Anderson localization in a two-dimensional random gap model

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1. Introduction

The classical approach to randomly scattered particles leads to diffusion, where random scattering originates either from particle–particle collisions (e.g., in a gas) or from collisions with (static) impurity scatterers. In quantum systems, however, diffusion appears only for weak disorder whereas it is destroyed due to Anderson localization at stronger randomness [1,2]. This effect is particularly strong in low-dimensional systems, such as two-dimensional graphene sheets or the surface of topological insulators. The scaling approach to generic random scattering [2] states that diffusion is entirely suppressed by Anderson localization for dimension $d \leq 2$. On the other hand, it has been argued that Anderson localization is prevented on the surface of topological insulators due to suppressed backscattering [3,4].

Inspired by the recent observation of metallic behavior (i.e. diffusive or even ballistic transport) in disordered two-dimensional systems (graphene) [5,6], a general discussion of diffusion and localization of quantum particle is required, which takes into account a spinor structure of the wavefunction. Two possibilities have been considered, namely ballistic transport for finite systems [7,8] and diffusive transport for infinite systems [9]. Diffusion is related to long ranged correlations, which is usually caused by spontaneous symmetry breaking [9,10]. This behavior might be restricted to the regime of weak scattering, since strong scattering is capable to localize particles. The aspect of weak localization is ignored here on purpose because it has its own problems [11,12]. This will be discussed in a separate paper. Instead, we will focus in the following mostly on the case of strong scattering. This is motivated by recent numerical studies, which have indicated that there is a transition to a localized phase at sufficiently strong disorder [13,14]. Here we will analyze details of the transition in terms of the scaling behavior of the localization length for strips of finite width. Moreover, the infinite system will be treated analytically within a strong scattering expansion. The latter provides a rigorous proof for exponential localization, supporting the numerical results at strong disorder. We study a random gap model with linear spectrum (2D Dirac fermions), but our methods can be easily applied to other systems as well.

2. Model

We consider the surface Hamiltonian of a topological insulator with bulk inversion symmetry of momentum $\mathbf{k}$ [3,14–16]

$$H = \begin{pmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(-\mathbf{k}) \end{pmatrix}, \quad h(\mathbf{k}) = \hbar \begin{pmatrix} C+M-(D+\delta)k^2 & v_F(k_x+ik_y) \\ v_F(k_x-ik_y) & C-M-(D-\delta)k^2 \end{pmatrix} \quad (1)$$

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This Hamiltonian consists of a pair of massive Dirac Hamiltonians $h(\mathbf{k})$, $h^*(-\mathbf{k})$. It should be noticed that this Hamiltonian reads in coordinate space

$$H = \begin{pmatrix} h & 0 \\ 0 & h^T \end{pmatrix}$$

with the matrix transposition ^{T}. We include disorder by a random variable M with mean $\overline{m}$. For our numerical transfer-matrix calculation we use a box distribution with width W. For simplicity we choose the Dirac point, where $C=0$ and $D=0$. The main feature is that there are two bands that touch each other at a spectral node $k=0$ if $M=0$, whereas $M\neq 0$ opens a gap $\Delta = 2|M|$. Thus, a random M creates a random gap. Our aim is to calculate the localization length Λ of the eigenstate ψ at energy $E=0$ which satisfies $h\psi = 0$ and the transition probability of a moving particle. The two block Hamiltonians $h(\mathbf{k})$, $h^*(-\mathbf{k})$ act on two separate spaces with the same localization properties. Therefore, it is sufficient to study just one of them.

2.1. Localization length

The localization length Λ of the eigenstates of Hamiltonian (1) can be calculated numerically within a transfer-matrix approach. For this purpose the continuous Hamiltonian must be discretized in space (cf. Appendix A). Then the transfer-matrix T_l of the eigenvalue problem $\psi_{l+1} = h^Y\psi_l + h^D\psi_{l-1}$ (cf. Eqs. (27) and (28)) reads

$$T_l = \begin{pmatrix} h^Y & h^D \\ 1 & 0 \end{pmatrix}, \tag{2}$$

which enables us to evaluate the Lyapunov exponents of the wavefunction [17,18]. With the initial values ψ_0 and ψ_1 the iteration of Eq. (28) provides the wavefunction ψ_L at site L by applying the product matrix

