Localization of electromagnetic waves in random media

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1. Introduction

This paper gives a brief review of some aspects relevant for scattering of electromagnetic waves in random media and a numerical study of a one-dimensional system of scatterers. To discuss the phenomenon of random scattering one can distinguish three different regimes: (I) *propagation*, (II) *diffusion* and (III) *localization* of waves. In case (I) very few scatterers are present and there is almost no effect on the free wave propagation: a locally created wave expands linearly in time. The diffusive regime occurs in the presence of intermediate randomness. The wave is scattered back and forth such that the vector of the wave front follows a random walk. Consequently, a locally created wave cannot expand linearly in time t, as in the case of free propagation, but only with $\sqrt{t}$. For strong randomness one expects destructive interference such that a locally created wave cannot expand at all but decays exponentially in space. This is known as Anderson localization [1] and will just be called localization subsequently. Regimes (I) and (II) can be observed in many realistic situations whereas regime (III) is more difficult to detect in experiments [2]. One of the reasons is that localization is a phenomenon based on elastic scattering in non-absorbing media. It turns out

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that it is not easy to distinguish localization from absorption because both are exponential effects when a wave expands in a medium [3,4].

2. Continuum models: wave equations

Localization was originally suggested in solid state physics for the wavefunction of electrons [1]. The idea for this effect was motivated by the experimental observation that some strongly disordered solids do not allow the conduction of an electric current whereas their less disordered counterparts do. In other words, due to disorder these materials are insulators. A description of this effect can be given in terms of the wavefunction $\Psi_E(r)$ of a quantum particle with energy E, using the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\,\Delta + V(r) - E\right]\Psi_E(r) = 0$$

with the three-dimensional Laplace operator Δ. There is a close formal analogy with electromagnetic waves, since each component of a stationary electromagnetic field $\Psi(r)$ at frequency ω, created by an electric current $j(\mathbf{r})$, is described by Helmholtz equation [6]

$$[\Delta + k_0^2(1 + \mu(\mathbf{r}))]\Psi(\mathbf{r}) = j(\mathbf{r}).$$

$k_0 = \omega/c$ is the wave vector with the speed of light c. The refractive index $1 + \mu$ is complex in general. Moreover, μ varies in space in the case of an inhomogeneous medium. A solution of the Helmholtz equation can be written as

$$\Psi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\, j(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}'$$

with the Green's function

$$G = [\Delta + k_0^2(1 + \mu)]^{-1}. \tag{1}$$

For a local current density at $\mathbf{r} = 0$ (i.e. $j(\mathbf{r}) = \delta(\mathbf{r})$), the field then reads

$$\Psi(\mathbf{r}) = G(\mathbf{r}, 0)$$

which implies for the intensity at $\mathbf{r}$

$$I(\mathbf{r}) = |\Psi(\mathbf{r})|^2 = G(\mathbf{r}, 0)G^*(\mathbf{r}, 0). \tag{2}$$

In the presence of random scatterers physical quantities like the intensity must be averaged with respect to their distribution.

3. Discrete models: random-matrix representation

With the new notation $k_0^2(1 + \mu) = k_0^2 + V + \mathrm{i}\varepsilon$, where $\mathrm{i}\varepsilon$ represents a small absorbing part and V the potential of the scatterers, the Green's function (1) reads

$$G = (\Delta + k_0^2 + V + \mathrm{i}\varepsilon)^{-1}$$

and for a homogeneous medium (i.e. without scattering: $V = 0$)

$$G_0 = (\Delta + k_0^2 + \mathrm{i}\varepsilon)^{-1}.$$

The latter can be evaluated, e.g., in three dimensions and is for a given pair of sites

$$G_0(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{\mathrm{e}^{\pm \mathrm{i}k_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|},$$

where the sign of the exponent depends on whether the limit $\varepsilon \to 0$ has been taken from positive or negative values.

