Dirac fermions with disorder in two dimensions: Exact results

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Dirac fermions on a two-dimensional lattice with disorder are considered. The Dirac mass, which controls the gap between the two bands of the fermions, is subject to random ¯uctuations. Another type of disorder is discussed presented by a random vector potential. It is shown that the imaginary part of the one-particle Green's function can be written as the imaginary part of another Green's function, which has only poles on the lower half plane. Therefore, it is possible to perform a Cauchy integration for a Lorentzian distribution in analogy with the Lloyd model. The results are compared with calculations performed in the continuum limit based on renormalization group and bosonization methods.

I. INTRODUCTION

The density of states (DOS) in two-dimensional electron systems with a pseudogap is a subject of interest for a number of physical situations discussed recently.[1±9] A typical model with a pseudogap is represented by the Dirac Hamiltonian in two dimensions

$$H = i\nabla_1\sigma_1 + i\nabla_2\sigma_2 + m\sigma_3, \tag{1}$$

where σ_μ are Pauli matrices including the 2×2 unit matrix σ_0. The Dirac equation for a state ψ then is $-\partial\psi/\partial t = H\psi$. The dispersion relation is $E(k_1,k_2) = \pm\sqrt{m^2+k_1^2+k_2^2}$ in the continuum limit. (The lattice will be considered later.) The two signs describe the particle and the hole band, respectively. Both bands touch each other if the Dirac mass vanishes, as one can see in the DOS $\rho(E) \propto |E|\Theta(E^2-m^2)$, where Θ is the step function. The touching bands are also a feature of a second-order phase transition because the decay length of the corresponding Green's function diverges as one goes to the special (critical) point $m=0$. This behavior is indeed formally related to a number of critical phenomena in two-dimensional systems like the ferromagnetic phase transition of the two-dimensional Ising model.[10] Another physical example, described by the Dirac Hamiltonian, is the degenerate semiconductor which exists for $m=0$.[1] Furthermore, the large-scale limit of a two-dimensional electron gas on a square lattice near the integer quantum Hall transition for certain commensurate ¯ux situations (e.g., half a ¯ux quantum per plaquette) is described by Dirac fermions.[3±7] A common feature of all these systems is that the DOS at the touching bands (i.e., at $m=0$) is zero, i.e., there is a pseudogap. This raises the question of whether there is a mechanism that creates states in the pseudogap, for instance, electron-electron interaction or quenched disorder. This is important in order to understand if there is a nonvanishing density of low-energy excitations created by interaction or disorder. In this paper only the effect of quenched disorder will be analyzed.

There are a number of studies for the effect of disorder in the pseudogap of Dirac fermions. A numerical calculation for an electron on a square lattice with half a ¯ux quantum per plaquette shows a nonzero density at low energies.[3] A coherent potential approximation (CPA) of the Dirac fermions with random energy term $E\sigma_0$ added to H also gives a nonzero DOS.[1] A similar result was found for a random mass term $m\sigma_3$ in a modi®ed model with N fermion levels per site, using the $N\to\infty$ limit.[11] However, these are essentially mean-®eld results that may be affected strongly by ¯uctuations in the two-dimensional system. It is possible that the CPA or $N\to\infty$ result are destroyed by ¯uctuations in $d=2$. Therefore, as an alternative approach a renormalization group treatment was applied to this problem.[10] From this it turned out that there is asymptotic freedom indicating that the pseudogap, which is controlled by large-scale degrees of freedom, is not affected by a random Dirac mass (marginally irrelevant perturbation). However, a rigorous estimation leads to a nonzero lower bound of the DOS in the pseudogap,[12] at least for a random Dirac mass. The renormalization group calculation indicates that the random energy term is a relevant perturbation, in agreement with the CPA result. A third type of disorder was studied recently by adding a random vector potential to H.[6] The renormalization group and bosonization treatment indicate a more complicated behavior of the pseudogap in this case: The average DOS vanishes like $|E|^\alpha$ with a nonuniversal exponent $\alpha>0$ if the randomness is weaker than a critical strength. On the other hand, the average DOS diverges if the randomness is stronger than the critical strength because of $\alpha<0$. (For more detailed results see Sec. III.)

