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Dynamical entanglement in coupled systems

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Abstract. We present a comparative analysis on the evolution of two coupled bosonic many-body quantum systems. Considering photons in two coupled cavities and bosonic atoms in a double-well potential, the effect of the boson-boson interaction on the spectral properties and the dynamical behavior is studied. In particular, we analyze the evolution of a Fock state to a N00N state, which is a superposition of two complementary Fock states. Such an entangled state appears in the evolution only if tunneling and many-body interaction are balanced.

1. Introduction

The dynamics of isolated many-body quantum systems has been a subject of intense research in atomic physics during recent years, in experiment [1] as well as in theory [2]. This interest has two major aspects. One is to prepare the system in a well-defined initial state and, secondly, to study its evolution for a period of time due to quantum tunneling and particle-particle interaction. The preparation of the initial state as the groundstate of a certain Hamiltonian H_0 and the evolution with $\exp(-iHt)$ for a different Hamiltonian H involves a sudden change $H_0 \to H$, which is usually called a quench. Such a quench can be realized in atomic systems by changing the potential wells in which the atoms are trapped [3]. For instance, bosonic atoms are prepared in a Fock state, where a definite number of atoms are localized in deep optical potential wells (right picture in Fig. 1). Then the potential barrier between a pair of neighboring wells is suddenly reduced such that the atoms can tunnel between these wells [4, 5]. From the theoretical point of view the statistical properties of this problem have been studied intensively using the Hubbard model and related models [6–9]. An essential element of the dynamical analysis is the evaluation of the spectral weights with respect to the initial state [10, 11], which links directly the quantum evolution with the spectral properties of the underlying Hamiltonian.

Besides systems of ultracold atoms an alternative approach of controllable bosons is based on photons. Then the role of the potential wells is played by microwave or optical cavities (left picture in Fig. 1). The experimental preparation of Fock states in a optical cavity has been achieved recently [12, 13]. This is a crucial step towards a systematic study of correlated many-body systems with photonic states. The interaction between the photons is indirectly mediated by atoms inside the optical cavities, which interact directly with the photons [14–16]. Once a Fock state with N photons has been prepared inside an optical cavity, we can couple the latter with another optical cavity by a waveguide or an optical fiber. Then the photons can tunnel between the two optical cavities, leading to a quantum evolution of the initial Fock state $|N, 0\rangle$ within the Hilbert space that is spanned by the eigenstates of the Hamiltonian of the new system. This new Hamiltonian can be approximated, for instance, by the Hubbard Hamiltonian, as suggested recently by several groups [17–20]. This type of system, including atomic degrees of freedom, was studied within a Hartree-Fock approximation [21].

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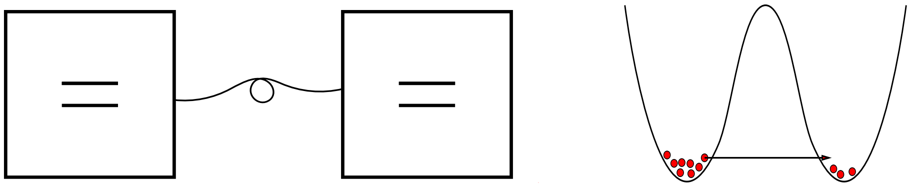

Figure 1. Two optical cavities with two-level atoms are coupled by an optical fiber (left picture). Bosonic atoms in a double-well potential present a similar system (right picture).

The evolution from the initial Fock state can, in principle, lead to an entangled state, such as the N00N state $(|N,0\rangle + \exp(i\phi)|0,N\rangle)/\sqrt{2}$. The N00N state has attracted much attention because it can be used for highly accurate interferometry and other precision measurements [22–25] and for technological application such as optical lithography [26]. Various methods for the creation of photonic N00N states have been suggested in the literature [27, 28] and indeed experimentally created up to $N = 5$ photons [25]. This clearly indicates that the creation of these entangled states is realistic. However, it still remains a problem to created N00N states for large N. The author discussed recently the dynamical creation of a N00N state from ultracold bosonic atoms in a double well [29], which shows that a balanced effect of inter-well tunneling and intra-well interaction can produce such a state with moderate probability. Since photon-photon interaction can also be mediated in a cavity by coupling the photons to atoms, we will analyze in the following the dynamical creation of entangled (N00N) states for a pair of anharmonic cavities, described by coupled Jaynes-Cummings models [30, 31], and compare the results with those of the Hubbard model.

The paper is organized as follows: In Sect. 2 we introduce the basic quantities for the dynamics of our systems and discuss the return probability, the transition probability and their relation with entangled states. As an example for non-interacting bosons we analyze two coupled harmonic cavities in Sect. 3. Then we discuss briefly the Jaynes-Cummings model for a single cavity with a two-level atom (Sect. 4.1) and derive an effective Hamiltonian for two coupled cavities (Sect. 4.2). After a brief review on bosonic atoms in a double-well potential (Sect. 5), the recursive projection method for calculating spectra and the dynamics of coupled systems is described in Sect. 6. The results of this approach for coupled anharmonic cavities and bosonic atoms in a double-well potential are discussed and compared with the results of the coupled harmonic cavities in Sect. 7. Finally, we summarize the work in Sect. 8.