A random medium is considered in which random scatterers are distributed on *fixed* sites $\mathbf{R}_1, \ldots, \mathbf{R}_n$. These are the sites of point-like scatterers, e.g. water droplets in fog or dust particles in the atmosphere or in the solar system. On each of these sites $\mathbf{R}_j$ there is a scattering potential such that

$$V(\mathbf{r}) = k_0^2 \sum_{j=1}^{n} \mu_j \phi_j(\mathbf{r}), \tag{3}$$

where $\phi_j(\mathbf{r})$ is a Wannier function, centered at $\mathbf{R}_j$ [5]. These functions are orthogonal with respect to different scattering sites:

$$\int \phi_j(\mathbf{r}) \phi_k(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} = \delta_{jk}.$$

μ_j and/or $\mathbf{R}_j$ may be chosen randomly, representing a random distribution of scatterers. There can be various situations of random scattering of which there are two special cases: (a) the positions of the random scatterers are on a regular lattice with random values of the scattering potential μ_j and (b) an array of *randomly positioned identical scatterers*. Case (a) will be called a *periodic array of random scatterers* (PARS) (Fig. 1).

G_0 can be used to express the Green's function for a system with scatterers as

$$G = (G_0^{-1} + V)^{-1} = G_0(\mathbf{1} + VG_0)^{-1}.$$

Since V in Eq. (3) implies a projection P onto the subspace of the sites of the scatterers, the Green's function should also be projected to this subspace:

$$G_P \equiv P(G_0^{-1} + V)^{-1}P = PG_0(\mathbf{1} + VG_0)^{-1}P = [[PG_0P]_P^{-1} + V_P]_P^{-1}, \tag{4}$$

where $[\ldots]_P^{-1}$ is the inverse with respect to the projected space and $V_P = PVP$. The last equality follows from the identity given in Appendix A. Thus, in the present case with n scatterers the Green's function can be reduced to an $n \times n$ matrix.

4. Perturbation theory: weak localization

Weak localization occurs in systems with moderate scattering. In the context of light scattering it is often called coherent backscattering because of its characteristic angular dependence, observed in systems like fog or dust clouds. This phenomenon connects the propagation of waves with diffusion, a concept originally established for classical particles. The idea may be very old but to my knowledge was introduced as a formal theory only in the 1970s for electrons in solid state physics. It is based on a perturbation theory with the small parameter λ/l, where λ is the wavelength of the scattered wave

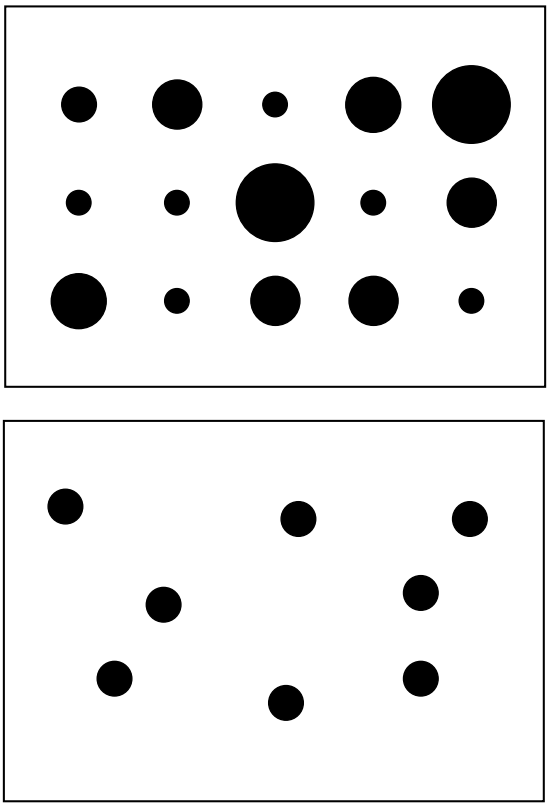

Fig. 1. A periodic array of random scatterers (PARS) and randomly positioned identical scatterers.

