

RELIABILITY ASSESSMENT OF DYNAMICAL SYSTEMS WITH RANDOM EXCITATION

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Abstract

During recent years, a variety of important interconnections between the theory of nonlinear dynamical systems, nonlinear stochastically excited systems, and nonlinear control systems have been obtained by mathematicians, physicists, and engineers. This paper is the beginning of a systematic study of reliability aspects within the emerging unified view of the different classes of nonlinear dynamical systems. We study the response behavior of dynamical systems with random excitation depending on two parameters: a bifurcation parameter $\alpha \in \mathbb{R}$, and a parameter $\rho \leq 0$ that measures the strength of the disturbance. The goal of this paper is to obtain precise stability and reliability diagrams in α - ρ -space. In particular, we characterize for each parameter combination the levels that the (maximal) system response will reach with probability 1, with positive probability, or with probability 0. Emphasis is placed on determining the parameter combinations for which the system behavior changes drastically, i.e. on stochastic bifurcation phenomena, to determine crucial system parameters and their critical values.

1. Introduction

Reliability assessment of dynamical systems with random excitation addresses the problems of maximal systems response (instantaneous collapse) and of sustained vibrations (fatigue). For nonlinear systems with non-Gaussian excitation the standard techniques of (statistical) reliability theory may not prove adequate, in particular if the system depends in a nonlinear fashion on parameters, or if a nonlinear control design is sought to prevent collapse and fatigue.

Recent advances in nonlinear control theory, ergodic theory of stochastic dynamical systems, stochastic bifurcation theory, and control of bifurcations allow the development of a dynamic reliability theory that takes nonlinear dynamics, non-Gaussian noise

and nonlinear control design into account. This paper presents an outline of such a theory, and reports first results.

Consider a nonlinear dynamical system depending on a real (bifurcation) parameter of the form

$$(1) \quad \dot{x} = X_0(x; \alpha)$$

in \mathbb{R}^d or on a smooth manifold M (of dimension $d < \infty$), where X_0 is a smooth vectorfield, and $\alpha \in I \subset \mathbb{R}$ is a bifurcation parameter. The system is subject to a random excitation, resulting in

$$(2) \quad \dot{x} = X_0(x; \alpha) + \sum_{i=1}^m \xi_i(t) X_i(x; \alpha) \quad \text{on } M,$$

where $\xi(t) = (\xi_i(t), i = 1 \dots m)$ is a stochastic process, and $X_i(x; \alpha), i = 1 \dots m$, are the (smooth) excitation dynamics. We assume that the excitation process $\xi(t)$ comes from an underlying nondegenerate stationary Markov diffusion process of the form

$$(3) \quad d\eta(t) = Y_0(\eta(t))dt + \sum_{i=1}^{\ell} Y_i(\eta(t)) \circ dW_i, \\ \xi(t) = f(\eta(t)),$$

where $\eta(t)$ lives on a smooth manifold N (of finite dimension), $Y_0 \dots Y_{\ell}$ are smooth vector fields, " \circ " denotes the symmetric (Stratonovič) differential and f is a C^∞ map. If Y_0 is linear and if $Y_1 \dots Y_{\ell}$ are constant on $N = \mathbb{R}^p$, then η is an Ornstein-Uhlenbeck process, with Gaussian statistics. We are interested in the realistic situation that N is compact, and hence $\eta(t)$ and $\xi(t)$ are non-Gaussian.

In order to study the behavior of system (2) depending on excitation range and on excitation statistics, we consider a family $\{U_\rho, \rho \geq 0\}$ of compact subsets of \mathbb{R}^m and a family $\{f_\rho, \rho \geq 0\}$ of onto maps $f_\rho : N \rightarrow U_\rho$, mapping the background noise $\eta(t)$

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onto the random excitation $\xi(t)$ in U_ρ . Thus system (2) depends on two parameters: the bifurcation parameter α and the range ρ of the disturbance.

In this paper we study the problem of level crossing, i.e. we characterize for each $\alpha - \rho$ combination the levels that the (maximal) system response will reach with probability 1, with positive probability, or with probability 0. Furthermore, we characterize the multistability regions of the system, consisting of those points, from which the response will reach multiple, distinct areas in the state space with positive probability.

2. Ergodic theory of nonlinear stochastic systems

Ergodic theory of nonlinear stochastic systems describes the systems' long term behavior and hence identifies almost sure maximal response and areas of sustained vibrations for large time, i.e. for $t \rightarrow \infty$. In the context of reliability theory the time it takes to reach certain levels of system response and to reach sustained oscillations are also of importance. Therefore we present a generalized classification of the states of the system that includes ergodic states as a special case.