2. Spectral density and the evolution of isolated systems

We consider a system which is isolated from the environment. In terms of photonic states this can be realized by an ideal optical cavity. With the initial state $|\Psi_0\rangle$ we obtain from the time evolution $|\Psi_t\rangle = e^{-iHt}|\Psi_0\rangle$ the evolution of the return probability $|\langle\Psi_0|\Psi_t\rangle|^2$ with the return amplitude $\langle\Psi_0|\Psi_t\rangle = \langle\Psi_0|e^{-iHt}|\Psi_0\rangle$. A Laplace transformation relates the return and the transition amplitude to another state $|Psi_1\rangle$ with the resolvent through the identities

$$\langle\Psi_0|\Psi_t\rangle = \int_\Gamma \langle\Psi_0|(z-H)^{-1}|\Psi_0\rangle e^{-izt}dz, \quad \langle\Psi_1|\Psi_t\rangle = \int_\Gamma \langle\Psi_1|(z-H)^{-1}|\Psi_0\rangle e^{-izt}dz \ , \tag{1}$$

where the contour Γ encloses all the eigenvalues E_j $(j = 0, 1, ..., N)$ of H, assuming that the underlying Hilbert space is $N + 1$ dimensional. With the corresponding eigenstates $|E_j\rangle$ the spectral representation of the resolvent is a rational function of z:

$$\langle\Psi_0|(z-H)^{-1}|\Psi_0\rangle = \sum_{j=0}^{N} \frac{|\langle\Psi_0|E_j\rangle|^2}{z-E_j} = \frac{P_N(z)}{Q_{N+1}(z)}, \quad Q_{N+1}(z) = \prod_{j=0}^{N}(z-E_j) \ , \tag{2}$$

where $P_N(z)$, $Q_{N+1}(z)$ are polynomials in z of order N, $N+1$, respectively. These polynomials can be evaluated by the recursive projection method [32].

The expression in Eq. (2) suggests the introduction of the spectral density $\rho_{0,0}(E)$ as the imaginary part of the resolvent:

$$\rho_{0,0}(E) = \lim_{\epsilon\to 0}\frac{1}{\pi}Im\langle\Psi_0|(E-i\epsilon-H)^{-1}|\Psi_0\rangle = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\sum_{j=0}^{N}\frac{|\langle\Psi_0|E_j\rangle|^2}{\epsilon^2+(E-E_j)^2} = \sum_{j=0}^{N}|\langle\Psi_0|E_j\rangle|^2\delta(E-E_j)\ , \tag{3}$$

where $|\Psi_0\rangle$ is a reference state. In other words, $\rho_{0,0}(E)$ is the diagonal element of the density matrix with respect to $|\Psi_0\rangle$. Then the return amplitude can be written as the Fourier transform of the spectral density

$$\langle\Psi_0|\Psi_t\rangle = \int\rho_{0,0}(E)e^{-iEt}dE = \sum_{j=0}^{N}|\langle\Psi_0|E_j\rangle|^2e^{-iE_jt}\ . \tag{4}$$

We can also evaluate other elements of the density matrix, such as the off-diagonal element

$$\rho_{1,0}(E) = \frac{1}{\pi}\lim_{\epsilon\to 0}Im\langle\Psi_1|(E-i\epsilon-H)^{-1}|\Psi_0\rangle = \sum_j\langle\Psi_1|E_j\rangle\langle E_j|\Psi_0\rangle\delta(E-E_j)\ , \tag{5}$$

whose Fourier transforms gives the transition amplitude between the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$:

$$\langle\Psi_1|\Psi_t\rangle = \int\rho_{1,0}(E)e^{-iEt}dE = \sum_{j=0}^{N}\langle\Psi_1|E_j\rangle\langle E_j|\Psi_0\rangle e^{-iE_jt}\ . \tag{6}$$

To characterize the entangled state $|\Psi_t\rangle = c_0|\Psi_0\rangle+c_1|\Psi_1\rangle$ that may appear during the evolution, we need to evaluate the amplitudes (4), (6) and count how often they realize certain values c_0, c_1 simultaneously during a long period of time. After normalization, this defines the conditional probability $P_t(c_0,c_1)$ for having $\langle\Psi_0|\Psi_t\rangle = c_0$ and $\langle\Psi_1|\Psi_t\rangle = c_1$ at a given time t.

