c \leqslant d$.

Remark 3.8. For hyperbolic matrices one has $d^c = 0$. Hence $m^s \in \{0, \ldots, d\}$ parametrizes the stability Lyapunov form L^s. Recall that the set of hyperbolic matrices is open and dense in $gl(d, \mathbb{R})$.

Remark 3.9. Let A be a hyperbolic matrix. Then there exists a neighborhood $U \subset gl(d, \mathbb{R})$ such that all $B \in U$ have the same stability Lyapunov form as A; compare [5], p. 54. In general, this is not true for the short zero-Lyapunov form, the short Lyapunov form, the Lyapunov form, or the Jordan form of matrices. Note that without hyperbolicity this property also does not hold for the stability Lyapunov form.

4. Some results on conjugacies and equivalences

For the following sections we need some simple facts on conjugacies and equivalences.

Proposition 4.1. *Let $X, Y \in \mathscr{X}(M)$ be two C^∞ vector fields on a C^∞ manifold M. Let $h : M \to M$ be a topological equivalence for X and Y.*

(i) *The point $p \in M$ is a fixed point of X iff $h(p)$ is a fixed point of Y.*
(ii) *The orbit $\varphi(\cdot, p)$ is closed iff $\psi(\cdot, h(p))$ is closed.*
(iii) *If $N \subset M$ is an α-(or ω-) limit set of X from $p \in M$, then $h[N]$ is an α-(or ω-) limit set of Y from $h(p) \in M$.*
(iv) *Given, in addition, two vector fields $Z_1, Z_2 \in \mathscr{X}(N)$ on a C^∞ manifold N. If the flows of X and Y, and of Z_1 and Z_2 are topologically conjugate, so are the product flows of $X \times Z_1$ and $Y \times Z_2$ on $M \times N$. This result is, in general not true for topological equivalence.*

The proofs of these results are straightforward from the definition of equivalences and conjugacies.

For a flow Φ on a compact metric space M and $\varepsilon, T > 0$ an $(\varepsilon, T)-$chain from $x \in M$ to $y \in M$ is given by

$$n \in \mathbb{N}, \quad x_0 = x, \ldots, x_n = y, \quad T_0, \ldots, T_{n-1} > T$$

with

$$d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon \quad \text{for all } i,$$

where d is the metric on M. A set K is chain transitive if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y. A set $\mathscr{M}$ is a chain recurrent component, if it is a maximal chain transitive set.

Lemma 4.2. *Topological conjugacies on a compact metric space M map chain transitive sets onto chain transitive sets.*

Proof. The equivalence map h is a homeomorphism and M is compact by assumption. Hence for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $z \in M$ it holds that $B(z, \varepsilon) \subset h^{-1}[B(h(z), \delta)]$ with $B(z, \varepsilon) = \{y \in M : d(z, y) < \varepsilon\}$. Let φ_1, φ_2 be flows on M with topological conjugacy h. For a chain transitive set $N_2 \subset M$ of φ_2, we claim that $N_1 := h^{-1}[N_2]$ is a chain transitive set of φ_1: Take $p_1, q_1 \in N_1$ and fix $\varepsilon > 0$, $T > 0$. Choose δ as above and let ξ_2 be a (δ, T)-chain from $p_2 = h(p_1)$ to $q_2 = h(q_1)$. Then $h^{-1}(\xi_2) =: \xi_1$ is an (ε, T)-chain from p_1 to q_1. $\square$

Lemma 4.3. *Topological equivalences on a compact metric space M map chain transitive sets onto chain transitive sets.*

Proof. We need to adjust the time of ξ_2 in the proof of Lemma 4.2. Since the time parametrization $\tau(\cdot)$ of φ_2 with respect to φ_1 is continuous in both variables, we can define $T_1 = \min_{p \in M} \tau_p(T)$. If we choose ξ_2 in the proof of Lemma 4.2 to be a (δ, T_1)-chain, then $h^{-1}(\xi_2)$ is an (ε, T)-chain of φ_1 from p_1 to q_1. $\square$

Lemma 4.4. *Topological equivalences map invariant sets onto invariant sets and minimal closed invariant sets onto minimal invariant sets.*

Proof. This follows, since h maps orbits onto orbits and closures of orbits onto closures of orbits. $\square$

5. Topological characterization of matrices in projective spaces

Proposition 2.4 characterizes topologically the stable (and unstable) dimensions of a hyperbolic matrix $A \in gl(d, \mathbb{R})$, i.e., the parameter m^s of the short zero-Lyapunov form $S^0L(A)$. We proceed now to determine the number m of different Lyapunov exponents in $S^0L(A)$.

