

## Quantum Hall transition in an array of quantum dots

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A two-dimensional array of quantum dots in a magnetic field is considered. The electrons in the quantum dots are described as unitary random matrix ensembles. The strength of the magnetic field is such that there is half a flux quantum per plaquette. This model exhibits the integer quantum Hall effect. For  $N$  electronic states per quantum dot the limit  $N \rightarrow \infty$  can be solved by a saddle-point integration of a supersymmetric field theory. The effect of level statistics on the density of states and the Hall conductivity is compared with the effect of temperature fluctuations. [S0163-1829(97)06716-7]

### I. INTRODUCTION

We consider a two-dimensional array of quantum dots in a homogeneous magnetic field perpendicular to the array. A quantum dot in an array is a complex finite system of electrons, subject to Coulomb interaction and a confining potential. Even if the number of electrons is small there is a large number of electronic states in a given energy interval. Therefore, we are forced to use a statistical description of the quantum dot. A typical feature of such a complicated nonintegrable system is level repulsion. The latter, also found in other complex many-particle systems such as atomic nuclei,<sup>1</sup> atoms,<sup>2</sup> or metallic particles,<sup>3</sup> can be conveniently described by random matrix ensembles.<sup>4</sup> Since the magnetic field breaks the time-reversal invariance in the dot, an appropriate model is the Gaussian unitary ensemble (GUE). Electrons can travel in the array of quantum dots due to tunneling between neighboring dots. On the square array, which will be considered in this paper, the tunneling rates are  $t$  and  $t'$  for nearest and next nearest neighbors, respectively (cf. Fig. 1). The coupling between the individual quantum dots due to these tunneling processes is weak. This allows us to assume that the statistical occupation of the electronic states in each dot is uncorrelated between different dots. Thus the quantum dots can be represented by independent random matrix ensembles. Moreover, we also assume for simplicity that the tunneling processes are independent, i.e., the tunneling electrons do not interact with each other.

For very weak tunneling rates the array should behave like an insulator because of the fluctuations of the energy levels. One would expect for increasing tunneling rates that a metallic regime can be reached where the array becomes conducting. However, due to the statistical fluctuations of the energy levels in the dots the effect of Anderson localization must play a crucial role in the array. Anderson localization prevents a two-dimensional system from becoming metallic, at least if no or only a weak magnetic field is present.<sup>5</sup> On the other hand, in the two-dimensional electron gas in a homogeneous magnetic field quantum Hall transitions (QHT) have been observed, which are accompanied by delocalized

electronic states.<sup>6</sup> A QHT occurs if a gap opens in a band of electronic states. This phenomenon is known, for instance, from electrons that are subject to a homogeneous magnetic field and a periodic potential.<sup>7,8</sup> Depending on the magnetic field the electrons form several subbands where each subband contributes  $e^2/h$  to the Hall conductivity.<sup>9-11</sup> As an approximation of the periodic potential one can use a tight-binding model where the lattice constant is given by the period of the potential. This approximation was used in a number of numerical studies<sup>12,13</sup> and in analytic work.<sup>14,15</sup> In this paper we will study the effect of the statistics of energy levels and the effect of thermal fluctuations on the QHT.

This paper is organized as follows: in Sec. II the tight-binding model with  $N$  levels per quantum dot and magnetic flux  $\phi = \phi_0/2$  is introduced. The Hall conductivity  $\sigma_{xy}$  is defined in Sec. III and discussed without level fluctuations. Then a functional integral representation is used to study the level fluctuations in the limit  $N \rightarrow \infty$  (Sec. IV).

### II. THE MODEL

There are two different approaches to the transport in quantum dots. One is based on the  $S$  matrix, the other one on the Hamiltonian. The former is very useful for numerical simulations because it describes directly the reflection and transmission of the electrons through the quantum dots.<sup>16,17</sup> The latter, however, requires the application of linear response theory to get a conductivity via Kubo's formula. In this paper the Hamiltonian representation will be used. The effective Hamiltonian of an array of quantum dots reads as a quadratic form  $\sum \tilde{H}_{r,r'}^{\alpha,\alpha'} c_r^\alpha c_{r'}^{\alpha'2}$  in the fermion creation and annihilation operators  $c^\alpha, c$  with the matrix elements

$$\tilde{H}_{r,r'}^{\alpha,\alpha'} = H_r^{\alpha,\alpha'} \delta_{r,r'} + H'_{r,r'} \delta_{\alpha,\alpha'} + V_r \delta_{r,r'} \delta_{\alpha,\alpha'}, \quad (1)$$

where  $\alpha, \alpha' = 1, \dots, N$  label the  $N$  electronic states in the quantum dots and  $r$  and  $r'$  label positions of the quantum dots in the two-dimensional array. In general, tunneling between all  $N$  states should be allowed with some probability,