3. Two coupled harmonic cavities

The Hamiltonian of two uncoupled harmonic cavities is $H_{hc} = \omega_0\sum_{j=1,2}a_j^\dagger a_j$, where the index $j=1,2$ refers to the two cavities with the photonic creation (annihilation) operators $a_j^\dagger$ (a_j). The eigenstates of this Hamiltonian are product Fock states $|N-k,k\rangle \equiv |N-k\rangle|k\rangle$ ($k=0,..,N$). Coupling two cavities by an optical fiber is described by the tunneling Hamiltonian

$$H_{HC} = -J(a_1^\dagger a_2 + a_2^\dagger a_1) + \omega_0(a_1^\dagger a_1 + a_2^\dagger a_2) \tag{7}$$

with eigenstates $|N-k;k\rangle$. Here we assume that N is even and calculate the overlap of the eigenstates $|N-k;k\rangle$ with the initial Fock state $|N,0\rangle$ [16]:

$$\langle N,0|N-k;k\rangle = 2^{-N/2}\binom{N}{k}^{1/2}\ , \tag{8}$$

which is non-zero for all eigenstates. The density-matrix elements with respect to $|\Psi_0\rangle = |N,0\rangle$ and $|\Psi_1\rangle = |0,N\rangle$ are

$$\rho_{0,0}(E) = 2^{-N}\sum_{k=0}^{N}\binom{N}{k}\delta(E+J(2k-N)),\quad \rho_{1,0}(E) = 2^{-N}\sum_{k=0}^{N}\binom{N}{k}(-1)^k\delta(E+J(2k-N))\ . \tag{9}$$

Thus, there is a binomial distribution for the spectral weight $|\langle N,0|N-k;k\rangle|^2$, with a maximal overlap for an equal number of photons in each cavity. For large N the binomial distribution becomes a Gaussian distribution, where the width of the envelope is related to the energy level spacings $\Delta E = 2J$. The Gaussian result resembles the Central Limit Theorem for independent photons. Such a behavior was also found previously for freely expanding bosons from an initial Fock state [33]. A Fourier transformation reveals a periodic behavior of the return and transition amplitudes as

$$\langle N,0|e^{-iHt}|N,0\rangle = \cos^N(Jt), \quad \langle 0,N|e^{-iHt}|N,0\rangle = (-i)^N \sin^N(Jt) \ . \tag{10}$$

Thus the evolution of the Fock state is periodic with period $2\pi/J$ but leads to a N00N state $c_0|N,0\rangle + c_N|0,N\rangle$ only with a probability that decays exponentially with N. For larger values of N the probability $P(c_0,c_N)$ indicates an anti-correlation: $P(c_0,c_N)$ vanishes as soon as both c_0 and c_N become nonzero. Therefore, the overlap of $|\Psi_t\rangle$ with a N00N state is strongly suppressed. This is a consequence of the fact that for an increasing N the photons disappear in the $(N+1)$–dimensional Hilbert space because there is no constraint due to interaction.

4. Cavities with two-level atoms

4.1. Single cavity with a two-level atom

An anharmonicity in a cavity can be created by adding an atom which interacts with the photons [14, 15, 34]. In the case of a single two-level atom we can describe the absorption and emission of photons by the atom approximately with the Jaynes-Cummings (JC) model [30, 31], whose Hamiltonian reads

$$H_{JC} = \omega_0 a^\dagger a + (\omega_0 + \Delta)c^\dagger c - g(a^\dagger c + c^\dagger a) \ . \tag{11}$$

Δ is the detuning between the atomic excitation energy and the photon energy, $c^\dagger$ (c) is the creation (annihilation) operator of the atomic excitation, and g is the coupling strength between the photons and the atom. The eigenvalues of this Hamiltonian are [30, 31]

$$E_{n,\pm} = \omega_0(n+1/2) \pm \sqrt{\Delta^2 + 4g^2(n+1)}, \quad n = 0,1,2,\dots \ , \tag{12}$$

where the eigenstates $|n,\pm\rangle$ are linear combinations of the two Fock states $|n:1\rangle$ and $|n+1:0\rangle$ with n and $n+1$ photons and an atomic state with $j=0$ (atomic groundstate) and $j=1$ (excited atom). The dynamics is characterized by the energy difference $E_{n,+} - E_{n,-} = 2\sqrt{\Delta^2 + 4g^2(n+1)}$ which creates Rabi oscillations between the two Fock states $|n:0\rangle$ and $|n-1:1\rangle$ with the Rabi frequency $\Omega_R = \sqrt{\Delta^2 + 4g^2(n+1)}$.

The JC model describes an extreme case of Hilbert-space localization, where the system is constraint to a two-dimensional subspace. It is enforced by the fact that the eigenstates of the JC model are linear combinations of only two Fock states. In other words, if this system is initially prepared in a Fock state, it will never escape from the related two-dimensional subspace. The drawback of the extreme localization in Hilbert space is that we are not able to create dynamically entangled Fock states whose difference in photon numbers is larger than 1. In a more general case, however, the eigenstates of the Hamiltonian may be a superposition of many Fock states. Then the overlap of the eigenfunctions with the initial Fock state plays a crucial role.

4.2. Two coupled cavities with two-level atoms

Now we prepare the anharmonic cavity of Sect. 4.1 in the eigenstate of the JC Hamiltonian $|N,+\rangle$ and connect it with another anharmonic cavity which is in the state $|0,+\rangle$. After the connection the photons start to tunnel between the two cavities. This system is now described by the Hamiltonian

$$H_{2JC} = -J(a_1^\dagger a_2 + a_2^\dagger a_1) + \sum_{j=1,2} [\omega_0 a_j^\dagger a_j + \omega_0 c_j^\dagger c_j - g(a_j^\dagger c_j + c_j^\dagger a_j)] \ , \tag{13}$$

where the first term describes the tunneling of photons between the cavities with rate J and the second term represents the cavity levels as well as the absorption and emission of photons by the two-level atom inside each cavity. Here we have assumed no detuning: $\Delta = 0$. For the initial state we choose a product of JC eigenstates $|N, \sigma; 0, \sigma'\rangle \equiv |N, \sigma\rangle|0, \sigma'\rangle$.