Denote by $\mathbb{P}^{d-1}$ the $(d-1)$-dimensional projective space. For $A \in gl(d, \mathbb{R})$ let φ be its linear flow in $\mathbb{R}^d$. The flow φ projects onto a flow $\mathbb{P}\varphi$ on $\mathbb{P}^{d-1}$, given by the differential equation

$$\dot{s} = h(s, A) = (A - s^{\mathrm{T}} A s I)s, \quad \text{with } s \in \mathbb{P}^{d-1}.$$

The Lyapunov spaces can be characterized by topological properties of the projective flow. Recall the following notions from topological dynamics (see e.g. [3] or [2, Appendix B]). For a flow Φ on a compact metric space Y a compact subset $K \subset Y$ is called *isolated invariant*, if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \text{int}N$, such that $\Phi(t, x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$. A *Morse decomposition* is a finite collection $\{\mathscr{M}_i, i = 1, \ldots, n\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that

(i) for all $x \in X$ one has $\omega(x), \alpha(x) \subset \bigcup_{i=1}^n \mathscr{M}_i$; and
(ii) suppose there are $\mathscr{M}_{j_0}, \mathscr{M}_{j_1}, \ldots, \mathscr{M}_{j_l}$ and $x_1, \ldots, x_l \in X \setminus \bigcup_{i=1}^n \mathscr{M}_i$ with $\alpha(x_i) \subset \mathscr{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathscr{M}_{j_i}$ for $i = 1, \ldots, l$;

then $\mathscr{M}_{j_0} \neq \mathscr{M}_{j_l}$.

The elements of a Morse decomposition are called *Morse sets*. Observe that $\mathscr{M}_i \preceq \mathscr{M}_j$, if $\alpha(x) \subset \mathscr{M}_i$ and $\omega(x) \subset \mathscr{M}_j$ for some x, defines an order on the Morse sets. A Morse decomposition is finer than another one, if all elements of the second one are contained in element of the first.

The following result is classical (compare, e.g., [2]).

Theorem 5.1. *Let $\mathbb{P}\varphi$ be the projection onto $\mathbb{P}^{d-1}$ of a linear flow $\varphi(t, x) = e^{At}x$. Then $\mathbb{P}\varphi$ has m chain recurrent components $\{\mathscr{M}_1, \ldots, \mathscr{M}_m\}$, where m is the number of different Lyapunov exponents of A. For each Lyapunov exponent λ_i we have $\mathscr{M}_i = \mathbb{P}L_i$, the projection of the ith Lyapunov space onto $\mathbb{P}^{d-1}$. Furthermore $\{\mathscr{M}_1, \ldots, \mathscr{M}_m\}$ defines the finest Morse decomposition of $\mathbb{P}\varphi$ and $\mathscr{M}_i \preceq \mathscr{M}_j$ iff $\lambda_i < \lambda_j$.*

Topologically equivalent flows $\mathbb{P}\varphi$ and $\mathbb{P}\psi$ have the same Morse decomposition:

Proposition 5.2. *For $A, B \in gl(d, \mathbb{R})$ let $\mathbb{P}\varphi$ and $\mathbb{P}\psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological equivalence h of $\mathbb{P}\varphi$ and $\mathbb{P}\psi$. Then the chain recurrent components $\mathscr{N}_1, \ldots, \mathscr{N}_n$ of $\mathbb{P}\varphi$ are of the form $\mathscr{N}_i = h[\mathscr{M}_i]$, where $\mathscr{M}_i$ is a chain recurrent component of $\mathbb{P}\varphi$. In particular the number of chain recurrent components of $\mathbb{P}\varphi$ and $\mathbb{P}\psi$ agree, and h maps the order on $\{\mathscr{M}_1, \ldots, \mathscr{M}_m\}$ onto the order on $\{\mathscr{N}_1, \ldots, \mathscr{N}_n\}$.*

Proof. The first part follows from Lemma 4.2. Correspondence of the orders follows immediately from the fact that h maps trajectories into trajectories, preserving their orientation. □

Indeed, much more can be said about the normal forms of A and B.

Proposition 5.3. *For A and B in $gl(d, \mathbb{R})$ let $\mathbb{P}\varphi$ and $\mathbb{P}\psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological equivalence h of $\mathbb{P}\varphi$ and $\mathbb{P}\psi$.*

Then the projective subspaces corresponding to real Jordan blocks of A are mapped onto projective subspaces corresponding to real Jordan blocks of B preserving the dimensions. Furthermore, h maps projective subspaces corresponding to real Jordan blocks for real eigenvalues, for complex eigenvalues whose imaginary part is rational modulo 2π, *and for eigenvalues whose imaginary part is irrational modulo* 2π, *onto projective subspaces of the same type.*

Proof. We may assume that A and B are in Jordan form. By Theorem 5.1 and Proposition 5.2 we can restrict our attention to one pair of Morse sets $\mathscr{N} = h[\mathscr{M}]$, with Lyapunov spaces $L_A, L_B \subset \mathbb{R}^d$.

