Delocalization of 2D Dirac Fermions: The Role of a Broken Supersymmetry

K. Ziegler
Max-Planck-Institut für Physik Komplexer Systeme, Außenstelle Stuttgart, Postfach 800665, D-70506 Stuttgart, Germany
and Institut für Physik, Universität Augsburg, D-86135 Augsburg, Germany
(Received 2 June 1997)

The mechanism of delocalization of two-dimensional Dirac fermions with random mass is investigated, using a superfield representation. Although localization effects are very strong, one fermion component can delocalize due to the spontaneous breaking of a special supersymmetry of the model. The delocalized fermion has a nonsingular density of states and is described by a diffusion propagator. Supersymmetry is restored if the mean of the random mass is sufficiently large. This is accompanied by a critical boson component. [S0031-9007(98)05786-X]

PACS numbers: 71.10.Pm, 72.15.Rn, 73.20.Jc, 71.55.Jv

The transition from localized to delocalized states of noninteracting quantum particles in a random potential is a phenomenon which is characterized by symmetries. In contrast to classical critical phenomena, where symmetries are either discrete or *compact* continuous, it was observed that the transition from localized to delocalized states of a particle, described by a discrete Schrödinger equation (tight-binding model), is related to *noncompact* symmetry groups [1]. Nonlinear σ models with the corresponding symmetries provide an effective large scale description, presenting the relevant degrees of freedom for localization or delocalization. They describe an effective diffusion of the quantum particle with diffusion coefficient $D \geq 0$. An important physical property of D in the nonlinear σ model is its flow under renormalization. In general, there are fixed points, one for delocalized states ($D > 0$) and one for localized states ($D = 0$) [2]. In two-dimensional systems the renormalization always drives the diffusion coefficient to zero [2], therefore reflecting the absence of delocalized states, at least in the absence of more complicated extensions of the Schrödinger equation like spin-orbit coupling.

It turned out that for a number of interesting physical systems the effective quantum theory is not defined by Schrödinger particles but by Dirac fermions. The main reason for this is a linear dispersion and a substructure, either given by a sublattice or a spin. For instance, the integer quantum Hall transition (QHT) in a 2D electron gas with magnetic field can be formulated with Dirac fermions without a magnetic field [3–10]. Other examples for Dirac fermions are the degenerate semiconductor [11] and quasiparticles in a 2D d-wave superconductor [12,13].

A Dirac fermion is a quantum particle with symmetry properties different from those of the Schrödinger particles. In particular, the symmetry of the 2D Dirac Hamiltonian is discrete in contrast to the continuous symmetries of the Schrödinger Hamiltonian. This fact has important consequences for the delocalization of the Dirac particle in $d = 2$ [9], and will be discussed in this Letter.

The Dirac Hamiltonian in 2D reads

$$H_D = i\nabla_1\sigma_1 + i\nabla_2\sigma_2 + M\sigma_3 . \quad (1)$$

∇_j is the lattice difference operator in the j direction, M is the mass of the particle, and σ_j is a Pauli matrix. The localization properties of massless Dirac fermions with random vector potential was recently studied [5,14]. It turned out that the low energy states are delocalized. Since the related Hamiltonian matrix has only off-diagonal elements, this result can be compared with similar observations in 1D systems: there are delocalized states at the band center with a singular density of states (DOS) if the Hamiltonian represents hopping between sublattices or different spin states [15]. In contrast to this, it is of interest to consider models where a diagonal (potential) term also appears in the Hamiltonian, and which have a nonsingular DOS.

The Dirac Hamiltonian H_D is an effective two-particle Hamiltonian because the Dirac theory includes particles and holes as the two components of the Dirac spinor. H_D is Hermitian and invariant under the transformation $H_D \to -\sigma_3 H_D \sigma_3$, provided the Dirac mass M is zero. However, this symmetry is not interesting here because it is always broken by the mass. Moreover, there is a space-dependent discrete transformation

$$H_D \to -SH_D'S \quad (2)$$

for which the massive H_D is invariant. The 2×2 matrix S_r is changing between σ_1 and σ_2 by going from one site to its nearest neighbor site, and H_D' is obtained from H_D by a space rotation of $\pi/2$ and a reflection of the y axis. (This is just an exchange of ∇_1 and ∇_2 in H_D.)

