ON THE CONVERGENCE OF MORTAR EDGE ELEMENT METHODS IN $\mathbb{R}^3$*

XUEJUN XU† AND R. H. W. HOPPE‡

Abstract. In this paper, we are concerned with mortar element methods for the numerical solution of the eddy currents equations based on domain decompositions on nonmatching grids using individual subdomain discretizations by the lowest order edge elements of Nédélec's first family. The main results are optimal a priori error estimates of the global discretization error and the Lagrange multipliers that take care of the weak continuity constraints on the tangential traces across interior subdomain boundaries. These estimates are derived under moderate regularity assumptions.

Key words. mortar edge elements, domain decomposition on nonmatching grids, eddy currents equations

AMS subject classifications. 65F10, 65N30

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1. Introduction.

Mortar element methods have attracted considerable attention in recent years, since they can handle situations where meshes on different subdomains need not align across interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. In [8], Bernardi, Maday, and Patera first introduced basic concepts of general mortar element methods, including the coupling of spectral elements with finite elements. Subsequently, they have been extensively used and analyzed by many authors. In [4], Ben Belgacem studied the mortar element method within a primal hybrid finite element formulation. Some extensions and convergence results in three dimensions have been considered in [5], [10], and [22].

In the framework of edge element discretizations, the mortar element method has been studied for two-dimensional problems in [3] and [6]. However, similar to second order elliptic problems (cf., e.g., [5], [10], [22]), the situation in the three-dimensional case is much more complicated, since it particularly requires a subtle specification of the multiplier space. Recently, the second author of this paper considered a mortar element method for three-dimensional Maxwell equations in [20], where the edge element of the first family has been studied (see also [21]). Related work for mortar edge elements has been proposed by Ben Belgacem, Buffa, and Maday in [7], but their result holds only for the lowest order edge elements of Nédélec's second family [26]. Furthermore, their error estimate of order $O(hlog(h))$ is not optimal and requires a somewhat high regularity of the solution, i.e., the solution is assumed to belong to $H^2(\mathbf{curl};\Omega)$.

In this paper, we will give an optimal error estimate for the mortar edge element method based on the lowest order edge elements of Nédélec's first family. Our convergence results are established under a weaker regularity assumption, i.e., the solution

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†LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, P.O. Box 2719, Beijing, 100080, People's Republic of China (xxj@lsec.cc.ac.cn). The work of this author was supported by the Alexander von Humboldt Foundation and special funds for major state basic research projects under 2005CB321701 as well as a grant from the National Science Foundation (NSF) of China (10471144).

‡Institut für Mathematik, Universität Augsburg, D-86159, Augsburg, Germany, and Department of Mathematics, University of Houston, Houston, TX 77204-3008 (rohop@math.uh.edu).

is assumed to belong to $H^1(\mathbf{curl};\Omega)$. On the other hand, on the basis of the discrete inf-sup condition constructed in [20], we also obtain an optimal error estimate for the Lagrange multiplier.

The paper is organized as follows. Section 2 describes the model problem under consideration. Section 3 introduces the mortar edge element method followed by the derivation of the optimal energy error estimate in section 4. Finally, section 5 is devoted to an optimal error estimate for the Lagrange multiplier.

2. Model problem. Given a bounded simply connected domain Ω in R^3 with polyhedral boundary $\partial\Omega$, we consider the following elliptic boundary value problem:

$$\begin{cases} \mathbf{curl}\ \mathbf{A}\ \mathbf{curl}\ \mathbf{j} + \mathbf{B}\ \mathbf{j} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{j} \wedge \mathbf{n} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathbf{n}$ denotes the exterior unit normal vector on $\partial\Omega$. We note that the above problem arises, for instance, in the computation of eddy currents and can be deduced from the time-dependent equations by using an implicit finite difference scheme (cf. [9], [18], [23]).

We assume $\mathbf{A} = \{a_{ij}\}_{i,j=1}^3$ and $\mathbf{B} = \{b_{ij}\}_{i,j=1}^3$ to be symmetric matrix-valued functions, with $a_{ij} \in C^1(\bar{\Omega}),\ b_{ij} \in L^\infty(\Omega),\ 1 \le i,j \le 3$, satisfying

$$c|\xi|^2 \le \sum_{i,j=1}^3 a_{ij}(x)\xi_i\xi_j \le C|\xi|^2, \qquad c|\xi|^2 \le \sum_{i,j=1}^3 b_{ij}(x)\xi_i\xi_j \le C|\xi|^2, \quad \xi \in R^3,$$

for almost all $x \in \Omega$. In this paper, the constants c and C with or without subscript always denote general positive constants independent of the mesh size. Moreover, we assume $\mathbf{f} \in L^2(\Omega)^3$ and suppose, for simplicity, that $\mathbf{g} = 0$.

We denote by $H(\mathbf{curl};\Omega)$ the Hilbert space

$$H(\mathbf{curl};\Omega) := \{\mathbf{q} \in L^2(\Omega)^3 \mid \mathbf{curl}\mathbf{q} \in L^2(\Omega)^3\}$$

equipped with the norm

$$\|\mathbf{q}\|_{\mathbf{curl},\Omega} := (\|\mathbf{q}\|_{0,\Omega}^2 + \|\mathbf{curl}\mathbf{q}\|_{0,\Omega}^2)^{\frac{1}{2}}.$$

Here and in what follows, $\|\cdot\|_{k,\Omega}, k \in \mathbb{N}_0$, stands for the norm of the Sobolev space $H^k(\Omega)^3$. Moreover, we define the space

$$H^1(\mathbf{curl};\Omega) := \{\mathbf{q} \in H^1(\Omega)^3 | \mathbf{curl}\mathbf{q} \in H^1(\Omega)^3\}$$

equipped with the norm

$$\|\mathbf{q}\|_{1,\mathbf{curl},\Omega} := (\|\mathbf{q}\|_{1,\Omega} + \|\mathbf{curl}\mathbf{q}\|_{1,\Omega})^{\frac{1}{2}}.$$

Similarly, if G is a subdomain of Ω, we can define the space $H^1(\mathbf{curl};G)$ over the subdomain G. The corresponding norm is denoted by $\|\mathbf{q}\|_{1,\mathbf{curl},G}$.

We refer to

$$\mathbf{V} := H_0(\mathbf{curl};\Omega) = \{\mathbf{q} \in H(\mathbf{curl};\Omega) \mid \mathbf{n} \wedge (\mathbf{q} \wedge \mathbf{n})|_{\partial\Omega} = 0\}$$

as the subspace of vector fields with vanishing tangential components trace on $\partial\Omega$.

Then, the variational formulation of (2.1) is to find $\mathbf{j} \in \mathbf{V}$ such that

$$a_\Omega(\mathbf{j},\mathbf{q}) \ = \ l(\mathbf{q}) \quad \forall \mathbf{q} \in \mathbf{V}, \tag{2.2}$$

where the bilinear form $a_\Omega(\cdot,\cdot) : H(\mathbf{curl};\Omega) \times H(\mathbf{curl};\Omega) \to \mathbb{R}$ and the functional $l(\cdot) : H(\mathbf{curl};\Omega) \to R$ are given by

$$a_\Omega(\mathbf{j},\mathbf{q}) := \int_\Omega (\mathbf{A}\ \mathbf{curl}\ \mathbf{j}\cdot \mathbf{curl}\ \mathbf{q} + \mathbf{B}\ \mathbf{j}\cdot\mathbf{q})\, dx,$$
$$l(\mathbf{q}) := \int_\Omega \mathbf{f}\cdot\mathbf{q}\, dx.$$

We further have to introduce the tangential traces of $H(\mathbf{curl};\Omega)$. In particular, we denote by div_τ and curl_τ the surfacic divergence and the adjoint of the surfacic rotational $\mathbf{curl}_\tau$ (cf. [1]). For $B \subset \partial\Omega$, the space $H_{00}^{\frac12}(B)$ is the subspace of functions $u \in H^{\frac12}(\Omega)$ whose extension $\tilde u$ by zero to $\partial\Omega\backslash B$ belongs to $H^{\frac12}(\partial\Omega)$ with norm $\|u\|_{H_{00}^{\frac12}(B)} := \|\tilde u\|_{\frac12,\partial\Omega}$. We refer to $H^{-\frac12}(B)$ as the dual space of $H_{00}^{\frac12}(B)$ (cf. [19] for details).

The tangential trace $(\mathbf{q}\wedge\mathbf{n})|_B$ belongs to the Hilbert space

$$H^{-\frac12}(\mathrm{div}_\tau;B) \ := \ \{\mathbf{q}\in H^{-\frac12}(B)^3 \mid \mathbf{n}\cdot\mathbf{q}|_B = 0 \text{ and } \mathrm{div}_\tau\mathbf{q}\in H^{-\frac12}(B)\}$$

equipped with the norm

$$\|\mathbf{q}\|_{-\frac12,\mathrm{div}_\tau,B} \ := \ (\|\mathbf{q}\|^2_{-\frac12,B} + \|\mathrm{div}_\tau\mathbf{q}\|^2_{-\frac12,B})^{1/2},$$

whereas the tangential components trace $(\mathbf{n}\wedge(\mathbf{q}\wedge\mathbf{n}))|_B$ lives in the Hilbert space

$$H^{-\frac12}(\mathrm{curl}_\tau;B) \ := \ \{\mathbf{q}\in H^{-\frac12}(B)^3 \mid \mathbf{n}\cdot\mathbf{q}|_B = 0 \quad\text{and}\quad \mathrm{curl}_\tau\mathbf{q}\in H^{-\frac12}(B)\}$$

equipped with the norm

$$\|\mathbf{q}\|_{-\frac12,\mathrm{curl}_\tau,B} \ := \ (\|\mathbf{q}\|^2_{-\frac12,B} + \|\mathrm{curl}_\tau\mathbf{q}\|^2_{-\frac12,B})^{1/2}.$$

The spaces $H^{-\frac12}(\mathrm{div}_\tau;B)$ and $H^{-\frac12}(\mathrm{curl}_\tau;B)$ are dual to each other with $\mathbf{L}^2_{\mathbf{t}}(B) := \{\mathbf{q}\in L^2(B)^3 \mid \mathbf{n}\cdot\mathbf{q}|_B = 0\}$ as the pivot space (cf. [13], [14], and [15] for details).

