Short note on the Rabi model

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Abstract

The spectral density of the Rabi model is calculated exactly within a continued-fraction approach. It is shown that the method provides a simple algorithm for the spectral density with convergent solutions. We compare these recursive solutions with the solutions of the Jaynes–Cummings model and discuss the effect of approximations on the spectral properties.

1. Introduction

A recent work by Braak [1] has renewed the interest in the old problem of coupling a photon field to a single spin-1/2 state, using the Rabi model [2]. The central statement of this work is that the eigenfunctions in Bargmann representation must be analytic functions in the entire complex plane. Based on this condition, a procedure is derived from the series expansion of the eigenstates which provides a recursive evaluation of the spectrum. It is claimed that the series expansion yields an exact solution of the Rabi model, which cannot be obtained from a direct continued-fraction (CF) approach, contrary to the previously suggested approaches by Schweber [3] and Swain [4][1]. In the following, it is shown that this statement is incorrect, and that the use of the extra condition of analyticity of the eigenfunction in Bargmann representation is not necessary. Therefore, the CF is directly applicable to the Rabi model and yields an exact solution for the spectral density, where the term 'exact' means that the CF is convergent and the spectral density can be evaluated by a simple algorithm to any desired accuracy.

Although the intention of this paper is to discuss the evaluation of the spectral density for the Rabi model, there is a more general aspect in terms of other physical systems: there exists a class of quantum models whose solutions can be obtained within a recursive algorithm. Examples are small quantum systems with Jahn–Teller electron–phonon coupling [6] and the double-well Bose–Hubbard model [7, 8]. The recursive approach, based on a CF, can provide more complex spectral properties and, therefore, a richer dynamics than we obtain

[1] This statement has been withdrawn in a more recent work by the author of [5].

from (integrable) quantum models whose solutions are given by orthogonal polynomials (e.g., harmonic oscillator or the hydrogen atom). Their spectra are rather regular, whereas models with CF solutions have a more complex spectrum. For instance, the spectrum of the two-site Jahn–Teller model [6] or the two-site Bose–Hubbard model [8] has more structural features and is very sensitive to small parameter changes.

2. Continued-fraction approach to the Rabi Hamiltonian

The Rabi model is defined by the Hamiltonian

$$H_R = \omega a^\dagger a + \Delta\sigma_3 + U(a^\dagger + a)\sigma_1, \tag{1}$$

where $a^\dagger$ (a) are creation (annihilation) operators of a photon and ω is the frequency of the photon field. The Pauli matrices σ_j ($j = 0, \ldots, 3$) describe operations on the spin-1/2 state, and Δ is a symmetry breaking field for the spin. U is the coupling strength between the photons and the spin states. This Hamiltonian maps a product state $|N\rangle \otimes |\sigma\rangle$ with N photons to $|N \pm 1\rangle \otimes |\sigma'\rangle$, where $\sigma, \sigma' = \downarrow, \uparrow$. Thus eigenstates of H are superpositions of product states, which can be derived by a recursive approach for the coefficients of the superposition [3]. In the following, we apply the recursive projection method (RPM) of [8]. This method is based on a random walk expansion in the underlying Hilbert space which visits each subspace at most once [9]. Consequently, there are no loops in the random walk, which leads directly to a CF [2].

The RPM, as described in [8], can be directly applied to the resolvent $\langle N; s|(z-H)^{-1}|N; s'\rangle$ and yields immediately the spectral density with respect to $|N; s\rangle \equiv |N\rangle \otimes |s\rangle$:

$$\rho_N(E) = \frac{1}{2} \sum_{s=\uparrow,\downarrow} \lim_{\epsilon\to 0} \mathrm{Im}(\langle N; s|(E - \mathrm{i}\epsilon - H)^{-1}|N; s\rangle). \tag{2}$$

The simplest case is the resolvent $\langle 0; s|(z - H)^{-1}|0; s'\rangle$ of states without photons $|0; s\rangle$ and with spin projection $s = \uparrow, \downarrow$:

$$(\langle 0; s|(z - H)^{-1}|0; s'\rangle) = \lim_{n\to\infty} \begin{pmatrix} g_n & 0 \\ 0 & h_n \end{pmatrix}, \tag{3}$$

where the eigenvalues of H are the poles of the resolvent. Then the matrix elements g_n, h_n are subject to the following recurrence relations:

$$g_k = \frac{1}{z - \omega(n-k) + \Delta - U^2(n-k+1)h_{k-1}} \tag{4}$$

$$h_k = \frac{1}{z - \omega(n-k) - \Delta - U^2(n-k+1)g_{k-1}}, \tag{5}$$

for $1 \leqslant k \leqslant n$ with the initial values

$$g_0 = \frac{1}{z - \omega n - \Delta}, \quad h_0 = \frac{1}{z - \omega n + \Delta}. \tag{6}$$

