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Renormalized transport properties of randomly gapped two-dimensional Dirac fermions

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We investigate the scaling properties of the recently acquired fermionic nonlinear σ model which controls gapless diffusive modes in a two-dimensional disordered system of Dirac electrons beyond charge neutrality. The transport on large scales is governed by a renormalizable nonlocal field theory. For zero-mean random gap, it is characterized by the absence of a dynamic gap generation and a scale-invariant diffusion coefficient. The β function of the dc conductivity, computed for this model, is in perfect agreement with numerical results obtained previously.

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I. INTRODUCTION

Transport in systems whose band structure has a node structure (e.g., Dirac points), as it appears in graphene and on the surface of three-dimensional (3D) topological insulators, has been the subject of intense research recently. The experimental observation of transport in graphene is characterized by a minimal conductivity at the charge neutrality (or Dirac) point and by a linearly increasing conductivity with increasing (electron or hole) charge density.[1,2] Thus, experimentally it is easy to distinguish the very robust minimal conductivity, which is contrasted by the disorder-dependent conductivity away from the Dirac point. The behavior at the Dirac point has also been predicted by field theory, showing that the minimal conductivity is quite independent (or very weakly dependent) on disorder. For instance, a nonlinear σ-model approximation[3–5] as well as perturbation theory in terms of disorder strength clearly shows a very weak disorder dependence.[6] This has led to the claim that ballistic transport cannot be distinguished from diffusive transport at the Dirac point. However, transport properties away from the Dirac point are theoretically not easily accessible. There have been several attempts, based on a classical Boltzmann approach, which predict the experimentally observed linearly increasing conductivity as we go away from the Dirac point. However, it was only recently that a more general field-theoretical approach, based on the Kubo formalism, was suggested to describe the transport of two-dimensional (2D) Dirac fermions within a unified theory,[3,7] using a four-body Hamiltonian.

In this paper, we focus on disorder due to a random gap. This case is particularly interesting because it can lead to a metal-insulator transition when the average gap is equal to a critical value.[5] Starting from a nonlinear σ model that controls the diffusive modes, we study the renormalization of the interaction of these modes as well as the renormalization of the diffusion coefficient and the conductivity at the Dirac point and away from it.

II. MODEL

Below we give a brief sketch of the field-theoretical approach to the conductivity. Here we are led by the representation given in Refs. [3–5,8]. The main quantity to be computed is the disorder averaged two-particle Green's function. The disorder potential v of the strength g is supposed to have zero mean $\langle v_r\rangle = 0$ and Gaussian correlator $\langle v_r v_{r'}\rangle = g\delta_{rr'}$. For random gap disorder, the disorder averaged two-particle Green's function reads

$$K_{rr'} = -\big\langle \mathrm{Tr}_n\big[G_{rr'}(i\epsilon)\sigma_1 G^{\mathrm{T}}_{r'r}(i\epsilon)\sigma_1\big]\big\rangle_v = \sum_{m,m',n,n'} [\sigma_1]_{mn}[\sigma_1]_{n'm'}\big\langle \phi^1_{r'm'}\bar\phi^1_{rm}\phi^2_{rn}\bar\phi^2_{r'n'}\big\rangle_\phi, \quad (1)$$

where Tr_n is taken on the extended Dirac space and ϕ is a four-component superfield $\phi = (\psi_{1,+},\psi_{1,-},\chi_{2,-},\chi_{2,+})$, consisting of a complex $\psi^*_{1,\pm}$ and a Grassmann $\chi^*_{2,\pm}$ field. In our notation, $\sigma_{1,2,3}$ are the usual Pauli matrices and σ_0 is the 2×2 unity matrix. The field averaging is defined as

$$\langle\cdots\rangle_\phi = \int \mathcal{D}[\phi]\cdots e^{-\mathcal{S}}, \quad (2)$$

with the action

$$\mathcal{S} = -i[\phi\cdot(\bar H_0 + i\bar\epsilon)\bar\phi] + g(\phi\cdot\bar\sigma_3\bar\phi)^2, \quad (3)$$

where $\bar\sigma_3 = 1_4\otimes\sigma_3$, $\bar\epsilon = \epsilon 1_4\otimes\sigma_0$, 1_4 is the 4×4 unity matrix, and

$$\bar H = \begin{pmatrix} H+\mu & 0 & 0 & 0 \\ 0 & H-\mu & 0 & 0 \\ 0 & 0 & H^{\mathrm{T}}-\mu & 0 \\ 0 & 0 & 0 & H^{\mathrm{T}}+\mu \end{pmatrix}, \quad (4)$$

with the chemical potential μ, the nonrandom Hamiltonian $H_0 = i\sigma\cdot\nabla$, and the random Hamiltonian

$$H = H_0 + v\sigma_3. \quad (5)$$

Then $\bar H$ is invariant under the global symmetry transformation

$$\bar H = e^{\bar S}\bar H e^{\bar S}, \quad (6)$$

where $\bar S$ is given by the following matrix:

$$\bar S = \begin{pmatrix} 0 & 0 & \varphi_1\sigma_1 & 0 \\ 0 & 0 & 0 & \varphi_2\sigma_1 \\ \varphi'_1\sigma_1 & 0 & 0 & 0 \\ 0 & \varphi'_2\sigma_1 & 0 & 0 \end{pmatrix}, \quad (7)$$

with two scalar fields φ_1 and φ_2, which obey Grassmann statistics, i.e., $\varphi_i\varphi'_i = -\varphi'_i\varphi_i$ and $\varphi_i\varphi_j = -\varphi_j\varphi_i$.

We decouple the interaction term in Eq. (3) by a Hubbard-Stratonovich transformation. Integrating out superfields ϕ

yields an action in terms of composite supersymmetric Hubbard-Stratonovich fields $\bar{Q}$,

$$\mathcal{S}' = \frac{1}{g}\mathrm{Trg}(\bar{Q})^2 + \log \mathrm{detg}(\bar{H}_0 + i\bar{\epsilon} + 2\bar{Q}\bar{\sigma}_3). \tag{8}$$

A nontrivial vacuum of this theory, $\bar{Q}_0$, is found from the saddle-point condition and turns out to be degenerated with respect to the transformation $\bar{S}$ defined in Eq. (7):

$$e^{\bar{S}}\bar{Q}_0 e^{-\bar{S}} = \bar{Q}_1 + \bar{Q}_2 e^{-2\bar{S}}, \tag{9}$$

where $\bar{Q}_1$ ($\bar{Q}_2$) commutes (anticommutes) with $\bar{S}$, and vanishes under the graded trace, $\mathrm{Trg}Q_0 = 0$. On the saddle-point manifold, the action represents a fermionic nonlinear σ model [3],

$$\mathcal{S}' = \log \mathrm{detg}(\bar{H}_0 + i\bar{\epsilon} + 2\bar{Q}_1\sigma_3 + 2\bar{Q}_2\bar{\sigma}_3 e^{2\bar{S}}). \tag{10}$$