(i) Consider first a Jordan block J corresponding to a real eigenvalue. Then the corresponding eigenvector is an equilibrium in $\mathbb{P}^{d-1}$. By Proposition 4.1(i) the same is true for the image under h, thus the projective eigenvector is mapped into a projective eigenvector corresponding to a real eigenvalue. Invoking the explicit solution formula one sees that for all other solutions starting in the corresponding subspace of $\mathbb{R}^d$ the projections to $\mathbb{P}^{d-1}$ tend for $t \to \pm\infty$ to this equilibrium. By continuity, the same is true for the images under h. Applying the same arguments to h^{-1}, one sees that the projective subspace corresponding to the Jordan block J is mapped onto a projective subspace corresponding to a Jordan block of B. Since h is a homeomorphism, both projective subspaces which are manifolds have the same dimension (invariance of domain theorem, [6]). By taking inverse images of the natural projection $\pi : \mathbb{R}^d \to \mathbb{P}^{d-1}$ it follows that the dimensions of the corresponding linear subspaces and hence of the Jordan blocks coincide.

(ii) Now consider a Jordan block corresponding to a complex eigenvalue whose imaginary part is rational modulo 2π. Then every solution in the corresponding projective real eigenspace is proper periodic and hence has a nontrivial closed orbit. By Proposition 4.1(ii) the same is true for the image under h which implies that it is a proper periodic orbit. Hence the projective eigenspace is mapped onto a projective eigenspace corresponding to an eigenvalue whose imaginary part is rational modulo 2π. Arguing as in (i) one sees that the projective subspace corresponding to the Jordan block J is mapped onto a projective subspace corresponding to a Jordan block of the same dimension.

(iii) Finally, consider a Jordan block corresponding to an eigenvalue whose imaginary part is irrational modulo 2π. Then every solution in the corresponding projective real eigenspace is dense and not closed. Again by Proposition 4.1(ii) the same is true for the image under h. Thus the image of the projective real eigenspace is a minimal closed invariant set which is not a periodic orbit. Hence it is contained in a projective eigenspace corresponding to an eigenvalue whose imaginary part is irrational modulo 2π. Arguing as in (ii) yields the assertion. □

Remark 5.4. We cannot give a complete characterization of matrices for which the projected linear flows on $\mathbb{P}^{d-1}$ are C^0-equivalent.

Proposition 5.3 shows that while C^0-equivalence of projected linear flows on $\mathbb{P}^{d-1}$ determines the number m of distinct Lyapunov exponents, it also characterizes the Jordan structure within each Lyapunov space (and, obviously, not the size of the Lyapunov exponents nor their sign). It imposes very restrictive conditions on the eigenvalues and the Jordan structure. Therefore, C^0-equivalences are not a useful tool to characterize m. The requirement of mapping orbits into orbits is too strong. A weakening leads us to the following characterization.

Theorem 5.5. *Two matrices A and B in $gl(d, \mathbb{R})$ have the same short Lyapunov form iff there exist a homeomorphism $h : \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ that maps the finest Morse decomposition of $\mathbb{P}\varphi$ onto the finest Morse decomposition of $\mathbb{P}\psi$, i.e., h maps Morse sets onto Morse sets and preserves their orders.*

Proof. Let $h : \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ be a homeomorphism that maps the finest Morse decomposition of $\mathbb{P}\varphi$ onto that of $\mathbb{P}\psi$, where φ is the linear flow of A and ψ is the linear flow of B. Let λ be a Lyapunov exponent of A with Lyapunov space $L_A(\lambda)$. The canonical projection $\pi : \mathbb{R}^d \to \mathbb{P}^{d-1}$ is a submersion, hence the Morse set $\mathscr{M}(\lambda) = \pi(L_A(\lambda))$ is a submanifold of $\mathbb{P}^{d-1}$; compare, e.g., [6]. Since h preserves the Morse decompositions, we have on the one hand that both Morse decompositions have the same cardinality. In particular, the numbers of Lyapunov exponents coincide by Theorem 5.1. On the other hand, the topological submanifold $h(\pi(L_A(\lambda))) = \pi(L_B(\mu))$ for some Lyapunov exponent μ and Lyapunov space $L_B(\mu)$ of B. Since h is a homeomorphism, $\pi(L_A(\lambda))$ and $\pi(L_B(\mu))$ have the same dimension (invariance of domain theorem, [6]). By taking inverse images of π it follows that the (linear) dimensions of $L_A(\lambda)$ and $L_B(\mu)$ coincide, and hence $SL(A) = SL(B)$.

For the converse, we order the Jordan blocks of A and B via

(a) order the real parts of the eigenvalues in increasing order, then within the blocks with the same real part,
(b) order the absolute values of the imaginary parts of the eigenvalues in increasing order, and within (a) and (b) by
(c) size of the Jordan block.