3. The mortar edge element method. We now introduce a mortar finite element method for the solution of (2.1). First, we partition Ω into nonoverlapping subdomains such that

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i \quad\text{and}\quad \Omega_i\cap\Omega_j = \emptyset, \quad i\neq j.$$

We assume this decomposition to be geometrically conforming in the sense that the intersection of $\bar\Omega_i\cap\bar\Omega_j$ for $i\neq j$ is either empty, a vertex, an edge, or a face. The skeleton of the decomposition

$$S = \bigcup_{i=1}^N \partial\Omega_i\backslash\partial\Omega$$

is partitioned into a set of disjoint open faces γ_m $(1\le m\le M)$ called mortars, i.e.,

$$S = \bigcup_{m=1}^M \bar\gamma_m, \quad \gamma_m\cap\gamma_n = \emptyset \ \text{ if } \ m\neq n.$$

We denote the common interface between Ω_i and Ω_j by γ_m. We refer to $\gamma_{m(i)}$ as the mortar associated with subdomain Ω_i, while the other face, which geometrically occupies the same place, is denoted by $\delta_{m(j)}$ and is called the nonmortar.

Let $\mathcal{T}_i$ be a regular and quasi-uniform triangulation of the subdomain Ω_i with mesh size $h_i := \max_{K\in\mathcal{T}_i} h_K$ made of tetrahedra. The triangulations generally do not align at the interfaces. We denote the global mesh $\cup_i \mathcal{T}_i$ by $\mathcal{T}_h$ with mesh size $h := \max_i h_i$. We refer to $\mathcal{T}_{\gamma_{m(i)}}$ and $\mathcal{T}_{\delta_{m(j)}}$ as the triangulations which are inherited from the triangulations $\mathcal{T}_i$ and $\mathcal{T}_j$ on the mortar and nonmortar sides, respectively. We further denote by $h_{\gamma_{m(i)}}$ and $h_{\delta_{m(j)}}$ the global mesh sizes with respect to the triangulations $\mathcal{T}_{\gamma_{m(i)}}$ and $\mathcal{T}_{\delta_{m(j)}}$. Moreover, for $\Sigma_i \subset \bar{\Omega}_i$ we define $\mathcal{F}_h(\Sigma_i)$ and $\mathcal{E}_h(\Sigma_i)$ as the sets of faces, respectively, edges, of $\mathcal{T}_i$ in Σ_i. Likewise, for $\Sigma_{\gamma_{m(i)}}$ and $\Sigma_{\delta_{m(j)}} \subset \gamma_m$ we refer to $\mathcal{E}_h(\Sigma_{\gamma_{m(i)}})$ and $\mathcal{E}_h(\Sigma_{\delta_{m(j)}})$ as the set of edges of $\mathcal{T}_{\gamma_{m(i)}}$, respectively, $\mathcal{T}_{\delta_{m(j)}}$, in $\Sigma_{\gamma_{m(i)}}$, respectively, $\Sigma_{\delta_{m(j)}}$.

We assume that there exist constants c, C independent of $h_{\gamma_{m(i)}}$ and $h_{\delta_{m(j)}}$ such that

$$c\, h_{\gamma_{m(i)}} \;\le\; h_{\delta_{m(j)}} \;\le\; C\, h_{\gamma_{m(i)}}. \tag{3.1}$$

For the discretization of $H(\mathbf{curl};\Omega_i)$, we introduce Nédélec's **curl**-conforming edge elements of the first family as described in [25], i.e., for a tetrahedron $K \in \mathcal{T}_i$ the lowest order edge element $\mathrm{ND}_1(\mathrm{K})$ is defined as

$$\mathrm{ND}_1(\mathrm{K}) \;:=\; \{\mathbf{q} = \mathbf{a} + \mathbf{b} \wedge \mathbf{x} \mid \mathbf{a},\ \mathbf{b} \in R^3,\ \mathbf{x} \in K\}.$$

Note that any $\mathbf{q} \in \mathrm{ND}_1(\mathrm{K})$ is uniquely determined by the degrees of freedom

$$l_e(\mathbf{q}) \;:=\; \int_e \mathbf{t_e} \cdot \mathbf{q}\, ds, \quad e \in \mathcal{E}_h(K), \tag{3.2}$$

where $\mathbf{t_e}$ stands for the tangential unit vector along e.

Then, the spaces $\mathrm{ND}_1(\Omega_i;\mathcal{T}_i)$ are given as follows:

$$\mathrm{ND}_1(\Omega_i;\mathcal{T}_i) \;:=\; \{\mathbf{q_h} \in H(\mathbf{curl};\Omega_i) \mid \mathbf{q_h}|_K \in \mathrm{ND}_1(K),\ K \in \mathcal{T}_i\}.$$

On the basis of the above definition, we consider the product space

$$\tilde{\mathbf{V}}_\mathbf{h} \;:=\; \{\mathbf{q_h} \in L^2(\Omega)^3 \mid \mathbf{q_h}|_{\Omega_i} \in \mathrm{ND}_{1,0}(\Omega_i;\mathcal{T}_i),\ 1 \le i \le n\},$$

where we refer to $\mathrm{ND}_{1,0}(\Omega_i;\mathcal{T}_i)$ as the subspace of vector fields with vanishing tangential component traces on $\partial\Omega \cap \partial\Omega_i$.

It is clear that we cannot expect $\tilde{\mathbf{V}}_\mathbf{h}$ to be a subspace of $H_0(\mathbf{curl};\Omega)$, since the tangential traces $(\mathbf{q_h} \wedge \mathbf{n})|_F, \mathbf{q_h} \in \tilde{\mathbf{V}}_\mathbf{h}$, are not continuous across the common face F of two adjacent subdomains. Therefore, in order to guarantee consistency of the approximation, we have to impose some weak continuity constraints on the tangential traces. We note that $(\mathbf{q_h} \wedge \mathbf{n})|_{\gamma_{m(i)}}$ and $(\mathbf{q_h} \wedge \mathbf{n})|_{\delta_{m(j)}}$ are elements of the lowest order Raviart–Thomas finite element spaces $\mathrm{RT}_0(\gamma_{m(i)};\mathcal{T}_{\gamma_{m(i)}})$ and $\mathrm{RT}_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$. We recall the definition of the lowest order Raviart–Thomas conforming finite element (cf. [12], [27]). For a triangle $T \in \mathcal{T}_{\gamma_{m(i)}}$, we define $\mathrm{RT}_0(\mathrm{T})$ by means of

$$\mathrm{RT}_0(\mathrm{T}) \;:=\; \{\mathbf{q} = \mathbf{a} + b\mathbf{x} \mid \mathbf{a} \in \mathbb{R}^2,\ b \in \mathbb{R},\ \mathbf{x} \in T\}.$$

Any $\mathbf{q} \in \mathrm{RT}_0(\mathrm{T})$ is uniquely defined by the degrees of freedom

$$l_e(\mathbf{q}) \;:=\; \int_e \mathbf{n_e} \cdot \mathbf{q}\, ds, \quad e \in \mathcal{E}_h(T), \tag{3.3}$$

where $\mathbf{n_e}$ stands for the exterior unit normal vector with respect to e.

Then, $\mathrm{RT}_0(\gamma_{m(i)};\mathcal{T}_{\gamma_{m(i)}})$ is given as

$$\mathrm{RT}_0(\gamma_{m(i)};\mathcal{T}_{\gamma_{m(i)}}) \ := \ \{\mathbf{q_h} \in H(\mathrm{div};\gamma_{m(i)}) \mid \mathbf{q_h}|_T \in \mathrm{RT}_0(T),\ T \in \mathcal{T}_{\gamma_{m(i)}}\},$$

and we can similarly define $\mathrm{RT}_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$.

For the Lagrange multiplier space we choose

$$\mathbf{M_h} := \prod_{\delta_{m(j)}} \mathbf{M_h}(\delta_{m(j)})$$

with

$$\dim\ \mathbf{M_h}(\delta_{m(j)}) = \dim\ \mathrm{RT}_{0,0}(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}}),$$

where $\mathrm{RT}_{0,0}(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$ denotes the subspace of vector fields with vanishing normal components along the boundary $\partial\delta_{m(j)}$.

For the proper definition of $\mathbf{M_h}(\delta_{m(j)})$ we need a more detailed specification of the basis fields of $\mathrm{RT}_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$. In view of (3.3), we specify the basis field $\mathbf{q}_\gamma$ associated with the edge $e_\gamma \in \mathcal{E}_h(\bar{\delta}_{m(j)})$ according to

$$\int_{e_\mu} \mathbf{n}_\mu \cdot \mathbf{q}_\gamma \ ds \ = \ h_{\delta_{m(j)}}\delta_{\gamma\mu}, \quad e_\mu \in \mathcal{E}_h(\bar{\delta}_{m(j)}). \tag{3.4}$$

We now define $\mathbf{M_h}(\delta_{m(j)})$ by an extension of the basis field $\mathbf{q_e} \in \mathrm{RT}_{0,0}(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$ with respect to those edges in $\delta_{m(j)}$ that have at least one neighboring edge on the boundary $\partial\delta_{m(j)}$. The precise specification requires some notation:

1. For an interior edge $e \in \mathcal{E}_h(\delta_{m(j)})$, we denote by

$$\mathcal{E}_h^{\partial\delta_{m(j)}}(e) \ := \ \{f \in \mathcal{E}_h(\partial\delta_{m(j)}) \mid f \subset \mathrm{supp}\ \mathbf{q}_e\} \tag{3.5}$$

the set of the neighboring edges on $\partial\delta_{m(j)}$.

2. For a boundary edge $f \in \mathcal{E}_h(\partial\delta_{m(j)})$, we refer to

$$\mathcal{E}_h^{\delta_{m(j)}}(f) \ := \ \{e \in \mathcal{E}_h(\delta_{m(j)}) \mid e \subset \mathrm{supp}\ \mathbf{q}_f\} \tag{3.6}$$

as the set of neighboring edges in the interior of $\delta_{m(j)}$.

Finally we define

$$\mathcal{E}_h^{\delta_{m(j)}}(\partial\delta_{m(j)}) \ := \bigcup_{f \in \mathcal{E}_h(\partial\delta_{m(j)})} \mathcal{E}_h^{\delta_{m(j)}}(f) \tag{3.7}$$

as the set of interior edges with a neighboring edge on $\partial\delta_{m(j)}$.