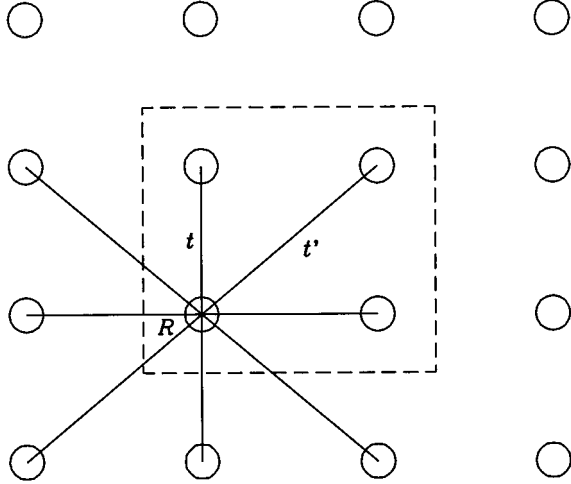


FIG. 1. Schematic picture of an array of quantum dots with nearest- ( $t$ ) and next-nearest-neighbor ( $t'$ ) tunneling. The square denotes the unit cell of the translational invariant array with magnetic flux  $\Phi = \Phi_0/2$ .

depending exponentially on the energy of the states  $\alpha$  and  $\alpha'$ . The inclusion of this dependence would require a detailed knowledge of these states. Therefore, we choose as a simplifying approximation the assumption that there is tunneling only between states with the same  $\alpha$  at nearest- or next-nearest-neighbor dots with fixed tunneling rates. The distance between neighboring dots is measured in units of  $(\phi_0/2B)^{1/2}$ . Typical distances are  $a = 100 \pm 500$  nm.<sup>18</sup> The magnetic field for the creation of one flux quantum per plaquette is  $B = \phi_0/a^2 \approx 0.016 \pm 0.4$  T. This regime is accessible in natural crystals ( $a \approx 0.5$  nm) only with astronomical magnetic fields.

The electron can occupy statistical states inside the quantum dot, which are represented by the matrix elements  $H_r^{\alpha,\alpha'}$ .  $H$  is the  $N \times N$  Hermitian Hamiltonian ( $H^2 = H$ ) of a quantum dot with  $N^2$  statistically independent matrix elements. They are Gaussian distributed with zero mean and  $\langle H_r^{\alpha,\alpha'} H_{r'}^{\alpha''\alpha'''} \rangle = (g/N) \delta^{\alpha\alpha''} \delta^{\alpha'\alpha'''} \delta_{r,r'}$ .  $g$  is the strength of the level fluctuations. This depends on the strength of the interaction between the electrons inside the dot. Therefore,  $g$  increases with the number of electrons per dot and with increasing confinement.

The tunneling is represented by the Hamiltonian  $H'$ . This reads in Landau gauge [with  $r = (x, y)$ ] for flux  $\phi$  per plaquette

$$H'_{r,r'} = t e^{2i\pi y \phi / \phi_0} \delta_{r', r+e_x} + t \delta_{r', r+e_y} + t' e^{2i\pi(y \pm 1/2) \phi / \phi_0} \delta_{r', r+e_x \pm e_y} + \text{H.c.} \quad (2)$$

The aim of this paper is to present a generic effect (the quantum Hall transition), including level statistics, in the simplest possible form. It seems that the case of a tight-binding model with  $\phi/\phi_0 = 1/2$  provides such a simple form because the corresponding Hamiltonian is translational invariant on the lattice with period 2. In principle, any commensurate (rational) flux  $\phi = (m/n)\phi_0$  could be studied. But this would require a bigger unit cell of size  $n$  in order to derive the effective large-scale properties. Nevertheless, it is expected that Dirac fermions (i.e., a linear spectrum) control the Hall transition whenever a gap opens.<sup>19</sup>

The potential term  $V_r$  represents an additional external (e.g., electric) field. Here we regard a staggered chemical potential  $V_r = (-1)^{x+y} \mu$ . The chemical potential controls the density of particles in the system. Therefore, the filling factor can be varied by varying the latter, and the system can be driven through a quantum Hall transition. The staggering of the potential favors a substructure of the electronic states. Formally, it opens a gap  $2\mu$  in the spectrum of the electrons, as will be explained below.<sup>20</sup> Another reason for the introduction of the staggered chemical potential is the fact that it allows a separation of the Hamiltonian into two independent Dirac fermions. This technical detail may not be crucial in the description of the physics on large scales, in particular for the transport properties. It gives the same behavior as one would obtain from other means of varying the filling factor  $\nu$ .

