PROBLEMS OF STATIONARY FLOW OF ELECTRORHEOLOGICAL FLUIDS IN A CYLINDRICAL COORDINATE SYSTEM*

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Abstract. We consider the general problem on stationary flow of the electrorheological fluid with the constitutive equation developed in [R. H. W. Hoppe and W. G. Litvinov, *Comm. Pure. Appl. Anal.,* 3 (2004), pp. 809–848] in the cylindrical coordinate system. The problem is studied under mixed boundary conditions wherein velocities are specified on one part of the boundary and surface forces are given on the other part. The existence of a solution to this problem and the convergence of Galerkin approximations are established. Then, we consider the occasion where the flow is axially symmetric and study a problem on an electrorheological clutch. This problem is solved numerically, and the results of calculations of the electric field and velocities are presented.

Key words. electrorheological fluid, generalized solution, existence theorem, approximate solutions, electrorheological clutch

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1. Introduction.

Electrorheological fluids are smart materials which are concentrated suspensions of polarizable particles in a nonconducting dielectric liquid. In moderately large electric fields, the particles form chains along the field lines, and these chains then aggregate to form columns [16]. These chainlike and columnar structures cause dramatic changes in the rheological properties of the suspensions. The fluids become anisotropic; the apparent viscosity (the resistance to flow) in the direction orthogonal to the direction of electric field abruptly increases, while the apparent viscosity in the direction of the electric field changes not so drastically.

The chainlike structures directed along the magnetic field lines are formed in magnetic suspensions whose behavior is similar to the behavior of electrorheological suspensions. It was shown experimentally that the apparent viscosity of the flow of magnetic suspensions in the direction orthogonal to the direction of the magnetic field is about three times greater than the apparent viscosity of the flow in the direction of the magnetic field; see [18, p. 85].

The chainlike and columnar structures are destroyed under the action of large stresses, and then the apparent viscosity of the fluid decreases and the fluid becomes less anisotropic.

The following constitutive equation of electrorheological fluids was developed in [8]:

(1.1)
$$\sigma_{ij}(p,u,E) = -p\delta_{ij} + 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u), \quad i,j=1,\dots,n, \quad n=2 \text{ or } 3.$$

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Here, $\sigma_{ij}(p,u,E)$ are the components of the stress tensor which depend on the pressure p, the velocity vector $u = (u_1,\dots,u_n)$, and the electric field strength $E = (E_1,\dots,E_n)$; δ_{ij} are the components of the unit tensor (the Kronecker delta); and $\varepsilon_{ij}(u)$ are the components of the rate of strain tensor

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.2}$$

where x_i are Cartesian coordinates of a point $x = (x_1,\dots,x_n)$.

Moreover, $I(u)$ is the second invariant of the rate-of-strain tensor

$$I(u) = \sum_{i,j=1}^{n} (\varepsilon_{ij}(u))^2, \tag{1.3}$$

and φ the viscosity function depending on $I(u)$, $|E|$, and $\mu(u,E)$.

The function μ is introduced into the constitutive equation (1.1) in order to take into account the anisotropy of the electrorheological fluid under which the viscosity of the fluid depends on the angle between the vector of the electric field and the vector of the velocity with respect to the charged electrode (the counter electrode is not charged usually). The electrode can move relative to the body of an electrorheological device, and hence we consider that the electrode can move relative to the reference frame under consideration.

Let $\breve{u}(x,t) = (\breve{u}_1(x,t), \breve{u}_2(x,t), \breve{u}_3(x,t))$ be a vector of transfer velocity; $\breve{u}(x,t)$ is the velocity of a point of the electrode which coincides with the point x of the frame at an instant t. It is assumed that $\breve{u}$ is a known function.

We define the function μ as the square of the cosine of the angle between the vector of the electric field and the vector of the velocity relative to the electrode, i.e.,

$$\mu(u,E) = \left(\frac{u-\breve{u}}{|u-\breve{u}|}, \frac{E}{|E|}\right)^2_{\mathbb{R}^3} = \frac{((u_i-\breve{u}_i)E_i)^2}{\left(\sum_{i=1}^3 (u_i-\breve{u}_i)^2\right)\left(\sum_{i=1}^3 E_i^2\right)}. \tag{1.4}$$

Here and below, the Einstein convention on summation over a repeated index is applied, and we denote by $(.,.)_{\mathbb{R}^3}$ the scalar product in $\mathbb{R}^3$.

If the electrode does not move relative to the reference frame, then $\breve{u} = 0$ and the function μ takes the form

$$\mu(u,E) = \left(\frac{u}{|u|}, \frac{E}{|E|}\right)^2_{\mathbb{R}^3}. \tag{1.5}$$

In the general case, the function $\breve{u}$ is defined as follows:

$$\breve{u}(x,t) = \mathring{u}(t) + w(x,t), \tag{1.6}$$

where $\mathring{u}(t) = (\mathring{u}_1(t), \mathring{u}_2(t), \mathring{u}_3(t))$ is a vector of the translation velocity and $w(x,t) = (w_1(x,t), w_2(x,t), w_3(x,t))$ is a vector of the rotational velocity.

The function $\mu(u,E)$ is an invariant which is independent of the choice of the reference frame and the motion of the frame with respect to the electrode.

The viscosity function φ is identified by approximation of flow curves (see [8]) and it was shown in [8] (see also the appendix) that it can be represented as follows:

$$\varphi(I(u), |E|, \mu(u,E)) = b(|E|, \mu(u,E))(\lambda + I(u))^{-\frac{1}{2}} + \psi(I(u), |E|, \mu(u,E)), \tag{1.7}$$

where λ is a small parameter, $\lambda \geq 0$.

The constitutive equation (1.1) with the viscosity function (1.7) allows one to describe the following main peculiarities of flow of electrorheological fluids:

(a) singular or almost singular viscosity function at zero value of the rate-of-strain tensor,
(b) an arbitrary nonlinear relationship between the shear rates and the shear stresses,
(c) an arbitrary dependence of the viscosity on the module of the vector of the electric field and on the angle between the vectors of the velocity and electric field (the anisotropy).

With some assumptions natural from a physical point of view, the constitutive equation (1.1) with the viscosity function (1.7) leads to well-posed mathematical problems (see sections 4 and 5 below and [8]).

The functions b and ψ in (1.7) can be identified so that a set of flow curves obtained for different electric fields E is approximated in an arbitrary range of the shear rates with an arbitrarily high degree of accuracy (for example, by splines).

The Bingham constitutive equation of electrorheological fluids, which is of considerable current use (see, e.g., [4], [16], [22]), gives no way to closely approximate a set of flow curves, especially at small shear rates (see Figure A-1 in the appendix). In addition, the Bingham constitutive equation takes no account of the anisotropy of electrorheological fluids.

We consider Maxwell's equations in the following form (see, e.g., [10]):

$$\begin{aligned} &\operatorname{curl} E + \frac{1}{c}\frac{\partial B}{\partial t} = 0, \qquad \operatorname{div} B = 0,\\ &\operatorname{curl} H - \frac{1}{c}\frac{\partial D}{\partial t} = 0, \qquad \operatorname{div} D = 0. \end{aligned} \tag{1.8}$$

Here E is the electric field, B the magnetic induction, D the electric displacement, H the magnetic field, and c the speed of light. One can assume that

$$D = \epsilon E, \qquad B = \mu H, \tag{1.9}$$

where ϵ is the dielectric permittivity and μ the magnetic permeability.

Since electrorheological fluids are dielectrics, the magnetic field H can be neglected. Then (1.8), (1.9) give the following relations:

$$\operatorname{curl} E = 0, \tag{1.10}$$

$$\operatorname{div}(\epsilon E) = 0. \tag{1.11}$$

It follows from (1.10) that there exists a function of potential θ such that

$$E = -\operatorname{grad}\theta, \tag{1.12}$$

and (1.11) implies

$$\operatorname{div}(\epsilon \operatorname{grad}\theta) = 0 \quad \text{in} \quad \Omega_1. \tag{1.13}$$

Here Ω_1 is the domain of the fluid flow in the Cartesian coordinate system.

The boundary conditions are the following:

$$\theta = U_i(t) \quad \text{on} \quad \Gamma_i, \quad i = 1, \dots, k, \tag{1.14}$$

$$\theta = 0 \quad \text{on} \quad \Gamma_{i0}, \tag{1.15}$$

$$\nu \cdot \epsilon \operatorname{grad}\theta = 0 \quad \text{on} \quad \Gamma \backslash \left(\bigcup_{i=1}^{k} \overline{(\Gamma_i \cup \Gamma_{i0})} \right). \tag{1.16}$$

Here Γ_i and Γ_{i0} are the surfaces of the ith control and null electrodes, respectively, and it is supposed that Γ_i, Γ_{i0} are open subsets of the boundary Γ of Ω_1.

Therefore, the equations for the functions E and (p, v) are separated. Because of this, we assume hereafter that the function of electric field E is known.

In the special case that the direction of the velocity relative to the electrode $u(x,t) - \check{u}(x,t)$ at each point (x,t) at which $E(x,t) \neq 0$ is known, the function $(x,t) \to \mu(u,E)(x,t)$ becomes well known, and the viscosity functions (1.7) takes the form

$$\varphi(I(u), |E|, x, t) = e(|E|, x, t)(\lambda + I(u))^{-\frac{1}{2}} + \psi_1(I(u), |E|, x, t), \tag{1.17}$$

where

$$\begin{aligned} e(|E|, x, t) &= b(|E|, \mu(u,E)(x,t)), \\ \psi_1(I(u), |E|, x, t) &= \psi(I(u), |E|, \mu(u,E)(x,t)). \end{aligned} \tag{1.18}$$

In many electrorheological devices the fluid flows in domains of which the boundaries are the surfaces of revolution. Problems on flow of electrorheological fluids in such domains are convenient to consider in cylindrical coordinates.

In section 2, we present governing equations. In section 3, we formulate a general boundary value problem on stationary flow of the electrorheological fluid in the cylindrical coordinate system and adduce some auxiliary results. Section 4 contains approximate solutions and existence theorems for the general boundary value problem. In section 5, we consider a problem on stationary axially symmetric flow. A problem on an electrorheological clutch is formulated and solved numerically in section 6.