$$M_L = \prod_{l=1}^{L} T_l. \tag{3}$$

For a random Hamiltonian this is a product of random matrices that satisfies Oseledec's theorem [19]. The latter states that there exists a limiting matrix

$$\Gamma = \lim_{L\to\infty} (M_L^\dagger M_L)^{1/2L}. \tag{4}$$

The eigenvalues of Γ are usually written as a diagonal matrix with exponential functions $\exp(\gamma_i)$, where γ_i is the Lyapunov exponent (LE). Adapting the numerical algorithm described in Ref. [18], the whole Lyapunov spectrum can be calculated and the smallest LE is identified with the inverse localization length $1/\Lambda$ [17]. Λ increases with the system width M according to a power law $\Lambda \propto M^\alpha$, where $\alpha > 1$ ($\alpha < 1$) in the regime of extended (localized) states, and $\alpha = 1$ in the critical regime. For the exponentially localized regime we expect $\Lambda \propto$ const. According to the one-parameter scaling theory by MacKinnon [20], the normalized localization length $\tilde{\Lambda} = \Lambda/M$, being a function of disorder strength W and system width M, depends only on a single parameter

$$\tilde{\Lambda}(M, W) = f(\xi(W)/M), \tag{5}$$

where ξ is a characteristic length of the system generated by disorder. Thus, any change of disorder strength W can be compensated by a change of the system width M. If there is a scale-invariant point W_c we can expand $\tilde{\Lambda}$ in its vicinity by assuming a power law with critical exponent ν of the correlation length as $\xi = |W - W_c|^{-\nu}$. Then we have [18]

$$\ln\tilde{\Lambda} = \ln\tilde{\Lambda}_c + \sum_{s=1}^{S} A_s(|W-W_c|M^{1/\nu})^s = \ln\tilde{\Lambda}_c + \sum_{s=1}^{S} A_s\left(\frac{\xi}{M}\right)^{-s/\nu}. \tag{6}$$

2.2. Transition probability

The motion of a quantum particle from site $\mathbf{r}'$ to site $\mathbf{r}$ during the time t is described by the transition probability

$$P_{\mathbf{r}\mathbf{r}'}(t) = |\langle\mathbf{r}|\exp(-iHt)|\mathbf{r}'\rangle|^2. \tag{7}$$

If we assume that $P_{\mathbf{r}\mathbf{r}'}(t)$ describes diffusion, we can obtain the mean square displacement with respect to $\mathbf{r}' = 0$ from the diffusion equation

$$\langle r_k^2\rangle = \sum_{\mathbf{r}} r_k^2 P_{\mathbf{r},0}(t) = Dt, \tag{8}$$

which, after applying a Laplace transformation, becomes

$$\sum_{\mathbf{r}} r_k^2 \int_0^\infty P_{\mathbf{r},0}(t) e^{-\varepsilon t}\, dt = \frac{D}{\varepsilon^2}. \tag{9}$$

Using the Green's function $G_{\mathbf{r}\mathbf{r}'}(z) = (H-z)^{-1}_{\mathbf{r}\mathbf{r}'}$, we obtain for large distances $|\mathbf{r}-\mathbf{r}'|$ and $\varepsilon \sim 0$

$$\int_0^\infty P_{\mathbf{r}\mathbf{r}'}(t)e^{-\varepsilon t}\, dt \sim \int_{E_0}^{E_F} \langle|G_{\mathbf{r}\mathbf{r}'}(E+i\varepsilon)|^2\rangle_d\, dE = \int_{E_0}^{E_F} \langle G_{\mathbf{r}\mathbf{r}'}(E+i\varepsilon)G_{\mathbf{r}'\mathbf{r}}(E-i\varepsilon)\rangle_d\, dE, \tag{10}$$

where $\langle\ldots\rangle_d$ is the average with respect to disorder that is causing scattering. E_0 is the lower band edge and $\mathrm{Tr}_4(\ldots)$ is the trace with respect to the 4 spinor components. The second equation is due to the fact that the Hamiltonian is Hermitean. Then we get with $\mathbf{r}' = 0$ from Eq. (10) for the diffusion coefficient at the energy E

$$D(E) \sim \lim_{\varepsilon\to 0} \varepsilon^2 \sum_{\mathbf{r}} r_k^2 \langle G_{\mathbf{r}0}(E+i\varepsilon)G_{0\mathbf{r}}(E-i\varepsilon)\rangle_d \tag{11}$$

with $D = \int_{E_0}^{E_F} D(E)\, dE$ in Eq. (9).