Then, for $e \in \mathcal{E}_h^{\delta_{m(j)}}(\partial\delta_{m(j)})$, we choose appropriate weighting factors $\lambda_{e,f} \in \mathbb{R}$, $f \in \mathcal{E}_h^{\partial\delta_{m(j)}}(e)$, and define the basis field $\tilde{\mathbf{q}}_e$, $e \in \mathcal{E}_h(\delta_{m(j)})$, according to

$$\tilde{\mathbf{q}}_e = \begin{cases} \mathbf{q}_e, & e \in \mathcal{E}_h(\delta_{m(j)}) \backslash \mathcal{E}_h^{\delta_{m(j)}}(\partial\delta_{m(j)}) \\ \mathbf{q}_e + \sum_{f \in \mathcal{E}_h^{\partial\delta_{m(j)}}(e)} \lambda_{e,f}\mathbf{q}_f, & e \in \mathcal{E}_h^{\delta_{m(j)}}(\partial\delta_{m(j)}), \end{cases} \tag{3.8}$$

where the weighting factors are assumed to satisfy

$$\begin{cases} \lambda_{e,f} \geq 0, \\ \sum_{e \in \mathcal{E}_h^{\delta_{m(j)}}(f)} \lambda_{e,f} = 1, \quad f \in \mathcal{E}_h(\partial\delta_{m(j)}). \end{cases} \tag{3.9}$$

The thus specified basis fields define

$$\mathbf{M_h}(\delta_{m(j)}) \ := \ \mathrm{span}\{\tilde{\mathbf{q}}_e | e \in \mathcal{E}_h(\delta_{m(j)})\}. \tag{3.10}$$

Remark 3.1. In view of (3.9) it is easy to check that $\mathbf{M_h}(\delta_{m(j)})$ contains the constant vectors.

Next, we introduce the L^2-projection $Q_h^{\delta_{m(j)}} : L^2(\gamma_m)^2 \to \mathbf{M_h}(\delta_{m(j)})$ as follows:

$$(Q_h^{\delta_{m(j)}}\mathbf{q}, \mathbf{w}) = (\mathbf{q}, \mathbf{w}), \quad \mathbf{w} \in \mathbf{M_h}(\delta_{m(j)}). \tag{3.11}$$

LEMMA 3.1. *Let* $Q_h^{\delta_{m(j)}}$ *be given by* (3.11). *Then there holds*

$$\|\mathbf{q} - Q_h^{\delta_{m(j)}}\mathbf{q}\|_{0,\gamma_m} \le C\ h_{\delta_{m(j)}}^{\frac{1}{2}}\ |\mathbf{q}|_{\frac{1}{2},\delta_{m(j)}}, \quad \mathbf{q} \in (H^{\frac{1}{2}}(\delta_{m(j)}))^2.$$

Proof. Let $\mathbf{I_h}$ denote the global interpolation operator associated with the space $\mathbf{M_h}(\delta_{m(j)})$, i.e.,

$$\mathbf{I_h}\mathbf{q} = \sum_{e\in\mathcal{E}_h(\delta_{m(j)})} l_e(\mathbf{q})\tilde{\mathbf{q}}_e,$$

where $l_e(\mathbf{q}) = \int_e \mathbf{n}_e \cdot \mathbf{q}\, ds\ \forall \mathbf{q} \in (H^1(\delta_{m(j)}))^2$.

In view of Remark 3.1 we know that $\mathbf{I_h}$ preserves constant vectors, i.e., for any $\mathbf{C} \in R^2$,

$$\mathbf{I}_h\mathbf{C} = \mathbf{C}.$$

Consequently, by the standard Bramble–Hilbert lemma and scaling argument we get

$$\begin{aligned}\|(\mathbf{I} - \mathbf{I_h})\mathbf{q}\|^2_{0,\gamma_m} &= \|(\mathbf{I} - \mathbf{I_h})(\mathbf{q} + \mathbf{C})\|^2_{0,\gamma_m}\\ &= \sum_{T\in\mathcal{T}_{\delta_{m(j)}}} \|(\mathbf{I} - \mathbf{I_h})(\mathbf{q} + \mathbf{C})\|^2_{0,T}\\ &\le\ Ch^2_{\delta_{m(j)}}|\mathbf{q}|^2_{1,\delta_{m(j)}}, \qquad \mathbf{q} \in (H^1(\delta_{m(j)}))^2,\end{aligned}$$

whence

$$\|(\mathbf{I} - \mathbf{I_h})\mathbf{q}\|_{0,\gamma_m}\ \le\ Ch_{\delta_{m(j)}}|\mathbf{q}|_{1,\delta_{m(j)}}, \quad \mathbf{q} \in (H^1(\delta_{m(j)}))^2.$$

It follows from the definition of $Q_h^{\delta_{m(j)}}$ that

$$\|(\mathbf{I} - Q_h^{\delta_{m(j)}})\mathbf{q}\|_{0,\gamma_m} \le \|(\mathbf{I} - \mathbf{I_h})\mathbf{q}\|_{0,\gamma_m}\ \le\ Ch_{\delta_{m(j)}}|\mathbf{q}|_{1,\delta_{m(j)}}, \quad \mathbf{q} \in (H^1(\delta_{m(j)}))^2.$$

On the other hand,

$$\|(\mathbf{I} - Q_h^{\delta_{m(j)}})\mathbf{q}\|_{0,\gamma_m} \le 2\|\mathbf{q}\|_{0,\delta_{m(j)}}.$$

The assertion then follows from a standard interpolation of the preceding inequalities. □

We now introduce the following mortar edge element space:

$$\begin{aligned}\mathbf{V_h} = \{\mathbf{q_h}\ |\ \mathbf{q_h} \in \tilde{\mathbf{V}}_\mathbf{h},\ &\text{and for any } \gamma_m = \gamma_{m(i)} = \delta_{m(j)},\\ Q_h^{\delta_{m(j)}}(\mathbf{q_h} \wedge \mathbf{n}|_{\gamma_{m(i)}}) &= Q_h^{\delta_{m(j)}}(\mathbf{q_h} \wedge \mathbf{n}|_{\delta_{m(j)}})\}.\end{aligned} \tag{3.12}$$

We define the bilinear form $a_h(\cdot,\cdot):\mathbf{V_h}\times\mathbf{V_h}\to\mathbb{R}$ by means of

$$a_h(\mathbf{j_h},\mathbf{q_h}) \;=\; \sum_{i=1}^{N}\int_{\Omega_i}(\mathbf{A}\ \mathbf{curl}\ \mathbf{j_h}\cdot\mathbf{curl}\ \mathbf{q_h}+\mathbf{B}\ \mathbf{j_h}\cdot\mathbf{q_h})\,dx. \tag{3.13}$$

Then the mortar finite element method for the solution of (2.4) can be stated as follows: Find $\mathbf{j_h}\in\mathbf{V_h}$ such that

$$a_h(\mathbf{j_h},\mathbf{q_h}) \;=\; l(\mathbf{q_h}),\quad \mathbf{q_h}\in\mathbf{V_h}. \tag{3.14}$$

4. Error estimates. We first recall the well-known Strang lemma (cf., e.g., [17]).

LEMMA 4.1 (Strang's lemma). *Let* $\mathbf{j},\mathbf{j_h}$ *be the solutions of* (2.2) *and* (3.14), *respectively. Then there holds*

$$\begin{aligned}\|\mathbf{j}-\mathbf{j_h}\|_{a_h} \;\le\; &\left(\inf_{\mathbf{q_h}\in\mathbf{V_h}/\{0\}}\|\mathbf{j}-\mathbf{q_h}\|_{a_h}+\sup_{\mathbf{q_h}\in\mathbf{V_h}\setminus\{0\}}\frac{|a_h(\mathbf{j},\mathbf{q_h})-(f,\mathbf{q_h})|}{\|\mathbf{q_h}\|_{a_h}}\right)\\ &:= C(E_a+E_c),\end{aligned}$$

where $\|\cdot\|_{a_h}=a_h(\cdot,\cdot)^{\frac{1}{2}}$.

We are now in a position to estimate the two terms on the right side of the above inequality. As usual, we refer to the first one as the approximation error and to the second one as the consistency error.

4.1. Consistency error. For $\mathbf{curl}\ \mathbf{j}\in(H^1(\Omega_i))^3$, $\mathbf{q_h}\in ND_1(\Omega_i;\mathcal{T}_i)$, by Stokes' theorem we get

$$\begin{aligned}&\int_{\Omega_i}\mathbf{curl}\cdot\mathbf{Acurl}\ \mathbf{j}\cdot\mathbf{q_h}\,dx\\ &\quad-\int_{\Omega_i}\mathbf{Acurl}\ \mathbf{j}\cdot\mathbf{curl}\ \mathbf{q_h}\,dx \;=\; (\mathbf{n}\wedge(\mathbf{Acurl}\ \mathbf{j}\wedge\mathbf{n}),\mathbf{q_h}\wedge\mathbf{n})_{0,\partial\Omega_i},\end{aligned}$$

where $\mathbf{n}\wedge(\mathbf{Acurl}\ \mathbf{j}\wedge\mathbf{n})$ is the tangential components trace of $\mathbf{Acurl}\ \mathbf{j}$. Rearranging the right-hand term in the above equality, for any $\mathbf{q_h}\in\tilde{\mathbf{V}}_\mathbf{h}$, and $\mathbf{curl}\ \mathbf{j}\in(H^1(\Omega_i))^3$, $i=1,\ldots,N$, we have (cf. [7] for details)

$$\begin{aligned}&\sum_{i=1}^{N}\left(\int_{\Omega_i}\mathbf{curl}\cdot\mathbf{Acurl}\ \mathbf{j}\cdot\mathbf{q_h}\,dx\;-\;\int_{\Omega_i}\mathbf{Acurl}\ \mathbf{j}\cdot\mathbf{curl}\ \mathbf{q_h}\,dx\right)\\ &\qquad=\sum_{m=1}^{M}(\mathbf{n}\wedge(\mathbf{Acurl}\ \mathbf{j}\wedge\mathbf{n}),[\mathbf{q_h}\wedge\mathbf{n}])_{0,\gamma_m},\end{aligned} \tag{4.1}$$

where $[\cdot]$ denotes the jump across the interface γ_m, i.e.,

$$[\mathbf{q_h}\wedge\mathbf{n}]=\mathbf{q_h}\wedge\mathbf{n}|_{\delta_{m(j)}}-\mathbf{q_h}\wedge\mathbf{n}|_{\gamma_{m(i)}}.$$