A similar system with a pseudogap is the d-wave superconductor. Nersesyan, Tsvelik, and Wenger[8] analyzed this system in $d=2$ and found for the pseudogap of the average DOS $\rho(E)\sim|E|^{1/7}$ in contrast to the linear behavior of the pure system. However, this result is in disagreement with others, which also ®nd a destruction of the pseudogap.[9] The effect of disorder in the d-wave superconductor will be discussed in a separate paper. The aim of this paper is to present an exact solution for the imaginary part of the single particle Green's function of disordered Dirac fermions.

There are several examples in the theory of a quantum particle in a random potential where the average one-particle Green's function can be calculated exactly. Apart from a number of one-dimensional examples,[13] there is the Lloyd model.[14] It is de®ned by the Hamiltonian $H=H_0+V$, where H_0 is a Hermitean matrix (e.g., a tight-binding Hamiltonian

for a particle on a lattice). V is a random potential distributed according to a Lorentzian (or Cauchy) distribution

$$P(V)dV=(\tau/\pi)[\tau^2+(V-V_0)^2]^{-1}dV. \quad (2)$$

The distribution density has two poles: $V=V_0\pm i\tau$. The Green's function $G(z)=(H+z)^{-1}$ must be averaged with respect to the random potential. $G(z)$ as a function of V_x for a ®xed site x is analytic in the upper (lower) complex half-plane if $\mathrm{Im}(z)$ is positive (negative), respectively. Therefore, the path of integration of V_x can be closed in that half-plane where G is analytic. As a result, only the pole of the Lorentzian density contributes to the integral $\int GP(V_x)dV_x$. This integration can be performed for all lattice sites leading eventually to the average Green's function

$$\langle G(z)\rangle=\{H_0+V_0+z+i\,\mathrm{sgn}[\mathrm{Im}(z)]\tau\}^{-1}$$
$$\equiv G\{z+i\,\mathrm{sgn}[\mathrm{Im}(z)]\tau\}. \quad (3)$$

The average DOS then reads

$$\langle\rho(E)\rangle=-(1/N\pi)\lim_{\epsilon\to 0}\mathrm{Im}\mathrm{Tr}\langle G(E+i\epsilon)\rangle, \quad (4)$$

where Tr is the trace operator and N is the number of lattice sites. Another example for an exact solution is the DOS of a particle in a homogeneous magnetic ®eld in two dimensions. If the corresponding Hilbert space of the particle is projected onto the lowest Landau level,[15] the average DOS for a white noise potential can be calculated exactly by summing up all terms of the perturbation theory with respect to the white noise potential. The exact solution is related to the fact that the lowest Landau level system is equivalent to a zero-dimensional model. It was discovered by Brézin, Gross, Itzykson[16] that the latter is a manifestation of the dimensional reduction of the two-dimensional system by 2 due to a supersymmetry of the lowest Landau level problem. Unfortunately, the simplicity of the average DOS of the lowest Landau level cannot be extended to higher Landau levels. It is also in sharp contrast to the complexity of the description of the localization properties.[17] There is some hope that the treatment of an electron on the square lattice in a strong magnetic ®eld is simpler than a continuum model. The lattice model is motivated by numerical simulations[18] and analytic calculations.[4±7] The reason for a simpli®cation is that the electron near a quantum Hall transition behaves like a Dirac fermion[4,5] because the excitations near the Fermi energy have a linear dispersion. The Hamiltonian of the Dirac fermions on a square lattice with unit lattice constant is

$$H+i\epsilon\sigma_0=(i\nabla_1+a)\sigma_1+i\nabla_2\sigma_2+m\sigma_3+i\epsilon\sigma_0. \quad (5)$$

The lattice gradient $i\nabla_\mu$, with $\nabla_\mu f(x)=(1/2)[f(x+e_\mu)-f(x-e_\mu)]$ and lattice unit vectors e_1, e_2, is Hermitean. Two types of disorder are discussed subsequently: a random Dirac mass m and a random vector potential a. The vector potential term is chosen in (5) in the same way as in Ref. 6. It can be considered as a weak disorder approximation of a ¯uctuating Peierls phase factor in Landau gauge. The Green's function now reads $G(m,i\epsilon)=(H+i\epsilon\sigma_0)^{-1}$, i.e., m and ϵ correspond with the real and imaginary parts of z in the Green's function of the Lloyd model, respectively. The treatment of this problem is rather technical, although the aim is always to ®nd a G with $\mathrm{Im}(G)=\mathrm{Im}(G')$, where the analytic properties of G and G' are different: $G=(H+i\epsilon\sigma_0)^{-1}$, as a function of a random variable at a given site, has poles on *both* complex half planes whereas $G'=(H'+i\epsilon\sigma_0)^{-1}$ has only a pole on *one* of the complex half planes. The Hamiltonian H' is obtained from H by multiplication with a diagonal matrix. The latter depends on the speci®c type of randomness.