2. Governing equations and assumptions. We consider the system of cylindrical coordinates r, α, z. An element of the length dl is defined in cylindrical coordinates as $dl = (dr^2 + r^2\, d\alpha^2 + dz^2)^{\frac{1}{2}}$. Denote the components of a vector v in the mobile orthonormal basis e_r, e_α, e_z by v_1, v_2, v_3; i.e., $v = (v_1, v_2, v_3)$.

Let $u = (u_1, u_2, u_3)$ be a velocity vector. The components of the rate-of-strain tensor have the following form in cylindrical coordinates,

$$\begin{aligned} &\epsilon_{11}(u) = \frac{\partial u_1}{\partial r}, \quad \epsilon_{22}(u) = \frac{1}{r}\frac{\partial u_2}{\partial \alpha} + \frac{u_1}{r}, \qquad \epsilon_{33}(u) = \frac{\partial u_3}{\partial z}, \\ &\epsilon_{12}(u) = \epsilon_{21}(u) = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_1}{\partial \alpha} + \frac{\partial u_2}{\partial r} - \frac{u_2}{r}\right), \\ &\epsilon_{23}(u) = \epsilon_{32}(u) = \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{1}{r}\frac{\partial u_3}{\partial \alpha}\right), \\ &\epsilon_{13}(u) = \epsilon_{31}(u) = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial r}\right), \end{aligned} \tag{2.1}$$

and the second invariant of the rate-of-strain tensor is defined by

$$I(u) = \sum_{i,j=1}^{3} (\epsilon_{ij}(u))^2. \tag{2.2}$$

We assume the following.

(A0) Ω_1 is a bounded domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary Γ.

Let $\mathcal{P}$ be the operator of translation from cylindrical coordinates to Cartesian ones,

$$\mathcal{P} : (r, \alpha, z) \to \mathcal{P}(r, \alpha, z) = (x_1, x_2, x_3),$$
$$x_1 = r\cos\alpha, \quad x_2 = r\sin\alpha, \quad x_3 = z, \quad r \in \mathbb{R}_+, \quad \alpha \in [0, 2\pi), \quad z \in \mathbb{R}, \tag{2.3}$$

where $\mathbb{R}_+ = \{y \in \mathbb{R}, y \geq 0\}$ and we identify the points $(0, \alpha, z)$ with the point $(0, 0, z)$, $\alpha \in [0, 2\pi)$. The inverse operator $\mathcal{P}^{-1}$ is defined by

$$\mathcal{P}^{-1} : \mathcal{P}^{-1}(x_1, x_2, x_3) = (r, \alpha, z),$$
$$r = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad \alpha = \arctan\frac{x_2}{x_1}, \qquad z = x_3. \tag{2.4}$$

Here, we consider that the mapping $(x_1, x_2) \to \arctan\frac{x_2}{x_1}$ is a multifunction at the point $x_1 = x_2 = 0$, namely, $\arctan\frac{0}{0} = [0, 2\pi)$.

Let

$$\Omega = \mathcal{P}^{-1}(\Omega_1), \qquad S = \mathcal{P}^{-1}(\Gamma). \tag{2.5}$$

We consider a stationary flow problem under the Stokes approximation; i.e., we ignore inertial forces, which are assumed to be small as compared with the internal forces caused by the viscous stresses. Then the motion equations take the following form:

$$\frac{\partial p}{\partial r} - 2\frac{\partial}{\partial r}(\varphi\epsilon_{11}(u)) - \frac{2}{r}\frac{\partial}{\partial\alpha}(\varphi\epsilon_{12}(u)) - 2\frac{\partial}{\partial z}(\varphi\epsilon_{13}(u)) - \frac{2\varphi}{r}(\epsilon_{11}(u) - \epsilon_{22}(u)) = K_1 \quad \text{in } \Omega, \tag{2.6}$$

$$\frac{1}{r}\frac{\partial p}{\partial\alpha} - 2\frac{\partial}{\partial r}(\varphi\epsilon_{21}(u)) - \frac{2}{r}\frac{\partial}{\partial\alpha}(\varphi\epsilon_{22}(u)) - 2\frac{\partial}{\partial z}(\varphi\epsilon_{23}(u)) - \frac{4}{r}\varphi\epsilon_{12}(u) = K_2 \text{ in } \Omega, \tag{2.7}$$

$$\frac{\partial p}{\partial z} - 2\frac{\partial}{\partial r}(\varphi\epsilon_{31}(u)) - \frac{2}{r}\frac{\partial}{\partial\alpha}(\varphi\epsilon_{32}(u)) - 2\frac{\partial}{\partial z}(\varphi\epsilon_{33}(u)) - \frac{2}{r}\varphi\epsilon_{13}(u) = K_3 \text{ in } \Omega. \tag{2.8}$$

Here the viscosity function φ is defined either by (1.7) or by (1.17), and K_1, K_2, K_3 are the components of the volume force vector K.

The velocity function u meets the incompressibility condition

$$\operatorname{div}_c u = \frac{\partial u_1}{\partial r} + \frac{1}{r}\frac{\partial u_2}{\partial\alpha} + \frac{\partial u_3}{\partial z} + \frac{u_1}{r} = 0 \quad \text{in } \Omega. \tag{2.9}$$

Here and below, we denote by div_c the operator of divergence in cylindrical coordinates.

Suppose that S_1 and S_2 are open subsets of S such that S_1 is nonempty, $S_1 \cap S_2 = \emptyset$, and $\overline{S}_1 \cup \overline{S}_2 = S$. We consider mixed boundary conditions, wherein velocities are specified on S_1 and surface forces are given on S_2, i.e.,

$$u = \hat{u} \quad \text{on } S_1, \tag{2.10}$$

$$[-p\delta_{ij} + 2\varphi\epsilon_{ij}(u)]\nu_j = F_i \quad \text{on } S_2, \qquad i, j = 1, 2, 3. \tag{2.11}$$

Here, by ν_j and F_i we denote the components of the unit outward normal to S_2 and the components of the vector of surface force with respect to the basis vectors e_r, e_α, e_z.

Let

$$\tilde{S} = \{(r, \alpha, z) | r = 0, \quad \alpha \in [0, 2\pi), \quad z \in \mathbb{R}_+\}, \qquad S_0 = \Omega \cap \tilde{S}. \tag{2.12}$$

In particular, S_0 can be an empty set. It follows from (2.9) that

$$u_1 = 0 \quad \text{on} \quad S_0, \tag{2.13}$$

and therefore $\frac{\partial u_1}{\partial z} = 0$ on S_0, and since $\epsilon_{13}(u) = 0$ on S_0 (see (2.8)), we obtain

$$\frac{\partial u_3}{\partial r} = 0 \quad \text{on} \quad S_0. \tag{2.14}$$

It follows also from (2.6), (2.7), and (2.1) that

$$\begin{aligned} &\lim_{r \to 0} (\epsilon_{11}(u) - \epsilon_{22}(u))(r, \alpha, z) = 0, \\ &\lim_{r \to 0} (\epsilon_{12}(u))(r, \alpha, z) = 0, \qquad u_2 = 0 \quad \text{on } S_0. \end{aligned} \tag{2.15}$$

In the case when the viscosity function is defined by (1.7), we assume the following.

(A1) $b : y_1, y_2 \to b(y_1, y_2)$ is a function continuous in $\mathbb{R}_+ \times [0, 1]$, and, in addition,

$$0 \leq b(y_1, y_2) \leq a_0, \qquad (y_1, y_2) \in \mathbb{R}_+ \times [0, 1], \tag{2.16}$$

where a_0 is a positive constant.

(A2) $\psi : (y_1, y_2, y_3) \to \psi(y_1, y_2, y_3)$ is a function continuous in $\mathbb{R}_+^2 \times [0, 1]$, and for an arbitrarily fixed $(y_2, y_3) \in \mathbb{R}_+ \times [0, 1]$ the partial function $\psi(., y_2, y_3) : y_1 \to \psi(y_1, y_2, y_3)$ is continuously differentiable in $\mathbb{R}_+$, and the following inequalities hold:

$$a_2 \geq \psi(y_1, y_2, y_3) \geq a_1, \tag{2.17}$$

$$\psi(y_1, y_2, y_3) + 2 \frac{\partial \psi}{\partial y_1}(y_1, y_2, y_3) y_1 \geq a_3, \tag{2.18}$$

$$\left| \frac{\partial \psi}{\partial y_1}(y_1, y_2, y_3) \right| y_1 \leq a_4, \tag{2.19}$$

where a_1–a_4 are positive constants.

In the case that the viscosity function is defined by (1.17), we suppose the following.

(A3) for an arbitrary fixed $(y_2, x, t) \in \mathbb{R}_+ \times \Omega_1 \times \mathbb{R}_+$, the partial function $\psi_1(., y_2, x, t) : y_1 \to \psi_1(y_1, y_2, x, t)$ is continuously differentiable in $\mathbb{R}_+$, and the following inequalities hold:

$$a_2 \geq \psi_1(y_1, y_2, x, t) \geq a_1, \tag{2.20}$$

$$\psi_1(y_1, y_2, x, t) + 2 \frac{\partial \psi_1}{\partial y_1} (y_1, y_2, x, t) y_1 \geq a_3, \tag{2.21}$$

$$\left| \frac{\partial \psi_1}{\partial y_1} (y_1, y_2, x, t) \right| y_1 \leq a_4. \tag{2.22}$$

As for the function e, we assume

(2.23)
$$e \in L_\infty(\mathbb{R}_+ \times \Omega_1 \times \mathbb{R}_+), \quad 0 \leq e(y, x, t) \leq a_0, \qquad y \in \mathbb{R}_+, \quad x \in \Omega_1, \quad t \in \mathbb{R}_+.$$

At $\lambda = 0$ the viscosity function φ defined by (1.7) is singular at $I(u) = 0$, $\varphi(0, |E|, \mu(u, E)) = \infty$, and flow problems for such viscosity function reduce to the solution of variational inequalities.