The field $\bar{Q}_2$ represents the order parameter for the spontaneous breaking of the symmetry generated by $\bar{S}$. Expanding Eq. (10) up to second order in $\bar{Q}_2$ as

$$\mathcal{S}' = \mathcal{S}_0 + \mathcal{S}'', \tag{11}$$

and using the exact relation for Grassmann fields,

$$e^{2\bar{S}} = 1 + 2\bar{S} + 2\bar{S}^2, \tag{12}$$

yields

$$\mathcal{S}'' = 4\mathrm{Trg}[\bar{G}_0\bar{Q}_2\bar{\sigma}_3\bar{S}^2 + 2(\bar{G}_0\bar{Q}_2\bar{\sigma}_3\bar{S})^2 + 2(\bar{G}_0\bar{Q}_2\bar{\sigma}_3\bar{S}^2)^2], \tag{13}$$

with matrix Green's function

$$\bar{G}_{0,rr'} = \mathrm{diag}\{g_+, g_-, g_-^{\mathrm{T}}, g_+^{\mathrm{T}}\}_{rr'}, \tag{14}$$

and

$$g_{\pm,rr'} = [H_0 + i(\epsilon + \eta \pm i\mu)\sigma_0]^{-1}_{rr'}, \tag{15}$$

where $\eta \sim \exp[-\pi/g]$ is the scattering rate.[4,5] Eventually, we rewrite Eq. (13) in terms of scalar Grassmann fields φ_j and obtain a nonlocal fermionic theory,

$$\mathcal{S}[\varphi] = \frac{g\eta}{2}\sum_{j=1,2}\sum_{rr'}\Bigg[\varphi_{jr'}\delta_{r'r}(i\epsilon - D\nabla^2)\varphi'_{jr} - 2\eta^2(-1)^j \times \sum_{s=\pm}\mathrm{Tr}_2\{s g_{s,rr'} g_{s,r'r}\}\varphi_{jr'}\varphi'_{jr'}\varphi_{jr}\varphi'_{jr}\Bigg], \tag{16}$$

which describes the diffusion of Dirac electrons. For $\mu < \eta$, the diffusion coefficient reads

$$D \approx \frac{1}{2\pi\eta} + O(\mu). \tag{17}$$

III. RENORMALIZATION-GROUP ANALYSIS

Below we investigate the scaling properties of action equation (16) at large distances. For this purpose, we expand the Fourier transform of the vertex function $\sum_{s=\pm}\mathrm{Tr}_2\{s g_{s,rr'} g_{s,r'r}\}$ to the leading order in momenta of the fields. This changes action equation (16) to

$$\mathcal{S}[\varphi] = \sum_{j,q}\varphi_{jq}(D'q^2 + i\epsilon')\varphi'_{jq} - i\lambda\sum_{j,kqpt}(-1)^j\delta_{k-q,p-t}(p-t)^2\varphi_{jk}\varphi'_{jq}\varphi_{jp}\varphi'_{jt}, \tag{18}$$

with the shorthands $\sum_q = \int d^2q/(2\pi)^2$, $\delta_{k,p} = (2\pi)^2\delta(k+p)$, $\epsilon' = g\eta\epsilon/2$, and $D' = g\eta D/2$. The interaction strength is defined as

$$\lambda = \frac{2}{3\pi}\frac{\mu\eta^3}{(\eta^2+\mu^2)^2} \approx \frac{2}{3\pi}\frac{\mu}{\eta}. \tag{19}$$

The zeroth-order term in field momenta of the interaction part is zero due to its locality, while the first-order term vanishes by the symmetry. The frequency ϵ is supposed to be small and sent to zero in the dc limit. For this reason, we do not distinguish between ϵ' and ϵ.

In order to find the infrared behavior of action equation (18), we follow the usual prescription of the Wilson renormalization-group (RG) transformation:[9] We decompose Grassmann fields into fast φ_f and slow φ_s modes. The idea is to integrate out fast modes and to obtain an action which mimics action equation (18), but contains solely slow fields. To the second order in dc perturbation theory, this action reads

$$\bar{\mathcal{S}}[\varphi_s] \approx \mathcal{S}_0[\varphi_s] + \mathcal{S}_{\mathrm{int}}[\varphi_s] + \langle\mathcal{S}_{\mathrm{int}}[\varphi_s,\varphi_f]\rangle_f^{\mathrm{dc}} - \tfrac{1}{2}\langle\mathcal{S}_{\mathrm{int}}[\varphi_s,\varphi_f]\mathcal{S}_{\mathrm{int}}[\varphi_s,\varphi_f]\rangle_f^{\mathrm{dc}}, \tag{20}$$

with

$$\mathcal{S}_{\mathrm{int}}[\varphi_s] = -i\lambda\sum_{j,kqpt}(-1)^j\delta_{k-q,p-t}(p-t)^2 \times \varphi_{sjk}\varphi'_{sjq}\varphi_{sjp}\varphi'_{sjt}, \tag{21}$$

and $\mathcal{S}_{\mathrm{int}}[\varphi_s,\varphi_f]$ represents terms which contain both slow and fast fields. The averaging operator reads

$$\langle\cdots\rangle_f^{\mathrm{dc}} = \frac{1}{\mathcal{Z}_0^{\mathrm{dc}}}\int\mathcal{D}[\varphi_f]\cdots e^{-\mathcal{S}_0^{\mathrm{dc}}[\varphi_f]}, \tag{22}$$

with the free dc action

$$\mathcal{S}_0^{\mathrm{dc}}[\varphi_f] = D'\sum_{j,r}\nabla\varphi_{jrf}\nabla\varphi'_{jrf}. \tag{23}$$

Obviously, this construction guarantees $\langle 1\rangle_f^{\mathrm{dc}} = 1$ and defines the dc propagator (in Fourier representation)

$$\langle\varphi_{iqf}\varphi'_{jkf}\rangle_f = \frac{\delta_{ij}\delta_{q,-k}}{D'q^2}. \tag{24}$$

The derivation of the renormalization-group equations for the frequency ϵ, "diffusion" coefficient D', and interaction strength λ of the action equation (18) is a challenging task. Diagrams, which have to be evaluated, arise by merging vertices shown in Figs. 1–3. The renormalization of the energy ϵ comes from one-loop diagrams, which emerge by averaging vertices depicted in Fig. 1 over fast fields. As shown in Appendix A, the evaluation of the diagrams yields a result that does not develop any divergences in the infrared, since the vertex is proportional to the squared loop momentum, and the

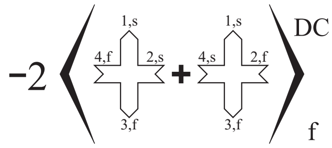

FIG. 1. Diagrams responsible for renormalization of ϵ.

propagator to inverse squared loop momentum:

$$\bar{\epsilon} = \epsilon + \frac{(-1)^j}{2\pi}\frac{\lambda}{D'}\Lambda_0^2, \tag{25}$$

where Λ_0 denotes an upper cutoff. This expression is not a renormalization-group equation in the strict sense, since it does not contain the running cutoff parameter $\ell = \log \Lambda_0/\Lambda$. It represents a kind of finite-size effect, which disappears in the continuous limit.

The renormalization of the diffusion coefficient can be obtained by integrating out fast fields in vertices depicted in Fig. 2. The evaluation of the functional integral is presented in Appendix B and yields the following renormalization of the free action:

$$\sum_{i,p} \gamma(p)\varphi_{ip}\varphi'_{ip} \approx \sum_{i,r} [\bar{D}'\nabla\varphi_{ir}\nabla\varphi'_{ir} + m\varphi_{ir}\varphi'_{ir}], \tag{26}$$

where m and $\bar{D}'$ are expansion coefficients of zeroth and second order in momentum p of the vertex function

$$\gamma(p) = 4\frac{(i\lambda)^2}{D'^3}\sum_{kq}\frac{(k+p)^2[(k+p)^2-(q+p)^2]}{k^2q^2(k+q+p)^2}. \tag{27}$$

There is no linear term in this expansion, since $\gamma(p)$ is symmetric with respect to the sign mirroring of p, i.e., $\gamma(-p) = \gamma(p)$. In the Appendices B1 and B2, it is shown that both coefficients m and $\bar{D}'$ vanish. This is a very important result, since it guarantees the preservation of the gapless diffusive mode and the reality of the diffusion coefficient even for the complex interaction strength. Therefore, the only running parameter is the interaction strength λ. Its renormalization is due to the one-loop diagrams, which emerge after integrating out fast fields in Fig. 3. The lengthy and elaborate calculations presented in Appendix C lead to the remarkably simple renormalization-group equation

$$\partial_\ell(i\lambda_j) = \frac{(-1)^j}{\pi}\frac{(i\lambda_j)^2}{D'^2}, \tag{28}$$

for each fermionic channel. This equation is easily solved with the same starting value $\lambda_{j0} = u_0$ in both channels, with u_0 given in Eq. (19),

$$\lambda_j = \frac{u_0}{1+\frac{u_0^2\ell^2}{\pi^2 D'^4}} + i\frac{(-1)^j}{\pi D'^2}\frac{u_0^2\ell}{1+\frac{u_0^2\ell^2}{\pi^2 D'^4}}. \tag{29}$$

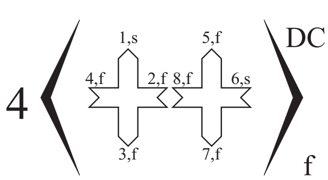

FIG. 2. Diagrams responsible for renormalization of D'.

