

## Condensation of a hard-core Bose gas

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A grand canonical system of hard-core bosons, subject to thermal fluctuations, is studied on a lattice. Starting from the slave-boson representation with fields for occupied and unoccupied sites, an effective field theory is derived in which a complex field corresponds with the order parameter of the condensate and a real field with the total density of bosons. Near the boundary between the normal and the superfluid phase, we obtain the Ginzburg-Landau functional for the superfluid order parameter. A mean-field calculation shows that the critical temperature  $T_c$  increases with increasing density up to a maximum and decreases with further increasing density.

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### I. INTRODUCTION

Interacting bosonic quantum systems are of special interest because of the effect of Bose condensation. The classical example is the Bose-Einstein condensation in an ideal Bose gas [1,2]. The analogous phenomenon in a real (i.e., interacting) Bose gas is the transition from a normal to a superfluid state (e.g., in  $^4\text{He}$ ). Another related phenomenon is the condensation of a cold gas of bosonic atoms in a magnetic trap [3,4]. There are a number of recent theoretical investigations of the effect of the interaction on the critical properties of the normal-superfluid transition based on Monte Carlo simulations [5,6] and analytic calculations [7,8]. The characteristic parameter for the effect of interaction is the ratio of the scattering length  $a$  and the typical interparticle distance  $n^{-1/3}$ , where  $n$  is the density of bosons. In the recent trapping experiments, this ratio is typically  $an^{1/3} \approx 10^{-2}$  [3,9] with a scattering length  $a \approx 10^{-7}$  cm for alkali-metal atoms. A significant depletion of the condensate due to interaction is expected for  $an^{1/3} > 0.1$ . This requires systems with much higher density and/or larger scattering length than observed in the experiments with alkali-metal atoms to date. However, since the depletion is about 90% in helium, there is a good reason to believe that the intermediate regime between the trapped alkali-metal atoms and helium can be approached by some new bosonic systems.

A fundamental model for the description of a dilute system of interacting bosons is the Gross-Pitaevskii functional [10,11]. On the other hand, for a dense system of bosons (like the classical  $^4\text{He}$  superfluid), a Ginzburg-Landau approach for the complex superfluid order parameter can be used. This is valid very close to the phase transition from the normal fluid to the superfluid, where the order parameter is small. The Ginzburg-Landau functional of superfluid phase transition is formally equivalent to the Gross-Pitaevskii functional, although the parameters are different. Away from the phase transition the interaction of the order-parameter field is more complicated than described by this theory. Moreover, the Ginzburg-Landau theory is only a description of the order parameter (which is related to the density of the superfluid) but does not take into account the interaction with the nonsuperfluid part of the system. Consequently, it does not provide reliable information, e.g., for the value of the critical

temperature. The question of the effect of the interaction on the latter has been discussed in the recent literature in great detail and with controversial results [5]. It seems that the critical temperature can be shifted by a variation of the density of bosons. In particular, at low density it was found that the shift  $\Delta T_c/T_0 \sim (na^3)^\alpha$ , where  $T_0$  is the critical temperature of the ideal (non-interacting) Bose gas,  $n$  is the total density of bosons, and  $a$  is the scattering radius of the hard-core interaction. Depending on the calculational method and approximations, the exponent  $\alpha$  varies between 1/3 [12] and 1/2 [13]. Recent Monte Carlo simulations [5,6] support  $\alpha = 1/3$ . This value was also obtained by a self-consistent calculation of the quasiparticle spectrum [7] and in a  $1/N$  expansion [8]. At a high density, the critical temperature reaches a maximum and decreases with even higher densities. The latter is a consequence of the depletion of the condensate due to interaction.

In order to give a complete overview of the properties of an interacting Bose gas, we need a model that takes fully into account all parts of the system of bosons. Close to the critical point, however, it should lead to the Ginzburg-Landau theory. Such a model was given by hard-core bosons, based on a slave-boson representation [14]. Here we will briefly discuss this model and evaluate its critical temperature for different densities.

The paper is organized as follows. In Sec. II the slave-boson representation of hard-core bosons on a lattice is introduced. Then in Sec. III, two collective fields are defined. One represents the superfluid condensate and the other, the total density of bosons, as discussed in Sec. IV. In Sec. V, the total density, the condensate density, and the critical temperature are calculated in mean-field approximation. Finally, concluding remarks and a discussion are given in Sec. VI.