This paper is organized as follows. In Sec. II the random Dirac mass and in Sec. III a random vector potential are analyzed. The problem of species multiplication due to the lattice is discussed in Sec. IV and the projection onto the homogeneous modes on the lattice is given.

II. RANDOM DIRAC MASS

The matrix $H+i\epsilon\sigma_0$ depends on the two complex variables $\pm m_x+i\epsilon$. Thus, in contrast to the Lloyd model, $G(m,i\epsilon)$ may have singularities in both complex half planes. Therefore, the Green's function of Dirac fermions is similar to the two-particle Green's function of a nonrelativistic particle. However, it will be shown subsequently that there is an alternative representation for the imaginary part of the Green's function that depends only on a single complex variable like the Green's function of the Lloyd model. As a ®rst step, $H+i\epsilon\sigma_0$ is multiplied by a diagonal matrix $D\sigma_3$ from the right [D is the staggered diagonal matrix $D_{x,x'}=(-1)^{x_1+x_2}\delta_{x,x'}$ with the two-dimensional space coordinates $x=(x_1,x_2)$]

$$H'=i(i\nabla_1)D\sigma_2+i(i\nabla_2)D\sigma_1+mD\sigma_0+i\epsilon D\sigma_3, \quad (6)$$

where $\nabla_\mu D$ is Hermitean, since D anticommutes with ∇_μ. Hermitean conjugation of H' yields

$$H'^{\dagger}=i(i\nabla_1)D\sigma_2+i(i\nabla_2)D\sigma_1+mD\sigma_0-i\epsilon D\sigma_3. \quad (7)$$

Moreover, σ_3 anticommutes with σ_1 and σ_2. Consequently, $i\epsilon D\sigma_3$ *commutes* with all other terms in H'. These properties lead to the product

$$H'H'^{\dagger}=[i(i\nabla_1)D\sigma_2+i(i\nabla_2)D\sigma_1+mD\sigma_0]^2+\epsilon^2\sigma_0. \quad (8)$$

From the de®nition of H' follows directly

$$H'H'^{\dagger}=[H+i\epsilon\sigma_0]\sigma_3D\sigma_3D[H+i\epsilon\sigma_0]^{\dagger}$$
$$=[H+i\epsilon\sigma_0][H-i\epsilon\sigma_0]$$
$$=[H-i\epsilon\sigma_0][H+i\epsilon\sigma_0]. \quad (9)$$

The right-hand side (rhs) of (8) can also be written $H''H''^{\dagger}$ with

$$H''=i(i\nabla_1)D\sigma_2+i(i\nabla_2)D\sigma_1+(mD+i\epsilon)\sigma_0. \quad (10)$$

As a result, H'' depends on only one complex variable $(-1)^{x_1+x_2}m_x+i\epsilon$ for a given site x. The imaginary part of the Green's function $(H+i\epsilon\sigma_0)^{-1}$ reads $(i/2)([H+i\epsilon\sigma_0]^{-1}-[H-i\epsilon\sigma_0]^{-1})=\epsilon([H-i\epsilon\sigma_0][H+i\epsilon\sigma_0])^{-1}$, i.e., it depends on the Hamiltonian only via H^2. Therefore, the identity $H^2+i\epsilon^2\sigma_0=H''H''^{\dagger}$ can be used to write

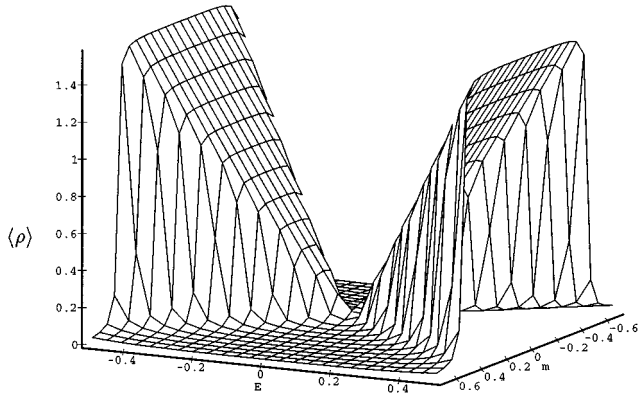

FIG. 1. Average density of states for disorder strength $\tau=0.01$.