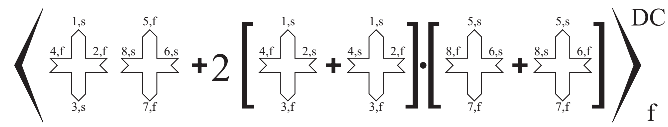

FIG. 3. Diagrams responsible for renormalization of λ.

Both real and imaginary parts of the interaction scale down to zero, but not equally fast. At large scales, the imaginary part of λ becomes dominant and therefore generates a genuine, real interaction. The RG flow for the Grassmann field φ_2 is depicted in Fig. 4. Figure 5 shows the RG landscape in the parametric space spanned by the real and imaginary part of the interaction λ_j. The RG trajectories represent a set of eccentric circles with diameter u_0, each attracted to the Gaussian fixed point at $\Re\lambda_j = 0$ and $\Im\lambda_j = 0$. This can be seen best if we substitute $\lambda_j = u_j + iv_j$ in Eq. (28). Then we get

$$\begin{aligned} \partial_\ell u_j &= (-1)^{1+j} 2u_j v_j, \\ \partial_\ell v_j &= (-1)^j (u_j^2 - v_j^2). \end{aligned} \tag{30}$$

The right-hand side of this system of differential equations represents, indeed, a parametrized circle.

Independent from the choice of the initial value, both the real and imaginary parts of λ_2 (analogously for λ_1) become equal at the length obtained from the condition

$$\frac{u_0\ell_*}{\pi D'^2} = 1 \quad \text{with} \quad \ell_* = \log\frac{\xi}{l},$$

with l denoting the mean free path, which gives

$$\xi = l\exp\left[\frac{\pi D'^2}{u_0}\right] \approx l\exp\left[\frac{3g^2}{32}\frac{\eta}{\mu}\right] \quad \text{for} \quad \mu < \eta. \tag{31}$$

Here we used definitions of the bare diffusion coefficient given by Eq. (17) and interaction strength given by Eq. (19). At half filling, i.e., for $\mu = 0$, this scale is infinite.

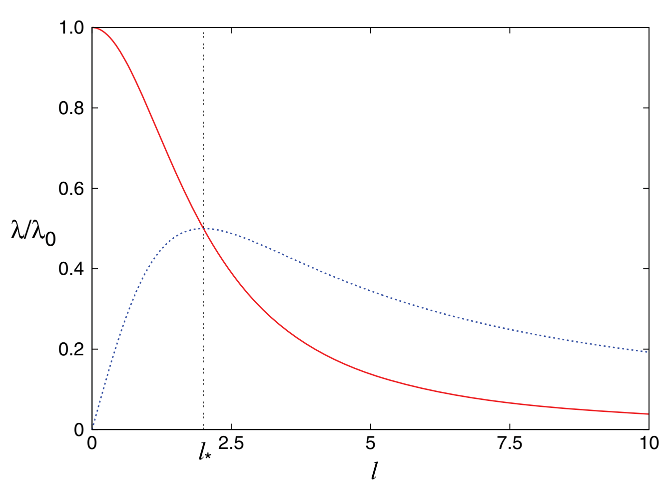

FIG. 4. (Color online) Renormalization of the interaction strength λ_2. Solid (red) line shows the real part and dashed (blue) line shows the imaginary part of λ_2. The crossover scale is shown by the vertical (black) dotted line at $\ell_* = \pi D'^2 u_0^{-1}$.

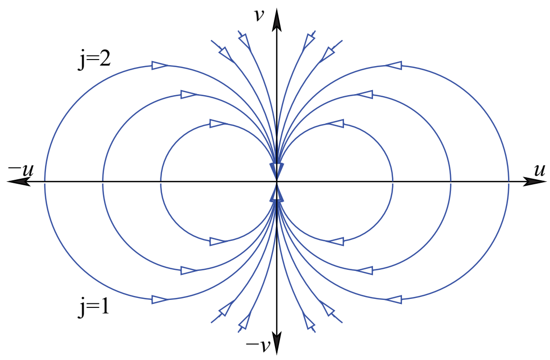

FIG. 5. (Color online) Renormalization-group flow of the interaction λ_j in the parametric space spanned by $u = \Re\lambda_j$ and $v = \Im\lambda_j$. The flow in the upper halfplane corresponds to the Grassmann field φ_2, and that in the lower plane corresponds to φ_1.

IV. SCALING PROPERTIES OF THE DC CONDUCTIVITY

Our ultimate task is to determine the scaling behavior of the dc conductivity. For this we need to compute the corrections to the conductivity which arise due to the doping. The dc conductivity is either determined from the Einstein relation $\bar{\sigma} \propto \rho D$ (ρ is the density of states at the Fermi level) or calculated from the Kubo formula,

$$\bar{\sigma} = 2\epsilon^2 \left.\frac{\partial}{\partial q^2}\bar{K}(q)\right|_{q=0}, \tag{32}$$

where the two-particle Green's function takes contributions from both channels $j = 1,2$ into account:

$$\bar{K}(q) = \frac{1}{g}\sum_{ij,p}\langle\varphi_{iq}\varphi'_{jp}\rangle. \tag{33}$$

Here, the functional integral should be performed over the full action equation (18):

$$\langle\cdots\rangle = \frac{1}{\mathcal{Z}}\int\mathcal{D}[\varphi]\cdots e^{-\mathcal{S}[\varphi]}, \tag{34}$$

with $\langle 1\rangle = 1$. To the leading order in λ_j, the two-particle Green's function is approximated as

$$\bar{K}(q) \approx \frac{1}{g}\sum_{ij,p}(\langle\varphi_{iq}\varphi'_{jp}\rangle^0 - \langle\varphi_{iq}\varphi'_{jp}\mathcal{S}_{\text{int}}\rangle^0), \tag{35}$$

where $\mathcal{S}_{\text{int}}$ is given in Eq. (18). The functional integration is to be performed over the free action only. As shown in Appendix D, we eventually arrive at the following one-loop RG equation for the conductivity:

$$\partial_\ell\sigma = -2i\frac{\sigma_0}{gD'}\sum_{j=1,2}[(-1)^j\lambda_j]. \tag{36}$$

Further progress can be made if we exploit Eq. (29),

$$\partial_\ell\sigma = \frac{4\sigma_0}{\pi g D'^3}\frac{u_0^2\ell}{1+\frac{u_0^2\ell^2}{\pi^2 D'^4}}. \tag{37}$$

Unpleasant constants can be eliminated by rescaling $u_0 \to \pi D'^2 u_0$ and using the definition of the diffusion coefficient

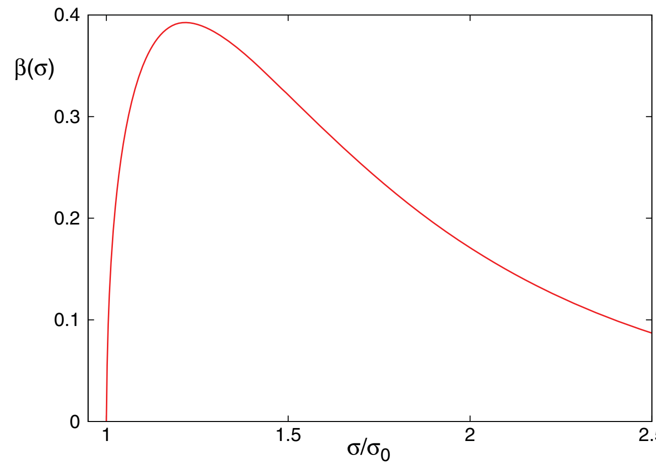

FIG. 6. (Color online) β function corresponding to the conductivity in Eq. (39).