In order to compare the Dirac Hamiltonian with the corresponding Hamiltonian $H = \nabla^2 + V$ of a Schrödinger particle in a random potential V, we extend the latter to $H_S = (\nabla^2 + V)\sigma_3$. This Hamiltonian describes particles *and* the corresponding holes, and can be used to express the two-particle Green's function for Anderson localization without a magnetic field. H_S is symmetric and invariant under a noncompact continuous symmetry under $H_S \to (c\sigma_0 + s_1\sigma_1 + s_2\sigma_2)H_S(c\sigma_0 + s_1\sigma_1 + s_2\sigma_2)$ with the condition $c^2 - s_1^2 - s_2^2 = 1$. The role of the chemical potential in the case of Dirac particles is played by the Dirac mass, as it was earlier discussed by Ludwig *et al.* [5].

Transport properties can be evaluated from the two-particle Green's function [9]

$$K(r,r';\epsilon) \equiv -\langle Tr_2[G(r,r';i\epsilon)\sigma_1 G^T(r',r;i\epsilon)\sigma_1]\rangle, \tag{3}$$

where $G(r,r';i\epsilon) \equiv (H + i\epsilon)^{-1}_{rr'}$ is the one-particle Green's function of H_D or H_S, and $\langle\cdots\rangle$ the average over random contributions in the Hamiltonian. For localized states the two-particle Green's function decays exponentially on the localization length.

H_S is invariant under the transposition T of the matrix elements, whereas the Dirac Hamiltonian H_D is not. It is convenient to write the two-particle Green's function as a functional integral

$$G_{jj'}(r,r';i\epsilon)G^T_{k'k}(r',r;i\epsilon) = \int \chi_{r'j'}\bar{\chi}_{rj}\Psi_{rk}\bar{\Psi}_{r'k'}\exp(-S_0)\,\mathcal{D}\Psi\,\mathcal{D}\chi\,. \tag{4}$$

S_0 is a quadratic form of the four-component superfield (χ_r, Ψ_r)

$$-i\,\mathrm{sgn}(\epsilon)\sum_{r,r'}\begin{pmatrix}\chi_r\\ \Psi_r\end{pmatrix}\cdot\begin{pmatrix}H + i\epsilon & 0\\ 0 & H^T + i\epsilon\end{pmatrix}_{r,r'}\begin{pmatrix}\bar{\chi}_{r'}\\ \bar{\Psi}_{r'}\end{pmatrix}, \tag{5}$$

with a complex component χ_r and a Grassmann component Ψ_r. The reason for introducing the superfield is that an extra normalization factor for the integral in Eq. (4) is avoided because of $\int \exp(-S_0)\,\mathcal{D}\Psi\,\mathcal{D}\chi = \det(H_D^T + i\epsilon)/\det(H_D + i\epsilon) = 1$. It is crucial that S_0 is *not* of the usual supersymmetric form [16], where both diagonal elements are $H + i\epsilon$, if $H^T \neq H$ [17]. This reflects a fundamental difference between the symmetric Schrödinger Hamiltonian H_S and the asymmetric Dirac Hamiltonian H_D for the construction of collective fields. In the following, we will concentrate on the Dirac Hamiltonian and refer to the literature for the case of the Schrödinger Hamiltonian [1,16,18].

In addition to the discrete symmetry of H_D, there is an invariance of the action S_0 for $\epsilon = 0$ under supersymmetry transformation

$$\mathbf{H}_D \equiv \begin{pmatrix}H_D & 0\\ 0 & H_D^T\end{pmatrix} \to U\begin{pmatrix}H_D & 0\\ 0 & H_D^T\end{pmatrix}U = \mathbf{H}_D$$

$$\text{for } U = \begin{pmatrix}(1 + \frac{1}{2}\psi\bar{\psi})\sigma_0 & \psi\sigma_1\\ \bar{\psi}\sigma_1 & (1 - \frac{1}{2}\psi\bar{\psi})\sigma_0\end{pmatrix}, \tag{6}$$

with Grassmann variables ψ and $\bar{\psi}$. It is important to notice that the Dirac mass does not break this symmetry but only the term proportional to ϵ in (5), since U^2 is not the unit matrix. Therefore, the field conjugate to the symmetry breaking field, which is quadratic in the superfield, must be studied in order to take the relevant symmetry properties into account. This leads to the collective field representation [9,16]

$$\begin{pmatrix}\chi_r\bar{\chi}_r & \chi_r\bar{\Psi}_r\\ \Psi_r\bar{\chi}_r & \Psi_r\bar{\Psi}_r\end{pmatrix} \leftrightarrow \mathbf{Q}_r = \begin{pmatrix}Q_r & \bar{\Theta}_r\\ \Theta_r & -iP_r\end{pmatrix}. \tag{7}$$

The matrix elements $Q_r, \ldots, P_r$ are 2×2 matrices, corresponding to the two components of Ψ_r and χ_r.