Then each similarity class in $gl(d, \mathbb{R})$ is uniquely determined. We may restrict our attention to the Lyapunov spaces L_A and L_B of corresponding Lyapunov exponents λ and μ of A and B, respectively. Take a basis $\mathscr{B}_A = \{x_1, \ldots, x_n\}$ of L_A adapted to the ordering above and similarly $\mathscr{B}_B = \{y_1, \ldots, y_n\}$ for L_B. Define $Tx_i = y_i$ for $i = 1, \ldots, n$. Using the same construction for all Lyapunov exponents of A, we arrive at $T \in GL(d, \mathbb{R})$. Its projection $\mathbb{P}T$ onto $\mathbb{P}^{d-1}$ is the desired homeomorphism. □

Corollary 5.6. *Assume that A and B are hyperbolic. Then A and B have the same short zero-Lyapunov form iff their linear flows in $\mathbb{R}^d$ are C^0-equivalent and there exists a homeomorphism $h : \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ that respects the finest Morse decomposition of their flows in $\mathbb{P}^{d-1}$.*

Proof. The proof combines Proposition 2.4 and Theorem 5.5. □

While Theorem 5.5 and Corollary 5.6 characterize the short (zero)−Lyapunov form of a matrix A in $gl(d, \mathbb{R})$, they are unsatisfactory in the sense that they are not constructive. The next section provides a constructive characterization using the induced flow on the flag manifold.

6. Topological characterization of matrices on flag manifolds

For a matrix $A \in gl(d, \mathbb{R})$ its linear flow φ on $\mathbb{R}^d$ induces flows on the Grassmannians and the flag manifold over $\mathbb{R}^d$. We first describe topological characteristics of these flows by specializing the results from Section 3 of [1] to the matrix case.

We denote by $\mathbb{G}_i$ the ith Grassmannian of i-dimensional subspaces of $\mathbb{R}^d$ (it may be identified with a subset of the projective space of the exterior product $\Lambda^i \mathbb{R}^d$). The kth flag of $\mathbb{R}^d$ is given by the following k-sequences of subspace inclusions,

$$\mathbb{F}_k = \{F_k = (V_1, \ldots, V_k),\ V_i \subset V_{i+1} \quad \text{and} \quad \dim V_i = i \text{ for } i = 1, \ldots, k\}.$$

For $k = d$ we obtain the complete flag $\mathbb{F} = \mathbb{F}_d$. For a matrix $A \in gl(d, \mathbb{R})$ we denote by $\mathbb{G}_i\varphi$ and $\mathbb{F}_k\varphi$ the induced flows on the Grassmannians and the flags, respectively.

The following result describes the Morse decomposition of $\mathbb{F}_k\varphi$; compare Theorem 5 of [1].

Theorem 6.1. *Let $A \in gl(d, \mathbb{R})$ with associated flows φ on $\mathbb{R}^d$ and $\mathbb{F}_k\varphi$ on the k-flag.*

(i) *For every $k \in \{1, \ldots, d\}$ there exists a unique finest Morse decomposition $\{{}_k\mathcal{M}_{i_j}\}$ of $\mathbb{F}_k\varphi$, where $i_j \in \{1, \ldots, d\}^k$ is a multiindex, and the number of chain transitive components in $\mathbb{F}_k$ is bounded by $\frac{d!}{(d-k)!}$.*

(ii) *Let $\mathcal{M}_i$ with $i \in \{1, \ldots, d\}^k$ be a chain recurrent component in $\mathbb{F}_{k-1}$. Consider the $(d - k + 1)$-dimensional vector bundle $\pi : \mathcal{W}(\mathcal{M}_i) \to \mathcal{M}_i$ with fibers*

$$\mathcal{W}(\mathcal{M}_i)_{F_{k-1}} = \mathbb{R}^d / V_{k-1} \quad \textit{for } F_k = (V_1, \ldots, V_{k-1}) \in \mathcal{M}_i \subset \mathbb{F}_{k-1}.$$

Then every chain recurrent component ${}_{\mathbb{P}}\mathcal{M}_{i_j}$, $j = 1, \ldots, k_i \leqslant d - k + 1$, of the projective bundle $\mathbb{P}\mathcal{W}(\mathcal{M}_i)$ determines a chain recurrent component ${}_k\mathcal{M}_{i_j}$ on $\mathbb{F}_k$ via

$${}_k\mathcal{M}_{i_j} = \{F_k = (F_{k-1}, V_k) \in \mathbb{F}_k : F_{k-1} \in \mathcal{M}_i \quad \textit{and} \quad \mathbb{P}(V_k/V_{k-1}) \in {}_{\mathbb{P}}\mathcal{M}_{i_j}\}.$$

Every chain recurrent component in $\mathbb{F}_k$ is of this form-this determines the multiindex i_j inductively for $k = 2, \ldots, d$.

Recall that the Grassmannian $\mathbb{G}_i$ is the subflag of the form $\{F = V_i, \dim V_i = i\}$. We obtain the following consequence of Theorem 6.1 ([1], Proposition 2).