$D' = g/4\pi$. This finally yields

$$\partial_\ell\sigma = \sigma_0\frac{u_0^2\ell}{1+u_0^2\ell^2}. \tag{38}$$

The integration of this equation is simple and we obtain the following asymptotic expression for the conductivity:

$$\sigma(u_0\ell) = \sigma_0 + \frac{\sigma_0}{2}\log\left[1+u_0^2\ell^2\right]. \tag{39}$$

At half filling, i.e., $u_0 = 0$, the conductivity does not flow, i.e., it is scale invariant. At large scales, i.e., for $L \gg \xi$, with ξ given in (31), the conductivity grows bilogarithmically as a function of the sample size $\sigma(L) \sim \sigma_0 \log\log L/\xi$. For this reason, ξ can be associated with an intermediate localization scale. Due to the infrared asymptotic freedom of the underlying model, this result should be asymptotically correct in all loops.

V. DISCUSSION AND CONCLUSIONS

The scaling properties of the conductivity are usually given by the β function,

$$\beta(\sigma) = \frac{d}{dl}\log\sigma, \tag{40}$$

with rescaled logarithmic length $l = u_0\ell$. The conductivity from Eq. (39) generates the β function, as depicted in Fig. 6. Its shape reveals a striking resemblance to the numerically determined β function of graphene at the Dirac point with random scalar potential disorder:[11] It starts at the value of the universal minimal conductivity; it is strictly positive; it reveals a distinct maximum related to the length ξ; and it does not have any fixed points besides $\sigma = \sigma_0$ and $\sigma = \infty$. Finally, it does not depend on any quantities apart from the conductivity σ itself, in line with the one-parameter scaling hypothesis. However, one might wonder whether the above-mentioned bilogarithmic asymptotics of the conductivity compares well with the predicted logarithmic growth at the Dirac point.[8,10,11] It is, indeed, not difficult to reproduce the β function of Ref. [11] by applying the Wilson RG transformation directly to the two-particle Green's function and exploiting the scaling properties of the disorder strength g. Instead, in our approach, we keep g scale invariant. This assumption suits well for weak disorder, provided the sample size is much smaller than disorder-generated intrinsic length $\sim\exp[1/g]$,[12] which corresponds to the common experimental situation. Under these circumstances, the scale invariance of the conductivity was demonstrated both numerically and analytically.[6,13] On the other hand, in finite samples, the conductivity grows logarithmically as a function of the chemical potential: $\sigma(u_0) \sim \sigma_0 \log(\mu/\eta)$, in accordance with Ref. [14].

In conclusion, we have presented a scaling analysis of the diffusion coefficient and the dc conductivity of doped graphene with a random gap. For this purpose, we have used an alternative field theory and investigated its scaling properties. On this basis, we derived an invariant diffusion coefficient and an astonishingly simple expression for the scaling of the conductivity that reproduces the distinct shape of the β function of disordered graphene, found previously in numerical calculations.[11]

ACKNOWLEDGMENT

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APPENDIX A: RENORMALIZATION OF THE ENERGY ϵ

For the sake of simplicity, we use below the following notation for the interaction part of the action equation (18):

$$\mathcal{S}_{\text{int}} = -i\lambda \sum_j (-1)^j \int d1d2d3d4\delta(1+3-2-4) \times (3-4)^2 \varphi_{j1}\varphi'_{j2}\varphi_{j3}\varphi'_{j4}.$$

The renormalization of ϵ is due to diagrams which are obtained by contracting fast fields in Fig. 1. Contracting is possible in the only way feasible, i.e., the diagrams have onefold degeneracy. To take Grassmann statistics correctly into account, it is necessary to permute fast fields through, such that they form "normal ordered" pairs $\langle\varphi\varphi'\rangle$:

$$\begin{aligned} &2i\lambda \sum_j (-1)^j \int d1d2d3d4(3-4)^2\delta(1+3-2-4) \\ &\times [\varphi_{j1s}\varphi'_{j2s}\dot{\varphi}_{j3f}\dot{\varphi}'_{j4f} + \varphi_{j1s}\dot{\varphi}'_{j2f}\dot{\varphi}_{j3f}\varphi'_{j4s}] \\ &= 2i\lambda \sum_j (-1)^j \int d1d2d3d4(3-4)^2\delta(1+3-2-4) \\ &\times [\varphi_{j1s}\varphi'_{j2s}\langle\dot{\varphi}_{j3f}\dot{\varphi}'_{j4f}\rangle - \varphi_{j1s}\varphi'_{j4s}\langle\dot{\varphi}_{j3f}\dot{\varphi}'_{j2f}\rangle]. \end{aligned}$$

The fields to be contracted are marked with black dots. Then, the contractions can be performed and we obtain

$$\begin{aligned} &2i\lambda \sum_j (-1)^j \int d1d2d3d4(3-4)^2\delta(1+3-2-4) \\ &\times [\varphi_{j1s}\varphi'_{j2s}\delta(3-4)\Pi(4) - \varphi_{j1s}\varphi'_{j4s}\delta(2-3)\Pi(3)], \end{aligned}$$

where $\Pi(q) = 1/D'q^2$. The first contribution is zero because

$$\int d3(3-4)^2\delta(3-4) = 0.$$

The second contribution is finite and cutoff dependent:

$$\begin{aligned} &-2i\lambda \int d3\Pi(3) \sum_j (-1)^j \int d1(3-1)^2 \varphi_{j1s}\varphi'_{j1s} \\ &\approx -2i\lambda \int d3\Pi(3)(3)^2 \sum_j (-1)^j \int d1\varphi_{j1s}\varphi'_{j1s}. \end{aligned}$$

The renormalization factor then reads

$$-2i\lambda \int d3\Pi(3)(3)^2 = -\frac{i}{2\pi}\frac{\lambda}{D'}(\Lambda_0^2 - \Lambda^2),$$

which reduces for $\Lambda \to 0$ to $-i\lambda\Lambda_0^2/2\pi D'$. Lifting it into the exponent and absorbing into the action gives Eq. (25).

APPENDIX B: RENORMALIZATION OF THE DIFFUSION COEFFICIENT

The renormalization of the diffusion coefficient comes from the diagrams constructed from vertices depicted in Fig. 2. Every diagram is twice degenerated, i.e., the analytical expression reads

$$\begin{aligned} &4(-i\lambda)^2 \sum_{ij} (-1)^{i+j} \int d1d2d3d4 \int d5d6d7d8 \\ &\times (3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8) \\ &\times [\varphi_{s1i}\dddot{\varphi}'_{f2i}\ddot{\varphi}_{f3i}\dot{\varphi}'_{f4i}\dddot{\varphi}_{f5j}\varphi'_{s6j}\dot{\varphi}_{f7j}\ddot{\varphi}'_{f8j} \\ &+ \varphi_{s1i}\dddot{\varphi}'_{f2i}\ddot{\varphi}_{f3i}\dot{\varphi}'_{f4i}\dot{\varphi}_{f5j}\varphi'_{s6j}\dddot{\varphi}_{f7j}\ddot{\varphi}'_{f8j}]. \end{aligned}$$

We permute fields through and perform functional integrations in order to get

$$\begin{aligned} &4(i\lambda)^2 \sum_{ij} (-1)^{i+j} \int d1d2d3d4 \int d5d6d7d8\varphi_{1si}\varphi'_{6sj} \\ &\times (3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8) \\ &\times [\langle\varphi_{f5j}\varphi'_{f2i}\rangle\langle\varphi_{f3i}\varphi'_{f8j}\rangle\langle\varphi_{f7j}\varphi'_{f4i}\rangle \\ &- \langle\varphi_{f7j}\varphi'_{f2i}\rangle\langle\varphi_{f3i}\varphi'_{f8j}\rangle\langle\varphi_{f5j}\varphi'_{f4i}\rangle] \\ &= 4(i\lambda)^2 \sum_{ij} (-1)^{i+j}\delta_{ij}\delta_{ij}\delta_{ij} \int d1d2d3d4 \int d5d6d7d8 \\ &\times (3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8) \\ &\times \varphi_{1si}\varphi'_{6sj}\Pi(2)\Pi(4)\Pi(8)[\delta(5-2)\delta(3-8)\delta(7-4) \\ &- \delta(7-2)\delta(3-8)\delta(5-4)] \\ &= 4(i\lambda)^2 \sum_i \int d1\varphi_{1i}\varphi'_{1i} \int d2d4(2-1)^2 \\ &\times [(2-1)^2 - (4-1)^2]\Pi(2)\Pi(4)\Pi(2+4-1) \\ &= \sum_{i,p} \gamma(p)\varphi_{ip}\varphi'_{ip}. \end{aligned}$$

The function $\gamma(p)$ is defined in Eqs. (26) and (27). The way in which the momenta in arguments of δ functions are integrated out is not unique. Therefore, we can shift integration variables when necessary.