The iteration of these recurrence relations gives a finite CF of the standard form [10]

$$g_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots}} \equiv b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_n|}{|b_n} \tag{7}$$

[2] A recurrence relation can also be established directly from the eigenvalue problem of the Rabi Hamiltonian, without projecting onto subspaces of the Hilbert space, by using a series expansion of the general solution [3, 4]. The recursive projection method is technically simpler because it deals with a finite number of photons and with the controllable limit of an infinite number of photons.

with coefficients

$$a_1 = 1, \quad a_k = (1-k)U^2, \quad b_0 = 0, \quad b_1 = z - \Delta,$$
$$b_k = z - (k-1)\omega + (-1)^k\Delta \quad (2 \leqslant k \leqslant n). \tag{8}$$

For the corresponding coefficients of h_n we must only replace Δ by $-\Delta$. Therefore, all considerations for g_n apply to h_n after replacing $\Delta \to -\Delta$. Due to $b_0 = 0$ the inverse of g_n can be obtained directly by inverting g_n in equation (7):

$$a_1/g_n = b_1 + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n}. \tag{9}$$

This means that the poles of g_n are obtained from the zeros of the related CF, and vice versa. g_n can also be expressed as the ratio (cf [10])

$$g_n = \frac{b_n A_{n-1} + a_n A_{n-2}}{b_n B_{n-1} + a_n B_{n-2}}, \tag{10}$$

where A_{n-l} (B_{n-l}) are polynomials in z of order $n-l-1$ ($n-l$) which are generated by the recurrence relations

$$A_k = b_k A_{k-1} + a_k A_{k-2}, \quad B_k = b_k B_{k-1} + a_k B_{k-2} \quad (k \geqslant 1), \tag{11}$$

with initial conditions $A_{-1} = 1, A_0 = b_0 = 0, B_{-1} = 0, B_0 = 1$.

By taking the limit $g = \lim_{n\to\infty} g_n$ the CF in equations (7) or (9) yields the spectral density of the Rabi model for the complex energy z. The existence of the limit $n \to \infty$ is a consequence of a theorem by Pringsheim (cf [10]): considering the tail t_n of the CF

$$t_n = \frac{a_{n+1}|}{|b_{n+1}} + \cdots, \tag{12}$$

we can write

$$g = b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_n|}{|b_n} + t_n \tag{13}$$

and estimate the error when we truncate g at $k = n$ (which gives g_n). Pringsheim's theorem states that for complex a_k, b_k with $|b_k| \geqslant |a_k| + 1$ ($k \geqslant n$) the truncated CF $t_{n,l}$,

$$t_{n,l} = \frac{a_{n+1}|}{|b_{n+1}} + \cdots + \frac{a_l|}{|b_l} \quad (n+1 < l), \tag{14}$$

is convergent for $l \to \infty$ and converges to a t_n with

$$0 \leqslant |t_n| \leqslant 1. \tag{15}$$

For the coefficients of the Rabi model in equation (8) the conditions for the coefficients are satisfied for a given z and for sufficiently large n if $U^2 < \omega$. Then (15) means that the ratio t_n/b_n is $O(1/n)$, which implies

$$t_{n-1} = \frac{a_n}{b_n + t_n} = \frac{a_n}{b_n} + O(1/n) = \frac{(1-n)U^2}{z - (n-1)\omega - (-1)^n\Delta} + O(1/n) = \frac{U^2}{\omega} + O(1/n). \tag{16}$$

This result enables us to approach the exact solution by a sequence of truncated CF's with a given error $O(1/n)$:

$$g = b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_{n-1}|}{|b_{n-1}} + \frac{a_n/b_n + O(1/n)|}{|1}. \tag{17}$$

Replacing a_n/b_n by $a_n/b_n + O(1/n)$ in the finite CF of equation (10) gives us

$$g = \frac{A_{n-1} + [a_n/b_n + O(1/n)]A_{n-2}}{B_{n-1} + [a_n/b_n + O(1/n)]B_{n-2}}. \tag{18}$$