We choose for the tunneling rate  $t = 1$ . Therefore,  $\mu$ ,  $t'$  are measured in units of  $t$ , and  $g$  is measured in units of  $t^2$ . If we identify fermions with the four corners of the unit cell (Fig. 1) the tunneling matrix  $H'$  can be diagonalized by a Fourier transformation. This gives a  $4 \times 4$  matrix in Fourier space.  $H$ , the Hamiltonian of a dot, is a diagonal matrix with respect to the four corners in the sublattice representation  $H = (H_1^{\alpha,\alpha'}, H_2^{\alpha,\alpha'}, H_3^{\alpha,\alpha'}, H_4^{\alpha,\alpha'})$ . A similar model with correlated randomness  $H_1^{\alpha,\alpha'} = H_3^{\alpha,\alpha'} = -H_2^{\alpha,\alpha'} = -H_4^{\alpha,\alpha'}$  was considered in Ref. 21

We begin the discussion of the model with the analysis of an array where the interaction of the electrons inside the quantum dots are neglected. It can be understood as a tight-binding model for noninteracting electrons in a metal with some electronic bands in a magnetic field.<sup>20,22</sup> The Fourier components of  $H'$  can be expanded around the four nodes  $k = (\pm\pi, \pm\pi)$  for  $k = (\pm\pi, \pm\pi) + ap$  with small  $p$  vectors. After a global orthogonal transformation the Hamiltonian reads

$$H''(p) = 2 \begin{pmatrix} \mu - t' & ip_x - p_y & -2t'(p_x + p_y) & 0 \\ -ip_x - p_y & -\mu + t' & 0 & -2t'(p_x - p_y) \\ -2t'(p_x + p_y) & 0 & \mu + t' & p_y + ip_x \\ 0 & -2t'(p_x - p_y) & p_y - ip_x & -\mu - t' \end{pmatrix} \equiv \begin{pmatrix} H''_{11} & H''_{12} \\ H''_{21} & H''_{22} \end{pmatrix}. \quad (3)$$

The last equation combines the  $4 \times 4$  structure to a  $2 \times 2$  structure with  $2 \times 2$  block matrices  $H''_{ij}$ . Neglecting terms  $O(p^2)$  the Green's function  $(\tilde{H} + i\omega)^{-1}$  decays into a diagonal block structure

$$\tilde{\mathcal{G}}(i\omega) = \begin{pmatrix} (H''_{11}\mathbf{1}_N + h_1 + i\omega)^{-1} & 0 \\ 0 & (H''_{22}\mathbf{1}_N + h_1 + i\omega)^{-1} \end{pmatrix}, \quad (4)$$

with the diagonal matrix  $h_1 = (H_1 + H_3, H_2 + H_4)$ . Thus the diagonal elements are statistically independent.  $\mathbf{1}_N$  is the  $N \times N$  unit matrix. It is interesting to notice that the matrices  $H''_{jj} = m_j \sigma_z + i\nabla_x \sigma_x \mp i\nabla_y \sigma_y$  represent two independent two-dimensional Dirac Hamiltonians with masses  $m_1 = \mu - t'$  and  $m_2 = \mu + t'$ , respectively.

### III. HALL CONDUCTIVITY

The current density in a Dirac model can be calculated from the response to an external vector potential  $q_y$ .<sup>23</sup> The introduction of this vector potential is similar to a change of the boundary conditions, a concept extensively used in numerical investigations of Anderson localization.<sup>24</sup> The response to the vector potential leads to the Hall conductivity  $\sigma_{xy}$  in terms of Green's functions

$$\sigma_{xy} = j_x / E_y = \frac{i}{q_y} \int \sum_{r'} \text{Tr}[\sigma_x (H_0 + h_1 - i\omega + E)_{r,r'}^{-1} (H_0 + h_1 - i\omega + E \mp q_y \sigma_y)_{r',r}^{-1}] \frac{d\omega}{2\pi}, \quad (5)$$

where the upper sign is for  $m_1$  and the lower sign for  $m_2$ . We obtain for  $q_y \sim 0$  the expression<sup>20,21</sup>

$$\sigma_{xy} \approx \pm i \sum_{r',r''} \int \text{Tr}[\sigma_x \tilde{\mathcal{G}}_{r,r'}(E - i\omega) \tilde{\mathcal{G}}_{r',r''}(E - i\omega) \times \sigma_y \tilde{\mathcal{G}}_{r'',r}(E - i\omega)] \frac{d\omega}{2\pi}. \quad (6)$$