### II. THE MODEL: SLAVE-BOSON REPRESENTATION

A continuous system of hard-core bosons with scattering length  $a$  is approximated by a lattice Bose gas with lattice constant  $a$ . Although this approximation is limited because it restricts configurations of bosons to be commensurate with the lattice structure, it is more suitable for the investigation of a dense system of bosons than the usually considered  $|\Phi|^4$  (Gross-Pitaevskii) theory. The representation of the model

uses the slave-boson approach to hard-core bosons [14]. [Originally, the slave-boson approach was invented for the (fermionic) Hubbard model [17,18].] The latter relies on a picture in which a particle trades its position with an empty site on the lattice. Both the particle as well as the empty site are described by corresponding creation and annihilation operators. In a functional integral representation of a grand canonical ensemble of bosons with chemical potential  $\mu$  at temperature  $T$ , this can be formulated in terms of a complex field  $b_x$  of bosons and a complex field  $e_x$  of empty sites with the action

$$S_{s.b.} = \frac{1}{T} \sum_{x,x'} b_x^* e_x t_{x,x'} b_{x'} e_{x'}^* - \frac{1}{T} \sum_x \mu |b_x|^2,$$

where the first term describes the exchange of bosons and empty sites in a hopping process at sites  $x$  and  $x'$  with rate  $t_{x,x'}$ . A local constraint  $|e_x|^2 + |b_x|^2 = 1$  takes care of the complementary character of the bosons and empty sites. In  $S_{s.b.}$ , we consider only the thermal fluctuations (i.e., a vanishing Matsubara frequency) because nonzero Matsubara frequencies are separated by a gap if the temperature  $T$  is nonzero. Here we assume that the temperature enters through the action  $S_{s.b.}$  but not through the constraint. Physical quantities can be calculated from the partition function

$$Z = \int e^{-S_{s.b.}} \prod_x \delta(|e_x|^2 + |b_x|^2 - 1) db_x db_x^* de_x de_x^*.$$

The local constraint implies

$$\langle |b_x|^2 \rangle_{s.b.} = 1 - \langle |e_x|^2 \rangle_{s.b.}$$

for the expectation value

$$\langle \dots \rangle_{s.b.} = \frac{1}{Z} \int e^{-S_{s.b.}} \dots \prod_x \delta(|e_x|^2 + |b_x|^2 - 1) \times db_x db_x^* de_x de_x^*.$$

Although both fields  $b_x$  and  $e_x$  are complex, the action depends only on their relative phase  $\bar{\phi}_x = \phi_x^e - \phi_x^b$ . If we are interested in the physics of the boson field  $b_x$  alone, the field  $e_x$  can be integrated out. This leads to the new action

$$S'_{s.b.} = \frac{1}{T} \sum_{x,x'} b_x^* \sqrt{1 - |b_x|^2} t_{x,x'} b_{x'} \sqrt{1 - |b_{x'}|^2} - \frac{1}{T} \sum_x \mu |b_x|^2,$$

where the integration is now restricted to  $|b_x| \leq 1$ . This expression can be compared with the Gross-Pitaevskii functional on the lattice

$$S_{GP} = \frac{1}{T} \sum_{x,x'} b_x t_{x,x'} b_{x'}^* - \frac{1}{T} \sum_x \left( \mu |b_x|^2 - \frac{u}{2} |b_x|^4 \right).$$

The main difference between the actions is the restriction of the fluctuations in the case of the slave-boson theory. The

latter has a direct consequence on the density of bosons. This quantity, which is given as the response to a change of the chemical potential  $\mu$

$$n = \frac{T}{N} \frac{\partial \ln Z}{\partial \mu} = \frac{T}{N} \frac{1}{Z} \frac{\partial Z}{\partial \mu} = \frac{1}{N} \sum_x \langle |b_x|^2 \rangle_{s.b.}, \quad (1)$$

where  $N$  is the number of lattice sites, increases monotonously with an increasing  $\mu$  due to the weight  $\exp(\mu \sum_x |b_x|^2 / T)$ . However, the growth saturates for  $S_{s.b.}$  because of the constraint  $|b_x| \leq 1$ . As  $|b_x|$  increases, the hopping term  $t_{x,x'}$  in  $S_{s.b.}$  decreases because of the square root factor  $\sqrt{1 - |b_x|^2} \sqrt{1 - |b_{x'}|^2}$ . This can be seen as a field-dependent mass renormalization that increases with the field. It represents the repulsive nature of the hard-core boson interaction. In the Gross-Pitaevskii functional  $S_{GP}$ , the interaction is separated from the hopping as a local term that depends only on  $|\Phi_x|$ . Therefore, the repulsive interaction of the Gross-Pitaevskii functional does not affect the kinetic energy of the bosons.

