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Overlapping Domain Decomposition methods with distributed Lagrange multipliers

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Abstract — In this paper, we consider an overlapping domain decomposition method for second order elliptic boundary value problems. The method is based on nonmatching simplicial triangulations of the subdomains and relies on an appropriate splitting of the variational formulation. The coupling conditions are taken care of by distributed Lagrange multipliers. The LBB condition is established and an efficient preconditioner is suggested.

Keywords: Overlapping Domain Decomposition, distributed Lagrange multipliers, spectral equivalence, block diagonal preconditioning.

1. INTRODUCTION

Given a bounded domain $\Omega \subset \mathbf{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, and a function $f \in L^2(\Omega)$, we consider the weak formulation of Poisson's equation with homogeneous Dirichlet boundary data: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x\,, \qquad v \in H_0^1(\Omega)\,. \tag{1.1}$$

We assume a decomposition

$$\Omega = \Omega_1 \cup \Omega_2\,, \qquad \Omega_{12} := \Omega_1 \cap \Omega_2 \neq \emptyset, \qquad \operatorname{meas}(\partial\Omega_{12} \cap \Gamma) > 0 \tag{1.2}$$

of the computational domain Ω into two partly overlapping subdomains Ω_i, $1 \leqslant i \leqslant 2$, with nonempty overlap Ω_{12} such that the intersection of $\partial\Omega_{12}$ and Γ has a positive measure. We set

$$\hat{\Omega}_i \;:=\; \Omega_i \setminus \bar{\Omega}_{12}\,, \qquad 1 \leqslant i \leqslant 2 \tag{1.3}$$

$$\Gamma_i^k \;:=\; \partial\Omega_i \cap \Omega_k\,, \qquad 1 \leqslant i \leqslant 2\,, \quad 1 \leqslant k \neq i \leqslant 2\,. \tag{1.4}$$

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For $D \subset \Omega$ with meas $(\partial D \cap \Gamma) > 0$ we define $H^1_{0,\Gamma}(D) := \{v \in H^1(D) : \ v|_{\partial D \cap \Gamma} = 0\}$. We remark that in this case $|\cdot|_{1,D}$ is a norm on $H^1_{0,\Gamma}(D)$ being equivalent to the standard Sobolev norm $\|\cdot\|_{1,D}$. With respect to the decomposition (1.2) we introduce the product space

$$V := H^1_{0,\Gamma}(\Omega_1) \times H^1_{0,\Gamma}(\Omega_2) \tag{1.5}$$

with norm $\|v\|_V := (\sum_{i=1}^2 \|v_i\|^2_{1,\Omega_i})^{1/2}, \ v = (v_1, v_2) \in V$, and the multiplier space

$$M := H^1_{0,\Gamma}(\Omega_{12}), \qquad \|\mu\|_M := \|\mu\|_{1,\Omega_{12}}, \qquad \mu \in M. \tag{1.6}$$

We further introduce [4] the bilinear form $a(\cdot,\cdot) : \ V \times V \to \mathbf{R}$

$$a(u,v) := \sum_{i=1}^{2} \int_{\Omega_i} a_i \operatorname{grad} u_i \cdot \operatorname{grad} v_i \, \mathrm{d}x, \qquad u, v \in V \tag{1.7}$$

with

$$a_i := \begin{cases} 1 & \text{in } \hat{\Omega}_i \quad 1 \leqslant i \leqslant 2 \\ \dfrac{1}{2} & \text{in } \bar{\Omega}_{12} \end{cases} \tag{1.8}$$

the bilinear form $b(\cdot,\cdot) : \ V \times M \to \mathbf{R}$

$$b(v,\mu) := \int_{\Omega_{12}} \operatorname{grad}(v_1 - v_2) \cdot \operatorname{grad} \mu \, \mathrm{d}x, \qquad v \in V, \quad \mu \in M \tag{1.9}$$

as well as the functional $\ell : V \to \mathbf{R}$

$$\ell(v) := \sum_{i=1}^{2} \ell_i(v_i) := \sum_{i=1}^{2} \int_{\Omega_i} a_i f v_i \, \mathrm{d}x, \qquad v = (v_1, v_2) \in V. \tag{1.10}$$