According to Eq. (9), diffusion requires a long range correlation for small ε in Eq. (10). Anderson localization, on the other hand, is characterized by an exponentially decaying correlation. A natural approach to study the latter for strong randomness would be a hopping expansion in Eq. (11). Unfortunately, such an expansion is plagued by poles on both sides of the real axis. This problem can be avoided if we focus on the most relevant contributions of the randomly fluctuating product of Green's functions $G_{\mathbf{r},\mathbf{r}'}(i\varepsilon)G_{\mathbf{r}',\mathbf{r}}(-i\varepsilon)$. They are associated with the underlying chiral symmetry. These fluctuations have been studied previously in Ref. [21], where the large scale behavior was found to be associated with the Grassmann integral

$$K_{\mathbf{r}\mathbf{r}'} = \langle G_{\mathbf{r}0}(E+i\varepsilon)G_{0\mathbf{r}}(E-i\varepsilon)\rangle_d \approx K_0 \int \varphi_{\mathbf{r}}\varphi'_{\mathbf{r}} J\mathcal{D}[\varphi,\varphi'] \tag{12}$$

with $\mathcal{D}[\varphi,\varphi'] = \prod_{\mathbf{r}} d\varphi\, d\varphi'$ and with the Jacobian

$$J = \frac{1}{detg(H_0 + i\varepsilon + i\eta\hat{U}^2)}, \quad H_0 = \langle H\rangle, \quad \hat{U}_{\mathbf{r}} = \begin{pmatrix} \mathbf{1}+2\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}} & -2\varphi_{\mathbf{r}}\sigma_1 \\ -2\varphi'_{\mathbf{r}}\sigma_1 & \mathbf{1}-2\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}} \end{pmatrix}. \tag{13}$$

The Jacobian appears since we have restricted the integration over randomness to those degrees of freedom which are associated with a global symmetry of the system. It is written in terms of a graded determinant *detg*, where the latter is expressed by conventional determinants in the relation

$$detg\begin{pmatrix} A & \Theta \\ \overline{\Theta} & B \end{pmatrix} = \frac{\det(A)}{\det(B)}\det(\mathbf{1} - \Theta B^{-1}\overline{\Theta}A^{-1}).$$

The parameter η is the scattering rate, which can be considered as an external parameter that is either calculated in self-consistent Born

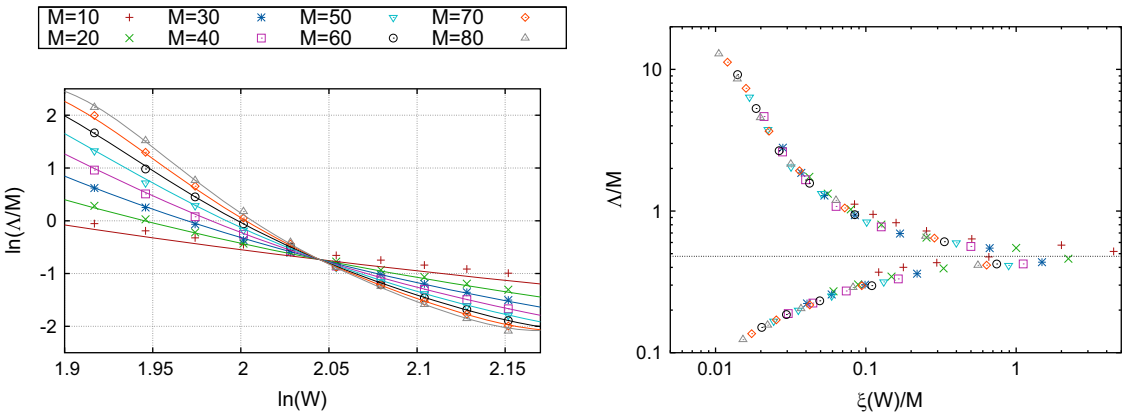

Fig. 1. Numerical evidence for a localization transition in two dimensions. The scaling behavior of the normalized localization length $\overline{\Lambda}$ as a function of increasing disorder W is plotted here for $\overline{m}=0.8$ and $\delta=0.5$. Left panel: fit to Eq. (6) near the critical point. Right panel: rescaled normalized localization length $\overline{\Lambda}$ near the critical point.