The probability for a dynamically changing of the atomic levels during the photonic tunneling process is characterized by the ratio [35]

$$\frac{\langle k-1, -|a^\dagger|k, +\rangle}{\langle k-1, -|a^\dagger|k, -\rangle} = \frac{\sqrt{k+2} - \sqrt{k+1}}{\sqrt{k+2} + \sqrt{k+1}} \approx 0 \ . \tag{14}$$

Thus, for sufficiently many photons k the change of the atomic level is strongly suppressed. With these approximations we decouple the dynamics of $\pm$ states into states where the atomic level is fixed. Then we are left with states $|k, N-k\rangle$, $|N-k, k\rangle$ which only depend on the number of photons in each cavity. The dynamics is described by a Hubbard-like model, where the n_j^2 interaction is replaced by a $\sqrt{n_j}$ photon-photon interaction [35]:

$$H_{eff} = -J(a_1^\dagger a_2 + a_2^\dagger a_1) + \omega_0(a_1^\dagger a_1 + a_2^\dagger a_2) \pm 2g(\sqrt{a_1^\dagger a_1} + \sqrt{a_2^\dagger a_2}) \ . \tag{15}$$

The sign of the coupling depends on whether both atoms are in the ground state (-) or in the excited state (+). This Hamiltonian has a two-fold degeneracy for $J = 0$ due to the equivalence of the two cavities. The interaction is weaker than the n_j^2 interaction of the corresponding Hubbard model in Eq. (16). This indicates that the properties of the coupled JC models may resemble the behavior of the Hubbard model in a double well [29], with less pronounced interaction features though.

5. Bosonic atoms in a double-well potential

Another interesting realization of coupled quantum systems are bosonic atoms which are trapped in a double-well potential. This has also been considered as a Josephson junction for ultracold gases. The main advantage of such a system is based on the fact that the trapping potential, including the barrier between the wells, is easy to control by external Laser fields. The dynamics has been studied experimentally in great detail [36–38] and has revealed several regimes, either dominated by tunneling or by the many-body interaction. From the theoretical point of view the physics is described by the Hubbard-Hamiltonian

$$H_{DW} = -J(a_1^\dagger a_2 + a_2^\dagger a_1) + g[(a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2] \ , \tag{16}$$

in contrast to the effective photon-photon interaction in Eq. (15). This indicates a strong many-body interaction for atoms. Some aspects can be described by a mean-field (Gross-Pitaevskii) approximation, other aspects are of genuine quantum nature. The formation of entangled states, such as a N00N state, belongs to the latter. Of particular interest for entanglement is the intermediate regime, where tunneling and many-body interaction are strongly competing, which restricts the available Hilbert space for the dynamical formation of new states but also mixes different regions of the space [29, 38].

6. Recursive projection method

The evaluation of the resolvents as rational functions presented in Eq. (2) can be performed analytically within the recursive projection method. This method is based on the idea that we divide the $N+1$–dimensional Hilbert space (N even) into two-dimensional subspaces spanned by the states $|k, N-k\rangle$, $|N-k, k\rangle$ ($k = 0, ..., N/2$), since we are interested in matrix elements with respect to the two states $|0, N\rangle$ and $|N, 0\rangle$. Then the recursive projection method is constructed as a directed walk through all subspaces, beginning with $k = N/2$ and terminating with $k = 0$. For this purpose we define

$$a_{N/2} = \langle N,0|(z-H)^{-1}|N,0\rangle = \langle 0,N|(z-H)^{-1}|0,N\rangle \tag{17}$$

and

$$b_{N/2} = \langle 0,N|(z-H)^{-1}|N,0\rangle = \langle N,0|(z-H)^{-1}|0,N\rangle \ . \tag{18}$$

Then $a_{N/2}$ and $b_{N/2}$ are obtained from the iteration of the recurrence relation (for details cf. Ref. [29])

$$a_{k+1} = \frac{z - \tilde{f}_{k+1} - J^2 a_k (N/2+k+1)(N/2-k)}{\left[z - \tilde{f}_{k+1} - J^2 a_k (N/2+k+1)(N/2-k)\right]^2 - J^4 b_k^2 (N/2+k+1)^2 (N/2-k)^2} \tag{19}$$

$$b_{k+1} = \frac{J^2 b_k (N/2+k+1)(N/2-k)}{\left[z - \tilde{f}_{k+1} - J^2 a_k (N/2+k+1)(N/2-k)\right]^2 - J^4 b_k^2 (N/2+k+1)^2 (N/2-k)^2} \tag{20}$$

with initial values $b_0 = 0$, $a_0 = 1/(z - \tilde{f}_0)$. The term $\tilde{f}_{k+1}$ depends on the interaction of the model:

$$\tilde{f}_{k+1} = \begin{cases} g[(N/2+k+1)^2 + (N/2-k-1)^2] & \text{Hubbard model in Eq. (16)} \\ \pm 2g[\sqrt{N/2+k+1} + \sqrt{N/2-k-1}] & \text{coupled JC model of Eq. (15)} \end{cases} . \tag{21}$$

The recurrence relation terminates after $N/2$ steps, giving us $a_{N/2}$ and $b_{N/2}$ in the form of ratios of polynomials $P_N(z)/Q_{N+1}(z)$, as described in Eq. (2) .