The equation (1.7) with a small positive λ defines a fluid with a finite but possibly large viscosity at $I(u) = 0$. From a physical point of view a fluid with bounded viscosity is more reasonable than the fluid with singular unbounded viscosity (all is bounded in actuality). It is shown in [8] that the solutions of the problems with bounded viscosities converge to the solution of the problem with the singular viscosity as λ tends to zero. Because of this, we assume that

$$\lambda > 0 \quad \text{in (1.7) and (1.17).} \tag{2.24}$$

Let us dwell on the physical sense of the inequalities (2.16)–(2.23). The inequalities (2.16) and (2.17) indicate that the viscosity is bounded from below and from above by positive constants. The inequality (2.18) implies that for fixed values of $|E|$ and $\mu(u, E)$ the derivative of the function $I(u) \to G(u)$ is positive, where $G(u)$ is the second invariant of the stress deviator

$$G(u) = \sum_{i,j=1}^{n} (\sigma_{ij}(p, u, E) + p\delta_{ij})^2 = 4[\varphi(I(u), |E|, \ \mu(u, E))]^2 I(u).$$

This means that in case of simple shear flow the shear stress increases with increasing shear rate. (2.19) is a restriction on $\frac{\partial \varphi}{\partial y_1}$ for large values of y_1.

The inequalities (2.20)–(2.23) are analogous to the inequalities (2.16)–(2.19).

All inequalities (2.16)–(2.23) are natural from a physical point of view.

The viscosity function is identified by approximation of a set of flow curves which are obtained experimentally by viscometric testing for different electric fields. The inequalities (2.16)–(2.23) are consistent with the shapes of the flow curves and enable one to approximate a set of flow curves over a wide range of shear rates with a high degree of accuracy (see the appendix below and [3], [8], [19]).

3. Generalized solution of the problem. We define the following sets:

$$J_0 = \Big\{ v | v = (v_1, v_2, v_3) \in C^\infty(\overline{\Omega})^3, \ v_1\big|_{S_0} = 0, \ v_2\big|_{S_0} = 0,$$

$$\frac{\partial v^k}{\partial \alpha^k}\Big|_{\alpha=0} = \frac{\partial v^k}{\partial \alpha^k}\Big|_{\alpha=2\pi}, \ k = 0, 1, 2, \ldots \Big\}, \tag{3.1}$$

$$J = \{v | v \in J_0, \, v\big|_{S_1} = 0\}, \tag{3.2}$$

$$J_1 = \{v | v \in J, \, \operatorname{div}_c v = 0\}. \tag{3.3}$$

Let H and H_1 be the closures of J and J_1 with respect to the norm

$$||v||_H = \left(\int_\Omega I(v) r \, dr \, d\alpha \, dz \right)^{\frac{1}{2}}, \tag{3.4}$$

and let H_0 be the closures of J_0 relative to the norm

$$||v||_{H_0} = \left(||v||_H^2 + \int_{S_1} |v|^2 \, ds \right)^{\frac{1}{2}}. \tag{3.5}$$

Let also Y be the space of scalar functions which are square integrable in Ω with respect to the measure $r\,dr\,d\alpha\,dz$. The norm in Y is defined by

$$||h||_Y = \left(\int_\Omega h^2\, r\, dr\, d\alpha\, dz \right)^{\frac{1}{2}}. \tag{3.6}$$

We define the operator G that maps H_0 into a set of vector valued functions determined in Ω_1 as follows:

$$\begin{aligned} v = (v_1, v_2, v_3) \in H_0, \qquad G(v) = \{G(v)_i\}_{i=1}^3, \\ G(v)_1 = (v_1 \cos\alpha - v_2 \sin\alpha) \circ \mathcal{P}^{-1}, \\ G(v)_2 = (v_1 \sin\alpha + v_2 \cos\alpha) \circ \mathcal{P}^{-1}, \quad G(v)_3 = v_3 \circ \mathcal{P}^{-1}. \end{aligned} \tag{3.7}$$

We assign also the following norm in H_0:

$$||v||_1 = ||G(v)||_{H^1(\Omega_1)^3}, \tag{3.8}$$

where $||\,\boldsymbol{.}\,||_{H^1(\Omega_1)^3}$ is the norm of the product of three Sobolev spaces $H^1(\Omega_1)$.

LEMMA 3.1. *Suppose that the condition* (A0) *is satisfied. Then the expressions* (3.4) *and* (3.8) *define equivalent norms in H, and the expressions* (3.5) *and* (3.8) *are equivalent norms in H_0. The operator G is an isomorphism of H_0 onto $H^1(\Omega_1)^3$, and the following equality holds:*

$$||h||_Y = ||h \circ \mathcal{P}^{-1}||_{L_2(\Omega_1)}. \tag{3.9}$$

Proof. The equivalence of the norms (3.4) and (3.8) in H, and the norms (3.5) and (3.8) in H_0, follows from the fact that $I(v)$ is the invariant, i.e.,

$$\sum_{i,j=1}^{3} [(\epsilon_{ij}(v))(r,\alpha,z)]^2 = \sum_{i,j=1}^{3} [(\varepsilon_{ij}(G(v)))(\mathcal{P}(r,\alpha,z))]^2, \tag{3.10}$$

and from the Korn inequality.

Therefore, $G(H_0) \subset H^1(\Omega_1)^3$. Let $g = (g_1, g_2, g_3) \in H^1(\Omega_1)^3$. We have

$$\begin{aligned} \left(\frac{\partial}{\partial r}(g_i \circ \mathcal{P}) \right)(r,\alpha,z) &= \frac{\partial g_i}{\partial x_1}(\mathcal{P}(r,\alpha,z)) \cos\alpha + \frac{\partial g_i}{\partial x_2}(\mathcal{P}(r,\alpha,z)) \sin\alpha, \\ \left(\frac{\partial}{\partial \alpha}(g_i \circ \mathcal{P}) \right)(r,\alpha,z) &= -\frac{\partial g_i}{\partial x_1}(\mathcal{P}(r,\alpha,z)) r \sin\alpha + \frac{\partial g_i}{\partial x_2}(\mathcal{P}(r,\alpha,z)) r \cos\alpha, \\ \left(\frac{\partial}{\partial z}(g_i \circ \mathcal{P}) \right)(r,\alpha,z) &= \frac{\partial g_i}{\partial x_3}(\mathcal{P}(r,\alpha,z)), \qquad i = 1,2,3. \end{aligned} \tag{3.11}$$

We define a vector-function $v = (v_1, v_2, v_3)$ as follows:

$$\begin{aligned} v_1 &= (g_1 \circ \mathcal{P}) \cos\alpha + (g_2 \circ \mathcal{P}) \sin\alpha, \\ v_2 &= (g_2 \circ \mathcal{P}) \cos\alpha - (g_1 \circ \mathcal{P}) \sin\alpha, \quad v_3 = g \circ \mathcal{P}. \end{aligned} \tag{3.12}$$

It follows from (3.7), (3.11), and (3.12) that $v = G^{-1} g \in H_0$, where G^{-1} is the inverse of G. Therefore, $G(H_0) = H^1(\Omega_1)^3$.

The equalities (3.11) imply $G^{-1} \in \mathcal{L}(H^1(\Omega_1)^3, H_0)$, and by the Banach theorem on closed range the operator G is an isomorphism of H_0 onto $H^1(\Omega_1)^3$. $\square$

Everywhere below, we use the following notations: if $\mathcal{H}$ is a normed space, we denote by $\mathcal{H}^*$ the dual of $\mathcal{H}$ and by (f,h) the duality between $\mathcal{H}^*$ and $\mathcal{H}$, where $f \in \mathcal{H}^*$, $h \in \mathcal{H}$. In particular, if $f \in Y$ or $f \in Y^n$, $n = 2$ or 3, then (f,h) is the scalar product in Y or in Y^n, respectively. That is, we identify the spaces Y and Y^n with their dual spaces Y^* and $(Y^n)^*$, respectively.

The sign $\rightharpoonup$ denotes weak convergence in a Banach space.

We suppose that $\hat{u}$ belongs to the space of traces on S_1 of the functions from H_0, i.e., $\hat{u} \circ \mathcal{P}^{-1} \in H^{\frac{1}{2}}(\mathcal{P}(S_1))$. Then, there exists a function $\tilde{u}$ such that

$$\tilde{u} \in H_0, \qquad \tilde{u}\big|_{S_1} = \hat{u}, \qquad \operatorname{div}_c \tilde{u} = 0. \tag{3.13}$$

We assume also that

$$K = (K_1, K_2, K_3) \in Y^3, \qquad F \circ \mathcal{P}^{-1} = (F_1, F_2, F_3) \circ \mathcal{P}^{-1} \in L_2(\mathcal{P}(S_2))^3. \tag{3.14}$$

We define operators $L : H \to H^*$ and $B \in \mathcal{L}(H, Y^*)$ as follows:

$$(L(v), h) = 2 \int_\Omega \varphi \epsilon_{ij}(\tilde{u} + v)\epsilon_{ij}(h) r\, dr\, d\alpha\, dz, \qquad v, h \in H, \tag{3.15}$$

$$(Bv, w) = \int_\Omega (\operatorname{div}_c v) w r\, dr\, d\alpha\, dz, \qquad v \in H,\ w \in Y. \tag{3.16}$$

In (3.15) the function φ is defined either by (1.7) or by (1.17).

We consider the problem: Find a pair of functions (v,p) satisfying

$$v \in H, \qquad p \in Y, \tag{3.17}$$

$$(L(v), h) - (B^* p, h) = (K + F, h), \qquad h \in H, \tag{3.18}$$

$$(Bv, w) = 0, \qquad w \in Y. \tag{3.19}$$

Here, B^* is the operator adjoint of B and

$$(K + F, h) = \int_\Omega K_i h_i\, r\, dr\, d\alpha\, dz + \int_{S_2} F_i h_i\, ds. \tag{3.20}$$

The pair $(u = v + \tilde{u}, p)$, where (v,p) is a solution of the problem (3.17)–(3.19), will be called the generalized solution of the problem (2.6)–(2.11), (2.13)–(2.15).

Indeed, by use of Green's formula, it can be seen that, if (v,p) is a solution of the problem (3.17)–(3.19), then the pair $(u = v + \tilde{u}, p)$ is a solution of the problem (2.6)–(2.11), (2.13)–(2.15) in the distributional sense. On the contrary, if (u,p) is a smooth solution of the problem (2.6)–(2.11), (2.13)–(2.15), then the pair $(v = u - \tilde{u}, p)$ is a solution of the problem (3.17)–(3.19).