To evaluate the density of the condensate  $\rho_s$ , we must add an external vector potential to the action  $S_{s.b.}$  (i.e., a Peierls factor to the hopping matrix  $t$ ) and measure the response to this potential [19]. We shall return to the corresponding expression subsequently.

Using the slave-boson theory as a starting point, an effective field theory will be derived that distinguishes between the condensate and the normal part. This can be understood as a field-theoretic version of the two-fluid theory [15,16]. Near the critical point, the condensate can be described by the usual Ginzburg-Landau theory.

### III. COLLECTIVE-FIELD REPRESENTATION

Since the fields  $b_x$  and  $e_x$  are subject to the local constraint, we cannot treat them in a conventional way as order-parameter fields. It is necessary to eliminate the constraint in the partition function, which can be achieved by integration over the fields. For this purpose we introduce a complex collective field  $\Phi_x$  and a real field  $\varphi_x$  that break up the bi-quadratic term in the action  $S_{s.b.}$ . This can be written as the (Hubbard-Stratonovich) transformation

$$S_{s.b.} \rightarrow S' = T \sum_{x,x'} \Phi_x (1-t)_{x,x'}^{-1} \Phi_{x'}^* + T \sum_x \varphi_x^2 + \sum_x \begin{pmatrix} e_x \\ b_x \end{pmatrix} \times \begin{pmatrix} 2\varphi_x + T^{-1} & \Phi_x \\ \Phi_x^* & -\mu T^{-1} \end{pmatrix} \begin{pmatrix} e_x^* \\ b_x^* \end{pmatrix}. \quad (2)$$

The new action  $S'$  gives the same partition function that can be seen by integrating over the collective fields. The field  $\varphi$  is necessary in order to invert the hopping term that is now the positive matrix  $1-t$ . Then we can perform the integration of the slave-boson fields. This becomes simple if the  $2 \times 2$  Hermitian matrix in Eq. (2) is diagonalized by a unitary transformation. The latter leaves the constraint  $|b_x|^2 + |e_x|^2 = 1$  invariant and gives the eigenvalues

$$\lambda_{x,\pm} = \varphi_x + 1/2T - \mu/2T \pm \sqrt{(\varphi_x + 1/2T + \mu/2T)^2 + |\Phi_x|^2}.$$

Now the partition function reads

$$Z = Z_0 \int e^{-S_b} \prod_x \delta(|b_x|^2 + |e_x|^2 - 1) e^{-T\varphi_x^2 - \lambda_{x,+}|e_x|^2 - \lambda_{x,-}|b_x|^2} \\ \times db_x db_x^* de_x de_x^* d\varphi_x d\Phi_x d\Phi_x^*,$$

where the only nonlocal term is

$$S_b = T \sum_{x,x'} \Phi_x (1-t)_{x,x'}^{-1} \Phi_{x'}^*.$$

The  $e_x$  and  $b_x$  integration can be carried out (see Appendix A), which yields

$$Z = Z_0 \int e^{-S_b} \prod_x e^{-T\varphi_x^2} \frac{e^{-\lambda_{x,+}} - e^{-\lambda_{x,-}}}{\lambda_{x,+} - \lambda_{x,-}} d\varphi_x d\Phi_x d\Phi_x^* \\ = Z_0 \int e^{-S_b} \prod_x e^{-1/4T + \mu/2T} \\ \times \int e^{-T\varphi_x^2} \frac{\sinh[\sqrt{(\varphi_x + \mu/2T)^2 + |\Phi_x|^2}]}{\sqrt{(\varphi_x + \mu/2T)^2 + |\Phi_x|^2}} \\ \times d\varphi_x d\Phi_x d\Phi_x^*.$$

The constant  $Z_0$  is the normalization factor of the Hubbard-Stratonovich transformation of Eq. (2). It is convenient to separate the field-independent factor  $\prod_x e^{-1/4T + \mu/2T}$  to define the partition function

$$\bar{Z} = \int e^{-S_b} \prod_x \\ \times \int e^{-T\varphi_x^2} \frac{\sinh[\sqrt{(\varphi_x + \mu/2T)^2 + |\Phi_x|^2}]}{\sqrt{(\varphi_x + \mu/2T)^2 + |\Phi_x|^2}} d\varphi_x d\Phi_x d\Phi_x^*.$$