Table 1
Critical values for $\delta=0.5$ obtained from fitting the data to Eq. (6).

Average gap $\overline{m}$	0	0.2	0.8
Exponent ν	1.299 ± 0.066	1.397 ± 0.069	1.451 ± 0.024
Critical disorder W_c	7.668 ± 0.008	7.629 ± 0.015	7.727 ± 0.01
Disorder range	$7.35\le W\le 7.8$	$7.1\le W\le 8.0$	$6.6\le W\le 8.4$
System sizes	$30\le M\le 80$	$20\le M\le 80$	$20\le M\le 80$

approximation [22] or is taken from experimental measurements [23]. In any case, the scattering rate increases with increasing disorder.

The relation between the correlation function $K_{\mathbf{r}\mathbf{r}'}$ and the integral in Eq. (12) is based on two facts. Firstly, we have a large freedom to choose a distribution of the random Green's function with the same expectation value. Secondly, by choosing a proper distribution we find a saddle-point approximation for the corresponding integration. This procedure was described in detail in Refs. [21,22], leading eventually to Eq. (12). As a result we have been able to avoid the spurious singularities, which appear when we apply a hopping expansion and integrate with respect to the random term of the Hamiltonian.

The expression in Eq. (13) enables us to rewrite J for weak scattering ($\eta\ll 1$) as

$$J=1/detg[\mathbf{1}+i\eta(\hat{H}_0+i\varepsilon)^{-1}\hat{U}^2]\equiv 1/detg(\mathbf{1}+i\eta\hat{G}_0\hat{U}^2) \tag{14}$$

and for strong scattering ($\eta\gg 1$) as

$$J=1/detg\left[\mathbf{1}+\frac{1}{i\eta}(\hat{H}_0+i\varepsilon)\hat{U}^{-2}\right] \tag{15}$$

since $detg(\hat{U}^2)=1$. These expressions can be used to employ an expansion in powers of η or $1/\eta$. The expression in Eq. (14) has been treated previously. It leads to diffusion, where the correlation function is a diffusion propagator [9]. In Section 4 we will extend the previous work to the regime of strong scattering, employing an expansion in powers of $1/\eta$ for the expression (15).

3. Numerical results: scaling of the localization length

Now we return to the method described in Section 2.1 and calculate the localization length Λ. Our calculation for strong randomness (i.e. large W) provides a critical value W_c, where the system is delocalized (localized) for $W<W_c$ ($W>W_c$). Around the critical value W_c we observe one-parameter scaling behavior for the normalized localization length $\overline{\Lambda}$, as described in Section 2.1. Some results are depicted in Fig. 1 and the results of the fitting procedure are listed in Table 1. This behavior is indicative of an Anderson transition.

4. Analytic results: strong scattering expansion

Eq. (12) is a convenient starting point to study transport properties with the Jacobian

$$J=\exp\left\{-\mathrm{Trg}\left[\log\left(\mathbf{1}+\frac{1}{i\eta}(\hat{H}_0+i\varepsilon)\hat{U}^{-2}\right)\right]\right\}, \tag{16}$$

where the graded trace Trg is with respect to the four-dimensional spinor space and the position $\mathbf{r}$. It is related to the conventional trace by

$$\mathrm{Trg}\begin{pmatrix}A & \Theta\\ \Theta & B\end{pmatrix}=\mathrm{Tr}\,A-\mathrm{Tr}\,B.$$