LEMMA 3.2. *Suppose that the condition* (A0) *is satisfied. Then, the following inf-sup condition,*

$$\inf_{g \in Y} \sup_{w \in H} \frac{(Bw, g)}{||w||_H ||g||_Y} \geq \beta_1 > 0, \tag{3.21}$$

holds true. The operator B is an isomorphism from $H_1^\perp$ onto Y, where $H_1^\perp$ is the orthogonal complement of H_1 in H, and the operator B^* is an isomorphism from Y onto the polar set

$$H_1^\circ = \{q \in H^*, \quad (q, v) = 0, \quad v \in H_1\}. \tag{3.22}$$

Moreover

$$||B^{-1}||_{\mathcal{L}(Y,H_1^{\perp})} \le \frac{1}{\beta_1}, \qquad ||(B^*)^{-1}||_{\mathcal{L}(H_1^{\circ},Y)} \le \frac{1}{\beta_1}. \tag{3.23}$$

This lemma follows from the corresponding result in Cartesian coordinates (see [2], [8], [13]), since G is an isomorphism of H_0 onto $H^1(\Omega_1)^3$ and $I(v)$ is invariant (see (3.10)) and $\operatorname{div}_c v$ is an invariant also; i.e.,

$$(\operatorname{div}_c v)(r,\alpha,z) = (\operatorname{div} G(v))(\mathcal{P}(r,\alpha,z)). \tag{3.24}$$

4. Approximate solutions and existence theorems. Let $\{X_m\}_{m=1}^{\infty}$ and $\{N_m\}_{m=1}^{\infty}$ be sequences of finite-dimensional subspaces in H and Y, respectively, such that

$$\lim_{m\to\infty} \inf_{h\in X_m} ||w-h||_H = 0, \qquad w \in H, \tag{4.1}$$

$$\lim_{m\to\infty} \inf_{g\in N_m} ||f-g||_Y = 0, \qquad f \in Y, \tag{4.2}$$

$$\inf_{g\in N_m} \sup_{h\in X_m} \frac{(Bh,g)}{||h||_H ||g||_Y} \ge \beta > 0, \tag{4.3}$$

$$X_m \subset X_{m+1}, \qquad N_m \subset N_{m+1}, \qquad m \in \mathbb{N}. \tag{4.4}$$

We seek an approximate solution of the problem (3.17)–(3.19) of the form

$$v_m \in X_m, \qquad p_m \in N_m, \tag{4.5}$$

$$(L(v_m),h) - (B^* p_m, h) = (K+F,h), \qquad h \in X_m, \tag{4.6}$$

$$(Bv_m, g) = 0, \qquad g \in N_m. \tag{4.7}$$

THEOREM 4.1. *Suppose that the function φ defining the operator L (see* (3.15)*) is given by* (1.7) *and that the conditions* (A0)*,* (A1)*,* (A2)*,* (2.24) *are satisfied. Let also* (3.13)*,* (3.14)*,* (4.1)*–*(4.4) *hold. Then there exists a solution (v,p) of the problem* (3.17)*–*(3.19)*, and for an arbitrary $m \in \mathbb{N}$ there exists a solution of the problem* (4.5)*–*(4.7)*, and a subsequence $\{v_k, p_k\}$ can be extracted from the sequence $\{v_m, p_m\}$ such that*

$$v_k \to v \quad \text{in } H, \qquad p_k \to p \quad \text{in } Y. \tag{4.8}$$

Indeed, we replace cylindrical coordinates r, α, z by Cartesian coordinates x_1, x_2, x_3 in the problems (3.17)–(3.19) and (4.5)–(4.7). Then, we use Lemma 3.1 and Theorem 5.1 from [8] for these problems in Cartesian coordinates and pass back to cylindrical coordinates. As a result, we obtain that there exists a solution of the problem (3.17)–(3.19), and there exists a solution of the problem (4.5)–(4.7) for any $m \in \mathbb{N}$, and a subsequence $\{v_k, p_k\}$ can be extracted from the sequence $\{v_m, p_m\}$ such that

$$v_k \rightharpoonup v \quad \text{in } H, \qquad p_k \rightharpoonup p \quad \text{in } Y. \tag{4.9}$$

(4.8) is proved by using (4.9) and the arguments of Theorem 2.1 from [2].

The next theorem follows from the results of [2].

THEOREM 4.2. *Suppose that the function φ defining the operator L (see* (3.15)*) is given by* (1.17) *and that the conditions* (A0)*,* (A3)*,* (2.23)*,* (2.24) *are satisfied. Let also* (3.13)*,* (3.14)*,* (4.1)*–*(4.4) *hold. Then there exists a unique solution (u,p) of the problem* (3.17)*–*(3.19)*, and for an arbitrary $m \in \mathbb{N}$ there exists a unique solution (v_m, p_m) of the problem* (4.5)*–*(4.7)*; moreover,*

$$v_m \to v \quad \text{in} \quad H, \qquad p_m \to p \quad \text{in} \quad Y.$$

5. Problem of axially symmetric flow.

5.1. Formulation of the problem. In the case of axially symmetric flow the components of a velocity vector u in the mobile orthonormal basis (e_r, e_α, e_z) of the cylindrical coordinate system depend on r, z only, i.e., $u(r,z) = (u_1(r,z), u_2(r,z), u_3(r,z))$; the components of the rate-of-strain tensor have the form

$$\epsilon_{11}(u) = \frac{\partial u_1}{\partial r}, \quad \epsilon_{22}(u) = \frac{u_1}{r}, \quad \epsilon_{33}(u) = \frac{\partial u_3}{\partial z},$$

$$\epsilon_{12}(u) = \epsilon_{21}(u) = \frac{1}{2}\left(\frac{\partial u_2}{\partial r} - \frac{u_2}{r}\right), \quad \epsilon_{23}(u) = \epsilon_{32}(u) = \frac{1}{2}\frac{\partial u_2}{\partial z},$$

$$\epsilon_{13}(u) = \epsilon_{31}(u) = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial r}\right); \tag{5.1}$$

and the second invariant of the rate-of-strain tensor is defined by

$$I(u) = \left(\frac{\partial u_1}{\partial r}\right)^2 + \left(\frac{u_1}{r}\right)^2 + \left(\frac{\partial u_3}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial r} - \frac{u_2}{r}\right)^2$$

$$+ \frac{1}{2}\left(\frac{\partial u_2}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial r}\right)^2. \tag{5.2}$$

We assume that the domain of flow of the electrorheological fluid Ω_1 satisfies the condition (A0) and has the following form:

$$\Omega_1 = \{x | x = (x_1, x_2, x_3),\ x_3 \in (0,l),\ (x_1^2 + x_2^2)^{\frac{1}{2}} < R_2(x_3),$$

$$(x_1^2 + x_2^2)^{\frac{1}{2}} > R_1(x_3) \text{ if } R_1(x_3) > 0,\ (x_1^2 + x_2^2)^{\frac{1}{2}} \geq R_1(x_3) \text{ if } R_1(x_3) = 0\}, \tag{5.3}$$

where R_1 and R_2 are functions given in $(0,l)$. The function R_1 takes nonnegative values, R_2 takes positive values, and $R_2(x_3) > R_1(x_3)$ for all $x_3 \in (0,l)$.

The condition (A0) imposes restrictions on the functions R_1 and R_2. The functions R_1 and R_2 can be Lipschitz continuous as well as discontinuous with a finite number of points of discontinuity. But in the second case the functions R_1 and R_2 must be Lipschitz continuous in between the points of discontinuity.

Let $\Omega_2 = \mathcal{P}^{-1}(\Omega_1)$; the mapping $\mathcal{P}^{-1}$ is defined by (2.4). Since the flow of the fluid is assumed to be axially symmetric—i.e., the functions of velocity, pressure, and electric field are independent of α in cylindrical coordinate system—we consider our problem in the domain Ω_3, which consists of points (r,z) such that $(r,\alpha,z) \in \Omega_2$, $\alpha \in [0, 2\pi)$.

According to (5.3), the domain Ω_3 is defined by

$$\Omega_3 = \{(r,z) | 0 < z < l,\ R_1(z) < r \text{ if } R_1(z) > 0,$$

$$R_1(z) \leq r \text{ if } R_1(z) = 0, \quad r < R_2(z)\}. \tag{5.4}$$

We consider the stationary flow problem under the neglect of the inertial forces. Taking into account (2.6)–(2.8) and (5.1), we obtain the following motion equations:

(5.5)
$$\frac{\partial p}{\partial r} - 2\frac{\partial}{\partial r}\left(\varphi\frac{\partial u_1}{\partial r}\right) - \frac{\partial}{\partial z}\left(\varphi\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial r}\right)\right) - \frac{2}{r}\varphi\left(\frac{\partial u_1}{\partial r} - \frac{u_1}{r}\right) = K_1 \quad \text{in } \Omega_3,$$

(5.6)
$$-\frac{\partial}{\partial r}\left(\varphi\left(\frac{\partial u_2}{\partial r} - \frac{u_2}{r}\right)\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial u_2}{\partial z}\right) - \frac{2}{r}\varphi\left(\frac{\partial u_2}{\partial r} - \frac{u_2}{r}\right) = K_2 \quad \text{in } \Omega_3,$$

(5.7)
$$\frac{\partial p}{\partial z} - \frac{\partial}{\partial r}\left(\varphi\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial r}\right)\right) - 2\,\frac{\partial}{\partial z}\left(\varphi\frac{\partial u_3}{\partial z}\right) - \frac{\varphi}{r}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial r}\right) = K_3 \quad \text{in } \Omega_3,$$

where the function φ is defined either by (1.7) or by (1.17).

The equation of incompressibility takes the form

$$\operatorname{div}_c u = \frac{\partial u_1}{\partial r} + \frac{\partial u_3}{\partial z} + \frac{u_1}{r} = 0 \quad \text{in } \Omega_3. \tag{5.8}$$

Let S be the boundary of Ω_3 and

$$\mathcal{T} = \{z | z \in (0,l),\ R_1(z) = 0\},$$
$$S_0 = \{(r,z) | r = 0,\ z \in \mathcal{T}\}. \tag{5.9}$$

In particular, S_0 can be an empty set.

Let also

$$S' = \{(r,\alpha,z) | (r,z) \in S\backslash S_0, \quad \alpha \in [0,2\pi)\}. \tag{5.10}$$

Then $\mathcal{P}(S') = \Gamma$, where Γ is the boundary of the domain Ω_1 defined by (5.3).