Since the Dirac Hamiltonian H_D is usually obtained from a large scale (or low energy) approximation of a nonrelativistic problem, there are several ways to introduce disorder which are motivated by the original condensed matter systems. One starting point is, for instance, the network model of Chalker and Coddington [19] for the QHT. This phenomenological description implies a random Dirac mass, a random energy term, and a random vector potential [10]. Thus, this model represents a complex situation which also includes fluctuations of the magnetic field. Here we are interested only in the simplest possible case for the QHT of a system in a homogeneous magnetic field. (Strong fluctuations of the random vector potential may drive the system into another universality class. At this point it is not clear if randomness in the vector potential is relevant in the experiments on a 2D electron gas.) The QHT can also be described by a tight-binding model with a homogeneous magnetic field [5] in a random chemical potential. The latter would lead to a random Dirac mass. However, there was the argument that the random Dirac mass alone does not present the generic situation for the QHT because the DOS is zero at low energy [5]; i.e., there are no bulk states even in the presence of disorder. It turned out though that these states exist if one goes beyond perturbation theory. This effect was also found in numerical calculations [20,21]. A consistent treatment of this nonperturbative contribution can be based on an effective field theory derived from the collective field $\mathbf{Q}$ [9]. This representation will be used in the following to discuss the breaking of the supersymmetry defined in (6) and its consequences for the existence of delocalized states.

Averaging over a Gaussian random Dirac mass M (where $\langle M_r\rangle = m$ and $\langle M_r M_{r'}\rangle = g\delta_{rr'}$) and transforming the functional integral to the collective field creates the new action [9,16]

$$S' = \frac{1}{g}\sum_r \mathrm{Trg}_4(\mathbf{Q}_r^2) + \ln\mathrm{detg}(\mathbf{H}_0 + i\epsilon - 2\tau\mathbf{Q}\tau), \tag{8}$$

with $\mathbf{H}_0 = \langle\mathbf{H}_D\rangle$ and the 4×4 diagonal matrix $\tau = [(\sigma_3)^{1/2}, (\sigma_3)^{1/2}]$. Trg_4 and detg are the "supertrace" and the "superdeterminant," respectively [16]. In particular, the two-particle Green's function at $r \neq r'$ then reads

$$K(r,r';\epsilon) = g^{-2}\langle(\Theta_{r,12} + \Theta_{r,21})(\bar{\Theta}_{r',12} + \bar{\Theta}_{r',21})\rangle_Q\,. \tag{9}$$

The functional integral $\langle \cdots \rangle_Q = \int \cdots \exp(-S') \mathcal{D}\mathbf{Q}$ can be approximated by a saddle point integration. A special saddle point is $\mathbf{Q}_0 = (m/4)\gamma_0 - i(\eta/2)\gamma_3$, where γ_j is the diagonal block matrix (σ_j, σ_j) and $\eta = \pi g \rho$ is proportional to the average DOS ρ [9]. The symmetry transformations are now applied to the saddle point solution $\tau \mathbf{Q}_0 \tau$. The discrete transformation (2) changes the sign of η; i.e., the discrete symmetry of the massive Dirac Hamiltonian is spontaneously broken if $\lim_{\epsilon \to 0} \eta \neq 0$. This is the case for $-m_c < m < m_c$ with $m_c = 2\exp(-\pi/g)$ [9]. The supersymmetry transformation (6), on the other hand, gives $\eta \to \eta U^2$. Thus, $\lim_{\epsilon \to 0} \eta \neq 0$ also indicates a spontaneously broken supersymmetry. This behavior is analogous to a Heisenberg ferromagnet, where η corresponds to the magnetization and ϵ plays the role of the external magnetic field. However, the situation is more complex for the Dirac fermions because the breaking of two symmetries is involved, a supersymmetric and a discrete one. As a consequence of the supersymmetry of S_0 at $\epsilon = 0$ there is not just an isolated saddle point but a whole saddle point manifold, created by the symmetry transformation U. Therefore, the field

$$\mathbf{Q}'_r = \tilde{U}_r \mathbf{Q}_0 \tilde{U}_r^{-1} = \frac{m}{4}\gamma_0 - i\frac{\eta}{2}\tau^* U_r^2 \tau^*$$
$$= \mathbf{Q}_0 - i\eta \begin{pmatrix} \psi_r \bar{\psi}_r \sigma_3 & -i\psi_r \sigma_1 \\ -i\bar{\psi}_r \sigma_1 & -\psi_r \bar{\psi}_r \sigma_3 \end{pmatrix} \qquad (10)$$