Corollary 6.2. *On every Grassmannian $\mathbb{G}_i$ there exists a finest Morse decomposition. Its Morse sets are given by the projection of the chain recurrent components from the complete flag $\mathbb{F}$.*

Theorem 6.1 describes the topological structure of $\mathbb{F}_k\varphi$. Its constructive part (ii) can be made more explicit for the Grassmannians.

Theorem 6.3. *Let $A \in gl(d, \mathbb{R})$ be a matrix with flow φ on $\mathbb{R}^d$. Let L_i, $i = 1, \dots, m$, be the Lyapunov spaces of A, i.e., their projections $\mathbb{P}L_i = \mathscr{M}_i$ are the finest Morse decomposition of $\mathbb{P}\varphi$ on the projective space. For $k = 1, \dots, d$ define the index set*

$$I(k) = \{(k_1, \dots, k_m) : k_1 + \cdots + k_m = k \quad \text{and} \quad 0 \leqslant k_i \leqslant d_i = \dim L_i\}.$$

Then the finest Morse decomposition on the Grassmannian $\mathbb{G}_k$ is given by the sets

$$\mathscr{N}^k_{k_1,\dots,k_m} = \mathbb{G}_{k_1} L_1 \oplus \dots \oplus \mathbb{G}_{k_m} L_m, \ (k_1, \dots, k_m) \in I(k).$$

For a proof of Theorem 6.3 see [1], Theorem 6. and Remark 7.

These results define an order (with associated graph) on the flag manifold $\mathbb{F}$ via the Grassmannians:

On each $\mathbb{G}_k$, $k = 1, \dots, d$ we use the order $\preceq_k$ related to the finest Morse decomposition of $\mathbb{G}_k\varphi$. And for Morse sets $\mathscr{N}^k$, $\mathscr{N}^{k-1}$ in $\mathbb{G}_k$ and in $\mathbb{G}_{k-1}$, respectively, we set $\mathscr{N}^{k-1} \sqsubseteq_{k-1} \mathscr{N}^k$ if $\mathscr{N}^k$ projects down to $\mathscr{N}^{k-1}$. Combined, $\preceq_k$ and $\sqsubseteq_{k-1}$ define the graph of an order relation.

Finite graphs that represent orders are directed graphs without loops. For these graphs one can define "elementary graphs" that only consider "nearest neighbors", i.e., without edges that result from transitivity. Here the situation is slightly more complicated, since these graphs represent the d different orders $\preceq_k$ and the $(d-1)$ different orders $\sqsubseteq_{k-1}$. Since the order $\sqsubseteq_{k-1}$ only involves the Grassmannian $\mathbb{G}_k$ and $\mathbb{G}_{k-1}$, all these edges of these graphs are "nearest neighbors". Hence edges of $\sqsubseteq_{k-1}$ cannot be used in a transitive way without destroying the order. On the other hand, the orders $\preceq_k$ on each Grassmannian $\mathbb{G}_k$ involve all Morse sets on $\mathbb{G}_k$, hence elementary versions on each level k make sense. More precisely: Let G be an order graph in $\mathbb{F}$ and G_k its subgraphs corresponding to level k. An edge (e_1, e_2) in G_k is called a transitivity edge, if there exist nodes $n_1, \dots, n_l$, $l \geqslant 3$ such that $(n_1 = e_2, \dots, n_l = e_2)$ is a path in G_k. The elementary graph $E(G_k)$ has the same nodes as G_k, but with all transitivity edges removed. We arrive at the following definition.

Definition 6.4. Let $A \in gl(d, \mathbb{R})$ and consider the graph corresponding to the order relations $\preceq_k$ and $\sqsubseteq_{k-1}$ on the flag $\mathbb{F}$. The Grassmann graph of A is the graph obtained from this graph by replacing on each level k (i.e., in each Grassmannian) the corresponding subgraph by its elementary version.

One easily checks that the Grassmann graph of a matrix A is unique.

Remark 6.5. Theorem 6.3 describes an indexing system for the finest Morse decomposition on each Grassmannian $\mathbb{G}_k$ and hence on the complete flag $\mathbb{F}$ that corresponds to the parametrization of the short Lyapunov form via Definition 3.3(i) and (iii), see comment after Definition 3.5.

For the following examples we will use a different indexing system that is a little more intuitive. Let $A \in gl(d, \mathbb{R})$ have the short Lyapunov form $(m, d_1, \ldots, d_m)$ with $\sum_{i=1}^{m} d_i = d$. Then the finest Morse decomposition associated to the flow $\mathbb{P}\varphi$ on $\mathbb{P}^{d-1}$ has m Morse sets $\{\mathscr{M}_1, \ldots, \mathscr{M}_m\}$ that are linearly ordered (Theorem 5.1). We associate the canonical basis $\{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$ with the Morse sets in such a way that each Morse set $\mathscr{M}_i$ corresponds to d_i basis vectors (in the respective orders), i.e.,

$$\mathscr{M}_1 \sim \{e_1, \ldots, e_{d_1}\},$$
$$\mathscr{M}_i \sim \{e_{\alpha_i}, \ldots, e_{\beta_i}\} \quad \text{where } \alpha_i = \sum_{j=1}^{i-1} d_j + 1,\ \beta_i = \sum_{j=1}^{i} d_j.$$