We refer to $B : V \to M^*$ as the operator associated with $b(\cdot,\cdot)$, i.e., $\langle Bv, \mu \rangle = b(v,\mu)$, $v \in V$, $\mu \in M$. Obviously, $\operatorname{Ker} B = \{v \in V : (v_1 - v_2)|_{\Omega_{12}} = 0\}$, so that $v \in \operatorname{Ker} B$ can be identified with a function in $H^1_0(\Omega)$. Hence, the macro-hybrid variational formulation of (1.1) with respect to the decomposition (1.2) is given by [5]: Find $(u,\lambda) \in V \times M$ such that

$$a(u,v) + b(v,\lambda) = \ell(v), \qquad v \in V \tag{1.11}$$

$$b(u,\mu) = 0, \qquad \mu \in M. \tag{1.12}$$

Theorem 1.1. *The bilinear form $a(\cdot,\cdot)$ is elliptic on* $\mathrm{Ker}B$ *and the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition*

$$\inf_{\mu\in M}\sup_{v\in V}\frac{b(v,\mu)}{\|v\|_V\,\|\mu\|_M}\geqslant\alpha>0\,. \tag{1.13}$$

Proof. The $\mathrm{Ker}B$-ellipticity of $a(\cdot,\cdot)$ is obvious. In order to verify (1.13), for a given $\mu\in M$ we split μ according to $\mu=\mu_1-\mu_2$ with $\mu_i\in H^1_{0,\Gamma}(\Omega_{12})$ and $\|\mu_i\|_{1,\Omega_{12}}\leqslant C\,\|\mu\|_{1,\Omega_{12}}$, $1\leqslant i\leqslant 2$. Here and in the sequel C denotes a generic positive constant not necessarily the same at each occurrence. We consider $v_i:=\mu_i|_{\Gamma_k^i}$, $1\leqslant i\leqslant 2$, $1\leqslant k\neq i\leqslant 2$, and denote by $w_i\in H_{0,\Gamma}(\Omega_i)$ the harmonic extension to $\hat{\Omega}_i$ so that

$$\|w_i\|_{1,\hat{\Omega}_i}\leqslant C\,\|v_i\|_{1/2,\Gamma_k^i}\leqslant C\,\|\mu_i\|_{1,\Omega_{12}}\leqslant C\,\|\mu\|_{1,\Omega_{12}}\,. \tag{1.14}$$

We set

$$v_i:=\begin{cases} w_i & \text{in } \hat{\Omega}_i\\ \mu_i & \text{in } \bar{\Omega}_{12}\,.\end{cases}$$

By construction $v_i\in H^1_{0,\Gamma}(\Omega_i)$. Taking (1.14) into account

$$\|v_i\|^2_{1,\Omega_i}=\|w_i\|^2_{1,\hat{\Omega}_i}+\|\mu_i\|^2_{1,\Omega_{12}}\leqslant C\,\|\mu_i\|^2_{1,\Omega_{12}}\leqslant C\,\|\mu\|^2_{1,\Omega_{12}}\,. \tag{1.15}$$

On the other hand

$$\begin{aligned} b(v,\mu) &= \int_{\Omega_{12}}\operatorname{grad}(v_1-v_2)\cdot\operatorname{grad}\mu\,\mathrm{d}x\\ &= \int_{\Omega_{12}}\operatorname{grad}(\mu_1-\mu_2)\cdot\operatorname{grad}\mu\,\mathrm{d}x=|\mu|^2_{1,\Omega_{12}}\leqslant C\,\|\mu\|^2_{1,\Omega_{12}}\,. \end{aligned} \tag{1.16}$$

Then, the assertion follows from (1.15), (1.16). □

2. DISCRETE LBB-CONDITION

For the overlapping decomposition (1.2) of the computational domain Ω we consider individual simplicial triangulations $\mathcal{T}_i$ of the subdomains Ω_i, $1\leqslant i\leqslant 2$, such that these triangulations induce both simplicial triangulations $\mathcal{T}_{12}$ and $\mathcal{T}_{21}$ of the overlap Ω_{12}, that in general do not match, as well as simplicial triangulations $\hat{\mathcal{T}}_i$ of $\hat{\Omega}_i$, $1\leqslant i\leqslant 2$.

We assume $\mathcal{T}_i$ to be shape regular and quasiuniform. These properties are inherited by the triangulations $\mathcal{T}_{12}$ and $\mathcal{T}_{21}$ of Ω_{12} and $\hat{\mathcal{T}}_i$ of $\hat{\Omega}_i$, $1\leqslant i\leqslant 2$. We refer to h_i, $1 \leqslant i \leqslant 2$, as the mesh sizes $h_i := \max\{h(T) : T \in \mathscr{T}_i\}$.