7. Results

The properties of two coupled anharmonic cavities in Sect. 4.2 and the bosonic atoms in the double-well potential in Sect. 5 are characterized by tunneling between the subsystems and by a many-body interaction. The iteration of Eqs. (19), (20) gives us the following four matrix elements of the resolvent

$$\langle N,0|(z-H)^{-1}|N,0\rangle, \quad \langle 0,N|(z-H)^{-1}|0,N\rangle, \quad \langle 0,N|(z-H)^{-1}|N,0\rangle = \langle N,0|(z-H)^{-1}|0,N\rangle \ , \tag{22}$$

which allows us to evaluate the spectral density $\rho_{0,0}(E)$ in Eq. (3) and the off-diagonal element $\rho_{1,0}(E)$ of the density matrix in (5). After a Fourier transformation we obtain the return amplitude $\langle\Psi_0|\Psi_t\rangle$ and the transition amplitude $\langle\Psi_1|\Psi_t\rangle$ from Eqs. (4), (6) as functions of time.

Here it should be noticed that there exists an invariance of the recurrence relation under the following simultaneous sign changes in Eqs. (19) and (20)

$$z \to -z, \quad g \to -g, \quad a_j \to -a_j, \quad b_j \to -b_j \ . \tag{23}$$

This implies that a change from $g \to -g$ in the initial states results in a mirror image with respect to energy of $\rho_{0,0}(E, g)$ and $\rho_{N,0}(E, g)$:

$$\rho_{0,0}(E, g) = \rho_{0,0}(-E, -g), \quad \rho_{N,0}(E, g) = \rho_{N,0}(-E, -g) \ . \tag{24}$$

This relation enables us to evaluate both atomic states of the coupled JC models with the same recurrence relation. Moreover, the density-matrix elements are invariant with respect to the harmonic frequency ω_0 of the cavities, except for a global energy shift. This reflects an important universality of the density matrix that allows us to separate the harmonic from the anharmonic properties of the cavities.

In the absence of tunneling ($J = 0$) the spectrum two harmonic cavities is completely degenerate for harmonic cavities

$$E_{N-k,k} = \omega_0(N-k) + \omega_0 k = \omega_0 N$$

but is only two-fold degenerate for anharmonic cavities

$$E_{N-k,k} = \omega_0 N \pm g(\sqrt{N-k} + \sqrt{k})$$

or for bosonic atoms in a double-well potential:

$$E_{N-k,k} = g[(N-k)^2 + k^2] \ .$$

After connecting the subsystems by a tunneling term the degeneracies are completely lifted. For harmonic cavities an equidistant spectrum appears with level spacing $\Delta E = 2J$ according to Eq. (9). Tunneling also lifts the two-fold degeneracies of the anharmonic cavities and the bosons in the double well, as depicted in Fig. 2. However, the levels are more irregularly distributed and their spacing is smaller than $2J$ for pairs of levels. Here it should be noticed that the spectrum of the bosonic atoms indicates a spectral fragmentation with an almost two-fold degenerate high energy part and a non-degenerate low energy part, which cannot be seen in the photonic spectrum.

The difference between non-interacting and interacting systems is even more pronounced for the dynamics of the return and transition amplitudes. While there is only a periodic behavior with $T = \pi/J$ in Eq. (10), anharmonic cavities have a more dynamic behavior (cf. Figs. 3, 4). In particular, on the time scale considered in Figs. 3, 4, there is no periodic behavior but oscillations on much shorter scales than $\pi/J \approx 4$. This is a consequence of the fact that the individual energy levels $E_k = J(2k - N)$ in Eq. (9) are invisible in the dynamics of the harmonic cavities due to

$$\sum_{k=0}^{N} \binom{N}{k} e^{iJ(2k-N)t} = (e^{iJt} + e^{-iJt})^N, \quad \sum_{k=0}^{N} \binom{N}{k} (-1)^k e^{iJ(2k-N)t} = (-e^{iJt} + e^{-iJt})^N \ . \tag{25}$$

Such kind of interference effect is accidental for the non-interacting system and does not occur in the presence of interaction. Therefore, we can distinguish the individual levels in the dynamics only for the latter.

The return amplitude decays for both types of systems rapidly (cf. Fig. 3) but it recovers much earlier for interacting systems, not to the full value though. Remarkable is also the behavior of the transition amplitude. The time T_t it takes to reach the state $|N, 0\rangle$ from $|0, N\rangle$ for the first time is about the same for all three types of systems, indicating that $T_t \approx \pi/J$ must be solely determined by the tunneling rate J. This provides a method to measure the tunneling rate J in the dynamics of the system, regardless of the interaction.