1. Computation of the mass term

First we evaluate the expression for the mass:

$$m \propto \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{k^2[k^2-q^2]}{k^2q^2(k+q)^2} = \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \left[\frac{k^2}{q^2(k+q)^2} - \frac{1}{(k+q)^2}\right].$$

Shifting in the second term $q \to q-k$, we have

$$\int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2} = \frac{\ell}{2\pi} \int \frac{d^2k}{(2\pi)^2},$$

where $\ell = \log \Lambda_0/\Lambda$. The first term is conveniently evaluated using Feynman parametrization:

$$\int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{k^2}{q^2(k+q)^2} = \int \frac{d^2k}{(2\pi)^2} k^2 \int_0^1 dx \int \frac{d^2q}{(2\pi)^2} \times \frac{1}{[(1-x)q^2 + x(k+q)^2]^2} = (*).$$

Performing shift $q \to q - xk$, we get

$$(*) = \int \frac{d^2k}{(2\pi)^2} k^2 \int_0^1 dx \int \frac{d^2q}{(2\pi)^2} \frac{1}{[q^2 + x(1-x)k^2]^2} = \frac{1}{4\pi} \int \frac{d^2k}{(2\pi)^2} k^2 \int_0^1 dx \frac{1}{k^2 x(1-x)} = \frac{1}{4\pi} \int_0^1 dx \left[\frac{1}{x} + \frac{1}{1-x}\right] \int \frac{d^2k}{(2\pi)^2} = \frac{1}{2\pi} \int_{e^{-\ell}}^1 \frac{dx}{x} \int \frac{d^2k}{(2\pi)^2} = \frac{\ell}{2\pi} \int \frac{d^2k}{(2\pi)^2},$$

i.e., the very same result. Therefore, the mass is zero, and the diffusive Goldstone mode is preserved.

2. Computation of the diffusion coefficient renormalization

Next we evaluate the renormalization of the diffusion coefficient:

$$\bar{D}' = 4\frac{(i\lambda)^2}{D'^3} \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \times \frac{\partial^2}{\partial p^2} \left[\frac{(q+p)^4}{k^2q^2(k+q+p)^2} - \frac{(q+p)^2(k+p)^2}{k^2q^2(k+p+q)^2}\right]\Bigg|_{p=0}.$$

We start with the first term. Using Feynman parametrization, we have

$$\mathrm{I} = \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{(q+p)^4}{k^2q^2(k+q+p)^2}\Bigg|_{p=0} = \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{(q+p)^4}{q^2} \int_0^1 dx \times \int \frac{d^2k}{(2\pi)^2} \frac{1}{[(1-x)k^2 + x(k+q+p)^2]^2}\Bigg|_{p=0}.$$

Next, shift $k \to k - x(q+p)$:

$$\frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{(q+p)^4}{q^2} \int_0^1 dx \times \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2 + x(1-x)(q+p)^2]^2}\Bigg|_{p=0} = \frac{1}{4\pi} \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{(q+p)^4}{q^2} \times \int_0^1 dx \frac{1}{(q+p)^2 x(1-x)}\Bigg|_{p=0} = \frac{1}{4\pi} \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{(q+p)^2}{q^2}\Bigg|_{p=0} \int_0^1 dx \frac{1}{x(1-x)} = \frac{1}{4\pi} \int \frac{d^2q}{(2\pi)^2} \frac{2}{q^2} \int_0^1 dx \frac{1}{x(1-x)} = 2\left(\frac{\ell}{2\pi}\right)^2.$$

It is important to recognize that shifting of the integration variables with respect to the external momentum p does not affect the final result. This is because the integration over the momentum-conserving δ function is not unique. Indeed, if we shift $q \to q - p$ in the above integral, we obtain the same result:

$$\frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{q^4}{(q-p)^2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2(k+q)^2}\Bigg|_{p=0} = \frac{\ell}{2\pi} \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{(q-p)^2}\Bigg|_{p=0},$$

where we skipped integration over the momentum k. After mirroring sign of q, we have

$$\frac{\ell}{2\pi} \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{(q+p)^2}\Bigg|_{p=0} = \frac{\ell}{2\pi} \int \frac{d^2q}{(2\pi)^2} \left[8\frac{(\hat{e}_p \cdot q)^2}{q^4} - \frac{2}{q^2}\right] = (*).$$

Here we have to shed some light on the structure of the first term:

$$\int \frac{d^2q}{(2\pi)^2} \frac{(\hat{e}_p \cdot q)^2}{q^4} = \int_\Lambda^{\Lambda_0} \frac{qdq}{(2\pi)^2} \frac{1}{q^4} \int_0^{2\pi} d\alpha (\hat{e}_p \cdot q)^2 = \int_\Lambda^{\Lambda_0} \frac{qdq}{(2\pi)^2} \frac{1}{q^4} \int_0^{2\pi} d\alpha q^2 \cos^2(\alpha - \varphi) = \int_\Lambda^{\Lambda_0} \frac{qdq}{(2\pi)^2} \frac{1}{q^4} \int_0^{2\pi} d\alpha \frac{q^2}{2}[1 + \cos 2(\alpha - \varphi)] = \int_\Lambda^{\Lambda_0} \frac{qdq}{(2\pi)^2} \frac{1}{2q^2} \int_0^{2\pi} d\alpha = \int \frac{d^2q}{(2\pi)^2} \frac{1}{2q^2}.$$

Thus we continue:

$$(*) = \frac{\ell}{2\pi} \int \frac{d^2q}{(2\pi)^2} \left[\frac{4}{q^2} - \frac{2}{q^2}\right] = 2\frac{\ell}{2\pi} \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2} = 2\left(\frac{\ell}{2\pi}\right)^2,$$

i.e., the very same result as before. This knowledge can now be used for evaluating the second term:

$$\mathrm{II} = \frac{\partial^2}{\partial p^2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{(q+p)^2(k+p)^2}{k^2q^2(k+p+q)^2}\bigg|_{p=0}.$$

Here we are allowed to shift $k \to k - p$, which yields

$$\begin{aligned}
&\int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{k^2}{q^2(k+q)^2}\ \frac{\partial^2}{\partial p^2}\frac{(q+p)^2}{(k-p)^2}\bigg|_{p=0} \\
&= \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{k^2}{q^2(k+q)^2} \\
&\quad \times \left[\frac{2}{k^2} + q^2\left(8\frac{(\hat{e}_p\cdot k)^2}{k^6} - \frac{2}{k^4}\right) + 8\frac{(\hat{e}_p\cdot q)(\hat{e}_p\cdot k)}{k^4}\right] \\
&= \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{2}{q^2(k+q)^2} \\
&\quad + \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k+q)^2}\left[8\frac{(\hat{e}_p\cdot k)^2}{k^4} - \frac{2}{k^2}\right] \\
&\quad + 8\int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{(\hat{e}_p\cdot q)(\hat{e}_p\cdot k)}{q^2k^2(k+q)^2}.
\end{aligned}$$