On the basis of the above equality, we can easily show that

$$E_c \;=\; \sup_{\mathbf{q_h}\in\mathbf{V_h}\setminus\{0\}}\left|\sum_{m=1}^{M}\frac{(\mathbf{n}\wedge(\mathbf{Acurl}\ \mathbf{j}\wedge\mathbf{n}),[\mathbf{q_h}\wedge\mathbf{n}])_{0,\gamma_m}}{\|\mathbf{q_h}\|_{a_h}}\right|.$$

THEOREM 4.1. *Assume* $\mathbf{j} \in H^1(\mathbf{curl};\Omega)$. *Then the consistency error can be estimated as follows*:

$$E_c \leq C \left(\sum_{j=1}^{N} h_j^2 \; \|\mathbf{curl}\; \mathbf{j}\|_{1,\Omega_j}^2 \right)^{\frac{1}{2}}.$$

Proof. It follows from Lemma 3.1, (3.12), and the trace inequality that

$$\begin{aligned}
&|(\mathbf{n} \wedge (\mathbf{A}\mathbf{curl}\; \mathbf{j} \wedge \mathbf{n}), [\mathbf{q_h} \wedge \mathbf{n}])_{0,\gamma_m}| \\
&\quad = |(\mathbf{n} \wedge (\mathbf{A}\mathbf{curl}\; \mathbf{j} \wedge \mathbf{n}) - Q_h^{\delta_{m(j)}}(\mathbf{n} \wedge (\mathbf{A}\mathbf{curl}\; \mathbf{j} \wedge \mathbf{n})), [\mathbf{q_h} \wedge \mathbf{n}])_{0,\gamma_m}| \\
&\quad \leq \|\mathbf{n} \wedge (\mathbf{A}\mathbf{curl}\; \mathbf{j} \wedge \mathbf{n}) - Q_h^{\delta_{m(j)}}(\mathbf{n} \wedge (\mathbf{A}\mathbf{curl}\; \mathbf{j} \wedge \mathbf{n}))\|_{0,\gamma_m} \; \|[\mathbf{q_h} \wedge \mathbf{n}]\|_{0,\gamma_m} \\
&\quad \leq C h_{\delta_{m(j)}}^{\frac{1}{2}} \; |\mathbf{n} \wedge (\mathbf{A}\mathbf{curl}\; \mathbf{j} \wedge \mathbf{n})|_{\frac{1}{2},\delta_{m(j)}} \; \|[\mathbf{q_h} \wedge \mathbf{n}]\|_{0,\gamma_m} \\
&\quad \leq C h_j^{\frac{1}{2}} \; \|\mathbf{curl}\; \mathbf{j}\|_{1,\Omega_j} \; \|[\mathbf{q_h} \wedge \mathbf{n}]\|_{0,\gamma_m}.
\end{aligned}$$

On the other hand, for $\mathbf{q_h} \in \mathbf{V_h}$, Theorem 3.2 in [20] yields

$$\|[\mathbf{q_h} \wedge \mathbf{n}]\|_{0,\gamma_m} \;\leq\; C\; h_{\delta_{m(j)}}^{\frac{1}{2}} \; (\|\mathbf{curl}\; \mathbf{q_h}\|_{0,\Omega_i} + \|\mathbf{curl}\; \mathbf{q_h}\|_{0,\Omega_j}). \tag{4.2}$$

On the basis of the preceding inequalities, we get

$$\begin{aligned}
E_c &\leq \left[\sum_{j=1}^{N} C\; h_j \; \|\mathbf{curl}\; \mathbf{j}\|_{1,\Omega_j} (\|\mathbf{curl}\; \mathbf{q_h}\|_{0,\Omega_i} + \|\mathbf{curl}\; \mathbf{q_h}\|_{0,\Omega_j}) \right] / \|\mathbf{q_h}\|_{a_h} \\
&\leq C \; \left[\|\mathbf{curl}\; \mathbf{q_h}\|_{0,\Omega} \left(\sum_{j=1}^{N} h_j^2 \; \|\mathbf{curl}\; \mathbf{j}\|_{1,\Omega_j}^2 \right)^{\frac{1}{2}} \right] / \|\mathbf{q_h}\|_{a_h} \\
&\leq C \; \left(\sum_{j=1}^{N} h_j^2 \; \|\mathbf{curl}\; \mathbf{j}\|_{1,\Omega_j}^2 \right)^{\frac{1}{2}}. \qquad \square
\end{aligned}$$

4.2. Approximation error. We first introduce the extension operator $E_h^{\delta_{m(j)}}$: $\mathrm{RT}_{0,0}(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}}) \to \mathrm{ND}_1(\Omega_j; \mathcal{T}_j)$, defined according to

$$(E_h^{\delta_{m(j)}} \lambda_h^j) \wedge \mathbf{n} = \lambda_h^j \text{ on } \delta_{m(j)}, \quad \lambda_h^j \in \mathrm{RT}_{0,0}(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}}),$$

where all degrees of freedom that are not located on $\delta_{m(j)}$ are set equal to zero.

In order to estimate $E_h^{\delta_{m(j)}} \lambda_h^j$, $\lambda_h^j \in \mathrm{RT}_{0,0}(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}})$, we need some auxiliary results.

LEMMA 4.2. *For any* $\mathbf{q_h} \in \mathrm{ND}_1(\Omega_i; \mathcal{T}_i)$, *there holds*

$$c h_i^3 \sum_{T \in \mathcal{F}_h(\bar{\Omega}_i)} |(\mathbf{n_T} \cdot \mathbf{curl} \mathbf{q_h})|_T|^2 \leq \|\mathbf{curl} \mathbf{q_h}\|_{0,\Omega_i}^2 \leq C h_i^3 \sum_{T \in \mathcal{F}_h(\bar{\Omega}_i)} |(\mathbf{n_T} \cdot \mathbf{curl} \mathbf{q_h})|_T|^2,$$

and

$$c h_i^3 \sum_{e \in \mathcal{E}_h(\bar{\Omega}_i)} |(\mathbf{t_e} \cdot \mathbf{q_h})(x_e^M)|^2 \leq \|\mathbf{q_h}\|_{0,\Omega_i}^2 \leq C h_i^3 \sum_{e \in \mathcal{E}_h(\bar{\Omega}_i)} |(\mathbf{t_e} \cdot \mathbf{q_h})(x_e^M)|^2,$$

where $\mathbf{n_T}$ *denotes the exterior unit normal vector with respect to* $T \in \mathcal{F}_h(\bar{\Omega}_i)$*, and* x_e^M *is the midpoint of the edge e. Similarly, for any* $\delta_{m(j)} \subset S$*, and any* $\mathbf{q_h} \in \mathrm{RT}_0(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}})$*, we have*

$$ch^2_{\delta_{m(j)}} \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} |(\mathrm{div}_\tau \mathbf{q_h})|_T|^2 \le \|\mathrm{div}_\tau \mathbf{q_h}\|^2_{0,\delta_{m(j)}} \le Ch^2_{\delta_{m(j)}} \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} |(\mathrm{div}_\tau \mathbf{q_h})|_T|^2,$$

and

$$ch^2_{\delta_{m(j)}} \sum_{e \in \mathcal{E}_h(\bar{\delta}_{m(j)})} |(\mathbf{n_e} \cdot \mathbf{q_h})(x_e^M)|^2 \le \|\mathbf{q_h}\|^2_{0,\delta_{m(j)}} \le Ch^2_{\delta_{m(j)}} \sum_{e \in \mathcal{E}_h(\bar{\delta}_{m(j)})} |(\mathbf{n_e} \cdot \mathbf{q_h})(x_e^M)|^2.$$

Proof. We first prove the second inequality. In the reference tetrahedron $\hat{K}$, it is easy to see that

$$\|\hat{\mathbf{q}}_h\|_{0,\hat{K}} \quad \text{and} \quad \left(\sum_{e \in \mathcal{E}_h(\bar{K})} |(\mathbf{t}_e \cdot \hat{\mathbf{q}}_\mathbf{h})(x_e^M)|^2 \right)^{\frac{1}{2}}$$

are equivalent norms over the finite dimension space. By a scaling argument and summing up all $e \in \mathcal{E}_h(\bar{\Omega}_i)$, we can get the second inequality. Similarly, the fourth inequality can be verified. Moreover, the first and third inequalities are easy consequences of the following fact:

$$\mathbf{curl}\ \mathbf{q_h}|_K \in P_0(K)^3,\ K \in \mathcal{T}_i, \quad \text{and} \quad \mathrm{div}_\tau \mathbf{q_h}|_T \in P_0(T),\ T \in \mathcal{T}_{\delta_{m(j)}}. \qquad \square$$

On the basis of Lemma 4.2 we can derive the following lemma.

LEMMA 4.3. *For* $\lambda_h^j \in \mathrm{RT}_{0,0}(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}})$ *there holds*

$$\|E_h^{\delta_{m(j)}} \lambda_h^j\|_{\mathbf{curl},\Omega_j} \le C\ h^{\frac{1}{2}}_{\delta_{m(j)}}\ \|\lambda_h^j\|_{\mathrm{div}_\tau,\delta_{m(j)}},$$

where $\|\mathbf{v}\|_{\mathrm{div}_\tau,\delta_{m(j)}} := (\|\mathbf{v}\|^2_{0,\delta_{m(j)}} + \|\mathrm{div}_\tau \mathbf{v}\|^2_{0,\delta_{m(j)}})^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathrm{RT}_{0,0}(\delta_{m(j)}; \mathcal{T}_{\delta_{m(j)}})$.