and l is the typical distance of the scatterers, often called the mean free path. It turned out that one has to take into account an infinite number of terms of this λ/l expansion in order to obtain diffusive behavior [8]. This can also be formulated in a self-consistent theory. From such a calculation one finds in particular, that the intensity satisfies a diffusion equation [7,8]. It is characterized by the diffusion coefficient

$$D(l) \propto \frac{l}{\lambda}. \tag{5}$$

Moreover, for an incoming planar wave the intensity of the scattered light is angular-dependent (albedo) with

$$\alpha(\Theta) = \frac{3}{4\pi}\left[1 + \frac{1 - \mathrm{e}^{-2\pi|\Theta|l/\lambda}}{2\pi|\Theta|l/\lambda}\right].$$

The angle Θ is measured with respect to backscattering. This angular behavior (see Fig. 2), with a pronounced maximum in backward direction, is the signature of coherent backscattering, and has been seen in many experiments [10].

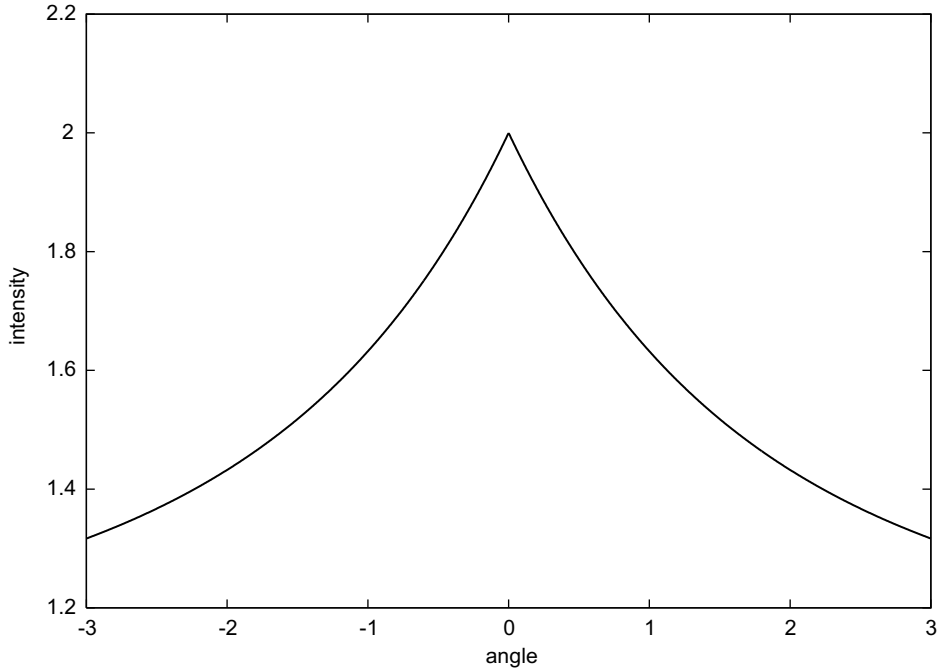

Fig. 2. The angular dependency of the intensity in units of l/λ.

5. Scaling theory: (strong) localization

Anderson introduced the idea of localization under the title "absence of diffusion" in 1958 [1]. A theoretical formulation was found later on the basis of a scaling theory [9]. It is clear from the result of the perturbation theory [5] that a vanishing diffusion coefficient cannot be reached within this method. Therefore, the scaling theory starts from the picture that a *finite* system with random scattering will always be diffusive. Thus a finite d-dimensional cube of size L^d can be characterized by the L-dependent diffusion coefficient $D(L)$. Under quite general conditions it can be assumed that this quantity behaves monotoneously: either $D(L)$ is monotoneously de- or increasing with the size L. A non-monotoneous behavior would indicate an extra length scale L_c at which a qualitative change of $D(L)$ takes place. There is no reason to believe that this is the case.