This partition function has the effective action for the collective field  $\Phi_x$

$$S = S_b + S_0,$$

where  $S_b$  is its nonlocal (i.e., hopping) part and

$$S_0 = \sum_x \ln[Z_1(|\Phi_x|^2)]$$

is its local (i.e., potential) part with

$$Z_1(|\Phi|^2) = \int e^{-T\varphi^2} \frac{\sinh[\sqrt{(\varphi + \mu/2T)^2 + |\Phi|^2}]}{\sqrt{(\varphi + \mu/2T)^2 + |\Phi|^2}} d\varphi.$$

#### IV. INTERPRETATION OF THE FIELDS

The introduction of the collective field has completely separated the hopping part  $S_b$  from the potential part  $S_0$ . The latter does not depend on the phase of the collective field. The hopping part alone describes free bosons (ideal Bose gas) with the usual complex field. This can be seen by writ-

ing the partition function of the ideal Bose gas as

$$Z_{IBG} = \prod_k \left[ \sum_{n_k \geq 0} e^{-n_k(\epsilon_k - \mu_0)/T} \right] = \prod_k [1 - e^{-(\epsilon_k - \mu_0)/T}]^{-1},$$

where  $k$  is a quantum number that characterizes the system of bosons. This can also be expressed in terms of an integral over a complex field as

$$Z_{IBG} = \int \exp\left(-\sum_k [1 - e^{-(\epsilon_k - \mu_0)/T}] |\Phi_k|^2\right) \\ \times \prod_k d\Phi_k d\Phi_k^* / \pi.$$

(Notice that a rescaling of the field yields only a factor to  $Z_{IBG}$ .) For the translational-invariant Bose gas,  $k$  is the wave vector and we have the energy

$$\epsilon_k = \frac{\hbar^2 k^2}{2m}$$

with the boson mass  $m$ .

Using the Fourier components  $(1 - \tilde{t}_k)^{-1}$  in the nonlocal term  $S_b$ , we can compare the latter with the corresponding expression of the ideal Bose gas

$$\sum_k [1 - e^{-(\epsilon_k - \mu_0)/T}] |\Phi_k|^2. \quad (3)$$

By setting the hopping term  $(1 - \tilde{t}_k)^{-1}$  equal to  $1 - e^{-(1/T)(\epsilon_k - \mu_0)}$ , we obtain

$$\tilde{t}_k = (1 - e^{(\epsilon_k - \mu_0)/T})^{-1}.$$

The chemical potential of the ideal Bose gas is restricted to  $\mu_0 \leq 0$  whereas it can have any real value in the interacting Bose gas. In particular, we can choose  $\mu_0 \geq 0$  such that  $\tilde{t}_k > 0$ . Moreover, in the Bose gas we can now apply the continuum approximation

$$S_b/T = \sum_{x,x'} \Phi_x (1-t)_{x,x'}^{-1} \Phi_{x'}^* \\ \approx \int \left[ -\frac{\hbar^2}{2m} \Phi_x (\nabla^2 \Phi_x^*)_x + \alpha |\Phi_x|^2 \right] d^3x, \quad (4)$$

where  $\alpha = \sum_x (1-t)_{x,x'}^{-1}$ . It is important to notice that  $\alpha = (1 - \tilde{t}_0)^{-1}$  is positive because  $1 - t$  was defined as a positive matrix. This implies that  $S_b$  is a positive quadratic form.  $S_b$  can be compared with the expression of the ideal Bose gas if we replace  $\alpha$  by  $-\mu_0$ . Then the expression in Eq. (4) is an approximation of Eq. (3) for small  $(\epsilon_k - \mu_0)T$ , which applies, e.g., to high temperatures. This approximation is very common in the literature [20,15] and can also be used in the case of hard-core bosons.