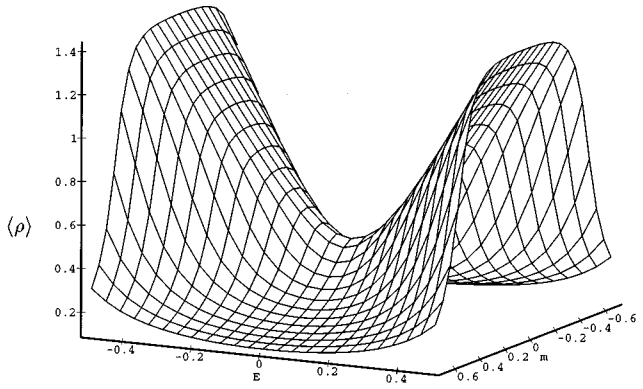

FIG. 2. Average density of states for disorder strength $\tau=0.1$.

$$\frac{i}{2}([H+i\epsilon\sigma_0]^{-1}-[H-i\epsilon\sigma_0]^{-1})$$
$$=\epsilon([H-i\epsilon\sigma_0][H+i\epsilon\sigma_0])^{-1}$$
$$=\epsilon(H''^{\dagger}H'')^{-1}=\tfrac{i}{2}[H''^{-1}-(H''^{\dagger})^{-1}]. \quad (11)$$

Thus the imaginary part of the average Green's function $(H-i\epsilon\sigma_0)^{-1}$ can be calculated exactly for a Lorentzian distribution due to (11), where m_x can be integrated out explicitly as in the Lloyd model. Only a pole of the distribution contributes leading to the replacements $\epsilon\to\tau+\epsilon$ and $m\to m_0$ in H. This implies

$$\tfrac{i}{2}\,\langle[H+i\epsilon\sigma_0]^{-1}-[H-i\epsilon\sigma_0]^{-1})\rangle=\tfrac{i}{2}\,[\hat{H}^{-1}-(\hat{H}^{\dagger})^{-1}], \quad (12)$$

with

$$\hat{H}=i\nabla_1\sigma_1+i\nabla_2\sigma_2+i(\epsilon+\tau)\sigma_0+m_0\sigma_3. \quad (13)$$

The imaginary τ term leads always to an exponential decay of the average Green's function with a typical decay length $\xi\sim(m_0^2+\tau^2)^{-1/2}$ at $\epsilon=0$. Moreover, from Eq. (4) follows for the average DOS,

$$\langle\rho(i\epsilon,m_0)\rangle=-\frac{1}{N\pi}\,\mathrm{Im}\,\mathrm{Tr}[\hat{H}^{-1}]. \quad (14)$$

The dependence on the energy E is obtained from an analytic continuation $i\epsilon\to i\epsilon+E$. The resulting average DOS is plotted in Fig. 1 for $\tau=0.01$ and in Fig. 2 for $\tau=0.1$. The nonvanishing DOS is in agreement with a rigorous proof[12] and a numerical result.[3] For Gaussian disorder with variance g there is a lower bound[12]

$$\langle\rho(0,0)\rangle\geqslant c_1 e^{-c_2/g}, \quad (15)$$

with some positive constants c_1, c_2, independent of g.

In the continuum limit it was argued, using a one-loop renormalization group calculation, that random fluctuations of the Dirac mass are irrelevant on large scales.[6,8] This implies a linearly vanishing DOS and a divergent correlation length at $E=m_0=0$.