Two different cases have to be considered separately:

Singular systems: The vector fields $X_0 \dots X_m$ have a common limit set, e.g. a fixed point $x^0 \in M$ with $X_0(x^0) = \dots = X_m(x^0) = 0$.

Regular systems: The system dynamics X_0 and the excitation dynamics $X_1 \dots X_m$ effect all components of the states $x \in M$, i.e. $\dim \mathcal{LA}\{X_0 \dots X_m\}(x) = \dim M$ for all $x \in M$. Here $\mathcal{LA}\{X_0 \dots X_m\}$ is the Lie algebra generated by $X_0 \dots X_m$, and $\dim \mathcal{LA}(x)$ denotes the dimension of the distribution generated by \mathcal{LA} in $T_x M$, the tangent space of M at the point $x \in M$.

If $\dim \mathcal{LA}(x) < \dim M$, then for integrable systems without common limit sets, the system can be considered on its maximal integral manifolds, where it is regular. Ergodic theory of singular systems needs additional techniques like linearization and Lyapunov exponents, which are beyond the scope of this short paper. Hence we restrict our attention to regular systems throughout this paper.

For the remainder of this section fix α and ρ in the randomly excited system (2). This system itself is not a Markov process, but the 'pair process' given by (2) and (3) is Markovian. The long term behavior of (2,3) can be described by associated control systems via the support theorem. Under our assumptions that $\eta(t)$ in (3) is nondegenerate and stationary, it suffices to describe the associated control structure of (2).

The control system associated to (2) is given by

(neglecting the fixed α and ρ)

$$(2C) \quad \dot{x} = X_0(x) + \sum_{i=1}^m u_i(t) X_i(x) \quad \text{on } M,$$

where $u(t) = (u_i(t), i = 1 \dots m) \in \mathcal{U} = \{u : \mathbb{R} \rightarrow U; \text{measurable}\}$ with $U \subset \mathbb{R}^m$ compact and $0 \in \text{int } U$. This last assumption guarantees that the undisturbed system (1) is a special case of (2) and (2C). The possible limit sets of the random system (2) are the invariant control sets of (2C), see [3]. Attraction and stability of these limit sets are described by the domains of attraction of the control sets:

Let $D \subset M$ be a control set of (2C) (compare e.g. [1] for the definition and basic properties of control sets) and define its domain of attraction by

$$A(D) = \{x \in M; \text{there exists } u \in \mathcal{U} \text{ and } t \geq 0 \text{ such that } \varphi(t, x, u) \in D\},$$

where $\varphi(t, x, u)$ is the solution of (2C) at time t with initial value $\varphi(0, x, u) = x$, under the control $u \in \mathcal{U}$. We obtain the following result:

1. Theorem. Let $C_j, j = 1 \dots n$ be the invariant control sets of (2C). Then for each $x \in M$ there exist $n+1$ numbers $p_j(x), p_\infty(x)$, with $j = 1 \dots n$ such that

$$p_\infty(x) + \sum_{j=1}^n p_j(x) = 1 \text{ and}$$

$$P\{\psi(t, x, \omega) \rightarrow C_j \text{ for } t \rightarrow \infty\} = p_j(x)$$

$$P\{\psi(t, x, \omega) \text{ leaves every compact set in } M \text{ for } t \rightarrow \infty\} = p_\infty(x).$$

Here $\psi(t, x, \omega)$ denotes the (pathwise) solutions of (2). If the invariant control sets are compact and if $p_\infty(x) = 0$, then $\psi(t, x, \cdot)$ converges in distribution to $\sum_{j=1}^n p_j(x) \mu_j$ for $t \rightarrow \infty$, where μ_j is the unique invariant probability measure of (2) in C_j . Furthermore, $p_j(x) > 0$ iff $x \in A(C_j)$.

While variant control sets of (2C) cannot carry invariant probabilities of (2), and hence do not contain limit sets of (2), the stochastic system will experience oscillations and slow down in these sets. Hence they are important from a reliability point of view, compare the next section. Furthermore, a certain type of variant control sets characterizes the multistability regions of the system, compare Section 4.