On $\mathbb{G}_1 = \mathbb{P}^{d-1}$ we index the Morse decomposition simply as $\mathscr{M}_i$. Each Morse set on $\mathbb{G}_2$ has two indices (j_1, j_2) and $\mathscr{M}_i \sqsubseteq \mathscr{M}_{j_1, j_2}$ iff $i \in \{j_1, j_2\}$. Several of the sets $\mathscr{M}_{j_1, j_2}$ on $\mathbb{G}_2$ may be identical. In this case we use the index pair with smallest numbers in each entry. Observe that the order relations are retained. Continuing for $\mathbb{G}_2, \ldots, \mathbb{G}_d$ we obtain unique indexes for all Morse sets on $\mathbb{G}_k$, and hence on the flag $\mathbb{F}$.

As described above, the order $\sqsubseteq_k$ can be read off the indexes immediately and $\sqsubseteq_k$ can be constructed explicitly. Furthermore, we have for the orders $\preceq_k$ on G_k:

$$\mathscr{M}_{(j_1,\ldots,j_k)} \preceq_k \mathscr{M}_{(j'_1,\ldots,j'_k)} \quad \text{iff} \quad j_i \leqslant j'_i \quad \text{for all } i = 1, \ldots, k.$$

The following simple examples illustrate some properties of Grassmann graphs. In particular, the first example shows that one cannot expect to determine the dimensions of the Lyapunov spaces from the Morse sets on the full flag This is the reason why we introduce Grassmann graphs (instead of graphs obtained from the Morse sets on the full flag).

Example 6.6. Consider the matrices

$$A = \mathrm{diag}(-1, -1, 1) \quad \text{and} \quad B = \mathrm{diag}(-1, 1, 1).$$

We obtain the following structure for the finest Morse decompositions for A:

$$\mathbb{G}_1: \quad \mathscr{M}_1 = ls\{e_1, e_2\},$$
$$\mathscr{M}_3 = ls\{e_3\}.$$

$$\begin{aligned}
\mathbb{G}_2\colon\quad & \mathscr{M}_{1,2} = ls\{e_1, e_2\},\\
& \mathscr{M}_{1,3} = \{ls\{x, e_3\},\ \text{with } x \in ls\{e_1, e_2\}\}.\\
\mathbb{G}_3\colon\quad & \mathscr{M}_{1,2,3} = ls\{e_1, e_2, e_3\}.
\end{aligned}$$

This leads to the Grassmann graph of A

$$\begin{array}{ccccc}
 & & \mathscr{M}_{1,2,3} & & \\
 & \nearrow & & \nwarrow & \\
\mathscr{M}_{1,2} & & \longrightarrow & & \mathscr{M}_{1,3}. \\
\uparrow & & \nearrow & & \uparrow \\
\mathscr{M}_1 & & \longrightarrow & & \mathscr{M}_3
\end{array}$$

In the same way one obtains the Grassmann graph of B

$$\begin{array}{ccccc}
 & & \mathscr{M}_{1,2,3} & & \\
 & \nearrow & & \nwarrow & \\
\mathscr{M}_{1,2} & & \longrightarrow & & \mathscr{M}_{2,3}. \\
\uparrow & & \nwarrow & & \uparrow \\
\mathscr{M}_1 & & \longrightarrow & & \mathscr{M}_2
\end{array}$$

On the other hand, the Morse sets in the full flag are given for A and B by

$$\begin{bmatrix} \mathscr{M}_{1,2,3} \\ \mathscr{M}_{1,2} \\ \mathscr{M}_1 \end{bmatrix} \preceq \begin{bmatrix} \mathscr{M}_{1,2,3} \\ \mathscr{M}_{1,3} \\ \mathscr{M}_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathscr{M}_{1,2,3} \\ \mathscr{M}_{1,2} \\ \mathscr{M}_1 \end{bmatrix} \preceq \begin{bmatrix} \mathscr{M}_{1,2,3} \\ \mathscr{M}_{2,3} \\ \mathscr{M}_2 \end{bmatrix},$$

respectively. Thus in the full flag the numbers and the orders of the Morse sets coincide, while the Grassmann graphs are different.