In the famous paper by Abrahams et al. [9] a scaling function was established for the conductance g of d-dimensional system. Their result can be translated into the scaling behavior of the diffusion coefficient D, since the latter is proportional to the conductance:

$$g = \rho D L^{d-2},$$

where ρ is the density of waves contributing to the diffusion. For its change under a change of scale L, the expression

$$\beta = \frac{\partial \log(g)}{\partial \log(L)}$$

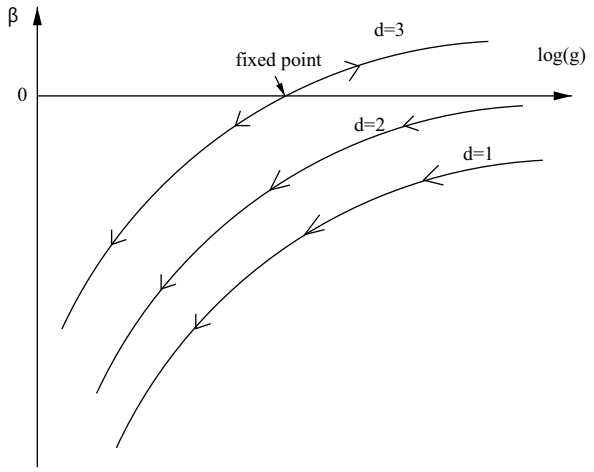

Fig. 3. Schematic picture of the scaling behavior. The upper curve shows a three-dimensional system with a fixed point, the lower two curves the one- and two-dimensional systems with localized asymptotic behavior (from Ref. [9]).

measures the localization: $\beta > 0$ means diffusion and $\beta < 0$ localization. In the asymptotic regime for large values of g the behavior is dominated by a macroscopic (i.e. L-independent) conductivity ρD such that $\beta \sim d - 2$. On the other hand, in the regime of very small g one expects localization. This means an exponential decay $g \sim \exp(-L/\xi)$ on the scale ξ. In the interpolating regime one finds that for $\beta > 0$ (< 0) the rescaled diffusion coefficient g will increase (decrease) under an increase of the size of the system L, indicating diffusion (localization). These qualitative arguments lead to the β functions shown in Fig. 3 [9]. It is interesting to notice that, according to this qualitative discussion, only systems with dimensionality larger than two can have a non-vanishing diffusion coefficient at infinite size. For systems with dimensionality $d \leqslant 2$ random scattering always leads to localization. A three-dimensional system can be diffusive if the randomness is weak but is localized for strong randomness. The former is the case when $\beta > 0$ at a given size L and the latter when $\beta < 0$, as indicated by the arrows in Fig. 3.

6. Numerical results in $d = 1$: the localization length

Localization is most likely to occur in low-dimensional systems. Although it cannot be treated in terms of a perturbation theory around the system without scatterers, it can easily be seen in numerical simulations.

According to the scaling picture in the previous section, there are only localized waves in an infinite one-dimensional systems. However, since we can only deal with finite systems in a computer simulation or in a real experiment the question is, whether the localization length is smaller or larger than the system length. In the latter case it would not be possible to observe the exponential decay of the wave. There are also non-generic models in which delocalized waves can survive even in an infinitely large one-dimensional system at special wavelengths. This was already discussed by Dyson who gave an exact argument for the absence of localization in a one-dimensional system with