3. Maximal response and oscillations of nonlinear stochastic systems

In this section we work under the assumptions of Theorem 1. In order to describe the maximal response of the nonlinear random dynamical system

(2), we need a distance function on the state space M . If $M = \mathbb{R}^d$, the Euclidean norm is often a natural choice. However, this norm weighs all components of the state x uniformly, which may not always be appropriate. E.g. if x measures position and velocity, different reliability bounds may be placed on the two components. Therefore, we introduce the concept of level sets:

2. Definition. A family $\{L_\rho, \rho \geq 0\}$ of sets in M is called a family of level sets, if $L_\rho = \{y \in M, r(x_0, y) \leq \rho\}$ for some reference point $x_0 \in M$. Here $r(\cdot, \cdot)$ denotes a metric on M .

3. Theorem. Under the conditions of Theorem 1., the maximal system response of (2) for $t \rightarrow \infty$ from $x \in M$ with respect to the family $\{L_\rho, \rho \leq 0\}$ of level sets is given by $\bar{\delta}(x) = \sup\{\delta \geq 0; L_\delta^c \cap \cup C_j \neq \emptyset \text{ where the union is taken over all } C_j \text{ with } p_j(x) > 0\}$. (c denotes the complement of a set.) The probability of reaching $\bar{\delta}(x)$ from x is $\sum p_j(x)$, where the sum is taken over all j with $cL_\delta^c(x) \cap C_j \neq \emptyset$. (c denotes the closure of a set.) Furthermore, the trajectories of (2), starting from x , undergo sustained oscillations in each C_j with $p_j(x) > 0$. If C_j is compact, these oscillations cover a dense set in C_j and their empirical measure approaches the invariant probability μ_j as $t \rightarrow \infty$.

Theorem 2. goes a long way in describing the maximal response and sustained oscillations for nonlinear dynamical systems with random excitation: It describes the long term behavior for $t \rightarrow \infty$ and the probability of its occurrence. The transient, i.e. finite time behavior of system (2) can be important for reliability assessment under at least the following circumstances: a) The system may reach a higher response level $\bar{\delta}$ before settling in the invariant control sets, and b) the time to reach the response level $\bar{\delta}$ in an invariant control set may be too long for practical considerations.

- a) The maximal system response from $x \in M$ with respect to $\{L_\delta, \delta \geq 0\}$ is given by the positive orbit of (2C) $O^+(x) = \{y \in M; \text{there exist } u \in U \text{ and } t \geq 0 \text{ with } \varphi(t, x, u) = y\}$. For general nonlinear stochastic systems like (2) there are no analytical methods to compute the probability of reaching the maximal systems response within $O^+(x)$, or the probability of reaching a predetermined level δ . This has to be done numerically.
- b) While it is relatively easy to obtain estimates for the system velocity outside of control sets, and hence estimates for the time the system takes to travel through sets outside of control sets, similar estimates for control sets are impossible to obtain analytically. The reason is that in variant control sets D there

is a positive probability that the system will oscillate through all of D before leaving the set: This slowing down effect may be strong enough to keep the system in D for a sufficiently long time period to ensure practical reliability. Again, numerical simulations are needed for estimates of the corresponding crossing times through D .

4. Multistability regions of nonlinear stochastic systems

Of particular importance for reliability assessment of dynamical systems with random excitation are those points in the state space M , from which different limit sets (invariant control sets of (2C)) are reached with positive probability: In these multistability regions small perturbations may lead to drastically different systems response. In order to describe multistability phenomena, we work in a compact, forward invariant subset $X \subset M$. Note that in X the system (2C) has only finitely many invariant control sets $C_1 \dots C_n$, each of which carries an invariant measure $\mu_j, j = 1, \dots, n$ of (2).

4. Definition. A point $x \in X$ is said to be multistable, if there exist at least two different invariant probability measures μ_i in X with $P\{\psi(t, x, \cdot) \rightarrow \text{supp } \mu_i\} > 0$.

Note that there exists a multistable point $x \in X$ iff (2C) has at least two invariant control sets in X . The multistability region is described by the 'relatively invariant control sets':

5. Definition. Let $MS \subset X$ be the set of multistable points. A control set $D \subset MS$ is called relatively invariant, if $x \in D$ and $\varphi(t, x, u) \notin D$ for some $t > 0, u \in U$ implies $\varphi(t, x, u) \notin MS$.

We obtain the following characterization of the multistability region MS :

6. Theorem. Consider the stochastic system (2), and let $X \subset M$ be a compact set such that all limit sets of (2C) have positive distance from the boundary of X . Then the set MS of multistable points is given

by $MS = \bigcup_{j=1}^k A(D_j)$, where $D_j, j = 1 \dots \ell$, are the relatively invariant control sets.

For a proof see [4]. Theorem 6. says in particular that the set MS is open and consists of at most finitely many components in X . The computation of MS reduces to the computation of the finitely many, relatively invariant control sets and their domains of attraction. In general, this has to be done numerically. For one-dimensional systems a control set is relatively invariant iff it is open, and hence coincides with its domain of attraction.