The integral representation of the correlation function $K_{\mathbf{r}\mathbf{r}'}$ in Eq. (12) with the Jacobian in Eq. (16) enables us to study the regime of strong scattering (i.e. $\eta\gg 1$) by applying a $1/\eta$ expansion. This allows us to rewrite the correlation function as

$$\begin{aligned}K_{\mathbf{r}\mathbf{r}'}&\approx K_0\int\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}'}\exp\left\{-\mathrm{Trg}\left[\log\left(\mathbf{1}+\frac{1}{i\eta}(\hat{H}_0+i\varepsilon)\hat{U}^{-2}\right)\right]\right\}\mathcal{D}[\varphi,\varphi']\\&=K_0\frac{\partial}{\partial\alpha}\int\exp\left\{\alpha\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}'}-\mathrm{Trg}\left[\log\left(\mathbf{1}+\frac{1}{i\eta}(\hat{H}_0+i\varepsilon)\hat{U}^{-2}\right)\right]\right\}\mathcal{D}[\varphi,\varphi']|_{\alpha=0}\end{aligned} \tag{17}$$

and to expand the exponential function as

$$\begin{aligned}&=K_0\frac{\partial}{\partial\alpha}\sum_{l\ge 0}\frac{1}{l!}\int\left(\alpha\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}'}-\mathrm{Trg}\left[\log\left(\mathbf{1}+\frac{1}{i\eta}(\hat{H}_0+i\varepsilon)\hat{U}^{-2}\right)\right]\right)^l\mathcal{D}[\varphi,\varphi']|_{\alpha=0}\\&=K_0\frac{\partial Z}{\partial\alpha}\Big|_{\alpha=0}.\end{aligned} \tag{18}$$

Here we have used the expression

$$Z=\sum_{l\ge 0}\frac{1}{l!}\left\langle\left(\sum_j A_j\right)^l\right\rangle,$$

where $\sum_j A_j$ is the expansion of $\alpha\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}'}-\mathrm{Trg}\left[\log\left(\mathbf{1}+\frac{1}{i\eta}(\hat{H}_0+i\varepsilon)\hat{U}^{-2}\right)\right]$:

$$\sum_j A_j=\alpha\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}'}+\sum_{j\ge 1}\frac{(-1)^j}{j(i\eta)^j}\mathrm{Trg}([(\hat{H}_0+i\varepsilon)\hat{U}^{-2}]^j)$$

and the average is with respect to the normalized integral

$$\langle\ldots\rangle=\frac{1}{\mathcal{N}}\int\ldots\mathcal{D}[\varphi,\varphi'].$$

Using the fact that the factors of the product $A_{j_1}A_{j_2}\cdots A_{j_l}$ can be reorganized as products of connected clusters $\{B_k\}$ (cf. Appendix B),

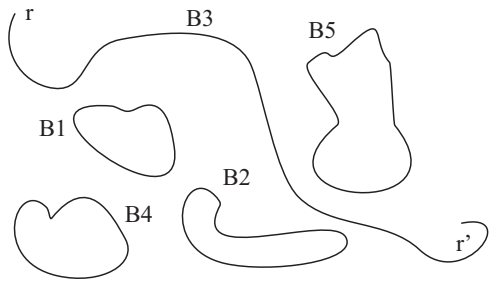

Fig. 2. Typical contribution to the $1/\eta$ expansion of Z. There are five connected clusters B1,…,B5 which are disconnected of each other. In particular, there is a random walk B3 from site $\mathbf{r}$ to site $\mathbf{r}'$, the only contribution to the correlation function $K_{\mathbf{rr}'}$ in Eq. (19).

we obtain from the Linked Cluster Theorem

$$\frac{\partial Z}{\partial \alpha} = Z\frac{\partial \log\, Z}{\partial \alpha} = Z\frac{\partial}{\partial \alpha}\sum_k \langle B_k\rangle. \tag{19}$$

Thus only those expressions $\langle B_k\rangle$ contribute that contain α. These contributions form random walks from site $\mathbf{r}$ to site $\mathbf{r}'$ with the discrete hopping term of Eqs. (21) and (22) (cf. Fig. 2). They can be estimated as

$$\left|\frac{\partial}{\partial \alpha}\sum_k \langle B_k\rangle\right|_{\alpha=0}\Bigg| \le \sum_{l\ge|\mathbf{r}-\mathbf{r}'|}\frac{1}{\eta^l}|Tr_4[(\hat{H}_0^l)_{\mathbf{r},\mathbf{r}'}]| \le \text{const.}(4/\eta)^{|\mathbf{r}-\mathbf{r}'|}, \tag{20}$$

where the factor 4 is due to the two dimensional random walk. Thus, we need $\eta > 4$ (in units of $\hbar v_F/a$ with lattice constant a) in order to have an absolutely convergent series and an exponential decay of the correlations. The latter describes Anderson localization, according to our discussion in Section 2.2.