Physical quantities can be expressed as expectation values of the new fields  $\varphi$  and  $\Phi$ . For instance, we obtain from Eq. (1) for the total density of bosons, the expression (see Appendix B)

$$n = \frac{1}{2} + T\langle\varphi_x\rangle,$$

where

$$\begin{aligned} \langle\cdots\rangle &= \frac{1}{Z} \int \cdots e^{-S_b} \\ &\times \prod_x e^{-T\varphi_x^2} \frac{\sinh[\sqrt{(\varphi_x + \mu/2T)^2 + |\Phi_x|^2}]}{\sqrt{(\varphi_x + \mu/2T)^2 + |\Phi_x|^2}} \\ &\times d\varphi_x \Phi_x d\Phi_x^*. \end{aligned} \quad (5)$$

Thus  $\varphi_x$ , which is conjugate to  $|e_x|^2$  according to Eq. (2), is related to the total density of bosons. Conversely,  $\Phi_x$ , which is conjugate to  $e_x b_x^*$  according to Eq. (2), corresponds to the density of the condensate [14,19] and can be expressed as the expectation value

$$\rho_s = T\alpha\langle|\Phi_x|^2\rangle = \frac{T\alpha}{Z} \int |\Phi_x|^2 e^{-S_b - S_0} \prod_x d\Phi_x d\Phi_x^*.$$

We notice that the potential part of the partition function  $S_0$  is symmetric with respect to  $\mu \rightarrow -\mu$ . This implies that the expectation value  $\langle\varphi_x\rangle$  is an *odd* function of  $\mu$ . Therefore, the density varies monotonously with  $\mu$ , as it should. The symmetry of  $S_0$  with respect to  $\mu \rightarrow -\mu$  implies that  $\rho_s$  is an *even* function of  $\mu$ . This is a characteristic feature of our lattice hard-core bosons in which bosons and empty sites are dual to each other.

#### Near the critical point

The  $\varphi$  integration in  $Z_1$  can be performed numerically in order to obtain an effective potential

$$-\sum_x \ln[Z_1(|\Phi_x|^2)].$$

It is interesting to notice that the expression  $\ln[Z_1(|\Phi_x|^2)]$  is linear for large values of  $|\Phi_x|$  (cf. Fig. 1). Therefore, the quadratic term in  $S_b$  suppresses the large fluctuations. This means that the  $|\Phi_x|^4$  approximation of the Ginzburg-Landau theory provides a stronger suppression of large fluctuations than the complete theory.

Near the critical point we can expand the free energy around  $|\Phi_x|=0$  because the order-parameter field  $\Phi$  is small. The result of this expansion is the Ginzburg-Landau functional for the collective field with

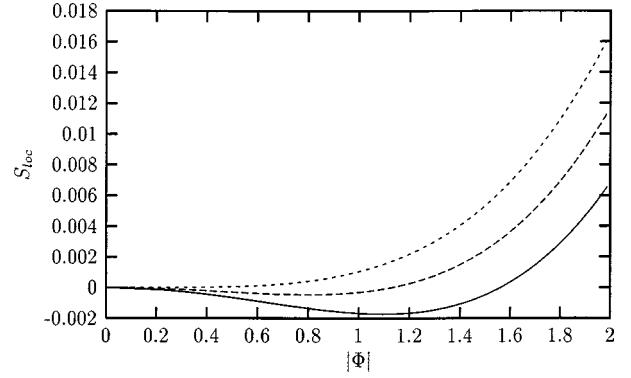


FIG. 1. Behavior of the local part of the action  $S_{loc} = \alpha T |\Phi|^2 - \ln[Z_1(|\Phi|^2)]$  for  $T=0.1$  K,  $\alpha=1.02$  K $^{-1}$  and three different values  $\mu/T=0,0.07,0.1$ . There is a critical point at  $\mu \approx 0$ .

$$\begin{aligned} &T \sum_{x,x'} \Phi_x (1-t)_{x,x'}^{-1} \Phi_{x'}^* - \sum_x (b_1 |\Phi_x|^2 + b_2 |\Phi_x|^4) \\ &\approx \int \left[ -\frac{T\hbar^2}{2m} \Phi_x (\nabla^2 \Phi^*)_x + (T\alpha - b_1) |\Phi_x|^2 \right. \\ &\quad \left. - b_2 |\Phi_x|^4 \right] d^3x, \end{aligned} \quad (6)$$

where

$$b_1 = \frac{Z_1'(0)}{Z_1(0)}, \quad b_2 = \frac{1}{2} \left[ \frac{Z_1''(0)}{Z_1(0)} - \left( \frac{Z_1'(0)}{Z_1(0)} \right)^2 \right].$$

Expression (6) is also known as the Ginzburg-Landau functional, which has the same form as the Gross-Pitaevskii functional, often used as a model of a dilute Bose system [10,11]. In our derivation we have  $\mu$ - and  $T$ -dependent coefficients. As one can see in Fig. 1, the term with  $|\Phi|^2$  can change its sign with a changing  $\mu$ . On the other hand, away from  $\Phi=0$  the behavior is controlled by the confining part  $|\Phi|^4$ . This is the typical Ginzburg-Landau picture of a second-order phase transition.