controls the fluctuations around the saddle point manifold with $\tilde{U}_r = \tau^* U_r \tau$ and $\tilde{U}_r^{-1} = \tau U_r \tau^*$. U_r is here the matrix U of Eq. (6) in which the Grassmann variable ψ is replaced by the Grassmann field ψ_r. That means for the large scale properties the integration with respect to $\mathbf{Q}_r$ can be restricted to an integration with respect to the field $\mathbf{Q}'_r$. Thus the critical (long-range) part of the random mass Dirac theory is controlled by the one-component fermion (Grassmann) field ψ_r. The bosonic (complex) field has only short-range correlations and, therefore, is localized by the disorder. The reason is that the bosonic field corresponds to the *discrete* symmetry transformation (2) which has a long-range mode only at the critical point where the order parameter η vanishes. The latter is indeed the case because the localization length of $Q_{11} - Q_{22}$ and $P_{11} - P_{22}$ increases like $(m_c - |m|)^{-1/2}$ as the critical value $\pm m_c$ is approached from $|m| < m_c$ [9]. This indicates a growing influence of these bosonic fields on the large scale properties.

The expansion of (8) up to second order in the gradients yields in general an action of the type [1,16,18,22]

$$i\epsilon \int d^2r \, \mathrm{Trg}_4(\gamma_3 \mathbf{Q}'_r) + \alpha \int d^2r \, \mathrm{Trg}_4(\nabla \mathbf{Q}'_r \cdot \nabla \mathbf{Q}'_r)$$
$$- \beta \int d^2r \sum_{\mu,\nu} \epsilon_{\mu\nu} \, \mathrm{Trg}_4(\mathbf{Q}'_r \nabla_\mu \mathbf{Q}'_r \nabla_\nu \mathbf{Q}'_r), \qquad (11)$$

where $\epsilon_{\mu\nu}$ is the antisymmetric unit tensor, and the parameters α and β are determined by the model. In particular, for the quantum Hall effect there is $\alpha = \sigma_{xx}$, the (unrenormalized) longitudinal conductivity, and $\beta = \sigma_{xy}$, the (unrenormalized) Hall conductivity [22]. The topological term $\int d^2r \sum_{\mu,\nu} \epsilon_{\mu\nu} \, \mathrm{Trg}_4(\mathbf{Q}'_r \nabla_\mu \mathbf{Q}'_r \nabla_\nu \mathbf{Q}'_r)$ takes care of the Hall plateaux because the latter are a consequence of the (topological) edge states in the presence of *localized* bulk states. At the QHT, however, transport is dominated by *delocalized* bulk states. Therefore, the topological term should not play a crucial role in this case. In fact, for the Dirac Hamiltonian H_D with $m = 0$, i.e., for the choice $\mathbf{Q}'$ of Eq. (10), the topological term vanishes. The only terms which remain in the action are the linear off-diagonal elements of $\mathbf{Q}'$

$$S'' = (1/\pi\rho) \int d^2r \, \bar{\psi}_r (\epsilon + D\nabla^2) \psi_r, \qquad (12)$$

where the average DOS ρ and the diffusion coefficient D can be evaluated from the saddle point equation. This surprisingly simple result, which satisfies the Ward identity $K(q = 0, \epsilon) = \pi\rho/\epsilon$, reflects the fact that only a one-component Grassmann field contributes to the massless fluctuations, created by the broken supersymmetry. That means there is a simple physical structure for the well-delocalized Dirac fermions in the vicinity of $m = 0$. The divergent localization length of two real boson components will eventually turn into a restoration of the supersymmetry, where $\lim_{\epsilon \to 0} \eta = 0$. Since the supersymmetric theory in 2D does not have delocalized states [16], the restoration of the supersymmetry must be accompanied by a transition into a localized regime. This is the regime characterized by the Hall plateaux. The critical behavior of the two real fields (which can be considered as the two components of one complex boson field) at $m = \pm m_c$ due to the spontaneously broken discrete symmetry (2) invalidates S'' near these points. It must be replaced by a more complicated field theory which includes both the critical boson field and the critical Grassmann field of (12). This would require an additional matrix field

$$\begin{pmatrix} q_r \sigma_3 & 0 \\ 0 & -ip_r \sigma_3 \end{pmatrix}, \qquad (13)$$

added to $\mathbf{Q}'_r$ in (10). The real field components q_r and p_r are related to $Q_{11,r} - Q_{22,r}$ and $P_{11,r} - P_{22,r}$, respectively.