5. Reliability of randomly excited systems depending on a parameter

Randomly excited system usually depend on a set of parameters that are not fixed, but can be used for design and/or control purposes in a way such that the resulting system has a desired behavior, e.g. with respect to reliability specifications. Design parameters can include e.g. damping (material constants) or kinetic constants (geometry of a system), control may involve feedback laws (for stabilization) etc. In this section we analyze reliability aspects of stochastic systems depending on a parameter $\alpha \in I \subset \mathbb{R}$ and draw some conclusions for optimal reliable design in Section 6.

In Sections 3. and 4. we saw that control sets of the associated control system (2C) play a crucial role in assessing reliability of nonlinear dynamical systems with random excitations of the form (2). Thus we will first analyze bifurcations of control sets for nonlinear control systems. The dependence of the control sets on two parameters α and ρ has to be taken into account: α describes the change in local behavior of the undisturbed system (1) and of the excitation vector fields $X_1 \dots X_m$, while ρ determines, how much of the global behavior of the system affects the change of control sets.

Fix α for now and consider the dependence of control sets on ρ : Assume that the undisturbed system (1) has a finest Morse decomposition $\{M_1, \dots, M_r\}$ with finitely many Morse sets, ordered by \prec (compare [2] for details). The following theorem was proved in [1] under the 'inner pair condition': The pair $(x, u) \in M \times U$ is called an inner pair of (2C) if there exist times $T, S > 0$ such that $\varphi(T, x, u) \in \text{int } \mathcal{O}_{\leq T+S}^+(x)$. Here $\mathcal{O}_{\leq T+S}^+(x)$ denotes the points of M that are reachable in (2C) up to time $T + S$, and "int" is the interior of a set.

7. Theorem. Assume that for all $x \in \bigcup_{k=1}^r M_k$ of (1) the pair $(x, 0)$ is an inner pair of $(2C_\rho)$ for $\rho > 0$. Then for each $M_k, k = 1 \dots r$ and each $\rho > 0$ there exists a control set D_k^ρ such that $M_k \subset \text{int } D_k^\rho$, and the limits $\lim_{\rho \downarrow 0} D_k^\rho$ are exactly the Morse sets of (1). Furthermore, for small ρ the order between the Morse sets of (1) and the control sets of $(2C_\rho)$ agree. In particular, the invariant control sets are in 1-1 correspondence with maximal Morse sets, i.e. attractors, of (1).

In the context of reliability theory Theorem 7. asserts that maximal system response levels are determined by the attractors of (1) for small random excitations, which is a rather intuitive result, but it need not hold for non-Markovian disturbances. As the excitation range grows, control sets belonging to differ-

ent Morse sets may merge. If invariant and variant control sets for $\rho < \rho_0$ merge into one control set for $\rho > \rho_0$, the resulting control set may be variant, and it cannot be a limit set for the random system (2). Hence the maximal system response will jump to a different value at this point, and a slowing down behavior will occur in the merged control set. Thus ρ_0 indicates a reliability bifurcation point, see Section 6. for an example.

In order to analyze the reliability behavior depending on the bifurcation parameter α , one has to study the specific bifurcation scenarios of system (1), because different scenarios affect the bifurcation of control sets and hence reliability aspects in different ways. If one is interested in the maximal response of (2) with respect to the Euclidian norm, (1) may be reduced to a one-dimensional system for $|x|$. The following theorem is an example of the possible bifurcation behavior of control sets in one-dimensional systems.

8. Theorem. Consider the randomly excited system (2) in \mathbb{R}^1 . Assume that the undisturbed system (1) undergoes a transcritical bifurcation at $\alpha_0 \in I$, and that the bifurcation branches are continuously differentiable in α around α_0 .

- (i) If the derivatives of the bifurcation branches at α_0 have opposite signs, then there exists $\rho_0 > 0$ such that the system (2C) has a variant and an invariant control set for all $\rho < \rho_0$ in a neighborhood of α_0 . The control sets depend continuously on ρ and α for $\rho < \rho_0$.
- (ii) If the derivatives of the bifurcation branches at α_0 have the same sign, then there exists $\rho_0 > 0$ such that the system (2C) has exactly one control set for all $\rho < \rho_0$ in a neighborhood of α_0 , and this control set is variant.