III. RANDOM VECTOR POTENTIAL

A calculation analogous to that of the random mass system can be performed for a random vector potential. For this purpose the orthogonal transformation $(\sigma_1+\sigma_3)/\sqrt{2}$ is applied to the Hamiltonian

$$H+i\epsilon\sigma_0\to(i\nabla_1+a)\sigma_3-i\nabla_2\sigma_2+m\sigma_1+i\epsilon\sigma_0. \quad (16)$$

The multiplication of the massless Hamiltonian (i.e., $m=0$) from the rhs with $D'\sigma_3$ [where $D'_{x,x'}=(-1)^{x_2}\delta_{x,x'}$] yields

$$H'=(i\nabla_1+a)D'\sigma_0-i(i\nabla_2)D'\sigma_1+i\epsilon D'\sigma_3. \quad (17)$$

The lattice difference operators $i\nabla_1 D'$ and $i(i\nabla_2)D'$ are Hermitean. Since $D'\sigma_3$ commutes with the first two terms of H', one obtains

$$[H+i\epsilon\sigma_0][H-i\epsilon\sigma_0]$$
$$=H'H'^{\dagger}=[(i\nabla_1+a)D'\sigma_0-i(i\nabla_2)D'\sigma_1]^2+\epsilon^2\sigma_0$$
$$=H''H''^{\dagger}, \quad (18)$$

with

$$H''=(i\nabla_1+a)D'\sigma_0-i(i\nabla_2)D'\sigma_1+i\epsilon\sigma_0, \quad (19)$$

which can be used to establish again Eq. (11). For $\langle a\rangle=0$ the average imaginary part of the Green's function and, therefore, the average DOS for $m_0=0$ is related to the Hamiltonian $\hat{H}$ as given in (13).

In contrast to this result, the bosonization of the Dirac fermions in the continuum limit leads to a different behavior.[6] For instance, the DOS reads

$$\langle\rho(E,0)\rangle\sim E^{(2-z)/z}, \quad (20)$$

where $z=1+\Delta_A/\pi$ (Δ_A is the variance of the fluctuations of the vector potential). For example, the DOS vanishes at $E=0$ for $z<2$ (weak disorder) and diverges for $z>2$ (strong disorder). The Green's function behaves like

$$\langle G_{0,x}(E,m=0)\rangle\sim e^{i|x|/\lambda}e^{-|x|/\xi_1}, \quad (21)$$

where for $E\sim 0$

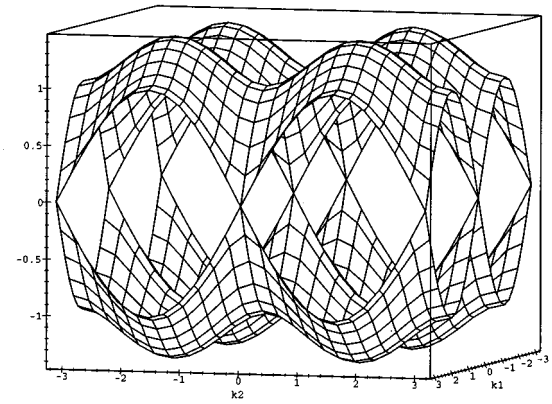

FIG. 3. Energy dispersion $E(k_1,k_2)$ of lattice Dirac fermions.

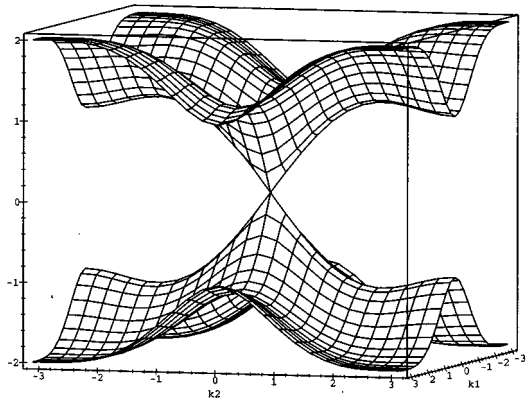

FIG. 4. Energy dispersion of lattice Dirac fermions after lifting the degeneracy of the low-energy properties.

$$\lambda \sim E^{-(1-\Delta_A/\pi)}, \tag{22}$$

$$\xi_1 \sim E^{-1}/\Delta_A . \tag{23}$$

Thus the Green's function decays exponentially for $E \neq 0$. There is a critical point $E=0$ where the correlation length diverges with E^{-1}. The difference of the results of Ref. 6 and the present work is probably related to the order of taking the continuum limit and averaging over disorder. It is not a consequence of the difference of disorder distributions (Gaussian in Ref. 6 versus Lorentzian distribution here) because the Gaussian distribution could also be treated for $H''^{-1}-H''^{2-1}$ in a strong-disorder expansion. The result of this expansion is also a ®nite nonzero DOS and a ®nite correlation length of G.