We denote by $S_1(\Omega_i;\mathscr{T}_i)$, $S_1(\hat{\Omega}_i;\hat{\mathscr{T}}_i)$, and $S_1(\Omega_{12};\mathscr{T}_{ik})$ the finite element spaces of continuous, piecewise linear finite elements on Ω_i, $\hat{\Omega}_i$, Ω_{ik} with respect to the triangulations $\mathscr{T}_i$, $\hat{\mathscr{T}}_i$, and $\mathscr{T}_{ik}$ and refer to $S_{1,\Gamma}(\Omega_i;\mathscr{T}_i)$, $S_{1,\Gamma}(\hat{\Omega}_i;\hat{\mathscr{T}}_i)$, and $S_{1,\Gamma}(\Omega_{ik};\mathscr{T}_{ik})$ as its subspaces of finite element functions vanishing on $\partial\Omega_i \cap \Gamma$, $\partial\hat{\Omega}_i \cap \Gamma$, and $\partial\Omega_{ik} \cap \Gamma$, respectively, $1 \leqslant i \leqslant 2$, $1 \leqslant k \neq i \leqslant 2$.

As in the continuous regime we consider the product space

$$V_h := S_{1,\Gamma}(\Omega_1;\mathscr{T}_1) \times S_{1,\Gamma}(\Omega_2;\mathscr{T}_2)$$

and choose

$$M_h := S_{1,\Gamma}(\Omega_{12};\mathscr{T}_{12})$$

as the space of discrete distributed Lagrange multipliers [1, 2].

Then, denoting by $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, and $\ell_h(\cdot)$ the restrictions of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $\ell(\cdot)$ to $V_h \times V_h$, $V_h \times M_h$ and V_h, respectively, and by B_h the operator associated with $b_h(\cdot,\cdot)$, the finite element approximation of (1.1) with respect to the overlapping domain decomposition (1.2) is given by: Find $(u_h,\lambda_h) \in V_h \times M_h$ such that

$$a_h(u_h,v_h) + b_h(v_h,\lambda_h) \;=\; \ell_h(v_h)\,, \qquad v_h \in V_h \tag{2.1}$$

$$b_h(u_h,\mu_h) \;=\; 0\,, \qquad \mu_h \in M_h\,. \tag{2.2}$$

Theorem 2.1. *The bilinear form $a_h(\cdot,\cdot)$ is elliptic on* $\mathrm{Ker}\,B_h$, *and the bilinear form $b_h(\cdot,\cdot)$ satisfies an* LBB*-condition uniformly in h_i, i.e., there exists a constant β depending only on the shape regularity and quasiuniformity of the triangulations $\mathscr{T}_i$ such that*

$$\inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b_h(v_h,\mu_h)}{\|v_h\|_V \, \|\mu_h\|_M} \geqslant \beta > 0\,. \tag{2.3}$$

Proof. Again, the $\mathrm{Ker}\,B_h$-ellipticity of $a_h(\cdot,\cdot)$ is obvious. The verification of the LBB-condition is the discrete analogue of the proof of Theorem 1.1. Referring to $P_h^{ik} : H^1_{0,\Gamma}(\Omega_{12}) \to S_{1,\Gamma}(\Omega_{12};\mathscr{T}_{ik})$, $1 \leqslant i \leqslant 2$, $1 \leqslant k \neq i \leqslant 2$, as the projections

$$\int\limits_{\Omega_{12}} \mathrm{grad}\, P_h^{ik}\mu \cdot \mathrm{grad}\,\mu_h \,\mathrm{d}x = \int\limits_{\Omega_{12}} \mathrm{grad}\,\mu \cdot \mathrm{grad}\,\mu_h \,\mathrm{d}x\,, \qquad \mu_h \in S_{1,\Gamma}(\Omega_{12};\mathscr{T}_{ik})$$

for a given $\mu_h \in M_h$ we split μ_h by means of $\mu_h = \mu_{h,1} - \mu_{h,2}$, where

$$\mu_{h,1} := P_h^{12}(I - P_h^{21})\mu_h\,, \qquad \mu_{h,2} := -P_h^{21}\mu_h$$

satisfying

$$\|\mu_{h,i}\|_{1,\Omega_{12}} \leqslant C\,\|\mu_h\|_{1,\Omega_{12}}\,. \tag{2.4}$$