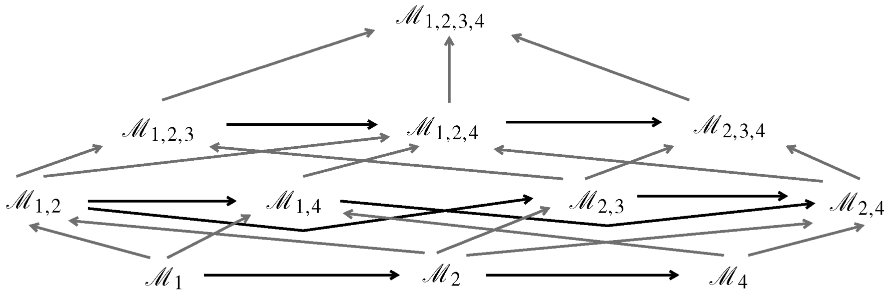
Example 6.7. Consider the matrix

$$B = \operatorname{diag}(-2, -1, -1, 1).$$

The finest Morse decomposition is given by

$$\begin{aligned}
\mathbb{G}_1\colon\quad & \mathscr{M}_1 = ls\{e_1\},\\
& \mathscr{M}_2 = ls\{e_2, e_3\},\\
& \mathscr{M}_4 = ls\{e_4\}.\\
\mathbb{G}_2\colon\quad & \mathscr{M}_{1,2} = \{ls\{e_1, x\}, \quad \text{with } x \in ls\{e_2, e_3\}\},\\
& \mathscr{M}_{1,4} = ls\{e_1, e_4\},\\
& \mathscr{M}_{2,3} = ls\{e_2, e_3\},\\
& \mathscr{M}_{2,4} = \{ls\{x, e_4\}, \quad \text{with } x \in ls\{e_2, e_3\}\}.\\
\mathbb{G}_3\colon\quad & \mathscr{M}_{1,2,3} = ls\{e_1, e_2, e_3\},\\
& \mathscr{M}_{1,2,4} = \{ls\{e_1, x, e_4\}, \quad \text{with } x \in ls\{e_2, e_3\}\},\\
& \mathscr{M}_{2,3,4} = ls\{e_2, e_3, e_4\}.\\
\mathbb{G}_4\colon\quad & \mathscr{M}_{1,2,3,4} = ls\{e_1, e_2, e_3, e_4\}.
\end{aligned}$$

This leads to the Grassmann graph of B



Note that on the level $\mathbb{G}_2$ the sets $\mathscr{M}_{1,4}$ and $\mathscr{M}_{2,3}$ are not comparable with respect to the order $\preceq_2$.

We proceed to discuss how one can regain information on the Lyapunov spaces from the Grassmann graph.

Definition 6.8. Let G be the Grassmann graph of a matrix $A \in gl(d, \mathbb{R})$. An *increasing path* p in G is a path from level $\mathbb{G}_1$ to level $\mathbb{G}_d$ that follows the order, $\sqsubseteq_1, \dots, \sqsubseteq_{d-1}$. The *in-order* of a node $n \in G$ is the number of edges that end in n and the *out-order* is the number of edges that begin in n. For an increasing path $p = (n_1, \dots, n_d)$ in G we define its *simple length*

$$\mathrm{sl}(p) = \max\{k,\ \text{in-order}(n_k) \leqslant 1\}.$$

For a node n on the level $\mathbb{G}_1 = \mathbb{P}^{d-1}$ we define its *multiplicity* as

$$\mathrm{mult}(n) = \max\{\mathrm{sl}(p),\ p \text{ is an increasing path with initial node } n\}.$$

Lemma 6.9. *Given a matrix $A \in gl(d, \mathbb{R})$ and a Lyapunov exponent λ_i of A with Lyapunov space $L_i \subset \mathbb{R}^d$. Denote the corresponding Morse set of the flow $\mathbb{P}\varphi$ by $\mathscr{M}_i = \mathbb{P}L_i \subset \mathbb{P}^{d-1}$. Then the multiplicity* mult$(\mathscr{M}_i)$ *of $\mathscr{M}_i$ in the Grassmann graph of A is equal to the* (*linear*) *dimension* $\dim L_i$.

Proof. Follows directly from Theorem 6.3. □

The lemma above says that one can recover the dimensions of the Lyapunov spaces from the orders $\sqsubseteq$ on the Grassmann graph of a matrix. Furthermore, the order of the Lyapunov exponents (and hence the Lyapunov spaces) can be recovered from the order $\preceq$ on level $\mathbb{G}_1$ of the graph, compare Theorem 5.1. Hence we can hope to use Grassmann graphs for the characterization of the short Lyapunov form of a matrix.

Definition 6.10. Let G and G' be finite directed graphs. A map $h : G \to G'$ is called a *graph homomorphism* if for all edges (n_1, n_2) in G, $(h(n_1), h(n_2))$ is an edge in

G'. Furthermore, h is a graph isomorphism if h is bijective and h and h^{-1} are graph homomorphisms.

Theorem 6.11. *The short Lyapunov form $SL(A)$ and $SL(B)$ are identical for two matrices $A, B \in gl(d, \mathbb{R})$ iff the Grassmann graphs of A and B are isomorphic.*

Proof. Let the Grassmann graphs $G(A)$ and $G(B)$ be isomorphic.