Proof. It follows from the definition of the extension operator $E_h^{\delta_{m(j)}}$ and Lemma 4.2 that

$$\begin{aligned}
\|\mathbf{curl}(E_h^{\delta_{m(j)}} \lambda_h^j)\|^2_{0,\Omega_j} &\le Ch_j^3 \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} |\mathbf{n_T} \cdot \mathbf{curl}(E_h^{\delta_{m(j)}} \lambda_h^j)|_T|^2 \\
&= Ch_j^3 \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} |\mathrm{div}_\tau (E_h^{\delta_{m(j)}} \lambda_h^j \wedge \mathbf{n})|_T|^2 \\
&= Ch_j^3 \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} |\mathrm{div}_\tau (\lambda_h^j)|_T|^2 \\
&\le Ch_j\ \|\mathrm{div}_\tau (\lambda_h^j)\|^2_{0,\delta_{m(j)}}.
\end{aligned}$$

Using Lemma 4.2 again, we have

$$\begin{aligned}
\|E_h^{\delta_{m(j)}} \lambda_h^j\|^2_{0,\Omega_j} &\le Ch_j^3 \sum_{e \in \mathcal{E}_h(\bar{\Omega}_j)} |(\mathbf{t_e} \cdot \mathbf{E_h^{\delta_{m(j)}}} \boldsymbol{\lambda}_\mathbf{h}^\mathbf{j})(x_e^M)|^2 \\
&= Ch_j^3 \sum_{e \in \mathcal{E}_h(\bar{\Omega}_j)} |\mathbf{n_e} \cdot (\mathbf{E_h^{\delta_{m(j)}}} \boldsymbol{\lambda}_\mathbf{h}^\mathbf{j} \wedge \mathbf{n})(x_e^M)|^2
\end{aligned}$$

$$= Ch_j^3 \sum_{e\in\mathcal{E}_h(\delta_{m(j)})} |(\mathbf{n_e}\cdot\lambda_h^j)(x_e^M)|^2$$
$$\le h_j \|\lambda_h^j\|_{0,\delta_{m(j)}}^2.$$

Then, Lemma 4.3 follows from the above two inequalities. ◻

LEMMA 4.4. *Let* $\Pi_h^j : H^1(\mathbf{curl};\Omega_j) \to \mathrm{ND}_1(\Omega_j;\mathcal{T}_j)$ *be the standard interpolation operator associated with subdomain* Ω_j. *Then there holds*

(i) $\|\mathbf{n_T}\cdot(\mathbf{curl}\ \Pi_h^j\mathbf{j} - \mathbf{curl}\ \mathbf{j})\|_{0,T} \le Ch_K^{\frac{1}{2}}\|\mathbf{curl}\ \mathbf{j}\|_{1,K},\quad K\in\mathcal{T}_j,$

(ii) $\|\Pi_h^j\mathbf{j} - \mathbf{j}\|_{0,T} \le Ch_K^{\frac{1}{2}}\|\mathbf{j}\|_{1,\mathbf{curl},K},\quad T\in\partial K.$

Proof. We first prove (i). For $K\in\mathcal{T}_i$ and $T\in\partial K$ let $F_K(\hat{x}) = B_K\hat{x}+b_K,\ \hat{x}\in\hat{K}$, be the affine transformation mapping the reference element $\hat{K}$ onto K. Further, choose $\hat{T}\in\partial\hat{K}$ such that $T = F_K(\hat{T})$ and denote by $F_T = F_K|_{\hat{T}}$ the associated affine transformation $F_T(\hat{x}) = B_T\hat{x} + b_T$, $\hat{x}\in\hat{T}$, mapping $\hat{T}$ onto T. Setting $\hat{\mathbf{j}} = B_K^*\mathbf{j}$, it is easy to check that

$$\mathbf{n_T}\cdot(\mathbf{curl}\ \Pi_h^j\mathbf{j} - \mathbf{curl}\ \mathbf{j})|_T = \mathrm{curl}_\tau\Pi_h^j\mathbf{j}|_T - \mathrm{curl}_\tau\mathbf{j}|_T.$$

We note (cf. Lemma 3.57 of [24] for details) that

$$\mathrm{curl}_\tau\ \mathbf{j}|_T \ = \ (B_T^*)^{-1}\ \mathrm{curl}_\tau\ \hat{\mathbf{j}}|_{\hat{T}}\ B_T^{-1},$$

where $\mathrm{curl}_\tau\ \mathbf{u}$ denotes the 2×2 matrix with entries

$$[\mathrm{curl}_\tau\ \mathbf{u}]_{i,j} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i},\quad \mathbf{u} := (u_1,u_2).$$

It follows that

$$\begin{aligned}(4.3)\qquad &\|\mathbf{n_T}\cdot(\mathbf{curl}\ \Pi_h^j\mathbf{j} - \mathbf{curl}\ \mathbf{j})\|_{0,T}^2\\ &= \|\mathrm{curl}_\tau\Pi_h^j\mathbf{j}|_T - \mathrm{curl}_\tau\mathbf{j}|_T\|_{0,T}^2\\ &\le C\ |\det B_T|\|B_T^{-1}\|^4\ \|\mathbf{n_{\hat{T}}}\cdot\mathbf{curl}(\hat{\Pi}_h^j\hat{\mathbf{j}} - \hat{\mathbf{j}})\|_{0,\hat{T}}^2\\ &\le C\ |\det B_T|\|B_T^{-1}\|^4\ \|\mathbf{curl}(\hat{\Pi}_h^j\hat{\mathbf{j}} - \hat{\mathbf{j}})\|_{0,\hat{T}}^2\\ &\le C|\det B_T|\|B_T^{-1}\|^4\ \|(I - \hat{W}_h^j)\mathbf{curl}\ \hat{\mathbf{j}}\|_{0,\hat{T}}^2.\end{aligned}$$

Here, we have used $\mathbf{curl}\ \hat{\Pi}_h^j\hat{\mathbf{j}} = \hat{W}_h^j\mathbf{curl}\ \hat{\mathbf{j}}$ with $\hat{W}_h^j$ being the L^2-projection onto the space of elementwise constants. It follows that

$$(4.4)\qquad \|(I - \hat{W}_h^j)\mathbf{curl}\ \hat{\mathbf{j}}\|_{0,\hat{T}}^2 \ \le\ C\ |\mathbf{curl}\ \hat{\mathbf{j}}|_{1,\hat{K}}^2.$$

We note that

$$\mathbf{curl}\ \hat{\mathbf{j}} \ = \ B_K^*\ \mathbf{curl}\ \mathbf{j}\ B_K,$$

where $\mathbf{curl}\ \mathbf{j}$ stands for the 3×3 matrix with entries

$$[\mathbf{curl}\ \mathbf{j}]_{i,j} = \frac{\partial j_i}{\partial x_j} - \frac{\partial j_j}{\partial x_i},\quad \mathbf{j} := (j_1,j_2,j_3).$$

Hence, by backtransformation we obtain (cf. Lemma 5.5 in [1] for details)

$$(4.5)\qquad |\mathbf{curl}\ \hat{\mathbf{j}}|_{1,\hat{K}}^2 \ \le\ C\ |\det B_K|^{-2}\ \|B_K\|^7\|B_K^*\|^2\ |\mathbf{curl}\ \mathbf{j}|_{1,K}^2.$$

Summarizing (4.3), (4.4), and (4.5), it follows that

$$
\begin{aligned}
(4.6)\quad & \|\mathbf{n_T}\cdot(\mathbf{curl}\,\Pi_h^j\,\mathbf{j}-\mathbf{curl}\,\mathbf{j})\|_{0,T}^2 \\
& \le\; C\frac{|\det\,B_T|}{|\det\,B_K|}\,(\|B_T^{-1}\|\;\|B_K\|)^4\;\|B_K\|^3\|B_K^*\|^2|\det\,B_K|^{-1}\;|\mathbf{curl}\,\mathbf{j}|_{1,K}^2.
\end{aligned}
$$

Finally, taking into account that $\mathcal{T}_i$ is a regular triangulation, we have

$$
\|B_T^{-1}\|\;\|B_K\|\;\le\;C,\quad \|B_K\|,\|B_K^*\|\;\le\;C\;h_K. \tag{4.7}
$$

Moreover,

$$
|\det\,B_T|\;=\;\frac{\mathrm{meas}(T)}{\mathrm{meas}(\hat{T})},\quad |\det\,B_K|\;=\;\frac{\mathrm{meas}(K)}{\mathrm{meas}(\hat{K})}. \tag{4.8}
$$

Using (4.7) and (4.8) in (4.6) gives the assertion.

We now prove (ii). Observing

$$
\mathbf{j}|_T=(B_T^*)^{-1}\hat{\mathbf{j}}|_{\hat{T}},
$$

we have

$$
\|\Pi_h^j\mathbf{j}-\mathbf{j}\|_{0,T}^2\le|\det B_T|\|(B_T^*)^{-1}\|^2\|\hat{\Pi}_h^j\hat{\mathbf{j}}-\hat{\mathbf{j}}\|_{0,\hat{T}}^2.
$$

Using the trace inequality and similar arguments as in the proof of Theorem 5.41 of [24], we can derive that

$$
\|\hat{\Pi}_h^j\hat{\mathbf{j}}-\hat{\mathbf{j}}\|_{0,\hat{T}}^2\le C(|\hat{\mathbf{j}}|_{1,\hat{K}}+|\mathbf{curl}\;\hat{\mathbf{j}}|_{1,\hat{K}}).
$$

On the other hand,

$$
|\hat{\mathbf{j}}|_{1,\hat{K}}^2\le\|B_K\|^5\|B_K^*\|^2|\det B_K^{-1}|^2|\mathbf{j}|_{1,K}^2.
$$

Combining the above three inequalities with (4.5), (4.7), and (4.8) yields Lemma 4.4(ii). ∎

We further introduce a special projection operator $\pi_h^{\delta_{m(j)}}$ which will play an important role in analyzing the approximate error of the mortar edge element method. We define $\pi_h^{\delta_{m(j)}}: L^2(\gamma_m)^2\to\mathrm{RT}_{0,0}(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$ according to

$$
\int_{\delta_{m(j)}}\pi_h^{\delta_{m(j)}}(\mathbf{p})\cdot\mathbf{q_h}\;dx\;=\;\int_{\delta_{m(j)}}\mathbf{p}\cdot\mathbf{q_h}\;dx,\quad \mathbf{q_h}\in\mathbf{M_h}(\delta_{m(j)}). \tag{4.9}
$$

The boundedness of $\pi_h^{\delta_{m(j)}}$ is a direct consequence of the following result.