Our main goal, the dynamical creation of an entangled state from a pure state, is also strongly affected by T_t, since entanglement in terms of a N00N state is not possible for times shorter than T_t. For times larger than T_t only the interacting systems can reach the state $|N, 0\rangle$ while maintaining a non-zero overlap with the initial state (cf. Figs. 3, 4). On the other hand, only for a small number of photons (e.g., $N = 2$) the harmonic cavities are capable to create

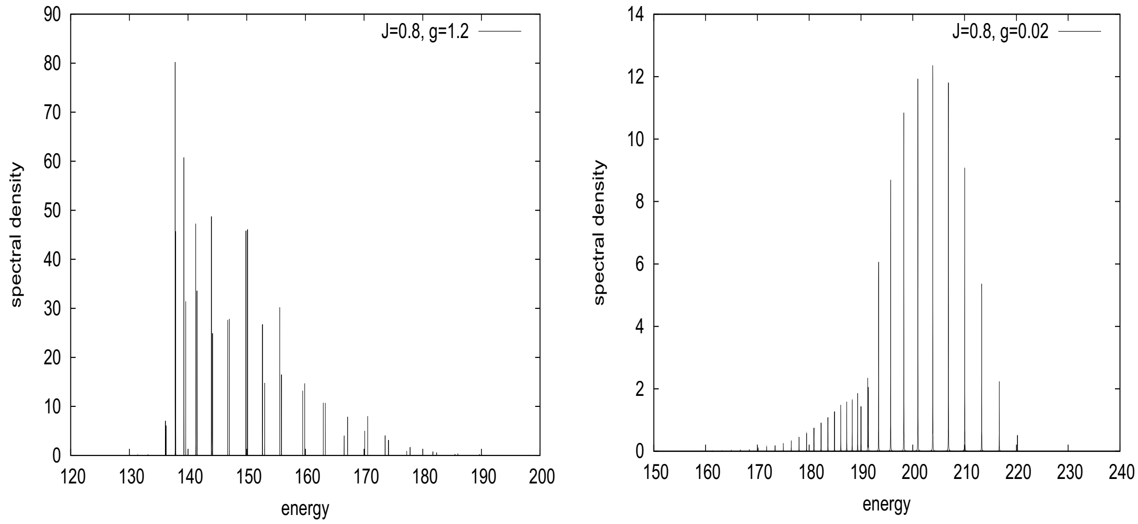

Figure 2. Spectral density $\rho_{0,0}$ for 100 bosons with tunneling rate $J = 0.8$. Left panel: Two coupled cavities where the photons are coupled with two-level atoms with coupling strength $g \approx 1.2$. Right panel: Bosonic atoms in a double-well potential with Hubbard coupling constant $g = 0.02$. All energies of the cavities are measured in units of the cavity frequency $\hbar\omega_0$, the energies of the bosonic atoms in units of the double-well potential.

a N00N state dynamically, as discussed in Sect. 3. For a discrete sequence of time steps we have counted the occurrence of certain values of the return amplitude ($\langle N, 0|e^{-iHt}|N, 0\rangle$ and the transition amplitude $\langle 0, N|e^{-iHt}|N, 0\rangle$) for interacting bosons. This yields the conditional probability $P_t(c_0, c_N)$, which was defined at the end of Sect. 2. Examples are plotted in Fig. 6. The plots demonstrate that the dynamical creation of a N00N state is feasible for anharmonic cavities even for $N = 100$. However, in comparison with bosonic atoms in a double well with Hubbard interaction this probability is quite small.

Phenomenological damping is described by an imaginary parameter $i\epsilon$ in the complex energy $z = E + i\epsilon$ of Eq. (1), which produces exponential damping on the time scale $T = \hbar/\epsilon$. In Fig. 5 the effect of damping on the transition dynamics is plotted for coupled cavities. As expected, the damping is crucial only on time scales larger that T. This implies that only strong damping may play a role in the suppression of dynamically created entanglement.

8. Discussion and Conclusions

We have considered a pair of optical cavities, where the photons in each cavity couple to a two-level atom and compared this system with bosonic atoms in a double-well potential. The cavities are described as JC models, while the atomic system is described by a Bose-Hubbard Hamiltonian. For a large number of photons the coupled JC models can be approximated by an effective Hamiltonian that resembles the Bose-Hubbard Hamiltonian, where the interaction term depends on the square root of the density.

After the initial preparation with all bosons in one subsystem the quantum system is allowed to evolve due to tunneling between the two subsystems. We have found that the boson-boson interaction plays a crucial role in terms of the spectral properties as well as for the dynamic properties: While the spectral density and the dynamics are very regular for harmonic cavities, the spectrum of the interacting systems are less regular with smaller level spacings, and the corresponding dynamics is also quite irregular. For harmonic cavities (i.e. in the absence