The effect of the  $|\Phi|^4$  interaction on  $T_c$  in the model of Eq. (6) was recently investigated in detail in a self-consistent calculation [7] and in a  $1/N$  expansion [8]. In this model it is assumed that the coefficient of the  $|\Phi|^4$  term depends on the scattering length  $a$  and the total density by a factor  $a^3 n$ . This is valid at low density. The calculation shows that the critical temperature of the interacting system  $T_c$ , normalized with the critical temperature of the ideal Bose gas, is shifted as (see Table I)

$$1 - T_0/T_c \sim c_0 a n^{1/3}. \quad (7)$$

TABLE I. The coefficients of the critical temperature shift in Eq. (7) from different works.

Reference	[5]	[6]	[7]	[8]
$c_0$	0.34	2.2	1.5	2.33

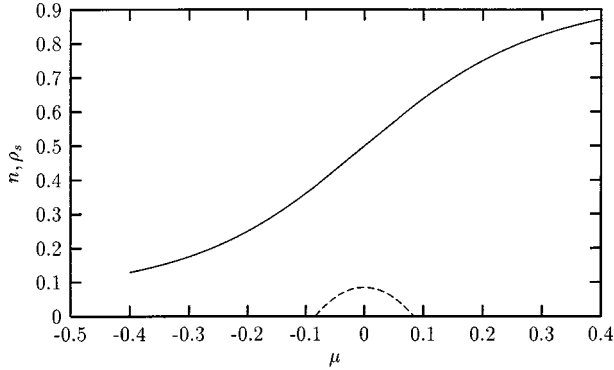


FIG. 2. Mean-field result of the total density  $n$  (full curve) and the superfluid density  $\rho_s$  (dashed curve) at  $\alpha T=0.1$ . The densities are measured in units of the scattering volume  $a^3$ .

Numerical investigations [5] show that the critical temperature must decrease at higher densities. This depletion effect, which cannot be seen in the Gross-Pitaevskii approach, will be discussed in the slave-boson approach subsequently.

### V. MEAN-FIELD APPROXIMATION

Since  $\varphi_x$  is a field that appears only in local terms, it can be integrated out at each site independently for a given value of the condensate field  $\Phi_x$ . The treatment of  $\Phi_x$  is more difficult since it appears in nonlocal term  $S_b$ . It can be studied in terms of the classical field equation

$$\left[ -\frac{\hbar^2 T}{2m} \nabla^2 + T\alpha - \frac{Z'_1(|\Phi_x|^2)}{Z_1(|\Phi_x|^2)} \right] \Phi_x = 0,$$

which is the extremum of the action  $S_b + S_0$ . A further simplification is the additional assumption that the condensate field varies only weakly in space:  $\nabla^2 \Phi \approx 0$ . This gives the mean-field or Thomas-Fermi approximation. Both densities,  $n$  and  $\rho_s$ , can be evaluated in mean-field approximation. The mean-field free energy reads

$$F_{MF} = -\frac{1}{N} \ln Z = \frac{1}{4} - \frac{\mu}{2T} + T\alpha |\Phi|^2 - \ln[Z_1(|\Phi|^2)], \quad (8)$$

where  $|\Phi|$  must be at the minimum of  $F_{MF}$ . For a given chemical potential  $\mu$  there is a critical value  $T_c$  that separates two regimes: one regime for  $T < T_c$  in which the minimum of the mean-field free energy is  $|\Phi|^2 > 0$  and another regime with  $|\Phi|^2 = 0$  for  $T \geq T_c$ . This can be seen by plotting  $T\alpha |\Phi|^2 - \ln[Z_1(|\Phi|^2)]$  (Fig. 1) which has, depending on  $\mu$ , either a minimum at  $|\Phi| = 0$  or another one at  $|\Phi| \neq 0$  [14]. The minimal value  $|\Phi|$  varies continuously as one goes through  $T_c \alpha$ . The behavior of the densities as functions of  $\mu$  at a fixed temperature is shown in Fig. 2.