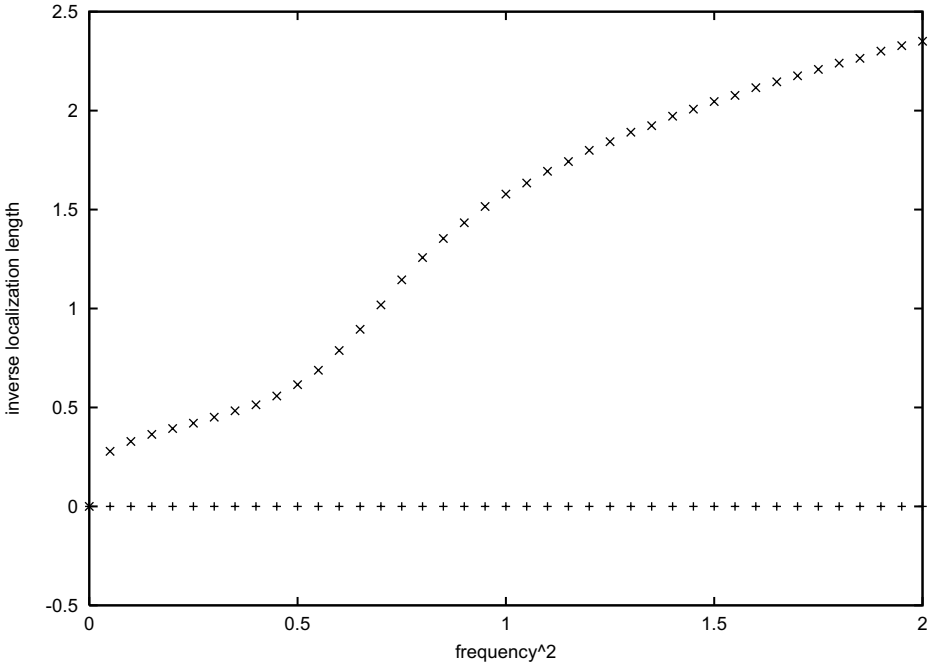

Fig. 4. Inverse localization length ξ^{-1} for a system with periodic scattering (+) and a PARS (×) with matrix elements uniformly distributed on $[0, 1]$ in one dimension.

random off-diagonal terms [11]. Such a model will be considered in the following as a realization of a one-dimensional PARS. The aim here is to study the behavior of the inverse localization length from the intensity [2]

$$\xi^{-1} = -\frac{1}{2|\mathbf{r}|}\log(|\Psi(\mathbf{r})|^2)$$

for large $|\mathbf{r}|$. This quantity is convenient for numerical studies because it is "self-averaging", i.e., there is no averaging over the random matrix elements required. Using the definition of the projected the Green's function in Eq. (4), for the PARS $[PG_0P]_P^{-1}$ is a translational-invariant symmetric matrix. Therefore, an orthogonal transformation can be applied to G_P to diagonalize $[PG_0P]_P^{-1}$:

$$[PG_0P]_P^{-1} \to k_0^2 + \mathrm{i}\varepsilon, \quad V_P \to H.$$

k_0 is not exactly the wave vector of the Green's function G_0 but a renormalized one. Thus the transformed model is given by a random matrix on a one-dimensional array of scatterers as

$$G = (H + k_0^2 + \mathrm{i}\varepsilon)^{-1}, \tag{6}$$

where H is a symmetric random matrix (i.e., $H_{j,k} = H_{k,j}$). For simplicity it is assumed that

$$H_{j,k} = \begin{cases} H_{j,j+1} \in [0, w] & \text{for } k = j + 1, \\ H_{j,j-1} \in [0, w] & \text{for } k = j - 1, \\ 0 & \text{for all other values of } k. \end{cases}$$