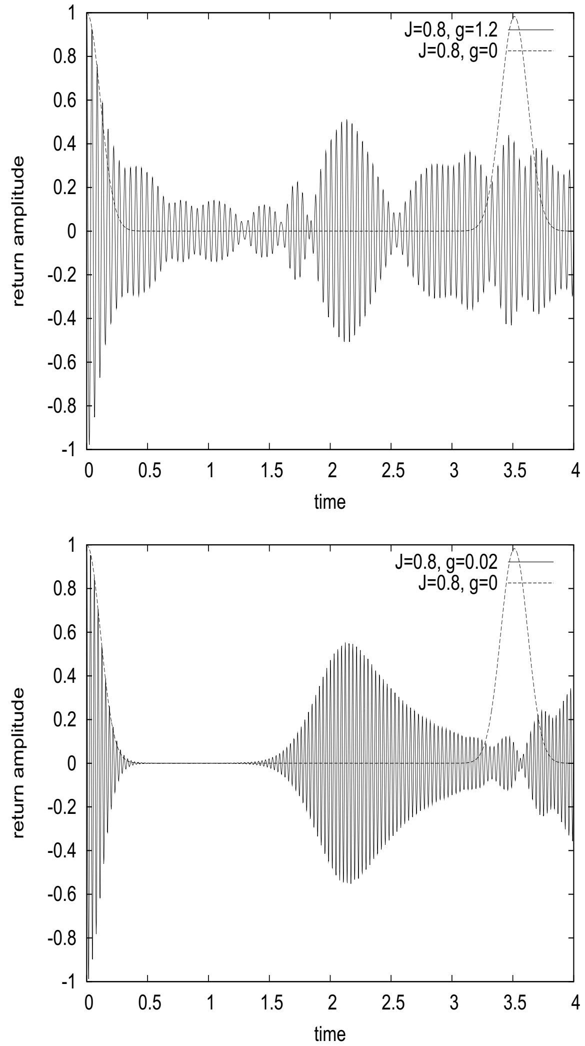

Figure 3. Return amplitude $\langle\psi_0|\psi_t\rangle$ as a function of time for two coupled harmonic cavities (dashed curve), for two coupled cavities with two-level atoms (full curve in the upper panel) and for bosonic atoms in a double-well potential (full curve in the lower panel). The parameters are the same as in the previous Figure, the time is measured in units of $1/\omega_0$.

of the two-level atoms) the distribution of the levels with spectral weights $p_j = |\langle\Psi_0|E_j\rangle|^2$ is binomial for equidistant energy levels with spacing $\Delta E = 2J$. The resulting evolution is periodic and is reminiscent of Rabi oscillations with a single frequency J. Such a behavior was observed experimentally for weakly interacting bosonic atoms [4, 5] and should also be accessible in experiments with photons in harmonic cavities.

The amplitudes for visiting the initial Fock state $|N,0\rangle$ or the complimentary Fock state $|0,N\rangle$ vary as $\cos^N(Jt)$ or $(-i)^N \sin^N(Jt)$, respectively. This implies for a large number N of bosons that (i) these states are visited only for a very short period of time and (ii) the two Fock states are visited at well separated times. Thus the dynamical creation of a N00N state from a Fock state $|N,0\rangle$ is very unlikely for harmonic cavities, unless the number of photons is small. The reason is that the photons can travel without seeing each other through the entire Hilbert space. A simultaneous overlap of $|\Psi_t\rangle$ with both Fock states $|N,0\rangle$ and $|0,N\rangle$ is very

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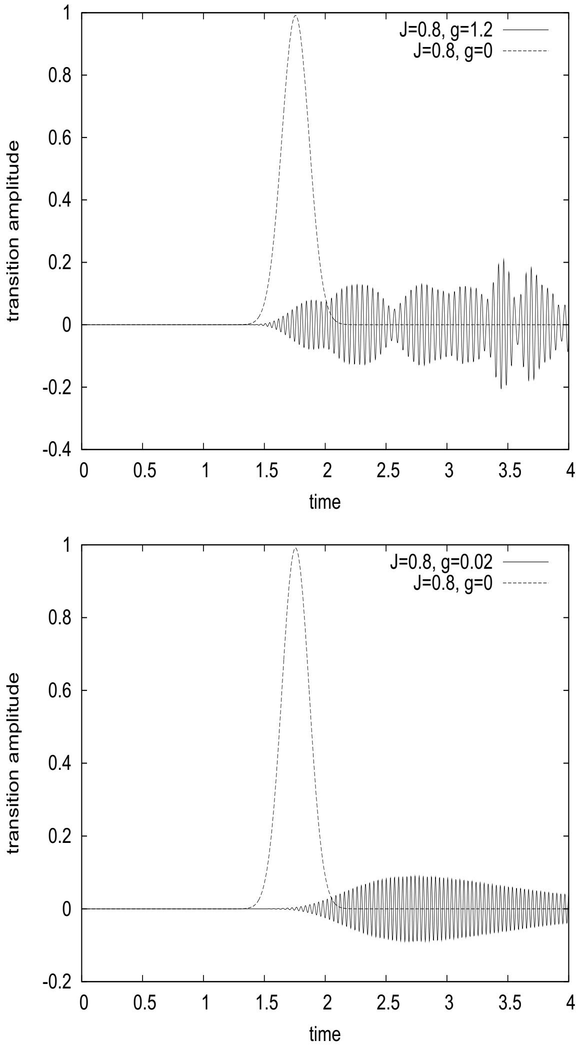