#### Near the critical point

For a very dilute system, the mean-field approximation is insufficient and the calculations of Refs. [7,8] should be applied. However, at higher densities (more than  $n \approx 0.2$ ) the

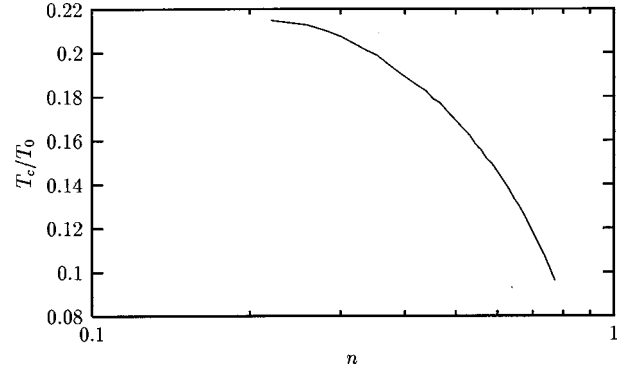


FIG. 3. Mean-field result of the critical temperature  $T_c$  of the hard-core bosons in arbitrary units, normalized with the critical temperature of the ideal Bose gas  $T_0 \propto n^{2/3}$ . This behavior is similar to the result found in a Monte Carlo simulation of Ref. [5].

mean-field approximation should be reliable. Then the critical temperature  $T_c$  of the mean-field calculation is

$$T_c = b_1 / \alpha \equiv Z'_1(0) / [\alpha Z_1(0)].$$

The decreasing behavior of  $T_c/T_0$  ( $T_0 \propto n^{2/3}$  is the condensation temperature of the ideal Bose gas) is shown in Fig. 3. Our mean-field result for  $T_c/T_0$  agrees qualitatively with the Monte Carlo result of Ref. [5]. However, we expect that fluctuations might reduce the critical temperature substantially close to  $n = 1$ .

We can expand the mean-field free energy (8) in powers of  $|\Phi|^2$  up to  $|\Phi|^4$ . Then the minimum of  $F$  must satisfy the mean-field equation

$$\begin{aligned} |\Phi|^2 &\sim -\frac{T\alpha Z_1(0) - Z'_1(0)}{T\alpha Z'_1(0) - Z''_1(0)} = \frac{T_c - T}{(T - T_c)b_1 - 2b_2/\alpha} \\ &\sim \frac{T_c - T}{-2b_2/\alpha} \quad (T \sim T_c) \end{aligned} \quad (9)$$

if the right-hand side is nonnegative and  $|\Phi|^2 = 0$  otherwise. Since  $b_2 \leq 0$  in a typical situation (cf. Fig. 1), there is a nonzero solution. The coefficient on the right-hand side of Eq. (9) can be evaluated numerically for a given value of  $\mu$ .

### VI. DISCUSSION

The slave-boson representation is given by two fields  $b_x$ ,  $e_x$  that are subject to a constraint: one represents empty, the other, singly occupied sites. These two fields are replaced by two collective fields  $\varphi_x$  and  $\Phi_x$  that have a direct physical interpretation. The former couples to  $|e_x|^2$  and the latter couples to the product  $e_x b_x^*$ . The introduction of the collective field  $\Phi_x$  enables us to integrate the slave-boson fields and to eliminate the constraint of these fields. The effective field theory provides a “two-liquid” theory that is represented by the fields  $\varphi_x$  and  $\Phi_x$ . These fields correspond with the total density  $n$  and the superfluid density  $\rho_s$ , respectively. The latter is the order parameter of the condensation whereas the total density  $n$  does not indicate a critical behavior at the condensation point (cf. Fig. 2). Therefore,  $n$  can be

fixed near the critical point and the theory can be expanded in terms of a small order-parameter field  $\Phi_x$ . This yields the well-known Ginzburg-Landau theory with density-dependent parameters. In other words, there is a phase boundary in the  $\mu$ - $T$  phase diagram that separates the normal from the superfluid phase. In the vicinity of this phase boundary we can apply the Ginzburg-Landau approach.

The condensate field can be treated in mean-field approximation, assuming a homogeneous order parameter. This reveals the phenomenon of depletion of the condensate due to a strong interaction among the bosons that has been observed in superfluid  $^4\text{He}$ . In the slave-boson representation, it is a consequence of the duality of empty and singly occupied lattice sites, which is reflected by the constraint  $|e_x|^2 + |b_x|^2 = 1$ . Thus, physical quantities are symmetric with respect to a half-filled system (i.e.,  $n = 1/2$  or  $\mu = 0$ ).