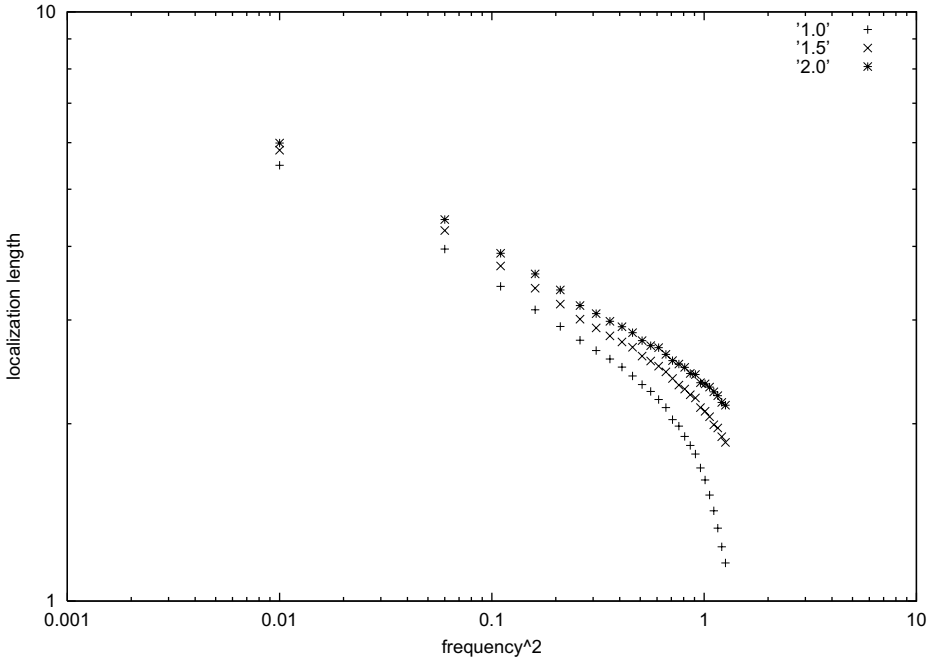

Fig. 5. Behavior of the localization length ξ in a one-dimensional PARS for uniformly distributed matrix elements on $[0, w]$ for $w = 1,\ 1.5$ and 2.

The distribution of the matrix elements is independent for each pair of $j, j+1$, $j, j-1$, except for the symmetric elements, and uniformly distributed on the interval $[0, w]$. The numerical calculation is performed on a chain with 10^5 sites.

First a periodic chain is studied with $H_{j,j+1} = 1$. Fig. 4 shows that the inverse localization length is always zero. Then matrix elements are considered which are uniformly distributed on the interval $[0, 1]$. As shown in Fig. 4, this leads to a non-zero inverse localization length which vanishes only at $k_0 = 0$ and grows with k_0. A more detailed numerical analysis, shown in Fig. 5, indicates a power law divergency for the localization length as one approaches $k_0 = 0$:

$$\xi \propto \xi_0 k_0^{-2\nu}, \quad \xi_0 \approx 1.8...2.2, \ \ \nu \approx 0.21$$

in units of the distance between the scatterers. It should be noticed that ν depends only weakly on w.

An experimental realization of such a system is, e.g., a waveguide which is randomly filled with alumina spheres [12,13]. The measurement of the transmission indicates the existence of localized waves, a clear distinction from absorption is difficult though.

The existence of delocalized states in the one-dimensional system at $k_0 = 0$ depends crucially on the fact that only the off-diagonal term H in the Green's function (Eq. (6)) is random. Adding a *diagonal* random term would immediately localize all states in one- and two-dimensional systems. It was found, however, that the Dirac equation in two dimensions, which describes a vector field (i.e., a Dirac spinor) rather than the scalar field $\Psi(\mathbf{r})$, can have a band of extended states even in the presence of a diagonal random term [14]. This indicates that a vector field may have qualitatively

different localization properties in low-dimensional systems. Since in general the electromagnetic field is also a vector field, replacing the scalar version of the Helmholtz equation by its vectorial counterpart [15] may lead to new effects in terms of localization.

7. Conclusions

Scattering of electromagnetic waves by random impurities is characterized by three different regimes: propagation, diffusion and localization of waves. Diffusion, related to coherent backscattering, is probably the most common form of light scattering, pure propagation is known in clean environments, whereas localization is extremely rare. Nevertheless, it is easily seen in numerical simulations. More work is necessary to determine the specific experimental conditions under which localization of light can be observed.

Appendix A.

The following identity for an operator A and the projection P on the a subspace holds:

$$(\mathbf{1} + PA)^{-1}P = [P + PAP]_P^{-1},$$

where $[\dots]_P^{-1}$ means the inverse with respect to the projected space. The validity of this identity will be shown in three steps. The first step is to show that the left-hand side of the identity is indeed zero outside the projected space by multiplying it from the right with $P(\mathbf{1} + PA)P$:

$$(\mathbf{1} - P)(\mathbf{1} + PA)^{-1}(\mathbf{1} + PA)P = (\mathbf{1} - P)P = 0.$$

Now we have to multiply the left-hand side of the identity with the inverse of the right-hand side, first from the right

$$(\mathbf{1} + PA)^{-1}P(\mathbf{1} + PAP) = (\mathbf{1} + PA)^{-1}(\mathbf{1} + PA)P = P$$

and then from the left

$$P(\mathbf{1} + PAP)(\mathbf{1} + PA)^{-1}P = P(\mathbf{1} + PA)(\mathbf{1} + PA)^{-1}P = P.$$

The first equality in this equation follows from the second equation of the appendix.

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