LEMMA 4.5. *The following* inf-sup *condition holds true:*

$$
\inf_{\mathbf{q_h}\in RT_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})}\;\sup_{\mu_h\in\mathbf{M_h}(\delta_{m(j)})}\;\frac{(\mathbf{q_h},\mu_h)_{0,\delta_{m(j)}}}{\|\mathbf{q_h}\|_{0,\delta_{m(j)}}\;\|\mu_h\|_{0,\delta_{m(j)}}}\;\ge\;C\;>\;0.
$$

Proof. Taking the construction (3.8) on the basis of $\mathbf{M_h}(\delta_{m(j)})$ into account, for $\mathbf{q_h}\in RT_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$ we determine $\mu_h\in\mathbf{M_h}(\delta_{m(j)})$ by specifying its degrees of freedom according to

$$
\ell_e(\mu_h)=\begin{cases}\ell_e(\mathbf{q_h}), & e\in\mathcal{E}_h(\delta_{m(j)})\setminus\mathcal{E}_h^{\delta_{m(j)}}(\partial\delta_{m(j)}),\\ \ell_e(\mathbf{q_h})+\sum_{f\in\mathcal{E}_h^{\delta_{m(j)}}(e)}\lambda_{e,f}\;\ell_f(\mathbf{q_h}), & e\in\mathcal{E}_h^{\delta_{m(j)}}(\partial\delta_{m(j)}).\end{cases}
$$

The assertion can then be verified by following lines of proof analogous to those of [20, Lemma 3.2]. ∎

Furthermore, by Lemma 3.2 in [20], we know that the following inf-sup condition also true

COROLLARY 4.6. *There holds*

$$\inf_{\mu_h \in \mathbf{M_h}(\delta_{m(j)})} \sup_{\mathbf{q_h} \in RT_{0,0}(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})} \frac{(\mathbf{q_h}, \mu_h)_{0,\delta_{m(j)}}}{\|\mathbf{q_h}\|_{0,\delta_{m(j)}} \ \|\mu_h\|_{0,\delta_{m(j)}}} \ \geq \ C \ > \ 0.$$

On the basis of Lemma 4.5, we have the following.

COROLLARY 4.7. *Let $\pi_h^{\delta_{m(j)}}$ be given by* (4.9). *Then there holds*

$$\|\pi_h^{\delta_{m(j)}}(\mathbf{p})\|_{0,\delta_{m(j)}} \ \leq \ C \ \|\mathbf{p}\|_{0,\gamma_m}, \quad \mathbf{p} \in L^2(\gamma_m)^2.$$

Proof. Using Lemma 4.5, straightforward computation reveals

$$\begin{aligned}
\|\pi_h^{\delta_{m(j)}}(\mathbf{p})\|_{0,\delta_{m(j)}} &\leq C \sup_{\mu_h \in \mathbf{M_h}(\delta_{m(j)})} \frac{(\pi_h^{\delta_{m(j)}}(\mathbf{p}), \mu_h)_{0,\delta_{m(j)}}}{\|\mu_h\|_{0,\delta_{m(j)}}} \\
&= C \sup_{\mu_h \in \mathbf{M_h}(\delta_{m(j)})} \frac{(\mathbf{p}, \mu_h)_{0,\delta_{m(j)}}}{\|\mu_h\|_{0,\delta_{m(j)}}} \\
&\leq C \ \|\mathbf{p}\|_{0,\gamma_m}. \qquad \square
\end{aligned}$$

As a further consequence of the inf-sup condition in Lemma 4.5, we obtain the following.

LEMMA 4.8. *Let $\Pi_h : H^1(\mathbf{curl};\Omega) \cap \mathbf{V} \rightarrow \tilde{\mathbf{V}}_\mathbf{h}$ be the standard interpolation operator. Then we have*

$$\|\mathrm{div}_\tau \pi_h^{\delta_{m(j)}}[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,\gamma_m} \ \leq \ C \ \|\mathrm{div}_\tau[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,\gamma_m}.$$

Proof. We denote by $P_h^{\delta_{m(j)}}$ the $\mathrm{RT}_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$-interpolation operator. Observing that $P_h^{\delta_{m(j)}}|_T$, $T \in \mathcal{T}_{\delta_{m(j)}}$, preserves constant tangential traces, by a Bramble–Hilbert argument we obtain

$$\begin{aligned}
&\|(I - P_h^{\delta_{m(j)}})[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,\gamma_m}^2 \\
&\quad \leq C h_{\delta_{m(j)}}^2 \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} \ \sum_{T' \cap T \neq \emptyset, T' \in \mathcal{T}_{\gamma_{m(i)}}} |[\Pi_h \mathbf{j} \wedge \mathbf{n}]|_{1,T' \cap T}^2 \\
&\quad = C h_{\delta_{m(j)}}^2 \sum_{T \in \mathcal{T}_{\delta_{m(j)}}} \|\mathrm{div}_\tau[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,T}^2 \\
&\quad = C \ h_{\delta_{m(j)}}^2 \ \|\mathrm{div}_\tau[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,\gamma_m}^2,
\end{aligned}$$

where we have used the fact that $\Pi_h \mathbf{j} \wedge \mathbf{n}|_{\gamma_m}$ belongs to the lowest order Raviart–Thomas space. Similar arguments for the proof of the first inequality can be found in [16] . So we get

$$\|(I - P_h^{\delta_{m(j)}})[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,\gamma_m} \leq C \ h_{\delta_{m(j)}} \ \|\mathrm{div}_\tau[\Pi_h \mathbf{j} \wedge \mathbf{n}]\|_{0,\gamma_m}. \tag{4.10}$$

Moreover, in view of

$$\mathrm{div}_\tau P_h^{\delta_{m(j)}}[\Pi_h \mathbf{j} \wedge \mathbf{n}] \ = \ W_h^{\delta_{m(j)}} \ \mathrm{div}_\tau[\Pi_h \mathbf{j} \wedge \mathbf{n}],$$

where $W_h^{\delta_{m(j)}}$ is the L^2-projection onto the elementwise constants, we obtain

$$\|\mathrm{div}_\tau P_h^{\delta_{m(j)}}[\Pi_h \mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m} \;\le\; C\,\|\mathrm{div}_\tau[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}. \tag{4.11}$$

We have $(\pi_h^{\delta_{m(j)}} - P_h^{\delta_{m(j)}})[\Pi_h\mathbf{j}\wedge\mathbf{n}] \in \mathrm{RT}_0(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$, and hence, by Lemma 4.5 and (4.10),

$$\begin{aligned} &\|(\pi_h^{\delta_{m(j)}} - P_h^{\delta_{m(j)}})[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}\\ &\quad\le\; C \sup_{\psi\in\mathbf{M_h}(\delta_{m(j)})} \frac{((\pi_h^{\delta_{m(j)}} - P_h^{\delta_{m(j)}})[\Pi_h\mathbf{j}\wedge\mathbf{n}],\psi)}{\|\psi\|_{0,\delta_{m(j)}}}\\ &\quad=\; C \sup_{\psi\in\mathbf{M_h}(\delta_{m(j)})} \frac{((I - P_h^{\delta_{m(j)}})[\Pi_h\mathbf{j}\wedge\mathbf{n}],\psi)}{\|\psi\|_{0,\delta_{m(j)}}}\\ &\quad\le\; C\, h_{\delta_{m(j)}}\,\|\mathrm{div}_\tau[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}. \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we get

$$\begin{aligned} &\|\mathrm{div}_\tau \pi_h^{\delta_{m(j)}}[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}\\ &\quad\le\; \|\mathrm{div}_\tau(\pi_h^{\delta_{m(j)}} - P_h^{\delta_{m(j)}})[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m} + \|\mathrm{div}_\tau P_h^{\delta_{m(j)}}[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}\\ &\quad\le C\, h_{\delta_{m(j)}}^{-1}\,\|(\pi_h^{\delta_{m(j)}} - P_h^{\delta_{m(j)}})[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m} + \|\mathrm{div}_\tau[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}\\ &\quad\le\; C\,\|\mathrm{div}_\tau[\Pi_h\mathbf{j}\wedge\mathbf{n}]\|_{0,\gamma_m}. \qquad \square \end{aligned}$$

We are now in a position to estimate the discretization error of the mortar edge element method.

THEOREM 4.2. *For any* $\mathbf{j}\in H^1(\mathbf{curl};\Omega)$ *there exists a function* $\mathbf{q_h}\in\mathbf{V_h}$ *such that*

$$\|\mathbf{j}-\mathbf{q_h}\|_{a_h} \le C\left(\sum_{j=1}^N h_j^2\,\|\mathbf{j}\|^2_{1,\mathbf{curl},\Omega_j}\right)^{\frac12}.$$

Proof. We define $\mathbf{q_h}$ as

$$\mathbf{q_h} = \Pi_h\mathbf{j} - \sum_{m=1}^M E_h^{\delta_{m(j)}}\{\pi_h^{\delta_{m(j)}}[(\Pi_h^j\mathbf{j}\wedge\mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i\mathbf{j}\wedge\mathbf{n})|_{\gamma_{m(i)}}]\}$$

and remark that $\mathbf{q_h}\in\mathbf{V_h}$ can be easily seen.

For each $\delta_{m(j)}$, by Lemma 4.3, Corollary 4.7, and Lemma 4.8, we get

$$\begin{aligned} &\|E_h^{\delta_{m(j)}}(\pi_h^{\delta_{m(j)}}((\Pi_h^j\mathbf{j}\wedge\mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i\mathbf{j}\wedge\mathbf{n})|_{\gamma_{m(i)}}))\|_{\mathbf{curl},\Omega_j}\\ &\quad\le C\, h_{\delta_{m(j)}}^{\frac12}\,\|\mathrm{div}_\tau\big(\pi_h^{\delta_{m(j)}}((\Pi_h^j\mathbf{j}\wedge\mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i\mathbf{j}\wedge\mathbf{n})|_{\gamma_{m(i)}})\big)\|_{0,\gamma_m}\\ &\qquad+ C\, h_{\delta_{m(j)}}^{\frac12}\,\|\pi_h^{\delta_{m(j)}}\big((\Pi_h^j\mathbf{j}\wedge\mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i\mathbf{j}\wedge\mathbf{n})|_{\gamma_{m(i)}}\big)\|_{0,\gamma_m}\\ &\quad\le\; C\, h_{\delta_{m(j)}}^{\frac12}\,\|\mathrm{div}_\tau\big((\Pi_h^j\mathbf{j}\wedge\mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i\mathbf{j}\wedge\mathbf{n})|_{\gamma_{m(i)}}\big)\|_{0,\gamma_m}\\ &\qquad+ C\, h_{\delta_{m(j)}}^{\frac12}\,\|(\Pi_h^j\mathbf{j}\wedge\mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i\mathbf{j}\wedge\mathbf{n})|_{\gamma_{m(i)}}\|_{0,\gamma_m}\\ &\quad:=\; I_1 \;+\; I_2. \end{aligned} \tag{4.13}$$