We set $v_{h,i} := \mu_{h,i}|_{\Gamma^i_k}$ and define $w_{h,i} \in S_{1,\Gamma}(\hat{\Omega}_i;\hat{\mathscr{T}}_i)$ as the discrete harmonic extension to $\hat{\Omega}_i$ so that [6]:

$$\|w_{h,i}\|_{1,\hat{\Omega}_i} \leqslant C\,\|v_{h,i}\|_{1/2,\Gamma^i_k} \leqslant C\,\|\mu_{h,i}\|_{1,\Omega_{12}}\,. \tag{2.5}$$

We set

$$v_{h,i} := \begin{cases} w_{h,i} & \text{in } \hat{\Omega}_i \\ \mu_{h,i} & \text{in } \bar{\Omega}_{12}\,. \end{cases}$$

Observing

$$\|v_{h,i}\|^2_{1,\Omega_i} = \|w_{h,i}\|^2_{1,\hat{\Omega}_i} + \|\mu_{h,i}\|^2_{1,\Omega_{12}}$$

as well as

$$\begin{aligned} b_h(v_h,\mu_h) &= \int_{\Omega_{12}} \operatorname{grad}(v_{h,1}-v_{h,2})\cdot\operatorname{grad}\mu_h\,\mathrm{d}x \\ &= \int_{\Omega_{12}} \operatorname{grad}\mu_{h,1}\cdot\operatorname{grad}\mu_h\,\mathrm{d}x - \int_{\Omega_{12}} \operatorname{grad}\mu_{h,2}\cdot\operatorname{grad}\mu_h\,\mathrm{d}x \\ &= \int_{\Omega_{12}} \operatorname{grad}\mu_h\cdot\operatorname{grad}\mu_h\,\mathrm{d}x - \int_{\Omega_{12}} \operatorname{grad}P_h^{21}\mu_h\cdot\operatorname{grad}\mu_h\,\mathrm{d}x \\ &\quad + \int_{\Omega_{12}} \operatorname{grad}P_h^{21}\mu_h\cdot\operatorname{grad}\mu_h\,\mathrm{d}x = |\mu_h|^2_{1,\Omega_{12}} \end{aligned}$$

and taking (2.4), (2.5) into account gives the assertion. □

3. ALGEBRAIC ANALYSIS AND PRECONDITIONING

The finite element problem (2.1) results in the saddle point algebraic system

$$\mathscr{A}\begin{pmatrix}\bar{u}\\ \bar{\lambda}\end{pmatrix} = \begin{pmatrix}\bar{f}\\ 0\end{pmatrix} \tag{3.1}$$

with the symmetric nonsingular matrix

$$\mathscr{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \tag{3.2}$$

and a vector $\bar{f} \in \mathbf{R}^n$, where $n = n_1 + n_2$ and $n_i = \dim S_{1,\Gamma}(\Omega_i; \mathscr{T}_i)$, $1 \leqslant i \leqslant 2$. Here,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{3.3}$$

is an 2×2 block diagonal positive definite matrix with $n_i \times n_i$ submatrices A_i, $1 \leqslant i \leqslant 2$, and

$$B = (B_1 \;\; B_2) \tag{3.4}$$

is an $m \times n$ matrix with $m \times n_i$ submatrices B_i, $1 \leqslant i \leqslant 2$, where $m = \dim M_h$. The matrices A_i are defined by

$$(A_i \bar{v}_i, \bar{w}_i) = \int_{\Omega_i} a_i \nabla v_{i,h} \cdot \nabla w_{i,h} \, \mathrm{d}x \tag{3.5}$$

where the functions $v_{i,h}, w_{i,h} \in S_{1,\Gamma}(\Omega_i; \mathscr{T}_i)$ are the finite element prolongations of vectors $\bar{v}_i, \bar{w}_i \in \mathbf{R}^{n_i}$, respectively, $1 \leqslant i \leqslant 2$. These matrices can be also presented in an 2×2 block form:

$$A_i = \begin{pmatrix} A_{ii} & A_{i\omega} \\ A_{\omega i} & A_{\omega}^{(i)} \end{pmatrix} \tag{3.6}$$

where the first block row stays for the degrees of freedoms corresponding to the grid nodes of $\mathscr{T}_i$ which belong to $\Omega_i \setminus \bar{\Omega}_{12}$, and the second row stays for the degrees of freedoms corresponding to the grid nodes of $\mathscr{T}_i$ which belong to $\bar{\Omega}_{12}$, $1 \leqslant i \leqslant 2$. The index ω is used to indicate the subdomain $\omega = \Omega_{12}$.