(i) We construct the orders $\preceq$ and $\sqsubseteq$ as follows. The only node with out-order 0 is the unique node n_l on the highest level l. All nodes n for which there is an edge (n, n_l) are on the level $l-1$. All nodes n' that are not on level $l-1$ and for which there is an edge (n', n) with n on level $l-1$, are on level $l-2$, etc.This algorithm stops after l' steps, i.e., after determining the nodes on level l', and all nodes are associated with some level. Then $l - l' + 1 = d$, the dimension of the underlying $\mathbb{R}^d$. We reindex the levels such that the smallest level is 1. Then, the edges between nodes on the same level k determines the order $\preceq_k$. And edges between nodes on levels $k-1$ and k determine the order $\sqsubseteq_{k-1}$. Note that the node corresponding to the Morse set $\mathscr{M}_1$ on $\mathbb{G}_1$ is the unique node with in-order 0.

(ii) The length of any increasing path $(n^1, n^2, \ldots, n^d)$ determines the dimension of the underlying space $\mathbb{R}^d$.

(iii) For each node in level $\mathbb{G}_1$, its multiplicity defines the dimension of the corresponding Lyapunov space.

(i)–(iii) mean that for any matrix its short Lyapunov form can be uniquely reconstructed from the Grassmann graph, hence isomorphic Grassmann graphs belong to matrices with identical short Lyapunov form. Vice versa, short Lyapunov forms determine Grassmann graphs by their construction (based on Theorems 6.1 and 6.3). □

Remark 6.12. Given a finite directed graph G without loops, there is a constructive algorithm to decide if G is the Grassmann graph of a matrix. G needs to have a unique edge n_l with out-order 0 (and a unique edge with in-order 0; this edge corresponds to the Morse set $\mathscr{M}_1$).Starting from the "maximal" edge g_l, we proceed as in steps (i)–(iii) from the proof of Theorem 6.11 to identify nodes on the different levels as well as the multiplicities of each edge on the lowest level 1. (Note that one can use the same procedure if the graphs within each level have not been replaced by their elementary versions; then one has to perform this replacement here.) With this information we use Theorem 6.3 to construct the Grassmann graph based on the order on level 1 and the multiplicity of the nodes. The graph so constructed is compared to the graph G to decide whether it is indeed the Grassmann graph of a matrix.

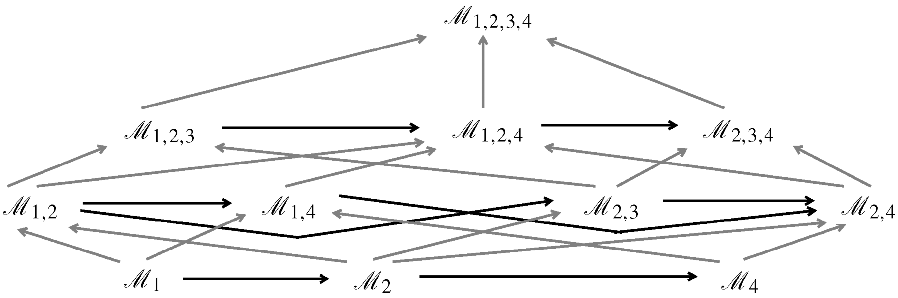
Remark 6.13. Graph isomorphisms define an equivalence relation on the set of all graphs. The corresponding equivalence classes of Grassmann graphs can be parametrized as in Remark 3.3(i) and (iii). This parametrization corresponds to the construction of the finest Morse decomposition on the Grassmannians $\mathbb{G}_k$ as in Theorem 6.3.

Example 6.14. Consider the two matrices

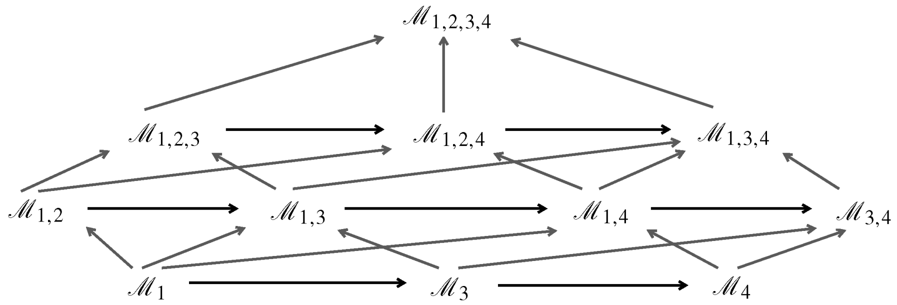
$$A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Their short Lyapunov form are $SL(A) = (3, 1, 2, 1)$ and $SL(B) = (3, 2, 1, 1)$. Hence the matrices are not Lyapunov equivalent. In their Grassmann graphs this is reflected in the following way.

The Grassmann graph $G(A)$ is given by:



On the other hand, the Grassmann graph $G(B)$ is given by:



Obviously $G(A)$ and $G(B)$ are not isomorphic, e.g., the out-orders of $\mathcal{M}_1$ in $G(A)$ and $G(B)$ are different.

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