5. Discussion and conclusion

Our analytic calculation supports the numerical result of a localized phase at sufficiently strong disorder. Here it should be noticed that the calculations are based on different quantities, namely the localization length Λ and the exponential decay of the average transition matrix $K_{\mathbf{rr}'}$. Since the localization length is self-averaging according to Eq. (4), it is expected that this quantity should be very robust in a real system. On the other hand, the localization length is difficult to measure directly in an experiment. Therefore, the transition matrix is more accessible because it is related to the conductivity by the Einstein relation

$$\sigma_{xx} \approx \varepsilon^2\frac{e^2}{h}\sum_{\mathbf{r}} x^2 K_{\mathbf{r}0},$$

where x is the direction of the position $\mathbf{r}$, in which the external electric field is applied. In the DC limit $\varepsilon\to 0$ the conductivity vanishes when $K_{\mathbf{r}0}$ decays exponentially. This is in stark contrast to the weak scattering case where the expansion in powers of η gives a diffusion propagator [22]

$$\tilde{K}_{\mathbf{q}} \propto \frac{1}{\varepsilon + Dq^2}$$

with the diffusion coefficient D. After Fourier transformation $\mathbf{q}\to\mathbf{r}$ this expression gives a correlation function that decays like $\sim r^{-1/2}$. Moreover, it gives a finite non-vanishing DC conductivity, since

$$\sum_{\mathbf{r}} x^2 K_{\mathbf{r}0} = -\frac{\partial \tilde{K}_{\mathbf{q}}}{\partial q_x^2}\Big|_{\mathbf{q}=0} \propto \frac{2D}{\varepsilon^2}.$$

On the surface of a typical topological insulator we expect substantial scattering due to disorder [3]. Our results indicate that the suppressed backscattering may not be able to prevent the localization of surface states. Therefore, it might be crucial for the appearance of a metallic behavior to reduce the disorder on the surface. In this case it could even be possible to observe an Anderson transition from extended to localized states, as our results indicate. Our calculation gives a rough estimate for the localized behavior in which the scattering rate must be larger than the bandwidth of the system without disorder. A similar transition was also observed in a numerical study of the conductivity in disordered graphene by Zhang et al. [29]. However, we cannot confirm their interpretation as a Kosterlitz–Thouless transition because we find a power law for the localization length.

In conclusion, we have studied a model for surface states on a topological insulator. Contrary to the assumption that suppressed backscattering may always create a metallic phase, we have found that the surface states are localized for strong scattering by disorder. For weak scattering, however, there is a metallic behavior and a phase transition from a delocalized to a localized phase when the disorder strength is increased. The transition is characterized by one-parameter scaling of the normalized localization length with a non-universal exponent.

Appendix A. Numerical transfer-matrix calculation

A numerical treatment of the Dirac Hamiltonian requires a discretization in space. However, the naive discretization through replacing the differential operator by a difference operator leads to additional new nodes, which is often called fermion doubling or multiplication [24]. In real space there are two methods to circumvent this problem [25–27]. One that we will adopt in this section goes back to an idea of Susskind. We start with discretizing the differential operator in an anti-symmetric way

$$\partial_x f(x) \approx \frac{1}{2\Delta}(f_{l+\Delta} - f_{l-\Delta}), \tag{21}$$

where Δ is the lattice constant which we set to one in the following. The discrete Dirac equation for $m=0$ and with $\hbar v_F = 1$ a then takes the form

$$-\frac{i}{2}\sigma_1\{\psi_{l+1,n}-\psi_{l-1,n}\} - \frac{i}{2}\sigma_2\{\psi_{l,n+1}-\psi_{l,n-1}\} = E\sigma_0\psi_{l,n} \tag{22}$$

with lattice points given by the integer coordinates (l,n). Fourier transformation leads to eigenvalues $E = \pm\sqrt{\sin(k_x)^2 + \sin(k_y)^2}$ which have four Dirac cones in the Brillouin zone corresponding to four Dirac fermions. In order to open a gap at three of them we introduce a lattice operator which acts on a wave function as [28]

$$\hat{B}\psi_{l,n} = \tfrac{1}{2}\{\psi_{l+1,n}+\psi_{l-1,n}+\psi_{l,n+1}+\psi_{l,n-1}\}. \tag{23}$$