Figure 4. Transition amplitude $\langle\psi_1|\psi_t\rangle$ as a function of time for two coupled harmonic cavities (dashed curve), for two coupled cavities with two-level atoms (full curve in the upper panel) and for bosonic atoms in a double-well potential (full curve in the lower panel). The parameters are the same as in the previous Figure.

unlikely then. This is a situation in which it is very difficult to control and follow the quantum evolution. On the other hand, applications of finite quantum systems, such as in quantum information processing [39, 40], require a controllable evolution, in which only certain parts of the available Hilbert space can be visited with reasonable probability. In terms of our two-cavity system this means that the spectral weight $p_j = |\langle\Psi_0|E_j\rangle|^2$ with respect to the initial state $|\Psi_0\rangle$ is small for most eigenstates $|E_j\rangle$ and has only a few pronounced maxima that can be used for information storage. We have found that such a structured spectral density appears for anharmonic cavities, created by coupling two-level atoms to the cavity photons. Then the photons experience a mutual influence which restricts their individual random walks in Hilbert space significantly and, what is even more important here, they can have a simultaneous overlap with both states $|N,0\rangle$ and $|0,N\rangle$. This effect enables the system to create dynamically a N00N state. The situation is even better for bosonic atoms in a double-well potential, where the

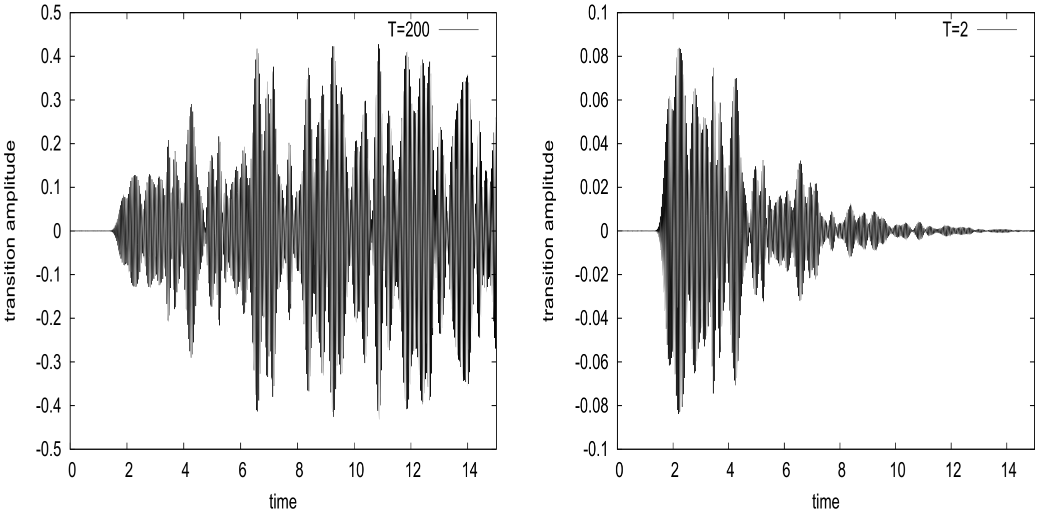

Figure 5. Effect of an increased damping on the transition amplitude $\langle\psi_1|\psi_t\rangle$ for coupled cavities with decay times $T = 200$ and $T = 2$. The other parameters are the same as in the previous Figures.

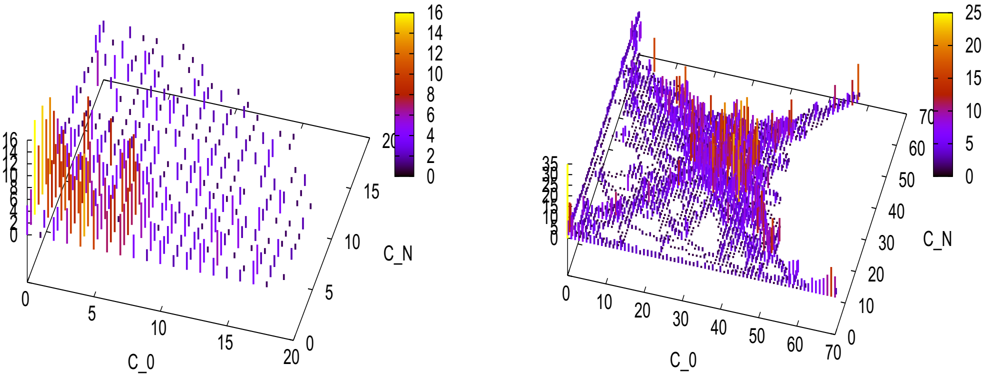

Figure 6. Probability $P_t(c_0, c_N)$ for creating a N00N state for large times in coupled anharmonic cavities with $N = 100$ photons, $J = 0.8$, $g = 1.2$ (left panel) and for $N = 20$ bosonic atoms in a double-well potential with $U/J \approx 0.026$ (right panel, from Ref [29]). The axes are scaled by a factor 100.

stronger interaction effects supports the formation of a N00N state. This allows us to conclude that the complex quantum dynamics of two coupled anharmonic optical cavities and bosonic atoms in a double-well potential offer an approach for quantum information processing as it has also been proposed for ultracold atoms [39] and cold trapped ions [40].

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