If there is only one electron per dot the energy spectrum has discrete levels that are well separated. For instance, with a harmonic oscillator potential for the isolated dot we have  $E_n = \hbar \omega_p (n + 1/2)$ . The separation of the energy levels in the single electron case allows us to neglect all levels with  $n > 0$ . Consequently, there is no statistics of energy levels and we can write  $h_1 = 0$ . For the Hall conductivity in units of  $e^2/h$  we find

$$\sigma_{xy} = (1/2) [-\text{sgn}(m_1) \Theta(|m_1| - |E|) + \text{sgn}(m_2) \Theta(|m_2| - |E|)], \quad (7)$$

where  $\Theta$  is the Heaviside step function. This result reflects correctly the qualitative behavior of the Hall conductivity at the QHT: the Hall conductivity of the original lattice fermion problem is the sum of the Hall conductivities from the light Dirac mass ( $m_1$ ) and the heavy Dirac mass ( $m_2$ ), such that the total  $\sigma_{xy}$  has a jump from 0 to 1 if the light mass changes the sign (i.e., exchange of particles and holes in the Dirac

model). Thus the Dirac fermions, together with the Hall conductivity of Eq. (6), represent a simple picture for a Hall transition. Special cases are  $\mu = 0$ , which gives  $\sigma_{xy} = \Theta(|m_1| - |E|)$  and the (unrealistic) case  $t' = 0$  with  $\sigma_{xy} = 0$ . The vanishing Hall conductivity reflects the unbroken time-reversal symmetry of the Hamiltonian if  $\phi = \phi_0/2$  and  $t' = 0$ . For example, the next-nearest-neighbor hopping term is important at half a flux quantum per plaquette in order to create a Hall current. For general flux  $\phi$ , however, the next-nearest-neighbor hopping term is not necessary to discuss the quantum Hall effect because the time-reversal symmetry is already broken by the complex nearest-neighbor hopping term.

The sharp steplike QHT is only possible in an ideal systems of noninteracting lattice electrons at zero temperature. In order to compare with real systems we have to include the statistical fluctuations of the energy levels as well as thermal fluctuations. The latter are taken into account by replacing the integral over  $\omega$  in Eq. (6) by a summation over discrete Matsubara frequencies  $\omega_n = (2n + 1)\pi T$  ( $n = 0, \pm 1, \pm 2, \dots$ ). This leads to a thermal broadening of the steplike behavior of  $\sigma_{xy}$ .

### IV. FUNCTIONAL INTEGRAL REPRESENTATION

The effect of the level fluctuations is evaluated by averaging  $\sigma_{xy}$  over the random matrix elements of  $h_1$ . In order to derive a simple expression for the limit of infinitely many energy levels per dot ( $N \rightarrow \infty$ ) it is convenient to write the product of Green's functions  $G = (H_0 + h_1 + z\sigma_0)^{-1}$  ( $H_0$  is either  $H''_{11}$  or  $H''_{22}$ ) in the expression of the Hall conductivity formally as a functional integral of a supersymmetric model<sup>21,25,26</sup>

$$G_{rr'}^{\alpha\alpha'} G_{r'r''}^{\beta\beta'} G_{r''r}^{\gamma\alpha} = \langle \Psi_r^{\alpha} \Psi_{r'}^{\alpha'} \chi_{r'}^{\beta'} \chi_{r''}^{\beta} \Psi_{r''}^{\gamma} \Psi_r^{\alpha} \rangle_S - \langle \chi_r^{\alpha'} \chi_{r'}^{\beta} \Psi_{r'}^{\beta'} \Psi_{r''}^{\beta} \chi_{r''}^{\alpha} \rangle_S, \quad (8)$$

with  $\langle \rangle_S = \int \dots \exp(-S_1) \Pi_r d\Phi_r d\tilde{\Phi}_r$  and with the supersymmetric action (sum convention for  $\alpha$ )