The matrices B_i are defined by

$$(B_i \bar{v}_i, \bar{\lambda}) = \int_{\Omega_{12}} \nabla v_{i,h} \cdot \nabla \lambda_h \, \mathrm{d}x \tag{3.7}$$

where $v_{i,h} \in S_{1,\Gamma}(\Omega_i; \mathscr{T}_i)$ and $\lambda_h \in M_h$. Here, λ_h is the finite element prolongation of $\bar{\lambda}_i \in \mathbf{R}^m$. We represent the matrices B_i in an 1×2 block form by

$$\begin{aligned} B_1 &= (0 \;\; B_{1\omega}) \\ B_2 &= (0 \;\; B_{2\omega}) \end{aligned} \tag{3.8}$$

where the matrices $B_{1\omega} \in \mathbf{R}^{m \times m}$ and $B_{2\omega} \in \mathbf{R}^{m \times n_3}$, $n_3 = \dim S_{1,\Gamma}(\Omega_1 2; \mathscr{T}_2)$ are defined by

$$(B_{i\omega} \bar{w}_i, \bar{\lambda}) = \int_{\omega} \nabla w_{i,h} \cdot \nabla \lambda_h \, \mathrm{d}x \tag{3.9}$$

with $w_{i,h} \in S_{1,\Gamma}(\Omega_{ik};\mathscr{T}_i)$, $1 \leqslant i \leqslant 2,\ 1 \leqslant k \neq i \leqslant 2$, and $\lambda_h \in M_h$. Due to the choice of M_h, the matrix $B_{1\omega}$ is symmetric and positive definite.

The LBB-condition (2.3) is equivalent to the statement that the minimal eigenvalue of the matrix $B_{1\omega}^{-1}S_\omega$ is bounded from below by a positive constant β_ω depending only on the shape regularity and quasiuniformity of the triangulations $\mathscr{T}_i$, $1 \leqslant i \leqslant 2$. Here,

$$\begin{aligned} S_\omega &= BA^{-1}B^T = B_1A_1^{-1}B_1^T + B_2A_2^{-1}B_2^T \\ &= B_{1\omega}[S_\omega^{(1)}]^{-1}B_{1\omega} + B_{2\omega}[S_\omega^{(2)}]^{-1}B_{2\omega}^T \end{aligned} \tag{3.10}$$

is the Schur complement matrix for $\mathscr{A}$, and

$$S_\omega^{(i)} = A_\omega^{(i)} - A_{\omega i}A_{ii}^{-1}A_{i\omega} \tag{3.11}$$

are the Schur complement for the matrices A_i, $1 \leqslant i \leqslant 2$.

It is obvious that

$$(S_\omega^{(i)}\bar{w}_i, \bar{w}_i) \geqslant \frac{1}{2}\int_\omega |\nabla w_{ih}|^2\,\mathrm{d}x \qquad \forall w_{i,h} \in S_{1,\Gamma}(\Omega_{ik};\mathscr{T}_i) \tag{3.12}$$
$$1 \leqslant i \leqslant 2, \quad 1 \leqslant k \neq i \leqslant 2.$$

To elaborate on the LBB-condition we consider the eigenvalue problem

$$S_\omega\bar{\mu} = \alpha B_{1\omega}\bar{\mu}. \tag{3.13}$$

Denoting by $\alpha_{\min}$ the minimal eigenvalue, we have

$$\alpha_{\min} \geqslant \min_{\bar{\mu}\in\mathbf{R}^m} \frac{(B[S_\omega^{(1)}]^{-1}B_{1\omega}\bar{\mu}, B_{1\omega}\bar{\mu})}{(B_{1\omega}\bar{\mu}, \bar{\mu})} =: \hat{\alpha}. \tag{3.14}$$

By using inequality (2.5) about the discrete harmonic extension for $i = 1$ we conclude that

$$([S_\omega^{(1)}]^{-1}B_{1\omega}\bar{\mu}, B_{1\omega}\bar{\mu}) \geqslant \frac{1}{1+C}(B_{1\omega}\bar{\mu}, \bar{\mu}) \tag{3.15}$$

and finally,

$$\alpha \geqslant \beta_\omega := \frac{1}{1+C}. \tag{3.16}$$

Here, C is the constant from the second inequality in (2.5) for $i = 1$.