IV. REMARK ON SPECIES MULTIPLICATION

The phenomenon of species multiplication in a fermion lattice theory is well known from lattice gauge ®eld theories.[19] It is due to several nodes in the energy dispersion of the lattice model, which indicate the existence of low-energy excitations on different length scales. The dispersion of the Dirac fermions considered in this paper for $m=a=0$ is $E(k_1,k_2)=\pm\sqrt{\sin^2 k_1+\sin^2 k_2}$. It has nine nodes at $k_j=0,\pm\pi$ (cf. Fig. 3). In contrast to the lattice model the corresponding continuum model, with $E(k_1,k_2)=\pm\sqrt{k_1^2+k_2^2}$, has low-energy excitations only for small wave vectors (i.e., on large scales) as discussed in the Introduction. It will be shown in this section, using the random mass model of Sec. II, that the species multiplication is not the reason for the smooth properties of the one-particle Green's function.

The degeneracy of the low-energy behavior of the lattice model can be lifted by introducing additional terms in the Hamiltonian.[19] A possible way is to replace the Hamiltonian H by the new Hamiltonian $H+\delta(\Delta-2)\sigma_3$, where δ is a positive number $(0<\delta\leqslant 1)$ and Δ is a lattice operator with $\Delta f(x)=[f(x+e_1)+f(x-e_1)+f(x+e_2)+f(x-e_2)]/2$. The dispersion of the new H is $E(k_1,k_2)=\pm\sqrt{\delta^2(\cos k_1+\cos k_2-2)^2+\sin^2 k_1+\sin^2 k_2}$ for $m=a=0$. This is shown in Fig. 4 for $\delta=1/2$. It is not clear to the author which tranformation can be applied to relate the imaginary-part Green's function with the new Hamiltonian in order to get the analytic behavior necessary to perform the Cauchy integration with respect to the randomness. However, this dif®culty can be circumvented by generalizing H to $\tilde{H}$ with

$$\tilde{H}=\begin{pmatrix} H+\delta(\Delta-2)\sigma_3 & m'\sigma_3 \\ m'\sigma_3 & H-\delta(\Delta+2)\sigma_3 \end{pmatrix}, \tag{24}$$

where m' is a random variable that is statistically independent of m with mean zero. Now the orthogonal transformation

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix} \tag{25}$$

rotates the diagonal part $(\Delta\sigma_3,-\Delta\sigma_3)$ in the off-diagonal positions and the off-diagonal part into the diagonal position $(m'\sigma_3,-m'\sigma_3)$ such that

$$\tilde{H}=\begin{pmatrix} H-(2\delta-m')\sigma_3 & \delta\Delta\sigma_3 \\ \delta\Delta\sigma_3 & H-(2\delta+m')\sigma_3 \end{pmatrix}. \tag{26}$$

The random variables $M_x\equiv -2\delta+m_x+m_x'$ and $M_x'\equiv -2\delta+m_x-m_x'$ in the diagonal part of $\tilde{H}$ can now be considered as new independent random variables.

The transformation

$$\tilde{H}\to\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}\tilde{H}\begin{pmatrix} D\sigma_3 & 0 \\ 0 & D\sigma_3 \end{pmatrix}=\begin{pmatrix} HD\sigma_3-(2\delta-m')D\sigma_0 & \delta\Delta D\sigma_0 \\ \delta D\Delta\sigma_0 & -HD\sigma_3+(2\delta+m')D\sigma_0 \end{pmatrix}=\tilde{H}' \tag{27}$$

generates the Hermitean matrix $\tilde{H}'$.

Using for the rhs of (27) the notation $T_0\tilde{H}T_1$ and applying the property

$$T_0\tilde{H}T_1 = T_1\tilde{H}T_0, \tag{28}$$

one obtains

$$\begin{aligned}(\tilde{H}'-i\epsilon D\gamma_3)(\tilde{H}'+i\epsilon D\gamma_3)&\\ = T_0(\tilde{H}-i\epsilon\gamma_0)T_1T_0(\tilde{H}+i\epsilon\gamma_0)T_1&\\ = T_0(\tilde{H}-i\epsilon\gamma_0)(\tilde{H}+i\epsilon\gamma_0)T_0&\end{aligned} \tag{29}$$