Suppose that S_1 and S_2 are open subsets of $S\backslash S_0$ such that S_1 is not empty, $S_1 \cap S_2 = \emptyset$, and $\overline{S}_1 \cup \overline{S}_2 = S\backslash S_0$. We consider mixed boundary conditions, wherein velocities are specified on S_1 and surface forces are given on S_2, i.e.,

$$u|_{S_1} = \check{u}, \tag{5.11}$$
$$[(-p + 2\varphi\epsilon_{11}(u))\nu_1 + 2\,\varphi\epsilon_{13}(u)\nu_3]|_{S_2} = F_1, \tag{5.12}$$
$$[2\varphi\epsilon_{21}(u)\nu_1 + 2\varphi\epsilon_{23}(u)\nu_3]|_{S_2} = F_2, \tag{5.13}$$
$$[(-p + 2\varphi\epsilon_{33}(u))\nu_3 + 2\varphi\epsilon_{31}(u)\nu_1]|_{S_2} = F_3, \tag{5.14}$$

where ν_1 and ν_3 are the components of the unit outward normal $\nu = (\nu_1, 0, \nu_3)$ to the boundary S'. By analogy with the above (see (2.13)–(2.15)), we obtain the following boundary conditions on S_0:

$$u_1\big|_{S_0} = 0, \qquad u_2\big|_{S_0} = 0, \qquad \frac{\partial u_3}{\partial r}\Big|_{S_0} = 0,$$
$$\lim_{r\to 0}\left(\frac{\partial u_1}{\partial r} - \frac{u_1}{r}\right)(r,z) = 0, \qquad z \in \mathcal{T},$$
$$\lim_{r\to 0}\left(\frac{\partial u_2}{\partial r} - \frac{u_2}{r}\right)(r,z) = 0, \qquad z \in \mathcal{T}. \tag{5.15}$$

5.2. Functional spaces and two lemmas. We introduce the following sets:

$$\mathcal{J}_0 = \{v | v = (v_1, v_2, v_3) \in C^\infty(\overline{\Omega}_3)^3,\ v_1\big|_{S_0} = 0,\ v_2\big|_{S_0} = 0\},$$
$$\mathcal{J} = \{v | v \in \mathcal{J}_0,\ v = 0 \quad \text{on } S_1\},$$
$$\mathcal{J}_1 = \{v | v \in \mathcal{J},\ \operatorname{div}_c v = 0\}, \tag{5.16}$$

where the operator div_c is defined by (5.8).

We denote by $\mathcal{H}$ and $\mathcal{H}_1$ the closures of $\mathcal{J}$ and $\mathcal{J}_1$ with respect to the norm

$$||v||_{\mathcal{H}} = \left(\int_{\Omega_3} I(v) r\, dr\, dz\right)^{\frac{1}{2}}, \tag{5.17}$$

where $I(v)$ is given by (5.2).

Let $\mathcal{H}_0$ be the closure of $\mathcal{J}_0$ relative to the norm

$$||v||_{\mathcal{H}_0} = \left(||v||^2_{\mathcal{H}} + \int_{S_1} |v|^2\, ds\right)^{\frac{1}{2}}, \tag{5.18}$$

where $ds = (dz^2 + dr^2)^{\frac{1}{2}}$.

Let also $\mathcal{Y}$ be the space of scalar functions which are square integrable in Ω_3 with respect to the measure $r\, dr\, dz$. The norm in $\mathcal{Y}$ is defined by

$$||h||_{\mathcal{Y}} = \left(\int_{\Omega_3} h^2 r\, dr\, dz\right)^{\frac{1}{2}}. \tag{5.19}$$

By analogy with the Lemma 3.1, we obtain the following result.

LEMMA 5.1. *Suppose that the domain* Ω_1 *defined by* (5.3) *satisfies the condition* (A0). *Then, the expressions* (5.17) *and* (3.8) *define equivalent norms* $\mathcal{H}$, *and the expressions* (5.18) *and* (3.8) *are equivalent norms in* $\mathcal{H}_0$; *moreover, the following equality holds:*

$$(2\pi)^{\frac{1}{2}} ||h||_{\mathcal{Y}} = ||\tilde{h} \circ \mathcal{P}^{-1}||_{L_2(\Omega_1)}, \tag{5.20}$$

with $\tilde{h}(r, \alpha, z) = h(r, z)$, $\alpha \in [0, 2\pi)$.

LEMMA 5.2. *Suppose that the domain* Ω_1 *defined by* (5.3) *satisfies the condition* (A0). *Denote by* $\mathcal{B}$ *the operator* div_c *acting in the space* $\mathcal{H}$, *i.e.,*

$$\mathcal{B}v = \frac{\partial v_1}{\partial r} + \frac{\partial v_3}{\partial z} + \frac{v_1}{r}. \tag{5.21}$$

Then, the following inf-sup condition,

$$\inf_{g \in \mathcal{Y}} \sup_{v \in \mathcal{H}} \frac{(\mathcal{B}v, g)}{||v||_{\mathcal{H}} ||g||_{\mathcal{Y}}} \geq \beta_2 > 0, \tag{5.22}$$

holds true.

The operator $\mathcal{B}$ *is an isomorphism from* $\mathcal{H}_1^\perp$ *onto* $\mathcal{Y}$, *where* $\mathcal{H}_1^\perp$ *is the orthogonal complement of* $\mathcal{H}_1$ *in* $\mathcal{H}$, *and the operator* $\mathcal{B}^*$ *that is the adjoint of* $\mathcal{B}$ *is an isomorphism from* $\mathcal{Y}$ *onto the polar set*

$$\mathcal{H}_1^\circ = \{q \in \mathcal{H}^*, \quad (q, v) = 0, \quad v \in \mathcal{H}_1\}. \tag{5.23}$$

Moreover,

$$||\mathcal{B}^{-1}||_{\mathcal{L}(\mathcal{Y}, \mathcal{H}_1^\perp)} \leq \frac{1}{\beta_2}, \qquad ||(\mathcal{B}^*)^{-1}||_{\mathcal{L}(\mathcal{H}_1^\circ, \mathcal{Y})} \leq \frac{1}{\beta_2}. \tag{5.24}$$

Lemma 5.2 does not follow from Lemma 3.2. For the proof of Lemma 5.2, see [14].

5.3. Generalized solution. We suppose that the function $\breve{u}$ (see (5.11)) belongs to the traces on S_1 of the functions from $\mathcal{H}_0$. Then there exists a function $\overset{*}{u}$ such that

$$\overset{*}{u} \in \mathcal{H}_0, \qquad \overset{*}{u}\big|_{S_1} = \breve{u}, \qquad \mathrm{div}_c\, \overset{*}{u} = 0. \tag{5.25}$$

We assume also that

$$K = (K_1, K_2, K_3) \in \mathcal{Y}^3, \qquad F = (F_1, F_2, F_3) \in L_2(S_2)^3. \tag{5.26}$$

Define an operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}^*$ as follows:

$$(\mathcal{M}(v), h) = 2 \int_{\Omega_3} \varphi \epsilon_{ij}(\overset{*}{u} + v)\epsilon_{ij}(h) r\, dr\, dz, \qquad v, h \in \mathcal{H}, \tag{5.27}$$

where the function φ is given either by (1.7) or by (1.17), and $\epsilon_{ij}(v)$ are defined by (5.1). We consider the problem: Find a pair of functions (v, p) satisfying

$$v \in \mathcal{H}, \qquad p \in \mathcal{Y}, \tag{5.28}$$

$$(\mathcal{M}(v), h) - (\mathcal{B}^*\, p, h) = (K + F, h), \qquad h \in \mathcal{H}, \tag{5.29}$$

$$(\mathcal{B}v, w) = 0, \qquad w \in \mathcal{Y}. \tag{5.30}$$

Here $\mathcal{B}^*$ is the operator adjoint of $\mathcal{B}$ and

$$(K + F, h) = \int_{\Omega_3} K_i h_i r\, dr\, dz + \int_{S_2} F_i h_i\, ds. \tag{5.31}$$

The pair $(u = \overset{*}{u} + v, p)$, where (v, p) is a solution of problem (5.28)–(5.30), will be called the generalized solution of the problem (5.5)–(5.8), (5.11)–(5.15).

Let $\{\mathcal{X}_m\}_{m=1}^{\infty}$ and $\{\mathcal{N}_m\}_{m=1}^{\infty}$ be sequences of finite-dimensional subspaces in $\mathcal{H}$ and $\mathcal{Y}$, respectively, such that

$$\lim_{m\to\infty} \inf_{h \in \mathcal{X}_m} ||w - h||_{\mathcal{H}} = 0, \qquad w \in \mathcal{H}, \tag{5.32}$$

$$\lim_{m\to\infty} \inf_{g \in \mathcal{N}_m} ||f - g||_{\mathcal{Y}} = 0, \qquad f \in \mathcal{Y}, \tag{5.33}$$

$$\inf_{g \in \mathcal{N}_m} \sup_{h \in \mathcal{X}_m} \frac{(\mathcal{B}h, g)}{||h||_{\mathcal{H}} ||g||_{\mathcal{Y}}} \geq \beta > 0, \tag{5.34}$$

$$\mathcal{X}_m \subset \mathcal{X}_{m+1}, \qquad \mathcal{N}_m \subset \mathcal{N}_{m+1}, \qquad m \in \mathbb{N}. \tag{5.35}$$

We seek an approximate solution of the problem (5.28)–(5.30) of the form

$$v_m \in \mathcal{X}_m, \qquad p_m \in \mathcal{N}_m, \tag{5.36}$$

$$(\mathcal{M}(v_m), h) - (\mathcal{B}^*\, p_m, h) = (K + F, h), \qquad h \in \mathcal{X}_m, \tag{5.37}$$

$$(\mathcal{B}\, v_m, g) = 0, \qquad g \in \mathcal{N}_m. \tag{5.38}$$

THEOREM 5.1. *Suppose that the function φ is given by* (1.7) *and that the conditions* (A1), (A2), (2.24) *are satisfied. Let the conditions* (A0), (5.3) *hold and Ω_3 be defined by* (5.4). *Assume also that* (5.25), (5.26), (5.32)–(5.35) *are fulfilled. Then there exists a solution v, p of the problem* (5.28)–(5.30), *and for an arbitrary $m \in \mathbb{N}$ there exists a solution of the problem* (5.36)–(5.38), *and a subsequence $\{v_k, p_k\}$ can be extracted from the sequence $\{v_m, p_m\}$ such that*

$$v_k \to v \quad \textit{in } \mathcal{H}, \qquad p_k \to p \quad \textit{in } \mathcal{Y}. \tag{5.39}$$

Indeed, by using the arguments of Theorem 5.1 from [8], we prove that there exists a solution of the problem (5.36)–(5.38) for any $m \in \mathbb{N}$, and a subsequence $\{v_k, p_k\}$ can be extracted from the sequence $\{v_m, p_m\}$ such that

$$v_k \rightharpoonup v \quad \text{in } \mathcal{H}, \qquad p_k \rightharpoonup p \quad \text{in } \mathcal{Y}. \tag{5.40}$$

(5.39) is proved by using (5.40) and the reasonings of the Theorem 2.1 from [2].