As far as the first term I_1 is concerned, applying Lemma 4.4 results in

$$
\begin{aligned}
I_1 \le C\ h_j^{\frac{1}{2}}\ &(\|\mathrm{div}_\tau\big((\Pi_h^j \mathbf{j} \wedge \mathbf{n})|_{\delta_{m(j)}} - (\mathbf{j} \wedge \mathbf{n})|_{\delta_{m(j)}}\big)\|_{0,\gamma_m} \\
&+\|\mathrm{div}_\tau\big((\Pi_h^i \mathbf{j} \wedge \mathbf{n})|_{\gamma_{m(i)}} - (\mathbf{j} \wedge \mathbf{n})|_{\gamma_{m(i)}}\big)\|_{0,\gamma_m}) \\
\le C\ h_j^{\frac{1}{2}}\ &\left(\sum_{T\in\mathcal{T}(\delta_{m(j)})} (\|\mathbf{n}_T\cdot(\mathbf{curl}\ \Pi_h^j\ \mathbf{j} - \mathbf{curl}\ \mathbf{j})|_T\|_{0,T}^2\right)^{\frac{1}{2}} \\
&+\left(\sum_{T\in\mathcal{T}(\gamma_{m(i)})} (\|\mathbf{n}_T\cdot(\mathbf{curl}\ \Pi_h^i\ \mathbf{j} - \mathbf{curl}\ \mathbf{j})|_T\|_{0,T}^2)^{\frac{1}{2}}\right) \\
\le C\ h_j^{\frac{1}{2}}\ &\big(h_j^{\frac{1}{2}}\|\mathbf{curl}\ \mathbf{j}\|_{1,\Omega_j} + h_i^{\frac{1}{2}}\|\mathbf{curl}\ \mathbf{j}\|_{1,\Omega_i}\big).
\end{aligned}
\tag{4.14}
$$

For the second term I_2, using Lemma 4.4, we obtain

$$
\begin{aligned}
I_2 \le C\ h_{\delta_{m(j)}}^{\frac{1}{2}}\ &\big(\|(\Pi_h^j \mathbf{j} \wedge \mathbf{n})|_{\delta_{m(j)}} - (\mathbf{j} \wedge \mathbf{n})|_{\delta_{m(j)}}\|_{0,\gamma_m}\big) \\
&+\|(\Pi_h^i \mathbf{j} \wedge \mathbf{n})|_{\gamma_{m(i)}} - (\mathbf{j} \wedge \mathbf{n})|_{\gamma_{m(i)}}\|_{0,\gamma_m} \\
\le C\ h_j^{\frac{1}{2}}\ &\big(h_j^{\frac{1}{2}}\|\mathbf{j}\|_{1,\mathbf{curl},\Omega_j} + h_i^{\frac{1}{2}}\|\mathbf{j}\|_{1,\mathbf{curl},\Omega_i}\big).
\end{aligned}
\tag{4.15}
$$

Observing the standard approximation property

$$
\|\mathbf{j} - \Pi_h \mathbf{j}\|_{a_h} \le C \left(\sum_{j=1}^N h_j^2\ \|\mathbf{j}\|_{1,\mathbf{curl},\Omega_j}^2\right)^{\frac{1}{2}}
$$

and using (4.13), (4.14), and (4.15) results in

$$
\begin{aligned}
\|\mathbf{j} - \mathbf{q_h}\|_{a_h}^2 \ &\le\ C\ \big(\|\mathbf{j} - \Pi_h \mathbf{j}\|_{a_h}^2 \\
&\quad + \sum_{m=1}^m \|E_h^{\delta_{m(j)}}(\pi_h^{\delta_{m(j)}}((\Pi_h^j \mathbf{j} \wedge \mathbf{n})|_{\delta_{m(j)}} - (\Pi_h^i \mathbf{j} \wedge \mathbf{n}))\|_{\mathbf{curl},\Omega_j}^2\big) \\
&\le\ C\ \sum_{j=1}^N h_j^2\ \|\mathbf{j}\|_{1,\mathbf{curl},\Omega_j}^2. \qquad \square
\end{aligned}
$$

Finally, Theorems 4.1 and 4.2 imply the main result of this paper.

THEOREM 4.3. *Let* $\mathbf{j} \in H^1(\mathbf{curl};\Omega)$ *and* $\mathbf{j_h} \in \mathbf{V_h}$ *be the solutions of* (2.2) *and* (3.14), *respectively. Then there holds*

$$
\|\mathbf{j} - \mathbf{j_h}\|_{a_h}\ \le\ C\ \left(\sum_{j=1}^N h_j^2\ \|\mathbf{j}\|_{1,\mathbf{curl},\Omega_j}^2\right)^{\frac{1}{2}}.
$$

5. Saddle point formulation. A saddle point formulation for mortar element methods associated with second order elliptic problems has been introduced in [4]. In particular, an a priori estimate for the Lagrange multiplier in the $(H_{00}^{\frac{1}{2}})'$-norm has been established there, whereas related estimates in mesh-dependent norms have been given in [28], [29], [30]. In this section, we will derive an a priori estimate for the Lagrange multiplier of the mortar edge element method.

First, we introduce a macrohybrid variational formulation for the continuous problem (2.1).

Using the domain decomposition as presented in the preceding section, we introduce the product space

$$\mathbf{X} \ := \ \{\mathbf{q} \in L^2(\Omega)^3 \mid \mathbf{q}|_{\Omega_i} \in H(\mathbf{curl};\Omega_i),\ (\mathbf{n}\wedge(\mathbf{q}\wedge\mathbf{n}))|_{\partial\Omega_i\cap\partial\Omega} = \mathbf{0}\}$$

equipped with the norm

$$\|\mathbf{q}\|_{\mathbf{X}} \ := \ \left(\sum_{i=1}^N \|\mathbf{q}\|^2_{\mathbf{curl},\Omega_i}\right)^{\frac{1}{2}}.$$

We further consider the subspace

$$\tilde{\mathbf{V}} \ := \ \left\{\mathbf{q}\in\mathbf{X} \mid [\mathbf{q}\wedge\mathbf{n}]|_{\gamma_m} \in (H_{00}^{\frac{1}{2}}(\gamma_m))^2\right\}$$

provided with the norm

$$\|\mathbf{q}\|_{\tilde{\mathbf{V}}} \ := \ \left(\|\mathbf{q}\|^2_{\mathbf{X}} + \|[\mathbf{q}\wedge\mathbf{n}]\|^2_{\frac{1}{2},S}\right)^{\frac{1}{2}},$$

where

$$\|[\mathbf{q}\wedge\mathbf{n}]\|_{\frac{1}{2},S} \ := \ \left(\sum_{\gamma_m\in S} \|[\mathbf{q}\wedge\mathbf{n}]\|^2_{(H_{00}^{\frac{1}{2}}(\gamma_m))^2}\right)^{\frac{1}{2}}.$$

A natural candidate for the multiplier space is then

$$\mathbf{M} \ := \ \prod_{\gamma_m} (H^{-\frac{1}{2}}(\delta_{m(j)}))^2$$

equipped with the norm

$$\|\mu\|_{\mathbf{M}} \ := \ \left(\sum_{\delta_{m(j)}\in S} \|\mu|_{\delta_{m(j)}}\|^2_{-\frac{1}{2},\delta_{m(j)}}\right)^{\frac{1}{2}},$$

where $H^{-\frac{1}{2}}(\delta_{m(j)}) := (H_{00}^{\frac{1}{2}}(\delta_{m(j)}))'$.

We introduce the bilinear form $a(\cdot,\cdot)\mathbf{X}\times\mathbf{X}\to\mathbb{R}$ as the sum of the bilinear forms associated with the subdomain problems according to

$$a(\mathbf{j},\mathbf{q}) \ := \ \sum_{i=1}^N a_{\Omega_i}(\mathbf{j}|_{\Omega_i},\mathbf{q}|_{\Omega_i}) = \sum_{i=1}^N \int_{\Omega_i} \big[\mathbf{A}\mathbf{curl}\ \mathbf{j}\cdot\mathbf{curl}\ \mathbf{q} + \mathbf{B}\mathbf{j}\cdot\mathbf{q}\big]dx.$$

Furthermore, we define the bilinear form $b(\cdot,\cdot):\tilde{\mathbf{V}}\times\mathbf{M}\to\mathbf{R}$ by means of

$$b(\mathbf{q},\mu) := \langle[\mathbf{q}\wedge\mathbf{n}],\mu\rangle_{\frac{1}{2},S},$$

where $\langle\cdot,\cdot\rangle_{\frac{1}{2},S} := \sum_{\delta_{m(j)}\in S}\langle\cdot,\cdot\rangle_{\frac{1}{2},\delta_{m(j)}}$.

Then the appropriate macrohybrid variational formulation of (2.1) can be formulated as follows:

Find $(\mathbf{j},\lambda)\in\tilde{\mathbf{V}}\times\mathbf{M}$ such that

$$\begin{aligned} a(\mathbf{j},\mathbf{q}) \ + \ b(\mathbf{q},\lambda) &= l(\mathbf{q}), \quad \mathbf{q}\in\tilde{\mathbf{V}}, \\ b(\mathbf{j},\mu) &= 0, \quad \mu\in\mathbf{M}. \end{aligned} \tag{5.1}$$

Denote by $B : \tilde{\mathbf{V}} \to \mathbf{M}$ the operator associated with the bilinear form $b(\cdot,\cdot)$, i.e.,

$$\langle B\mathbf{q}, \mu\rangle_{\frac{1}{2},S} = b(\mathbf{q},\mu), \quad \mu \in \mathbf{M}.$$

It is proved in Theorem 2.1 of [20] that the bilinear form $a(\cdot,\cdot)$ is Ker$B$-elliptic and the bilinear form $b(\cdot,\cdot)$ satisfies the LBB condition. So the saddle point problem (5.1) admits a unique solution. For $\mathbf{q} \in \mathbf{V} \subset \tilde{\mathbf{V}}$, the first equation of (5.1) reduces to (2.2). Hence, the solution $\mathbf{j}$ of (5.1) is also the solution of (2.2). Finally, by (4.1) we know that $\lambda|_{\gamma_m} = \mathbf{n} \wedge (\mathbf{A}\,\mathbf{curl}\,\mathbf{j} \wedge \mathbf{n})|_{\gamma_m}$.