As a direct consequence of these results the value of the conductivity at $m = 0$ (the "conduction peak") can be evaluated from the Einstein relation $\sigma_{xx} = (e^2/\hbar) D\rho$ [23]

$$\sigma_{xx} = \frac{e^2}{\pi h} \frac{1}{1 + g/2\pi}. \qquad (14)$$

This is in agreement with experimental results [24,25] and other theoretical work [26,27]. For weak disorder the second factor can be neglected. In this case the peak value is just the universal constant $\sigma_{xx} = e^2/\pi h$. The latter was obtained for Dirac fermions in a random vector potential [5] and for the lowest Landau level with random spin scattering [28], regardless of the strength g. Thus, the extra factor in Eq. (14) indicates that the random Dirac mass, representing a random scalar potential, has a stronger effect on the Dirac particle than the random vector potential or a random spin scattering.

In conclusion, the large scale behavior of the two-particle Green's function of 2D Dirac fermions with random mass is characterized by a single massless Grassmann field on the interval $[-m_c, m_c]$ of the average Dirac m. It describes delocalized states with nonsingular DOS due to a broken supersymmetry by a diffusion propagator. There is a two-component real bosonic field which has a divergent localization length as the critical points $\pm m_c$ are approached. It corresponds to a broken discrete symmetry of the Dirac Hamiltonian. This mechanism of delocalization is different from the one which is responsible for delocalized states in the random vector potential or for random spin scattering.

[1] L. Schäfer and F. Wegner, Z. Phys. B **38**, 113 (1980).
[2] E. Abrahams *et al.,* Phys. Rev. Lett. **42**, 673 (1979).
[3] M. P. A. Fisher and E. Fradkin, Nucl. Phys. **B251 [FS13]**, 457 (1985).
[4] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[5] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B **50**, 7526 (1994).
[6] Y. Hatsugai, Phys. Rev. B **48**, 11 851 (1993).
[7] M. Oshikawa, Phys. Rev. B **50**, 17 357 (1994).
[8] K. Ziegler, Europhys. Lett. **28**, 49 (1994).
[9] K. Ziegler, Phys. Rev. B **55**, 10 661 (1997).
[10] C.-M. Ho and J. T. Chalker, Phys. Rev. B **54**, 8708 (1996).
[11] E. Fradkin, Phys. Rev. B **33**, 3257 (1986); **33**, 3263 (1986).
[12] A. A. Nersesyan, A. M. Tsvelik, and F. Wenger, Phys. Rev. Lett. **72**, 2628 (1994); Nucl. Phys. **B438**, 561 (1995).
[13] K. Ziegler, M. H. Hettler, and P. J. Hirschfeld, Phys. Rev. Lett. **77**, 3013 (1996).
[14] Y. Morita and Y. Hatsugai, cond-mat/9705192.
[15] I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
[16] K. B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, 1996).
[17] It should be noticed, however, that the standard version of the quadratic form in Eq. (5) can be obtained by renaming the Grassmann field as $\Psi_r \rightarrow i\bar{\Psi}_r$, $\bar{\Psi}_r \rightarrow i\Psi_r$.
[18] B. L. Altshuler and B. D. Simons, *Physique Quantique Mésoscopique,* edited by E. Akkermans *et al.* (Elsevier, Amsterdam, 1995).
[19] J. T. Chalker and P. D. Coddington, J. Phys. C **21**, 2665 (1988).
[20] Y. Hatsugai and P. A. Lee, Phys. Rev. B **48**, 4204 (1993).
[21] D.-H. Lee and Z. Wang, Philos. Mag. Lett. **73**, 145 (1996).
[22] H. Levine *et al.,* Phys. Rev. Lett. **51**, 1915 (1983); A. M. M. Pruisken, in *The Quantum Hall Effect,* edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990).
[23] K. Ziegler and G. Jug, Z. Phys. B **104**, 5 (1997).
[24] P. L. McEuen *et al.,* Phys. Rev. Lett. **64**, 2062 (1990).
[25] L. P. Rokhinson, B. Su, and V. J. Goldman, Solid State Commun. **96**, 309 (1995).
[26] I. M. Ruzin, N. R. Cooper, and B. I. Halperin, Phys. Rev. B **53**, 1558 (1996).
[27] N. R. Cooper, B. I. Halperin, C.-K. Hu, and I. M. Ruzin, Phys. Rev. B **55**, 4551 (1997).
[28] S. Hikami, M. Shirai, and F. Wegner, Nucl. Phys. **B408**, 415 (1993).