For the reliability behavior of system (2) this theorem has the following consequences: In case (i) the maximal systems response and the areas of sustained oscillations vary continuously with α in a neighborhood of α_0 . They are determined by the bifurcation branches of the undisturbed system (1), i.e. by the sets of stable and unstable fixed points. In case (ii) the system exhibits in a neighborhood of α_0 two reliability bifurcation points for small excitation ranges ρ . In particular, the maximal response jumps at these two points and there is one area of sustained oscillations, given by a variant control set, with large crossing time, i.e. further reliability assessment from a practical point of view has to take estimates of this crossing time into account.

6. An Example

We consider a dynamical system in $M = \mathbb{R}^1$, whose bifurcation diagram is given in Figure 1. Let x_0 be a critical threshold, below which the state x (e.g. the norm of the system response) is not allowed to fall. In

the deterministic case for $\alpha \leq \alpha_0$, the system will fulfill this requirement for initial values above the upper bifurcation branch.

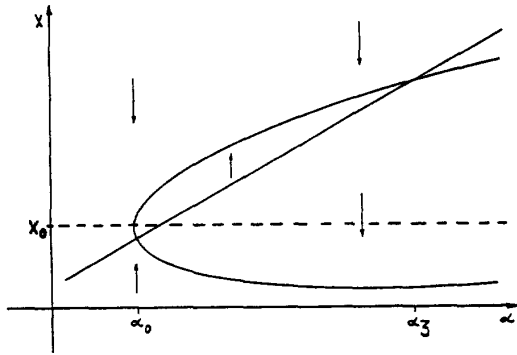


Figure 1. Deterministic bifurcation diagram with pitchfork and transcritical bifurcation.

Now consider the same system, but with a bifurcation parameter α that is randomly excited by external or internal noise. Figure 2a. indicates the behavior of the stochastic system for an excitation range 2ρ , i.e. α_t is a stochastic process with values in $[\alpha - \rho, \alpha + \rho]$. For $\alpha \leq \alpha_1$ the stochastic system response is not surprising: The system will almost surely converge towards the region A around the lower bifurcation branch, thus crossing the critical threshold x_0 with probability 1. In order to retain reliability under stochastic excitation of size 2ρ , the system has to be designed in such a way that $\alpha > \alpha_1$. What is surprising, is the behavior for $\alpha \in (\alpha_2, \alpha_4)$: there the randomness of α will turn the region around the upper bifurcation branches into a transient region, i.e. after a relatively long transience time the system will drop to the lower bifurcation branch, thus crossing the threshold x_0 with probability 1. The reason is that the noise exploits some of the nonlocal dynamics around $\alpha = \alpha_3$, resulting in a transient region for the random system. Figure 2b. shows the minimal levels that will be reached by the system response with probability one for initial values above the curve.

The area B in Figure 2a. is a bistability region. For initial values in B there is a positive probability that the system response will take values in the upper and in the lower A-areas. (Once the solution enters one of these areas, its distribution will converge towards the invariant measure whose support is the corresponding A- set). Hence, for each $x \in B$ there

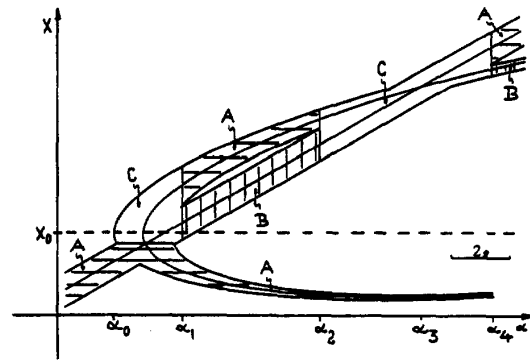
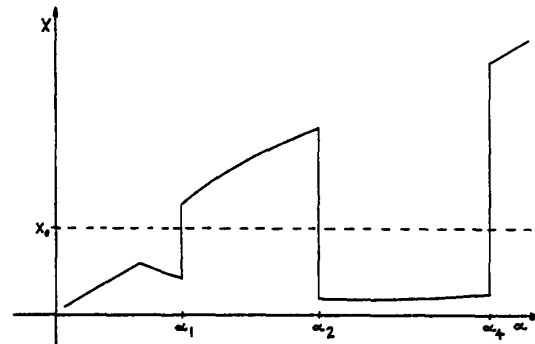


Figure 2: a) system of Figure 1. with stochastic excitation of size 2ρ



b) Minimal levels that are reached with probability 1

exists a probability $q(x)$ with $0 < q(x) < 1$ such that the system response from x crosses the level x_0 with probability $q(x)$.

Effects like the ones discussed here show how the interaction between deterministic dynamics and stochastic excitation can drastically alter the response behavior of a system.

7. References

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