The first and second terms can be easily evaluated after shifting $k \to k - q$ and $q \to q - k$, respectively. Both give the same contribution I calculated above. The last term is more cumbersome:

$$\begin{aligned}
&8\int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{(\hat{e}_p\cdot q)(\hat{e}_p\cdot k)}{q^2k^2(k+q)^2} \\
&= 8\int \frac{d^2q}{(2\pi)^2} \frac{\hat{e}_p\cdot q}{q^2} \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \\
&\quad \times \frac{\hat{e}_p\cdot k}{[(1-x)k^2 + x(k+q)^2]^2}.
\end{aligned}$$

Again, we shift $k \to k - xq$, which gives

$$\begin{aligned}
&\int \frac{d^2q}{(2\pi)^2} \frac{\hat{e}_p\cdot q}{q^2} \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{8\hat{e}_p\cdot(k-xq)}{[k^2 + x(1-x)q^2]^2} \\
&= -\int \frac{d^2q}{(2\pi)^2} \frac{(\hat{e}_p\cdot q)^2}{q^2} \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{8x}{[k^2 + x(1-x)q^2]^2} \\
&= -8\int \frac{d^2q}{(2\pi)^2} \frac{(\hat{e}_p\cdot q)^2}{q^4} \frac{1}{4\pi} \int_0^1 \frac{dx}{1-x} \\
&= \frac{1}{\pi}\int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2} \int_1^{e^{-\ell}} \frac{dy}{y} = -2\left(\frac{\ell}{2\pi}\right)^2 = -\mathrm{I},
\end{aligned}$$

where at some point we have substituted $y = 1 - x$. Summing over all contributions to II, we obtain

$$\mathrm{II} = \mathrm{I} + \mathrm{I} - \mathrm{I} = \mathrm{I}.$$

The renormalization of the diffusion coefficient to order λ^2 is therefore zero:

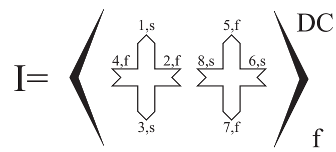
$$\bar{D}' \propto \mathrm{I} - \mathrm{II} = 0.$$

APPENDIX C: RENORMALIZATION OF THE INTERACTION STRENGTH

The renormalization of the interaction is due to one-loop diagrams which arise after merging vertices depicted in Fig. 3. We consider both contributions separately.

1. Contribution I

Contribution I is as follows:



and has a twofold degeneration, i.e., there are two possibilities for contracting fast fields:

$$\begin{aligned}
\mathrm{I} = (-i\lambda)^2 &\sum_{ij}(-1)^{i+j}\int d1d2d3d4\int d5d6d7d8 \\
&(3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8) \\
&\times[\varphi_{s1i}\dot{\varphi}'_{f2i}\varphi_{s3i}\ddot{\varphi}'_{f4i}\dot{\varphi}_{f5j}\varphi'_{s6j}\ddot{\varphi}_{f7j}\varphi'_{s8j} \\
&+\varphi_{s1i}\ddot{\varphi}'_{f2i}\varphi_{s3i}\dot{\varphi}'_{f4i}\dot{\varphi}_{f5j}\varphi'_{s6j}\ddot{\varphi}_{f7j}\varphi'_{s8j}].
\end{aligned} \tag{C1}$$

Next, we permute and contract Grassmann fields:

$$\begin{aligned}
\mathrm{I} = (i\lambda)^2 &\sum_{ij}(-1)^{i+j}\int d1d2d3d4\int d5d6d7d8 \\
&\times(3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8) \\
&\times\big[(-1)^{3+2}\varphi_{s1i}\varphi_{s3i}\langle\varphi_{f5j}\varphi'_{f2i}\rangle\varphi'_{s6j}\langle\varphi_{f7j}\varphi'_{f4i}\rangle\varphi'_{s8j} \\
&+(-1)^{1+5}\varphi_{s1i}\langle\varphi_{f7j}\varphi'_{f2i}\rangle\varphi_{s3i}\langle\varphi_{f5j}\varphi'_{f4i}\rangle\varphi'_{s6j}\varphi'_{s8j}\big] \\
= (i\lambda)^2 &\sum_{ij}(-1)^{i+j}\delta_{ij}\delta_{ji}\int d1d2d3d4\int d5d6d7d8 \\
&(3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8) \\
&\times[\delta(7-2)\delta(5-4) - \delta(5-2)\delta(7-4)] \\
&\varphi_{s1i}\varphi_{s3i}\varphi'_{s6j}\varphi'_{s8j}\Pi(2)\Pi(4).
\end{aligned}$$

Performing summations and integrations, we arrive at

$$\begin{aligned}
\mathrm{I} = (i\lambda)^2 &\int d1d3d6d8\delta(1+3-6-8) \\
&\times\sum_i\varphi_{s1i}\varphi_{s3i}\varphi'_{s6i}\varphi'_{s8i}\int d4(4-3)^2 \\
&\times[(4-6)^2 - (4-8)^2]\Pi(4)\Pi(4-3-1).
\end{aligned}$$

After reordering fields in the first term and renaming variables, we finally obtain

$$\begin{aligned}
\mathrm{I} = -2(i\lambda)^2 &\int d1d2d3d4\delta(1+3-2-4) \\
&\times\sum_i\varphi_{i1}\varphi'_{i2}\varphi_{i3}\varphi'_{i4}\int d5(5-3)^2 \\
&\times(5-2)^2\Pi(5)\Pi(5-3-1).
\end{aligned}$$

Now we expand the vertex function up to the second order in field momenta:

$$\int d5(5-3)^2(5-2)^2\Pi(5)\Pi(5-3-1)$$
$$= \frac{1}{D'^2}\int \frac{d^2q}{(2\pi)^2}\frac{(q-k)^2(q-p)^2}{q^2(q-k-t)^2}$$
$$\approx \left[kp\frac{\partial^2}{\partial k\partial p}+kt\frac{\partial^2}{\partial k\partial t}+pt\frac{\partial^2}{\partial p\partial t}+\frac{k^2}{2}\frac{\partial^2}{\partial k^2}+\frac{p^2}{2}\frac{\partial^2}{\partial p^2}+\frac{t^2}{2}\frac{\partial^2}{\partial t^2}\right]\frac{1}{D'^2}\int \frac{d^2q}{(2\pi)^2}\frac{(q-k)^2(q-p)^2}{q^2(q-k-t)^2}\bigg|_{k,p,t=0}. \quad \text{(C2)}$$

(i) Order kp: The factor is zero, since

$$\frac{\partial^2}{\partial k\partial p}\int \frac{d^2q}{(2\pi)^2}\frac{(q-k)^2(q-p)^2}{q^2(q-k)^2}\bigg|_{k,p=0} = \frac{\partial^2}{\partial k\partial p}\int \frac{d^2q}{(2\pi)^2}\frac{(q-p)^2}{q^2}\bigg|_{k,p=0} = 0$$

after differentiation with respect to k.