$$S_1 = -i\sigma_z \sum_{r,r',\mu,j,j'} \Phi_{r,\mu,j}^{\alpha} (H_0 + z\sigma_0)_{r,j;r',j'} \tilde{\Phi}_{r',\mu,j'}^{\alpha} - i\sigma_z \sum_{r,\mu,j} (\Phi_{r,\mu,j}^{\alpha'} h_r^{\alpha'} \tilde{\Phi}_{r,\mu,j}^{\alpha}), \quad (9)$$

where  $\sigma_z = \text{sgn}(\text{Im}z)$  and the field  $\Phi_{r,j}^{\alpha} = (\Psi_{r,j}^{\alpha}, \chi_{r,j}^{\alpha})$ . The first component is Grassmann and the second complex.  $\mu = 1, 2$  labels the complex and the Grassmann components, and  $j = 1, 2$  labels the two components of the Dirac model. This choice guarantees a normalized functional. Consequently, the averaging with respect to the Gaussian distributed matrix elements of  $h_1$  can be performed in the functional integral as  $\langle \exp(-S_1) \rangle_{h_1} = \exp(-S_2)$  with the effective action  $S_2$ . The latter is obtained from  $S_1$  by replacing the second term with  $(g/N) \sum_{r,\mu,j} (\Phi_{r,\mu,j}^{\alpha} \tilde{\Phi}_{r,\mu,j}^{\alpha})^2$ . Thus we have derived an effective field theory for  $\Phi$ , which serves as a generating functional for the average product of Green's functions. It is important to notice that *not only*  $h_1$  creates the

interaction in  $S_2$  but also other types of random terms in  $S_1$ . For instance, the interaction can also be created by a term  $(N/g)(Q_{r,\mu,j})^2 - 2i\sigma_z Q_{r,\mu,j} \Phi_{r,\mu,j}^\alpha \Phi_{r,\mu,j}^\alpha$  as the second term in  $S_1$ , followed by an integration over the matrix  $Q$ . This  $Q$ , in contrast to the random matrix  $h_1$ , does not depend on the index  $\alpha$  of the electronic states inside the quantum dot. This means that the distribution  $h_1$  can be transformed into another distribution with a new "random variable"  $Q$  (which does not have a probability measure but some generalized distribution including Grassmann variables). In other words, we can write, after integrating out the  $\Phi$ ,

$$\begin{aligned} \langle [(H_0 + h_1 + z\sigma_0)^{-1}]^{\alpha\alpha} \dots \rangle_{h_1} \\ = \langle [(H_0 + 2Q + z\sigma_0)^{-1}]^{\alpha\alpha} \dots \rangle_Q. \end{aligned} \quad (10)$$

The distribution that belongs to  $\langle \rangle_Q$  was investigated in detail in Ref. 26. Here we need only the result for leading order in  $N$ :  $\langle \rangle_Q = \int \dots \exp[-NS(Q,P)] \prod_r dP_r dQ_r$  with diagonal matrix  $Q_r$ ,  $P_r$ , and

$$\begin{aligned} S(Q,P) = \frac{1}{g} \sum_r [\text{Tr}(Q_r^2) + \text{Tr}(P_r^2)] + \ln \det(H_0 + 2Q + z\sigma_0) \\ - \ln \det(H_0 - 2iP + z\sigma_0). \end{aligned} \quad (11)$$

The number of levels  $N$  appears in front of the action. Thus the effect of the statistics of the energy levels can be evaluated for  $N \rightarrow \infty$  in saddle-point (SP) approximation. The SP equation reads

$$\frac{\delta}{\delta Q} \left[ \frac{1}{g} \text{Tr}(Q_r^2) + \ln \det(H_0 + 2Q + z\sigma_0) \right] = 0. \quad (12)$$