THEOREM 5.2. *Suppose that the function* φ *is given by* (1.17) *and that the conditions* (A0), (A3), (2.23), (2.24) *are satisfied. Let also* (5.25), (5.26), (5.32)–(5.35) *hold. Then, there exists a unique solution of the problem* (5.28)–(5.30), *and there exists a unique solution of the problem* (5.36)–(5.38) *for any* $m \in \mathbb{N}$; *in addition,* $v_m \to v$ *in* $\mathcal{H}$, $p_m \to p$ *in* $\mathcal{Y}$.

The proof of this theorem is analogous to the proof of Theorem 2.1 from [2].

6. Electrorheological clutch.

6.1. Problem on an electric field. Figure 1 (left) displays a scheme of an electrorheological clutch consisting of two coaxial cylinders. The gap between the cylinders is filled with an electrorheological fluid. The inner cylinder hosts a high voltage lead supplying the lateral surface, which serves as the electrode, whereas the lateral surface of the outer cylinder acts as the counter electrode.

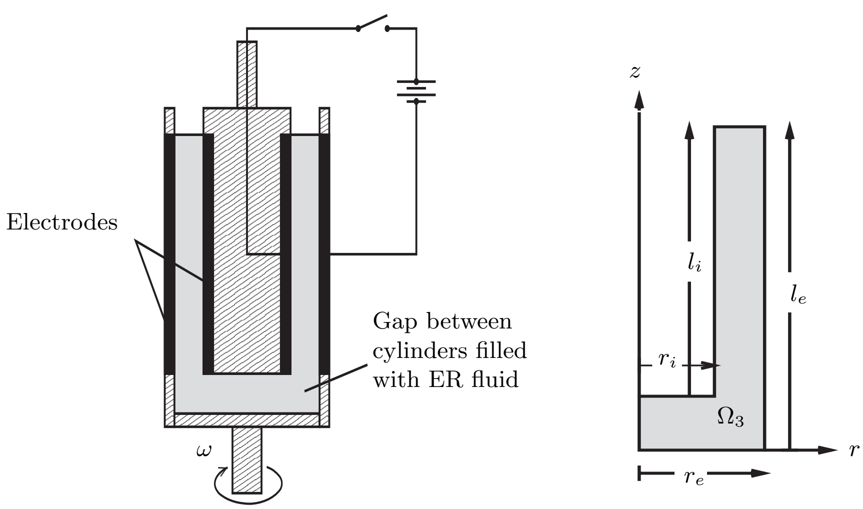

FIG. 1. *Simple model for an electrorheological fluid clutch (left) and the computational domain (right).*

By applying a voltage, one enhances the viscosity of the fluid. Under sufficiently large voltage the inner and external cylinders are almost rigidly bound and rotate practically at the same angular velocity. By varying the voltage, one obtains various slippage of the cylinders, i.e., various transmission ratio of the clutch.

The flow in the clutch is axially symmetric. According to Figure 1 (left) and (5.3), (5.4), the domain Ω_3 corresponding to the flow in the clutch has the form shown in Figure 1 (right).

The vector function of electric field E is defined as $E = -\operatorname{grad}\theta$, where θ is the function of the electric potential that meets the following equation:

$$\operatorname{div}(\chi \operatorname{grad}\theta) = 0, \tag{6.1}$$

where χ is the dielectric permittivity. In our case $\operatorname{grad}\theta = (\frac{\partial\theta}{\partial r}, \frac{\partial\theta}{\partial z})$, and (6.1) takes the form

$$\frac{\partial}{\partial r}\left(\chi\frac{\partial\theta}{\partial r}\right) + \frac{\chi}{r}\frac{\partial\theta}{\partial r} + \frac{\partial}{\partial z}\left(\chi\frac{\partial\theta}{\partial z}\right) = 0 \quad \text{in} \quad \Omega_3, \tag{6.2}$$

and θ satisfies the following boundary conditions:

$$\theta = U \quad \text{on } D_1, \qquad \theta = 0 \quad \text{on } D_2, \qquad \frac{\partial\theta}{\partial r} = 0 \quad \text{on } S_0,$$
$$\nu_1\chi\frac{\partial\theta}{\partial r} + \nu_3\chi\frac{\partial\theta}{\partial z} = 0 \quad \text{on } S\backslash(\overline{D}_1 \cup \overline{D}_2 \cup \overline{S}_0). \tag{6.3}$$

Here, $U = \text{constant} > 0$, S is the boundary of Ω_3, and

$$\begin{aligned} D_1 &= \{(r,z)|\, r = r_i,\ z \in (l_e - l_i, l_e)\},\\ D_2 &= \{(r,z)|\, r = r_e,\ z \in (l_e - l_i, l_e)\},\\ S_0 &= \{(r,z)|\, r = 0,\ z \in (0, l_e - l_i)\}. \end{aligned} \tag{6.4}$$

Let

$$\mathcal{Z} = \left\{w | w \in C^\infty(\overline{\Omega}_3),\ \frac{\partial w}{\partial r} = 0 \quad \text{on } S_0\right\}, \tag{6.5}$$

and let Z_0 be the closure of $\mathcal{Z}$ with respect to the norm

$$||w||_{Z_0} = \left(\int_{\Omega_3}\left[w^2 + \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] r\,dr\,dz\right)^{\frac{1}{2}}. \tag{6.6}$$

Again, we consider the following space:

$$Z = \{w | w \in Z_0,\ w = 0 \quad \text{on} \quad D_1 \cup D_2\}. \tag{6.7}$$

The expression

$$||w||_Z = \left(\int_{\Omega_3}\left[\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] r\,dr\,dz\right)^{\frac{1}{2}} \tag{6.8}$$

defines a norm in Z being equivalent to the norm of Z_0 determined by (6.6). Let θ_0 be a function such that

$$\theta_0 \in Z_0, \quad \theta_0 = U \quad \text{on } D_1, \qquad \theta_0 = 0 \quad \text{on } D_2. \tag{6.9}$$

We assume that χ is a function that is integrable in Ω_3 with respect to the measure $r\,dr\,dz$, and in addition,

$$b_1 \geq \chi \geq b_0 > 0 \quad \text{a.e. in} \quad \Omega, \tag{6.10}$$

where b_0 and b_1 are positive constants.

Define a bilinear form $a : Z_0 \times Z \to \mathbb{R}$ as follows:

$$a(q,h) = \int_{\Omega_3} \chi\left(\frac{\partial q}{\partial r}\frac{\partial h}{\partial r} + \frac{\partial q}{\partial z}\frac{\partial h}{\partial z}\right) r\,dr\,dz, \quad q \in Z_0, \quad h \in Z. \tag{6.11}$$

Consider the following problem: Find θ_1 satisfying

$$\theta_1 \in Z, \qquad a(\theta_1, h) = -a(\theta_0, h), \qquad h \in Z. \tag{6.12}$$

The function $\theta = \theta_0 + \theta_1$ is a generalized solution of the problem (6.2), (6.3).

The Riesz theorem implies the following result.

THEOREM 6.1. *Suppose that* (6.9) *and* (6.10) *are satisfied. Then there exists a unique solution of the problem* (6.12)*, and there exists a unique generalized solution* θ *of the problem* (6.2), (6.3). *The function* θ *is represented in the form* $\theta = \theta_0 + \theta_1$, *where* θ_0 *is a function satisfying* (6.9) *and* θ_1 *is the solution of the problem* (6.12).

6.2. Problem on the fluid flow. We assume that the velocity vector u and the pressure p depend only on r, z in the mobile orthonormal basis e_r, e_α, e_z of cylindrical coordinate system r, α, z and, in addition, $u(r,z) = (0, u_2(r,z), 0)$. We denote the function u_2 by u.

According to (2.1), we have

$$\begin{aligned} &\epsilon_{12}(u) = \epsilon_{21}(u) = \frac{1}{2}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right), \qquad \epsilon_{23}(u) = \epsilon_{32}(u) = \frac{1}{2}\frac{\partial u}{\partial z}, \\ &\epsilon_{11}(u) = \epsilon_{22}(u) = \epsilon_{33}(u) = \epsilon_{13}(u) = \epsilon_{31}(u) = 0, \end{aligned} \tag{6.13}$$

and

$$I(u) = \frac{1}{2}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2. \tag{6.14}$$

In line with (5.5)–(5.7), the motion equations take the form

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial z} = 0, \tag{6.15}$$

$$\frac{\partial}{\partial r}\left(\varphi\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right) + \frac{\partial}{\partial z}\left(\varphi\frac{\partial u}{\partial z}\right) + \frac{2}{r}\varphi\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = 0, \tag{6.16}$$

where volume force vector is ignored.

In the case under consideration, the condition of incompressibility (2.9) is satisfied.