with the diagonal matrices $\gamma_0=(\sigma_0,\sigma_0)$ and $\gamma_3=(\sigma_3,-\sigma_3)$. Moreover, one has for the imaginary part of the one-particle Green's function as before $(i/2)[(\tilde{H}+i\epsilon\gamma_0)^{-1}-(\tilde{H}-i\epsilon\gamma_0)^{-1}] = \epsilon[(\tilde{H}-i\epsilon\gamma_0)(\tilde{H}+i\epsilon\gamma_0)]^{-1}$. Due to (29) and $T_0^{-1}=T_0$ this can be rewritten as

$$\epsilon T_0[(\tilde{H}'-i\epsilon D\gamma_3)(\tilde{H}'+i\epsilon D\gamma_3)]^{-1}T_0. \tag{30}$$

The left-hand side of (29) reads

$$\begin{aligned}(\tilde{H}'-i\epsilon D\gamma_3)(\tilde{H}'+i\epsilon D\gamma_3)&\\ =(\tilde{H}')^2+\epsilon^2\gamma_0=(\tilde{H}'-i\epsilon\gamma_0)(\tilde{H}'+i\epsilon\gamma_0)&\end{aligned} \tag{31}$$

because $\tilde{H}'$ and $D\gamma_3$ commute. This implies for (30)

$$\begin{aligned}\epsilon T_0[(\tilde{H}'-i\epsilon\gamma_0)(\tilde{H}'+i\epsilon\gamma_0)]^{-1}T_0&\\ =\tfrac{i}{2}[(\tilde{H}'+i\epsilon\gamma_0)^{-1}-(\tilde{H}'-i\epsilon\gamma_0)^{-1}].&\end{aligned} \tag{32}$$

Consequently, the imaginary part of the Green's function satisfies

$$\begin{aligned}\tfrac{i}{2}[(\tilde{H}+i\epsilon\gamma_0)^{-1}-(\tilde{H}-i\epsilon\gamma_0)^{-1}]&\\ =\tfrac{i}{2}T_0[(\tilde{H}'+i\epsilon\gamma_0)^{-1}-(\tilde{H}'-i\epsilon\gamma_0)^{-1}]T_0,&\end{aligned} \tag{33}$$

analogously to (11). At a given site x the matrix $\tilde{H}'+i\epsilon\gamma_0$ depends on the random variables in the combinations $(-1)^{x_1+x_2}M_x+i\epsilon$ and $(-1)^{x_1+x_2}M_x'+i\epsilon$. Assuming a Lorentzian distribution for M_x and M_x', the integration can be performed again as in Sec. II. As a result the imaginary part of the average Green's function is

$$\mathrm{Im}\begin{pmatrix}\mathring{H}+\delta(\Delta-2)\sigma_3 & 0\\ 0 & \mathring{H}-\delta(\Delta+2)\sigma_3\end{pmatrix}^{-1}, \tag{34}$$

where $\mathring{H}$ is the average Hamiltonian (13). Thus the lifting of the degeneracy of the nodes in the dispersion relation does not change the analytic behavior of the average one-particle Green's function.

V. CONCLUSION

An exact expression for the average imaginary part of the one-particle Green's function and the average DOS of two-dimensional lattice Dirac fermions have been derived for a random Dirac mass and for a random vector potential. We have shown that there is a nonzero DOS due to disorder and there is a finite decay length for the average one-particle Green's function. This implies the creation of a nonvanishing density of low-energy excitations due to disorder in a vicinity of $E=M=0$. These lattice results are in agreement with numerical simulation.[3] However, they are in disagreement with the results of a renormalization group calculation and a bosonization approach for a continuous system of Dirac fermions,[6,8] where the DOS vanishes or diverges at $E=M=0$. Moreover, the lattice model does not exhibit the critical properties of the Green's function and the DOS found in the renormalization group calculation and in the bosonization approach. It is possible to take the continuum limit of the lattice model after the averaging over disorder, for instance, in the Hamiltonian (13). This, however, does not lead to a critical behavior. It seems that the critical behavior of the DOS is a consequence of taking the continuum limit first and performing the averaging over disorder afterwards. This is plausible because the effect of randomness is much stronger in the continuum due to statistically independent fluctuations on arbitrarily short scales. It is shown in Sec. IV that species multiplication, which is a special effect of the lattice model, is not the reason for the smooth behavior of the average DOS.

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