A second SP equation appears from the variation of  $P$  by making the replacement  $Q \rightarrow -iP$ . As an ansatz we take a uniform SP solution  $Q_0 = -iP_0 = (1/2)[i\eta\sigma_0 + M_s\sigma_3]$ . Then Eq. (12) leads to the conditions  $\eta = (\eta + \omega - iE)gI$ ,  $M_s = -m_1 g I / (1 + gI)$  with the integral  $I = \int [(m_1 + M_s)^2 + (\eta + \omega - iE)^2 + k^2]^{-1} d^2k / 2\pi^2$ . This result means that disorder shifts the frequency  $\omega \rightarrow \omega + \eta$  and the Dirac mass  $m_1 \rightarrow M' = m_1 + M_s$ , where  $\eta(m_1, \omega)$  and  $M_s(m_1, \omega)$  are solutions of the SP equation. For instance, with  $\omega = 0$  we have  $\eta^2 = (1/4)(M_c^2 - m_1^2)\Theta(M_c^2 - m_1^2)$ , where  $M_c = 2e^{-\pi/g}$ . The sign of  $\eta$  is fixed by the condition that  $\eta$  must be analytic in  $\omega$ . This implies  $\text{sgn}(\eta) = \text{sgn}(\omega)$ . The average density of states (DOS) is proportional to  $\eta$  in the  $N \rightarrow \infty$  limit. Thus we have a narrow DOS for the array of quantum dots of width  $2M_c$  in contrast to the isolated dot, which has a semi-circular density of width  $2\sqrt{g}$ . The DOS vanishes for  $E = 0$  in the absence of level fluctuations. The creation of a non-zero DOS due to level fluctuations is a nonperturbative effect. The vanishing DOS only means that there are no electronic states in the bulk of the system. However, there can be edge states at the boundary of the lattice that contribute to the Hall conductivity. This phenomenon was already observed in the system without level fluctuations, where the DOS vanishes completely at  $E = 0$  but the Hall conductivity is nonzero according to Eq. (7).

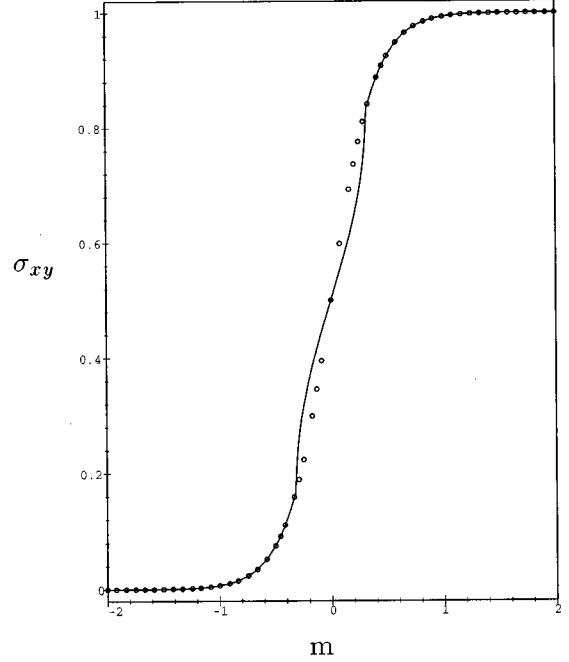


FIG. 2. Hall conductivity  $\sigma_{xy}$  in units of  $e^2/h$  as a function of the effective chemical potential  $m = \mu - t'$  at temperature  $T = 0.1$ . The circles are without level fluctuations and the full curve includes level fluctuations with variance  $g = 1.36$ .

At  $T = 0$  and  $E = 0$  the Hall conductivity per fermion level reads in the limit  $N \rightarrow \infty$  and with the approximation that  $M'$  and  $\eta$  do not depend on  $\omega$ :

$$\begin{aligned} \sigma_{xy} \approx 1/2 - \text{sgn}(m_1) [1/2 - (1/\pi) \arctan(\sqrt{M_c^2/m_1^2 - 1})] \\ \times \Theta(M_c^2 - m_1^2). \end{aligned} \quad (13)$$

The Hall conductivities are plotted in Fig. 2 for  $T = 0.1$  with and without level fluctuations. It is remarkable that the Hall conductivity is enhanced by the level fluctuations for  $\sigma_{xy} < 1/2$  whereas it is suppressed for  $\sigma_{xy} > 1/2$ . The effect of these fluctuations is strictly constrained to the interval  $2M_c$ . Thus the contribution of the edge states to the Hall conductivity, where the average DOS vanishes, is not affected.

## V. CONCLUSIONS

In a square array of quantum dots with  $N$  electronic states per dot we have investigated the DOS and the Hall conductivity. Both quantities are significantly affected by the statistical fluctuations of the energy levels. In particular, the Hall conductivity, which is steplike at the QHT in the absence of fluctuations, has a more complicated behavior in the presence of level fluctuations. Thermal fluctuations have a different effect on the Hall conductivity; they lead to a simple broadening of the steplike behavior.

Only the average quantities have been considered. However, it is possible within the same method described in this paper to study also higher moments of these quantities.

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