We prescribe velocities on the surfaces of the internal and external cylinders S_1 and specify surface forces on the top boundary of the electrorheological fluid S_2. In this case, we have (see Figure 1 (right))

$$S_1 = \bigcup_{i=1}^{4} S_{1i}, \tag{6.17}$$

where

$$\begin{aligned} &S_{11} = \{(r,z) | z = 0,\, r \in (0, r_e)\}, \qquad S_{12} = \{(r,z) | r = r_e,\, z \in (0, l_e)\}, \\ &S_{13} = \{(r,z) | z = l_e - l_i,\, r \in (0, r_i)\}, \qquad S_{14} = \{(r,z) | r = r_i,\, z \in ((l_e - l_i), l_e)\}, \end{aligned} \tag{6.18}$$

and

$$S_2 = \{(r,z) | z = l_e,\, r \in (r_i, r_e)\}. \tag{6.19}$$

In the case that the inner cylinder is leading, we deal with the following boundary conditions:

$$u(r,z) = \begin{cases} 0 & \text{on } S_{11} \cup S_{12} \cup S_0, \\ \omega r & \text{on } S_{13}, \\ \omega r_i & \text{on } S_{14}, \end{cases} \tag{6.20}$$

$$\varphi \frac{\partial u}{\partial z} = 0 \quad \text{on } S_2, \qquad p = \tilde{c} \quad \text{on } S_2. \tag{6.21}$$

Here ω is the angular velocity of the internal cylinder, and we assume that $F_1 = F_2 = 0$, $F_3 = -\tilde{c}$, $\tilde{c} = \text{constant} > 0$; see (5.12)–(5.14). S_0 is given in (6.4), and according to (5.15), we have

$$u\big|_{S_0} = 0, \qquad \lim_{r \to 0} \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right)(r,z) = 0, \qquad z \in (0, l_e - l_i). \tag{6.22}$$

In the case that the external cylinder is leading, we consider the boundary conditions of this type:

$$u = \begin{cases} \omega r & \text{on } S_{11}, \\ \omega r_e & \text{on } S_{12}, \\ 0 & \text{on } S_{13} \cup S_{14} \cup S_0, \end{cases} \tag{6.23}$$

where ω is the angular velocity of the external cylinder, and, in addition, (6.21) and (6.22) hold.

In the case under consideration the set $\mathcal{J}_0$ has the form

$$\mathcal{J}_0 = \{ v | v \in C^\infty(\overline{\Omega}_3), \quad v = 0 \quad \text{on } S_0 \}, \tag{6.24}$$

and $\mathcal{H}_0$ is the closure of $\mathcal{J}_0$ relative to the norm (compare with (5.16)–(5.18), (6.14))

$$||v||_{\mathcal{H}_0} = \left(\int_{\Omega_3} \left[v^2 + \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right] r \, dr \, dz \right)^{\frac{1}{2}}. \tag{6.25}$$

The space $\mathcal{H}$ appears as

$$\mathcal{H} = \{ v | v \in \mathcal{H}_0, \, v = 0 \quad \text{on } S_1 \}, \tag{6.26}$$

and the norm in $\mathcal{H}$ is given by

$$||v||_{\mathcal{H}} = \left(\int_{\Omega_3} \left[\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right] r \, dr \, dz \right)^{\frac{1}{2}}. \tag{6.27}$$

It follows from Lemma 5.1 that the expressions (6.25) and (3.8) with $v_1 = 0$, $v_2 = v$, $v_3 = 0$ define equivalent norms in $\mathcal{H}_0$, whereas (6.27) and (3.8) are equivalent norms in $\mathcal{H}$.

Equation (6.15) implies $p = c = \text{constant}$, and (6.21) yields $c = \tilde{c}$.

Let $\overset{*}{u}$ be a function from $\mathcal{H}_0$ that satisfies either (6.20) or (6.23) according to which cylinder, inner or external, is leading.

The operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}^*$ is defined as follows:

$$(\mathcal{M}(v), h) = \frac{1}{2} \int_{\Omega_3} \varphi \left[\left(\frac{\partial(\overset{*}{u} + v)}{\partial r} - \frac{\overset{*}{u} + v}{r} \right) \left(\frac{\partial h}{\partial r} - \frac{h}{r} \right) + \frac{\partial(\overset{*}{u} + v)}{\partial z} \frac{\partial h}{\partial z} \right] r\, dr\, dz. \tag{6.28}$$

In the case under consideration the velocity vector is orthogonal to the vector of the electric field at each point $(r, z) \in \overline{\Omega}_3$. Therefore, in (6.28) the function φ is defined by (see (1.7))

$$\varphi = b(|E|, 0)(\lambda + I(\overset{*}{u} + v))^{-\frac{1}{2}} + \psi(I(\overset{*}{u} + v), |E|, 0), \tag{6.29}$$

where the function I is given by (6.14).

We consider the following problem: Find a function v such that

$$v \in \mathcal{H}, \qquad (\mathcal{M}(v), h) = 0, \qquad h \in \mathcal{H}. \tag{6.30}$$

The pair $(u = \overset{*}{u} + v, p)$, where v is a solution of the problem (6.30) and $p = \tilde{c}$, is a generalized solution of the problem (6.15), (6.16), (6.21), (6.22), and (6.20) or (6.23).

Let $\{\mathcal{V}_m\}_{m=1}^{\infty}$ be a sequence of finite-dimensional subspaces in $\mathcal{H}$ such that

$$\lim_{m \to \infty} \inf_{h \in \mathcal{V}_m} ||w - h||_{\mathcal{H}} = 0, \qquad w \in \mathcal{H}, \tag{6.31}$$

$$\mathcal{V}_m \subset \mathcal{V}_{m+1}, \qquad m \in \mathbb{N}. \tag{6.32}$$

We define an approximate solution of the problem (6.30) of the form

$$v_m \in \mathcal{V}_m, \qquad (\mathcal{M}(v_m), h) = 0, \qquad h \in \mathcal{V}_m. \tag{6.33}$$

It follows from Theorem 5.2 that for the function φ defined by (6.29) there exists a unique solution of the problems (6.30) and (6.33); in addition, $v_m \to v$ in $\mathcal{H}$.

6.3. Simulation results. The nonlinear problem (6.30) is solved through solving a sequence of linear problems. Given $v^0 \in \mathcal{H}$, find $v^k \in \mathcal{H}$, $k = 1, 2, \dots$, such that

$$d \in \mathcal{H}, \quad (\hat{\mathcal{M}}(v^{k-1})d, h) = -(\mathcal{M}(v^{k-1}), h) \quad \forall h \in \mathcal{H}, \tag{6.34}$$

$$v^k = v^{k-1} + \alpha\, d. \tag{6.35}$$

Here α is a relaxation parameter, and $\hat{\mathcal{M}}$ is the linearized version of the operator $\mathcal{M}$ (cf. [2]), defined as

$$\begin{aligned}(\hat{\mathcal{M}}(w)v, h) = \frac{1}{2} \int_{\Omega_3} &(b(|E|, 0)(\lambda + I(\overset{*}{u} + w))^{-\frac{1}{2}} + \psi(I(\overset{*}{u} + w), |E|, 0)) \\ &\times \left[\left(\frac{\partial(\overset{*}{u} + v)}{\partial r} - \frac{\overset{*}{u} + v}{r} \right) \left(\frac{\partial h}{\partial r} - \frac{h}{r} \right) + \frac{\partial(\overset{*}{u} + v)}{\partial z} \frac{\partial h}{\partial z} \right] r\, dr\, dz.\end{aligned} \tag{6.36}$$

Note that $(\hat{\mathcal{M}}(w)v, h) = (\mathcal{M}(v), h)$ whenever $w = v$. The algorithm can be termed the Birger–Kachanov method with relaxation; see [5] for the analysis of the original Birger–Kachanov method.

We consider the electrorheological fluid called the Rheobay TP AI 3656, a product of Bayer [1]. The experimentally obtained flow curves (relating the shear stress to the shear rate) of this product, corresponding to different electric field strengths

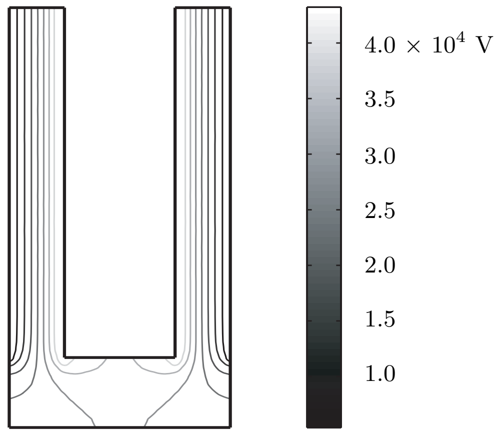

Fig. 2. *Contour plot of the electric potential: wide gap configuration.*

orthogonal to the velocities, have been approximated by cubic splines. The viscosity function φ is then calculated from these splines; see the appendix for details.

In order to understand the behavior of our electrorheological fluid in our model clutch, we study the flow in two different geometrical configurations of the clutch, the *wide*- and the *narrow* gap configurations. In the *wide* gap configuration, we take $r_i = 35$ mm, $r_e = 70$ mm, and $l_i = 250$ mm, $l_e = 300$ mm. During this test the cylinder (outer or inner, whichever is leading) rotates with an angular velocity of 125 rad sec^{-1}. For the *narrow* gap configuration, we take a much narrower gap between the cylinders by setting $r_i = 24$ mm and $r_e = 25$ mm. In this case $l_i = 25$ mm, $l_e = 30$ mm, and the angular velocity of the leading cylinder (outer or inner) is 5 rad sec^{-1}.

The function of the electric field potential θ was calculated approximately by using the Galerkin method with continuous and piecewise linear finite elements for the problem (6.12).

Figure 2 shows a contour plot of the electric potential calculated on the *wide* gap configuration for an applied voltage of 10 kV on the inner electrode. The distribution of this electric potential is linear along any cross-section inside the gap between the electrodes.

Angular velocity profiles for different applied voltages, calculated at one cross section of the gap, are shown in Figures 3 and 4.