(ii) Order kt: The expansion factor is also zero, because

$$\frac{\partial^2}{\partial k\partial t}\int \frac{d^2q}{(2\pi)^2}\frac{(q-k)^2q^2}{q^2(q-k-t)^2}\bigg|_{k,t=0}$$
$$= -\frac{\partial}{\partial k}\int \frac{d^2q}{(2\pi)^2}\frac{2\hat{e}_t\cdot(q-k)}{(q-k)^2}\bigg|_{k=0}$$
$$= -\int \frac{d^2q}{(2\pi)^2}\left[4\frac{(\hat{e}_k\cdot q)(\hat{e}_t\cdot q)}{q^4}-2\frac{\hat{e}_t\cdot\hat{e}_k}{q^2}\right](*).$$

Rewrite the first term as

$$\int_0^{2\pi} d\varphi(\hat{e}_t\cdot q)(\hat{e}_k\cdot q)$$
$$= q^2\int_0^{2\pi} d\varphi\cos(\alpha-\varphi)\cos(\beta-\varphi)$$
$$= \frac{q^2}{2}\cos(\alpha-\beta)\int_0^{2\pi}d\varphi = (\hat{e}_k\cdot\hat{e}_t)\frac{q^2}{2}\int_0^{2\pi}d\varphi.$$

Plugging this back into $(*)$, we see that

$$-\int \frac{d^2q}{(2\pi)^2}\left[4\frac{(\hat{e}_k\cdot q)(\hat{e}_t\cdot q)}{q^4}-2\frac{\hat{e}_t\cdot\hat{e}_k}{q^2}\right]$$
$$= -\int \frac{d^2q}{(2\pi)^2}\left[2\frac{\hat{e}_k\cdot\hat{e}_t}{q^2}-2\frac{\hat{e}_t\cdot\hat{e}_k}{q^2}\right] = 0.$$

(iii) Order pt: The expansion factor reads

$$\frac{\partial^2}{\partial p\partial t}\int \frac{d^2q}{(2\pi)^2}\frac{q^2(q-p)^2}{q^2(q-t)^2}\bigg|_{p,t=0}$$
$$= \int \frac{d^2q}{(2\pi)^2}\frac{\partial}{\partial p}(q-p)^2\frac{\partial}{\partial t}\frac{1}{(q-t)^2}\bigg|_{p,t=0}$$
$$= -4\int \frac{d^2q}{(2\pi)^2}\frac{(\hat{e}_t\cdot q)(\hat{e}_p\cdot q)}{q^4} = -2\int \frac{d^2q}{(2\pi)}\frac{\hat{e}_t\cdot\hat{e}_p}{q^2}.$$

Thus, the order pt in expansion is

$$-\frac{2(p\cdot t)}{2\pi}\ell.$$

(iv) Order k^2: The corresponding factor is zero,

$$\frac{1}{2}\frac{\partial^2}{\partial k^2}\int \frac{d^2q}{(2\pi)^2}\frac{q^2(q-k)^2}{q^2(q-k)^2}\bigg|_{k=0} = \frac{1}{2}\frac{\partial^2}{\partial k^2}\int \frac{d^2q}{(2\pi)^2} = 0.$$

(v) Order t^2: The factor in expansion is

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\int \frac{d^2q}{(2\pi)^2}\frac{q^2}{(q-t)^2}\bigg|_{t=0}.$$

Here, it is possible to shift $q\to q+t$ and we obtain

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\int \frac{d^2q}{(2\pi)^2}\frac{(q+t)^2}{q^2}\bigg|_{t=0} = \frac{\ell}{2\pi}.$$

(vi) Order p^2: The corresponding expansion coefficient reads

$$\frac{1}{2}\frac{\partial^2}{\partial p^2}\int \frac{d^2q}{(2\pi)^2}\frac{q^2(q-p)^2}{q^4}\bigg|_{p=0} = \frac{\ell}{2\pi}.$$

In conclusion, the leading-order momentum expansion of the vertex function reads

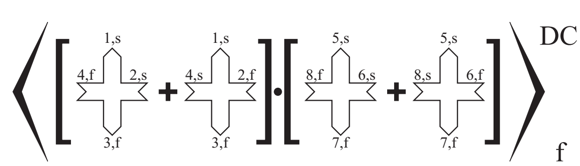
$$\text{Eq. (C2)} = \frac{(t^2-2t\cdot p+p^2)}{D'^2}\frac{\ell}{2\pi} \to \frac{(2-1)^2}{D'^2}\frac{\ell}{2\pi},$$

and the contribution to the renormalization of the interaction strength is

$$\text{I} \approx -\frac{(i\lambda)^2}{D'^2}\frac{\ell}{\pi}\sum_i\int d1d2d3d4\delta(1+3-2-4)\times(2-1)^2\varphi_{i1}\varphi'_{i2}\varphi_{i3}\varphi'_{i4}. \quad \text{(C3)}$$

2. Contribution II

Contribution II is as follows:

II=2 

where each diagram has the degeneracy one, and we may write corresponding expressions as

$$\text{II} = 2(-i\lambda)^2\sum_{ij}(-1)^{i+j}\int d1d2d3d4\int d5d6d7d8$$
$$\times(3-4)^2(7-8)^2\delta(1+3-2-4)\delta(5+7-6-8)$$
$$\times[\varphi_{s1i}\dot{\varphi}'_{f2i}\ddot{\varphi}_{f3i}\varphi'_{s4i}\varphi_{s5j}\varphi'_{s6j}\dot{\varphi}_{f7j}\ddot{\varphi}'_{f8j}$$
$$+\varphi_{s1i}\varphi'_{s2i}\ddot{\varphi}_{f3i}\dot{\varphi}'_{f4i}\varphi_{s5j}\varphi'_{s6j}\dot{\varphi}_{f7j}\ddot{\varphi}'_{f8j}$$
$$+\varphi_{s1i}\varphi'_{s2i}\dot{\varphi}_{f3i}\ddot{\varphi}'_{f4i}\varphi_{s5j}\dot{\varphi}'_{f6j}\ddot{\varphi}_{f7j}\varphi'_{s8j}$$
$$+\varphi_{s1i}\dot{\varphi}'_{f2i}\ddot{\varphi}_{f3i}\varphi'_{s4i}\varphi_{s5j}\ddot{\varphi}'_{f6j}\dot{\varphi}_{f7j}\varphi'_{s8j}]. \quad \text{(C4)}$$

After permuting fields, performing summations and integrations, and renaming variables, we obtain the following expressions:

$$\mathrm{IIa} = 2(i\lambda)^2 \sum_i \int d1d2d3d4(3-4)^2\delta(1+3-2-4) \times \varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i} \int d5(5-1)^2\Pi(5)\Pi(5+2-1),$$

$$\mathrm{IIb} = -2(i\lambda)^2 \sum_i \int d1d2d3d4(3-4)^4\delta(1+3-2-4) \times \varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i} \int d5\Pi(5)\Pi(5+3-4),$$

$$\mathrm{IIc} = 2(i\lambda)^2 \sum_i \int d1d2d3d4(3-4)^2\delta(1+3-2-4) \times \varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i} \int d5(5-4)^2\Pi(5)\Pi(5-2-4),$$

$$\mathrm{IId} = -2(i\lambda)^2 \sum_i \int d1d2d3d4\delta(1+3-2-4) \times \varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i} \int d5(5-4)^2(5-1)^2\Pi(5)\Pi(5+2-1).$$

While evaluating contributions IIa and IIc, it suffices to take only the most divergent part from the integral over loop momentum 5 into account. This yields

$$\mathrm{IIa} = \mathrm{IIc} \approx 2(i\lambda)^2 \int \frac{d^2q}{(2\pi)^2} q^2\Pi^2(q) \times \sum_i \int d1d2d3d4(3-4)^2\delta(1+3-2-4)\varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i} = \frac{\ell}{\pi}\frac{(i\lambda)^2}{D'^2} \sum_i \int d1d2d3d4\delta(1+3-2-4) \times (3-4)^2\varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i}.$$

For contribution Ib, it is necessary to perform a full integration over the loop momentum with the help of the Feynman parametrization:

$$\int \frac{d^2q}{(2\pi)^2}\Pi(q)\Pi(q+t) = \frac{1}{D'^2}\int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2(q+t)^2} = \frac{\ell}{2\pi}\frac{1}{D'^2t^2},$$

where we replace momentum $(3-4) \to t$. Hence, contribution IIb reads

$$\mathrm{IIb} = -\frac{\ell}{\pi}\frac{(i\lambda)^2}{D'^2} \sum_i \int d1d2d3d4\delta(1+3-2-4) \times (3-4)^2\varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i} = -\mathrm{IIa}.$$

The evaluation of the contribution IId goes analogously to the evaluation of the contribution I with the result

$$\mathrm{IId} = -2\frac{(i\lambda)^2}{D'^2}\frac{\ell}{2\pi} \sum_i \int d1d2d3d4\delta(1+3-2-4) \times (2+4)^2\varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i}.$$