Let us return to the eigenvalue problem (3.13). We shall prove that the eigenvalues α of (3.13) are bounded from above by positive constant independent of the mesh. Due to (3.12), the maximal eigenvalue of (3.13) is bounded from above by the maximal eigenvalue of the eigenvalue problem

$$2\left[B_{1\omega}[A_\omega^{(1)}]^{-1}B_{1\omega}+B_{2\omega}[A_\omega^{(2)}]^{-1}B_{2\omega}^T\right]\bar{w}=\alpha B_{1\omega}\bar{w} \tag{3.17}$$

which is equivalent to the eigenvalue problem

$$B_{2\omega}[A_\omega^{(2)}]^{-1}B_{2\omega}^T\bar{w}=\tilde{\alpha}B_{1\omega}\bar{w} \tag{3.18}$$

where $\tilde{\alpha}=\alpha/2-1$.

Let $\tilde{\alpha}$ be an eigenvalue of (3.18) and $\bar{w}$ be a corresponding eigenvector. Then

$$\tilde{\alpha}=\frac{(\bar{\psi},B_{2\omega}^T\bar{w})}{(B_{1\omega}\bar{w},\bar{w})}\leqslant\frac{\left[\int_\omega|\nabla\psi_h|^2\,\mathrm{d}x\right]^{1/2}}{\left[\int_\omega|\nabla w_h|^2\,\mathrm{d}x\right]^{1/2}} \tag{3.19}$$

where $\psi_h\in S_{1,\Gamma}(\Omega_{12};\mathscr{T}_2)$, $w_h\in S_{1,\Gamma}(\Omega_{12};\mathscr{T}_1)$, and

$$\bar{\psi}=[A_\omega^{(2)}]^{-1}B_{2\omega}^T\bar{w}\,. \tag{3.20}$$

The last equality results in

$$\int_\omega|\nabla\psi_h|^2\,\mathrm{d}x=\int_\omega\nabla\psi_h\cdot\nabla w_h\,\mathrm{d}x\leqslant\left[\int_\omega|\nabla\psi_h|^2\,\mathrm{d}x\right]^{1/2}\left[\int_\omega|\nabla w_h|^2\,\mathrm{d}x\right]^{1/2}. \tag{3.21}$$

Substituting (3.21) to (3.19) we obtain the inequality

$$\tilde{\alpha}\leqslant 1$$

whence

$$\alpha\leqslant 4\,.$$

Thus, we have proved the following statement.

Theorem 3.1. *Under the assumptions made in Section 2, the matrix S_ω is spectrally equivalent to the matrix $B_{1\omega}$, i.e., the eigenvalues of (3.13) belong to the segment $[1/(1+C);4]$, where C is a positive constant depending only of the shape regularity and quasiuniformity of the triangulations $\mathscr{T}_i$, $1\leqslant i\leqslant 2$.*

The latter result is a good motivation to propose an 3×3 block diagonal matrix for the preconditioning of system (3.1) with the matrix

$$\mathscr{A}=\begin{pmatrix} A_1 & 0 & B_1^T\\ 0 & A_2 & B_2^T\\ B_1 & B_2 & 0\end{pmatrix}. \tag{3.22}$$

Let $H_i \in \mathbf{R}^{n_i \times n_i}$ be symmetric positive definite matrices, which are spectrally equivalent to the matrices A_i^{-1}, respectively, $1 \leqslant i \leqslant 2$, and $H_\lambda \in \mathbf{R}^{m\times m}$ be a symmetric positive definite matrix which is spectrally equivalent to the matrix $B_{1\omega}^{-1}$. The preconditioner $\mathscr{H}$ for the matrix $\mathscr{A}$ in (3.22) is defined by [3]:

$$\mathscr{H} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_\lambda \end{pmatrix}. \tag{3.23}$$

Theorem 3.2. *Under the assumptions made above, the matrix $\mathscr{H}$ in (3.23) is spectrally equivalent to the matrix $\mathscr{A}^{-1}$ in (3.22), i.e., the eigenvalues of the matrix $\mathscr{H}\mathscr{A}$ belong to the union of two segments $[d_1; d_2]$ and $[d_3; d_4]$ with end points $d_1 \leqslant d_2 < 0 < d_3 \leqslant d_4$, where d_1, d_2, d_3, and d_4 depend only on the shape regularity and quasiuniformity of the triangulations $\mathscr{T}_i$, $1 \leqslant i \leqslant 2$.*

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