From the calculations performed we arrive at the following conclusions:

1. The electric field $E = (E_r, E_z)$ in the gap between the cylinders is close to a constant vector $(U/(r_i - r_e), 0)$ for the narrow and wide gap configurations. At each point between the electrodes, with the exception of points in a very small zone by the ends of the electrodes, the electric field (E_r, E_z) tends to $(U/(r_i - r_e), 0)$ as $(r_e - r_i)/r_i$ tends to zero. The electric field decays sharply as the distance to the electrodes increases (see Figure 2).
2. In the case when the gap between the cylinders is wide and the outer cylinder is leading, a zone with a constant angular velocity is formed near the outer cylinder, and this zone increases with the increase of voltage (see Figure 3 left).
3. In the case when the gap between the cylinders is wide and the inner cylinder is leading, a zone with a constant angular velocity is formed near the outer

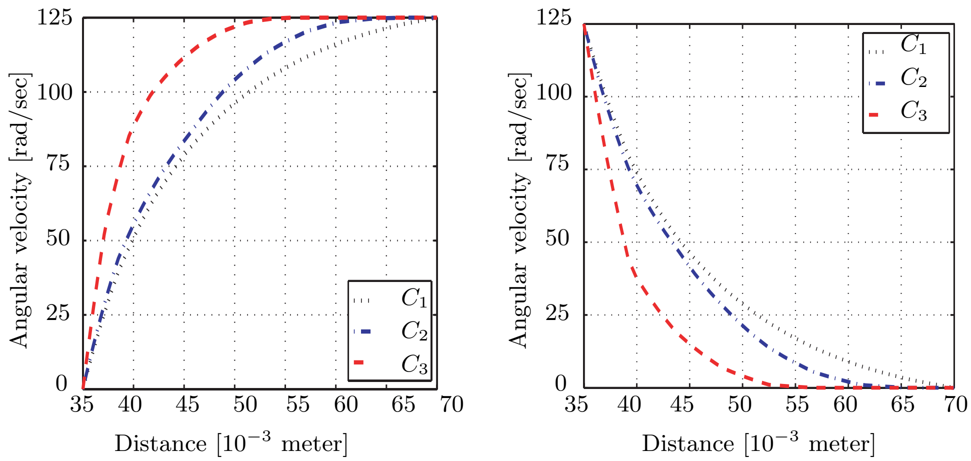

Fig. 3. *Angular velocity profile. Wide gap configuration with leading outer cylinder (left) and leading inner cylinder (right). The curves C_1, C_2, and C_3 correspond to $U = 0$ V, $U = 50$ kV, and $U = 100$ kV, respectively.*

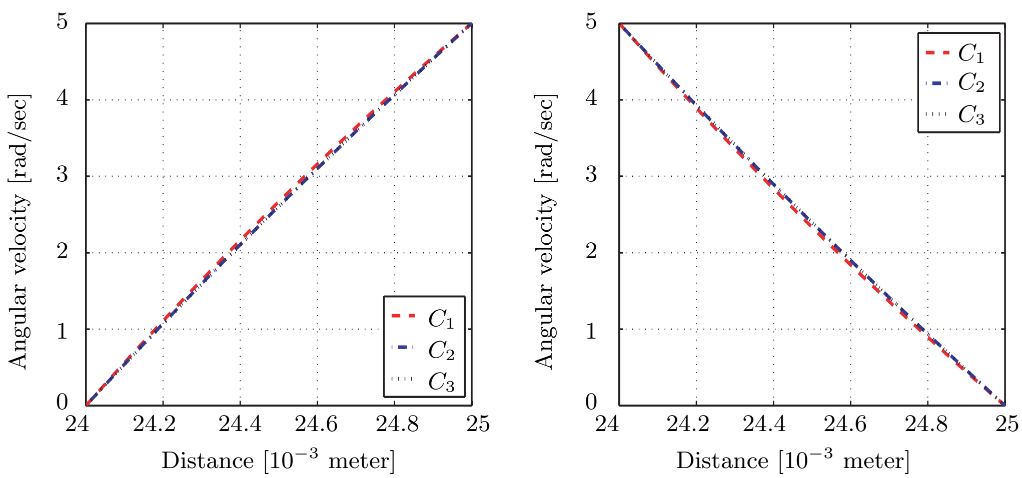

Fig. 4. *Angular velocity profile. Narrow gap configuration with leading outer cylinder (left) and leading inner cylinder (right). The curves C_1, C_2, and C_3 correspond to $U = 0$ V, $U = 2$ kV, and $U = 3$ kV, respectively.*

cylinder, as in the case when the outer cylinder is leading. This zone increases under the increase of voltage (see Figure 3 right).

4. In the case of a narrow gap between the cylinders, the zone with a constant angular velocity is not formed. The velocity profiles are almost linear at various voltages. No matter what cylinder is leading and what voltage is applied, the velocity profile tends to linear as $(r_e - r_i)/r_i$ tends to zero (see Figure 4). In this case essentially the velocity profile does not depend on the shape of a flow curve, and the shear rate is a constant.

We note that in the case of a wide gap between the cylinders, the zone with a constant angular velocity is also formed under the flow of the Bingham fluid. The proximity of the Bingham velocity profiles to the profiles presented in Figure 3 depends on the proximity of approximations of the flow curves by the affine functions $\tau_0+b_0\gamma$, where τ_0 and b_0 are the yield stress and the viscosity of the Bingham fluid and γ is the shear rate. (About the proximity of solutions for close flow curves, see [13, section 6.2].)

As may be seen from the appendix Figure A-1, one cannot obtain good approximations of the flow curves by affine functions, especially for small shear rates.

Appendix. Identification of the viscosity function. In the following, we present a set of cubic splines (flow splines) approximating a set of experimentally obtained flow curves and show how the viscosity function φ is calculated from these splines. These flow curves (splines) are for the electrorheological fluid called Rheobay TP AI 3656, a product of Bayer, based on a water-free dispersion of polymer particles in silicone oil (Baysilone Oil M); see [1] for specifications. The application of such a product can be found in various devices, such as shock absorbers, vibration dampers, clutches, and so on.

A.1. The flow splines. The set of cubic splines approximating experimentally obtained flow curves corresponding to a set of different electric field strengths, which are orthogonal to the velocity, are shown in Figure A-1. Complete information for the reconstruction of these splines, i.e., the sample points representing the shear rates γ, the data representing the shear stress τ, and the end slopes (derivatives), are provided in Table A-1.

Each flow curve has been approximated within the interval $[\gamma_0, \gamma_1]$ (in our case $\gamma_0 = 100\ \mathrm{sec}^{-2}$, $\gamma_1 = 2000\ \mathrm{sec}^{-2}$) by a cubic spline with the given end slopes. Outside of the interval $[\gamma_0, \gamma_1]$ the splines have been extended on $\mathbb{R}_+$ by straight lines (see the dotted lines in Figure A-1), so that the obtained function $\gamma \to \tau(\gamma)$ becomes continuously differentiable in $\mathbb{R}_+$.

A linear interpolation is used to calculate the function $\tau(\gamma)$ for values of $|E|$ intermediate between the values given in the Table A-1.

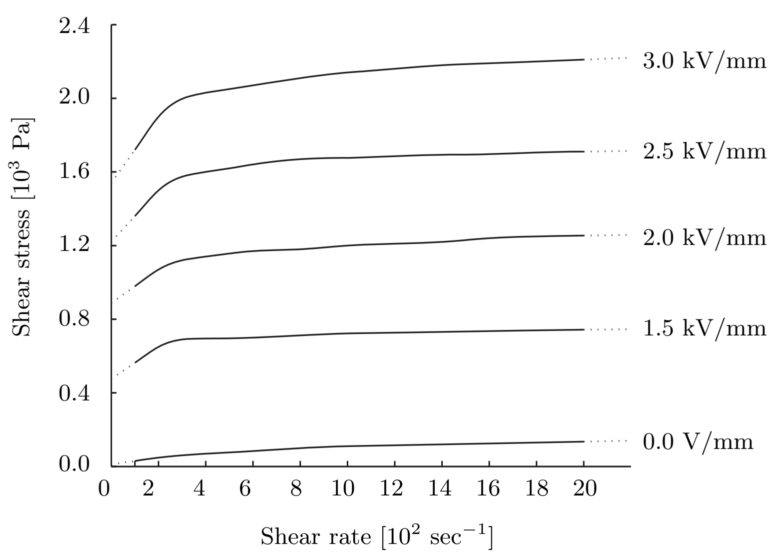

FIG. A-1. *Flow splines showing the effect of field strength (50Hz, AC) and shear rate γ on shear stress τ at $40^o C$. The cubic splines are constructed using the data provided in Table* A-1.

TABLE A-1

The table contains complete information for the reconstruction of the five cubic splines displayed in Figure A-1. Each of the last five columns in the table corresponds to a spline approximating a flow curve, containing two end slopes and eleven data values corresponding to the eleven sample points in the first column of the table.

	Shear stress (Pa)				
Shear rate γ [per sec]	0.0 V/mm	1.5 kV/mm	2.0 kV/mm	2.5 kV/mm	3.0 kV/mm
1.0×10^2	30.2	563.0	979.0	1360.0	1720.0
2.0×10^2	48.0	650.0	1070.0	1500.0	1900.0
4.0×10^2	69.3	695.0	1140.0	1600.0	2030.0
6.0×10^2	83.5	700.0	1170.0	1640.0	2070.0
8.0×10^2	100.0	712.0	1180.0	1670.0	2110.0
1.0×10^3	110.0	723.0	1200.0	1676.0	2140.0
1.2×10^3	115.0	727.0	1210.0	1686.0	2160.0
1.4×10^3	120.0	731.0	1220.0	1693.0	2180.0
1.6×10^3	125.0	735.0	1240.0	1696.0	2190.0
1.8×10^3	130.0	740.0	1250.0	1706.0	2200.0
2.0×10^3	135.0	743.0	1254.0	1710.0	2210.0
Slope at left end	0.180	0.870	0.910	1.400	1.800
Slope at right end	0.025	0.015	0.020	0.020	0.050

A.2. The viscosity function. We now show how the viscosity function φ is calculated from the function $\tau(\gamma)$. Let τ_0 be the point of intersection of a left dotted line with the shear stress axis. This dotted line is the continuation of the flow curve in the interval $[0, \gamma_0)$, and τ_0 is the yield stress.

The viscosity function, in the case of a simple shear flow, is determined as follows:

$$\varphi = \frac{1}{2}\frac{\tau}{\gamma}, \quad \text{where} \quad \gamma = \left(\frac{1}{2}I(u)\right)^{\frac{1}{2}}.$$

Generalizing it to an arbitrary flow, we get

$$b = \frac{\tau_0}{\sqrt{2}} \quad \text{and} \quad \psi = \frac{\tau - \tau_0}{2\gamma} = \frac{\tau - \tau_0}{(2I(u))^{\frac{1}{2}}}. \tag{A-1}$$

For a fixed value of $|E|$ and $I(u)$, we can thus find the value of φ using (A-1) and (6.29), i.e.,

$$\varphi = \frac{b}{(\lambda + I(u))^{\frac{1}{2}}} + \psi. \tag{A-2}$$

The parameter λ was chosen equal to $1.011e^{-11}$ sec^{-2}.

In the general case, when there are given flow curves for different values of $|E|$ and $\mu(u, E)$, one obtains expressions (A-2) for different values of $|E|$ and $\mu(u, E)$; see (1.7).

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