An apparent problem with the momentum dependence of the vertex function can be cured if we remember that in the dc limit, φ and φ' are not independent but $\varphi_{-p} = \varphi'_p$. Using this property, we can mirror momenta $2 \to -2$ and $3 \to -3$ and obtain

$$\mathrm{IId} = 2\frac{(i\lambda)^2}{D'^2}\frac{\ell}{2\pi} \sum_i \int d1d2d3d4\delta(1+2-3-4) \times (2-4)^2\varphi_{1i}\varphi'_{3i}\varphi_{2i}\varphi'_{4i}.$$

The sign change is due to permuting Grassmann variables. After renaming the variables, we get the topological structure of the initial interaction term. Summing up all contributions gives

$$\mathrm{I} + \mathrm{IIa} + \mathrm{IIb} + \mathrm{IIc} + \mathrm{IId} = \frac{\ell}{\pi}\frac{(i\lambda)^2}{D'^2} \sum_i \int d1d2d3d4(3-4)^2\delta(1+3-2-4) \times \varphi_{1i}\varphi'_{2i}\varphi_{3i}\varphi'_{4i}.$$

Therefore, the RG equation for the interaction strength acquires the form

$$i\bar{\lambda} = i\lambda + (-1)^j\frac{\ell}{\pi}\frac{(i\lambda)^2}{D'^2}. \tag{C5}$$

It is therefore convenient to distinguish between interactions in each channel:

$$i\bar{\lambda}_j = i\lambda_j + (-1)^j\frac{\ell}{\pi}\frac{(i\lambda_j)^2}{D'^2}.$$

In the continuous limit, this gives Eq. (28).

APPENDIX D: SCALING OF THE DC CONDUCTIVITY TO ONE-LOOP ORDER

The dc conductivity is calculated from Kubo formula (32) with the two-particle Green's function defined in Eqs. (33) and (34). The action is slightly changed using the acquired knowledge, as

$$\mathcal{S}[\varphi] = \mathcal{S}_0[\varphi] + \mathcal{S}_{\mathrm{int}}[\varphi],$$

with

$$\mathcal{S}_0[\varphi] = \sum_{ij=1,2} \int d1d2\delta_{ij}\delta(1-2)\varphi_{i1}(i\epsilon + D'\nabla^2)\varphi'_{j2},$$

and

$$\mathcal{S}_{\mathrm{int}}[\varphi] = \sum_{j=1,2}(-1)^j\lambda_j \int d1d2d3d4\delta(1+3-2-4) \times (3-4)^2\varphi_{j1}\varphi'_{j2}\varphi_{j3}\varphi'_{j4}.$$

We evaluate Eq. (35), denoting

$$K_0(q) \sim \sum_{ij,p}\langle\varphi_{iq}\varphi'_{jp}\rangle^0, \quad K_1(q) \sim \sum_{ij,p}\langle\varphi_{iq}\varphi'_{jp}\mathcal{S}_{\mathrm{int}}\rangle^0,$$

where

$$\langle\cdots\rangle^0 = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}[\varphi]\cdots e^{-\mathcal{S}_0[\varphi]}.$$

Here, the proportionality factor is $1/g$. The zeroth-order contribution to the conductivity is calculated as usual:

$$K_0(q) \sim \sum_{ij} \int \frac{d^2 p}{(2\pi)^2} \langle \varphi_{iq} \varphi'_{jp} \rangle^0$$
$$= \sum_{ij} \int \frac{d^2 p}{(2\pi)^2} \frac{\delta_{ij}(2\pi)^2 \delta(q-p)}{D' q^2 + i\epsilon} = \frac{2}{D' q^2 + i\epsilon},$$

and further with $D' = g/4\pi$:

$$\sigma_0 = 2\frac{\epsilon^2}{g} \frac{\partial}{\partial q^2} \frac{2}{D' q^2 + i\epsilon}\bigg|_{q=0} = \frac{2D'}{g} = \frac{1}{\pi}, \tag{D1}$$

i.e., the usual universal dc conductivity of Dirac electron gas. The evaluation of the second term is more cumbersome. Respecting all possible (four in total) contraction combinations yields

$$K_1(q) \sim -\sum_{ij} \int \langle \varphi_{iq} \varphi'_{jp} \mathcal{S}_{\rm int}[\varphi] \rangle^0$$
$$= \sum_{ij\alpha} (-1)^\alpha i \lambda_\alpha \int \frac{d^2 p}{(2\pi)^2}$$
$$\times \int d1d2d3d4 (3-4)^2 \delta(1+3-2-4)$$
$$\times [\dot\varphi_{iq} \ddot\varphi'_{jp} \ddot\varphi_{\alpha 1} \dot\varphi'_{\alpha 2} \dddot\varphi_{\alpha 3} \dddot\varphi'_{\alpha 4} + \dot\varphi_{iq} \ddot\varphi'_{jp} \ddot\varphi_{\alpha 1} \dddot\varphi'_{\alpha 2} \dddot\varphi_{\alpha 3} \dot\varphi'_{\alpha 4}$$
$$+ \dot\varphi_{iq} \ddot\varphi'_{jp} \dddot\varphi_{\alpha 1} \dddot\varphi'_{\alpha 2} \ddot\varphi_{\alpha 3} \dot\varphi'_{\alpha 4} + \dot\varphi_{iq} \ddot\varphi'_{jp} \dddot\varphi_{\alpha 1} \dot\varphi'_{\alpha 2} \ddot\varphi_{\alpha 3} \dddot\varphi'_{\alpha 4}]$$
$$= i \sum_{ij\alpha} (-1)^\alpha \lambda_\alpha \delta_{i\alpha} \delta_{j\alpha} \delta_{\alpha\alpha}$$
$$\times \int \frac{d^2 p}{(2\pi)^2} \int d1d2d3d4 (3-4)^2 \delta(1+3-2-4)$$
$$\times [-\delta(2-q)\delta(1-p)\delta(3-4) K_0(q) K_0(p) K_0(3)$$
$$+ \delta(4-q)\delta(1-p)\delta(3-2) K_0(q) K_0(p) K_0(3)$$
$$- \delta(4-q)\delta(3-p)\delta(1-2) K_0(q) K_0(p) K_0(2)$$
$$+ \delta(q-2)\delta(3-p)\delta(1-4) K_0(q) K_0(p) K_0(4)].$$

While contributions from the first and third terms vanish, both of the other terms give equal finite contributions:

$$K_1(q) \sim 2i \sum_{j=1,2} \frac{[(-1)^j \lambda_j]}{(D' q^2 + i\epsilon)^2} \int \frac{d^2 k}{(2\pi)^2} \frac{k^2 + q^2}{D' k^2 + i\epsilon}. \tag{D2}$$

The integral

$$\int \frac{d^2 k}{(2\pi)^2} \frac{k^2}{D' k^2 + i\epsilon}$$

behaves well in the dc limit ($\epsilon \to 0$) as it does not develop any infrared divergences. The corresponding contribution disappears in the continuous limit and does not affect the conductivity at large scales. Therefore, the main contribution arises from the following expression:

$$K_1(q) \sim \frac{2iq^2}{(i\epsilon)^2} \sum_{j=1,2} \int \frac{d^2 k}{(2\pi)^2} \frac{[(-1)^j \lambda_j]}{D' k^2 + i\epsilon}.$$

In the dc limit, the integral diverges logarithmically. The corresponding conductivity correction reads

$$\sigma_1 = 2\epsilon^2 \frac{\partial}{\partial q^2} K_1(q)\bigg|_{q=0} = -2i \frac{\sigma_0}{g D'} \sum_{j=1,2} [(-1)^j \lambda_j] \ell,$$

and the full expression for the renormalized conductivity is

$$\bar\sigma = \sigma_0 - 2i \frac{\sigma_0}{g D'} \sum_{j=1,2} [(-1)^j \lambda_j] \ell. \tag{D3}$$

The continuous limit of this expression yields Eq. (36).

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