The discretized form of the Hamiltonian (1) for uniform gap now reads

$$h = \sin(k_x)\sigma_1 - \sin(k_y)\sigma_2 + [m + \delta(\cos(k_x)+\cos(k_y)-2)]\sigma_3, \tag{24}$$

which gives $h(\mathbf{k})$ of Eq. (1) in the continuum limit and has the dispersion

$$E = \pm\sqrt{\sin(k_x)^2 + \sin(k_y)^2 + (m + \delta\cos(k_x) + \delta\cos(k_y) - 2\delta)^2}. \tag{25}$$

For $m=0, \delta\neq 0$ there is a node at $k_x = k_y = 0$ and three additional nodes, for $m=0$, $\delta=0$ at k_x, $k_y = \pm\pi$. Using this model node degeneracy can be lifted via the parameter δ.

We absorb the index n with the help of matrix representation and write for the wave function

$$\psi_{l+1} = h^Y\psi_l + h^D\psi_{l-1}. \tag{26}$$

Each spinor component is now a M-component vector, where M is the width of a strip and thus $n = 1, 2, \ldots, M$. The matrices h^Y, h^D read

$$h^Y_{n,n} = 2S^{-1}[E\sigma_0 + (2\delta - m)\sigma_3] \quad h^Y_{n,n+1} = S^{-1}[i\sigma_2 - \delta\sigma_3]$$

$$h^Y_{n,n-1} = -S^{-1}[i\sigma_2 + \delta\sigma_3] \quad h^D_{n,n} = -S^{-1}[i\sigma_1 + \delta\sigma_3] \tag{27}$$

with $S = -i\sigma_1 + \delta\sigma_3$ and where h^Y has periodic boundary conditions in the y-direction. This matrix structure allows us to construct a transfer matrix T_l through Eq. [18]

$$\begin{pmatrix} \psi_{l+1} \\ \psi_l \end{pmatrix} = \begin{pmatrix} h^Y & h^D \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_l \\ \psi_{l-1} \end{pmatrix} \equiv T_l \begin{pmatrix} \psi_l \\ \psi_{l-1} \end{pmatrix}. \tag{28}$$

Appendix B. Linked cluster theorem

We must organize the $1/\eta$ expansion in order to extract the spatial decay of the correlation function $K_{\mathbf{rr}'}$. For this purpose we employ the Linked Cluster Theorem [30]. The latter can be formulated for the expression

$$\frac{1}{l!}\langle(\sum_j A_j)^l\rangle = \frac{1}{l!}\sum_{j_1 j_2 \ldots j_l} \langle A_{j_1} A_{j_2} \cdots A_{j_l}\rangle. \tag{29}$$

The product of the $A_i A_j$ is called disconnected (unlinked) if the two factors do not share any Grassmann variable. This would lead to $\langle A_i A_j\rangle = \langle A_i\rangle\langle A_j\rangle$. Otherwise they are called connected (linked) and we would have $\langle A_i A_j\rangle \neq \langle A_i\rangle\langle A_j\rangle$. In the sum (29) we combine for a given set $j_1, j_2, \ldots, j_l$ all connected factors in products $\{B_k\}$ such that

$$\langle A_{j_1} A_{j_2} \cdots A_{j_l}\rangle = \langle B_{k_1}\rangle\langle B_{k_2}\rangle \cdots \langle B_{k_n}\rangle \quad (n \le l), \tag{30}$$

where the new indices $k_1, \ldots, k_n$ refer to the indices $j_1, \ldots, j_l$ of the combined factors A_j. Now we must reorganize the summation. A permutation of the $j_1, j_2, \ldots, j_l$ gives the same expression for Eq. (30). Therefore, the summation with respect to the permutations contributes only a factor $l!$. On the other hand, we allow also a permutation of the $k_1, k_2, \ldots, k_n$, which would also leave the expression (30) invariant. Consequently, we must divide the summation with respect to these n permutations by $n!$. This gives us eventually

$$Z = \sum_{l \ge 0} \frac{1}{l!}\langle(\sum_j A_j)^l\rangle = \sum_{n \ge 0} \frac{1}{n!}(\sum_k \langle B_k\rangle)^n = \exp(\sum_k \langle B_k\rangle),$$

which is the Linked Cluster Theorem, since the B_k are connected according to our construction.