The lattice theory is not very accurate at densities  $n \approx 0.5$  because it reduces the motion of bosons significantly in comparison with a continuous system. Moreover, the mean-field approximation neglects vortices and fluctuations of the order parameter. From this point of view, we can only expect a qualitative agreement of our results with those, e.g., obtained from experiments. Nevertheless, the mean-field approximation should be reasonable if the density is not too low. However, the critical exponent of the order parameter should be renormalized due to fluctuations, using the renormalization group for the three-dimensional  $|\Phi|^4$  model.

The main result is that the phase diagram of the slave-boson theory of the strongly interacting bose system has two normal phases (i.e., a dense and a dilute one) and a superfluid phase for intermediate densities. Near the transition points a Ginzburg-Landau approach can be used to describe the physics of a small order-parameter field. The superfluid density is low in our hard-core system. This might be a consequence of the strong interaction that suppresses the superfluid component.

In conclusion, we established an effective field theory that enables us to evaluate the properties of a strongly interacting system of bosons. It takes into account the order parameter of the condensate and the total density by interacting fields. It describes the phase transition between a normal phase and a condensed phase. The phase transition was studied in mean-field approximation. We evaluated the density-dependent critical temperature at densities  $n > 0.2$  in which the mean-field approximation of the order parameter is reliable.

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#### APPENDIX A

The integration of the  $e$  and the  $b$  fields can be performed in  $Z$  for each point  $x$  independently

$$\int \delta(|b_x|^2 + |e_x|^2 - 1) e^{-\lambda_{x,+}|e_x|^2 - \lambda_{x,-}|b_x|^2} db_x db_x^* de_x de_x^*. \quad (\text{A1})$$

The integrand does not depend on the phases of the field. Therefore, the phase integration contributes a factor  $4\pi^2$ . Moreover, we set  $s := |b_x|^2$  and  $t := |e_x|^2$ , which yields for Eq. (A1)

$$\begin{aligned} & \pi^2 \int_0^\infty \int_0^\infty \delta(s+t-1) e^{-\lambda_{x,+}t - \lambda_{x,-}s} ds dt \\ &= \pi^2 \int_0^1 e^{-\lambda_{x,+}t - \lambda_{x,-}(1-t)} dt \\ &= -\pi^2 \frac{e^{-\lambda_{x,+}} - e^{-\lambda_{x,-}}}{\lambda_{x,+} - \lambda_{x,-}}. \end{aligned}$$

#### APPENDIX B

To write the total density of bosons we have to evaluate

$$\frac{T}{N} \frac{\partial \ln Z}{\partial \mu} = \frac{1}{2} + \frac{T}{N} \frac{1}{Z} \frac{\partial \bar{Z}}{\partial \mu}.$$

Differentiation yields

$$\frac{\partial \bar{Z}}{\partial \mu} = \int \sum_x \frac{\partial Z_1(|\Phi_x|^2)}{\partial \mu} \left[ \prod_{x \neq x} Z_1(|\Phi_x|^2) \right] \prod_x d\Phi_x d\Phi_x^*.$$

Since

$$\frac{\partial Z_1(|\Phi_x|^2)}{\partial \mu} = \int \varphi e^{-T\varphi^2} \frac{\sinh[\sqrt{(\varphi + \mu/2T)^2 + |\Phi_x|^2}]}{\sqrt{(\varphi + \mu/2T)^2 + |\Phi_x|^2}} d\varphi,$$

we can write for the previous expression

$$\begin{aligned} \frac{\partial \bar{Z}}{\partial \mu} &= \int \sum_x \varphi_x^- e^{-S_b} \\ &\times \prod_x e^{-T\varphi_x^2} \frac{\sinh[\sqrt{(\varphi_x^- + \mu/2T)^2 + |\Phi_x|^2}]}{\sqrt{(\varphi_x^- + \mu/2T)^2 + |\Phi_x|^2}} \\ &\times d\varphi_x d\Phi_x d\Phi_x^*. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{Z} \frac{\partial \bar{Z}}{\partial \mu} &= \frac{1}{Z} \sum_x \int \varphi_x^- e^{-S_b} \\ &\times \int e^{-T\varphi_x^2} \frac{\sinh[\sqrt{(\varphi_x^- + \mu/2T)^2 + |\Phi_x|^2}]}{\sqrt{(\varphi_x^- + \mu/2T)^2 + |\Phi_x|^2}} d\varphi_x^- \\ &\times \left[ \prod_{x \neq x} Z_1(|\Phi_x|^2) \right] \prod_x d\Phi_x d\Phi_x^* \\ &= \sum_x \langle \varphi_x^- \rangle. \end{aligned}$$

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