Next, we consider the discrete version of (5.1). On $\tilde{\mathbf{V}}_\mathbf{h}$, we define the norm

$$\|\mathbf{q_h}\|_{\tilde{\mathbf{V}}_\mathbf{h}} := \left(\|\mathbf{q_h}\|^2_{\mathbf{X}} + \|[\mathbf{q_h} \wedge \mathbf{n}]|_S\|^2_{\frac{1}{2},h,S}\right)^{\frac{1}{2}}, \quad \mathbf{q_h} \in \tilde{\mathbf{V}}_\mathbf{h},$$

where $\|\cdot\|_{\frac{1}{2},h,S}$ is given by

$$\|[\mathbf{q_h} \wedge \mathbf{n}]|_S\|_{\frac{1}{2},h,S} := \left(\sum_{\gamma_m \subset S} \|[\mathbf{q_h} \wedge \mathbf{n}]\|^2_{\frac{1}{2},h,\gamma_m}\right)^{\frac{1}{2}}$$

and $\|\cdot\|_{\frac{1}{2},h,\gamma_m}$ stands for the mesh-dependent norm:

$$\|[\mathbf{q_h} \wedge \mathbf{n}]\|_{\frac{1}{2},h,\gamma_m} := h^{-\frac{1}{2}}_{\delta_{m(j)}} \|[\mathbf{q_h} \wedge \mathbf{n}]\|_{0,\gamma_m}.$$

The Lagrange multiplier space $\mathbf{M_h}$ will be provided with the following mesh-dependent norm:

$$\|\mu_h\|_{\mathbf{M_h}} := \|\mu_h\|_{-\frac{1}{2},h,S}, \quad \mu_h \in \mathbf{M_h},$$

where

$$\|\mu_h\|_{-\frac{1}{2},h,S} := \left(\sum_{\delta_{m(j)} \subset S} \|\mu_h\|^2_{-\frac{1}{2},h,\delta_{m(j)}}\right)^{\frac{1}{2}}$$

and $\|\cdot\|_{-\frac{1}{2},h,\delta_{m(j)}}$ is given by

$$\|\mu_h|_{\delta_{m(j)}}\|_{-\frac{1}{2},h,\delta_{m(j)}} := h^{\frac{1}{2}}_{\delta_{m(j)}} \|\mu_h\|_{0,\delta_{m(j)}}.$$

In addition to the bilinear form $a_h(\cdot,\cdot) : \tilde{\mathbf{V}}_\mathbf{h} \times \tilde{\mathbf{V}}_\mathbf{h} \to \mathbb{R}$ as defined by (3.13), we introduce the bilinear form $b_h(\cdot,\cdot) : \tilde{\mathbf{V}}_\mathbf{h} \times \mathbf{M_h} \to \mathbb{R}$ according to

$$b_h(\mathbf{q_h}, \mu_h) := \sum_{\gamma_m \in S} ([\mathbf{q_h} \wedge \mathbf{n}]|_{\gamma_m}, \mu_h)_{0,\delta_{m(j)}}.$$

Then the mortar edge element approximation of (5.1) amounts to the solution of the following problem: Find $(\mathbf{j_h}, \lambda_h) \in \tilde{\mathbf{V}}_\mathbf{h} \times \mathbf{M_h}$ such that

$$\begin{aligned} a_h(\mathbf{j_h}, \mathbf{q_h}) + b_h(\mathbf{q_h}, \lambda_h) &= l(\mathbf{q_h}), \quad \mathbf{q_h} \in \tilde{\mathbf{V}}_\mathbf{h}, \\ b_h(\mathbf{j_h}, \mu_h) &= 0, \quad \mu_h \in \mathbf{M_h}. \end{aligned} \tag{5.2}$$

The saddle point problem (5.2) admits a unique solution which follows from the following LBB condition for the bilinear form $b_h(\cdot,\cdot)$.

LEMMA 5.1. *The bilinear form* $b_h(\cdot,\cdot) : \tilde{\mathbf{V}}_\mathbf{h} \times \mathbf{M}_\mathbf{h} \to \mathbf{R}$ *satisfies a discrete* inf-sup *condition (LBB condition) uniformly in* h_i*, i.e., there exists a constant* $c > 0$ *independent of the mesh size* h_i *such that*

$$\sup_{\mathbf{q_h}\in\tilde{\mathbf{V}}_\mathbf{h}} \frac{b_h(\mathbf{q_h},\mu_h)}{\|\mathbf{q_h}\|_{\tilde{\mathbf{V}}_\mathbf{h}}} \;\geq\; c\; \|\mu_h\|_{\mathbf{M_h}}.$$

Proof. For any $\mu_h \in \mathbf{M_h}(\delta_{m(j)})$ we define $\mathbf{p_h^j} \in \mathrm{RT}_{0,0}(\delta_{m(j)};\mathcal{T}_{\delta_{m(j)}})$ according to

$$\ell_e(\mathbf{p_h^j}) \;=\; \ell_e(\mu_h), \quad e\in\mathcal{E}_h(\delta_{m(j)})$$

and refer to $\mathbf{q_h^j} \in \mathrm{ND}_1(\Omega_j;\mathcal{T}_j)$ as the trivial extension, i.e.,

$$\mathbf{q_h^j}\wedge\mathbf{n} \;=\; \mathbf{p_h^j} \;\text{ on }\; \delta_{m(j)},$$

where all degrees of freedom that are not located on $\delta_{m(j)}$ are set equal to zero, especially $[\mathbf{q_h^j}\wedge\mathbf{n}] = \mathbf{p_h^j}$. On the basis of Lemma 4.3, we have

$$\begin{aligned}\|\mathbf{q_h^j}\|_{\mathbf{curl},\Omega_j} &\leq C\; h_j^{\frac12}\; \|\mathbf{p_h^j}\|_{\mathrm{div}_\tau,\delta_{m(j)}}\\ &\leq C\; h_j^{-\frac12}\; \|\mathbf{p_h^j}\|_{0,\delta_{m(j)}}\\ &= C\; h_j^{-\frac12}\; \|[\mathbf{q_h^j}\wedge\mathbf{n}]\|_{0,\delta_{m(j)}}.\end{aligned}$$

By Corollary 4.6 and the above inequality, we obtain

$$\begin{aligned}(\mu_h,[\mathbf{q_h^j}\wedge\mathbf{n}]|_{\delta_{m(j)}})_{0,\delta_{m(j)}} &\geq C\; \|\mu_h\|_{0,\delta_{m(j)}}\|[\mathbf{q_h^j}\wedge\mathbf{n}]\|_{0,\delta_{m(j)}}\\ &\geq C\; h_j^{\frac12}\; \|\mu_h\|_{0,\delta_{m(j)}}\|\mathbf{q_h^j}\|_{\mathbf{curl},\Omega_j}\\ &\geq C\; \|\mu_h\|_{-\frac12,h,\delta_{m(j)}}\; \|\mathbf{q_h^j}\|_{\mathbf{curl},\Omega_j}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mu_h,[\mathbf{q_h^j}\wedge\mathbf{n}]|_{\delta_{m(j)}})_{0,\delta_{m(j)}} &\geq C\; \|\mu_h\|_{0,\delta_{m(j)}}\; \|[\mathbf{n}\wedge\mathbf{q_h^j}]\|_{0,\delta_{m(j)}}\\ &= C\; h_j^{\frac12}\; \|\mu_h\|_{0,\delta_{m(j)}}\; h_j^{-\frac12}\; \|[\mathbf{q_h^j}\wedge\mathbf{n}]\|_{0,\delta_{m(j)}}\\ &= C\; \|\mu_h\|_{-\frac12,h,\delta_{m(j)}}\; \|[\mathbf{q_h^j}\wedge\mathbf{n}]\|_{\frac12,h,\delta_{m(j)}}.\end{aligned}$$

Adding the above inequalities and summing over all $\delta_{m(j)} \subset \Gamma$ gives the assertion. ∎

Finally, we obtain the following.

THEOREM 5.2. *Let* $\mathbf{j}\in H^1(\mathbf{curl};\Omega)$ *and* $(\mathbf{j_h},\lambda_h)\in\tilde{\mathbf{V}}_\mathbf{h}\times\mathbf{M_h}$ *be the solutions of* (2.2) *and* (5.2), *respectively. Then there holds*

$$\|\lambda-\lambda_h\|_{-\frac12,h,S} \leq C\left(\sum_{j=1}^N h_j^2\; \|\mathbf{j}\|^2_{1,\mathbf{curl},\Omega_j}\right)^{\frac12}.$$

Proof. On the basis of the inf-sup condition developed in Lemma 5.1 and arguments similar to those in [12] for the mixed finite element methods and [30] for the saddle point method for mortar element methods, we get

$$\|\lambda-\lambda_h\|_{-\frac12,h,S} \leq C(\|\mathbf{j}-\mathbf{j_h}\|_{a_h} + \inf_{\mu_h\in\mathbf{M_h}} \|\lambda-\mu_h\|_{-\frac12,h,S}).$$

By Theorem 4.3, we have

$$\|\mathbf{j}-\mathbf{j_h}\|_{a_h} \le C \left(\sum_j^N h_j^2 \|\mathbf{j}\|^2_{1,\mathbf{curl},\Omega_j} \right)^{\frac{1}{2}}. \tag{5.3}$$

Moreover, by Lemma 3.1

$$\begin{aligned}\inf_{\mu_h \in \mathbf{M_h}(\delta_{m(j)})} \|\lambda - \mu_h\|_{-\frac{1}{2},h,\delta_{m(j)}} &= h^{\frac{1}{2}}_{\delta_{m(j)}} \inf_{\mu_h \in \mathbf{M_h}(\delta_{m(j)})} \|\lambda-\mu_h\|_{0,\delta_{m(j)}} \\ &\le C\, h_j \ \|\mathbf{n}\wedge(\mathbf{A}\ \mathbf{curl}\ \mathbf{j}\wedge\mathbf{n})\|_{\frac{1}{2},\delta_{m(j)}} \\ &\le C\, h_j \ \|\mathbf{curl}\ \mathbf{j}\|_{1,\Omega_j}.\end{aligned}$$

Summing over all $\delta_{m(j)}$ results in

$$\inf_{\mu_h \in \mathbf{M_h}} \|\lambda-\mu_h\|_{-\frac{1}{2},h,S} \le C \left(\sum_j^N h_j^2 \|\mathbf{curl}\ \mathbf{j}\|^2_{1,\Omega_j} \right)^{\frac{1}{2}}. \tag{5.4}$$

Finally, combining (5.3) and (5.4) gives the assertion. □

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