References

[1] P.W. Anderson, Physical Review 109 (1958) 1492.
[2] E. Abrahams, P.W. Anderson, D.C. Licciardello, T.V. Ramakrishnan, Physical Review Letters 42 (1979) 673.
[3] X.-L. Qi, S.-C. Zhang, Reviews of Modern Physics 83 (2011) 1057.
[4] D. Culcer, Physica E 44 (2012) 860.
[5] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos, A.A. Firsov, Nature 438 (2005) 197.
[6] Y. Zhang, Y.-W. Tan, H.L. Stormer, P. Kim, Nature 438 (2005) 201.
[7] B. Rosenstein, M. Lewkowicz, H.C. Kao, Y. Korniyenko, Physical Review B 81 (2010) 041416. (R).
[8] O. Morandi, F. Schürrer, Journal of Physics A: Mathematical and Theoretical 44 (2011) 265301.
[9] K. Ziegler, Physical Review B 55 (1997) 10661;
K. Ziegler, Physical Review Letters 80 (1998) 3113.
[10] L. Schäfer, F. Wegner, Zeitschrift für Physik B 38 (1980) 113.
[11] T. Ando, Y. Zheng, H. Suzuura, Journal of the Physical Society of Japan 71 (2002) 1318;
H. Suzuura, T. Ando, Physical Review Letters 89 (2002) 266603;
E. McCann, et al., Physical Review Letters 97 (2006) 146805.
[12] D.V. Khveshchenko, Physical Review Letters 97 (2006) 036802.
[13] A. Hill, K. Ziegler arXiv:1211.2329.
[14] A. Yamakage, K. Nomura, K. Imura, Y. Kuramoto arXiv:1211.5026.
[15] B.A. Bernevig, T.L. Hughes, S. Zhang, Science 314 (2006) 1757.
[16] A. Yamakage, K. Nomura, K. Imura, Y. Kuramoto, Journal of the Physical Society of Japan 80 (2011) 053703.
[17] J.L. Pichard, G. Sarma, Journal of Physics C: Solid State Physics 14 (1981) L127.
[18] A. MacKinnon, B. Kramer, Zeitschrift für Physik B: Condensed Matter 13 (1983) 1546.
[19] V. Oseledec, Transactions of the Moscow Mathematical Society 19 (1968) 197.
[20] A. MacKinnon, B. Kramer, Physical Review Letters 47 (1981) 21.
[21] K. Ziegler, Journal of Physics A: Mathematical and Theoretical 45 (2012) 335001.
[22] K. Ziegler, Physical Review Letters 102 (2009) 126802;
[b] K. Ziegler, Physical Review B 79 (2009) 195424.
[23] E. Pallecchi, A.C. Betz, J. Chaste, G. Fève, B. Huard, T. Kontos, J.-M. Berroir, B. Placais, Physical Review B 83 (2011) 125408.
[24] L. Susskind, Physical Review D 16 (1977) 3031.
[25] R. Stacey, Physical Review D 26 (1982) 468.
[26] J. Tworzydło, C.W. Groth, C.W.J. Beenakker, Physical Review B 78 (2008) 235438.
[27] M.V. Medvedyeva, J. Tworzydło, C.W.J. Beenakker, Physical Review B 81 (2010) 214203.
[28] K. Ziegler, Physical Review B 53 (1996) 9653.
[29] Y.Y. Zhang, J. Hu, B.A. Bernevig, X.R. Wang, X.C. Xie, W.M. Liu, Physical Review Letters 102 (2009) 106401.
[30] J. Glimm, A. Jaffe, Quantum Physics, Springer-Verlag, 1981.