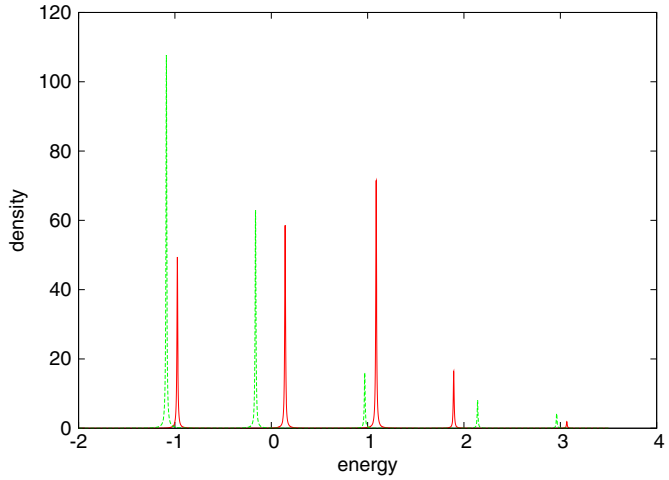

Figure 1. Spectral density $\rho_0(E)$ for maximally $n = 500$ photons and $\Delta = 0.4$, $U = 0.99$, $\epsilon = 0.005$ in units of the photon frequency ω. The green (red) curves are contributions from g_n (h_n).

3. Conclusions

Equation (18), together with the coefficients in equations (8) and (11), allows us to approach successively the spectral density of the Rabi Hamiltonian, where the accuracy is improved with increasing n. An example is depicted in figure 1. The error is $O(1/n)$ and estimated by the tail t_n of (16). According to our definition this provides an exact solution of the Rabi spectrum, where the poles of g and h are the corresponding eigenvalues. The evaluation of $\rho_N(E)$ for $N > 0$ can also be performed within the same approach.

The individual matrix elements g, h avoid level crossing due to parity conservation, since eigenstates of consecutive eigenvalues have different parity. However, the sum of g_n and h_n in the spectral density $\rho_0(E)$ allows level crossing by shifting their levels relative to each other, for instance, by changing Δ.

The RPM provides g_n, h_n with the unique initial expressions g_0, h_0 of equation (6). It has been claimed in [5] that this leads to the same large n asymptotics as the result in [1] by comparing the two spectra. It should be noted that a series representation of the solution of the full problem [3, 4] may cause some difficulty for the initial condition of the recurrence relation due to the implicit limit $n \to \infty$. Therefore, the RPM is easier to control because the initial expressions g_0, h_0 are well defined for finite n, where the limit $n \to \infty$ is only taken at the end.

The 'rotating-wave' approximation of the Rabi Hamiltonian yields the Jaynes–Cummings (JC) Hamiltonian [11]:

$$H_{\rm JC} = \omega a^\dagger a + \Delta\sigma_3 + U(a^\dagger\sigma_- + a\sigma_+), \tag{19}$$

with $\sigma_\pm = (\sigma_1 \pm {\rm i}\sigma_2)/2$. This Hamiltonian does not belong to the class of models with recursive solutions because the recurrence relation of the RPM terminates already after the first step for each k:

$$g_k = \frac{z - \omega(n-k+1) - \Delta}{[z-\omega(n-k)+\Delta][z-\omega(n-k+1)-\Delta] - U^2(n-k+1)}$$
$$h_k = \frac{1}{z-\omega(n-k)-\Delta}. \tag{20}$$

Here the evaluation of the eigenvalues requires only the solution of a quadratic equation for each k, in agreement with the original work on this model [11, 12]. The asymmetry of g_k and h_k reflects the fact that the JC Hamiltonian can only absorb (emit) a photon in the low (high)-energy spin state.

A comparison of the Rabi and the JC model in terms of spectral properties sheds some light on the accuracy of the 'rotating-wave' approximation for small quantum systems. It is obvious that the energy levels $E_k = \omega(n-k+1/2) \pm \sqrt{(\omega+2\Delta)^2/4 + U^2(n-k+1)}$ of the JC model do not reproduce the more complex energy-level spacings of the Rabi Hamiltonian as visible, for instance, in figure 1. This might not be very significant for a qualitative picture though. More important, however, is the spectral density ρ_0, which, in contrast to figure 1, would include only the two levels $E_{n\pm} = \omega/2 \pm \sqrt{(\omega+2\Delta)^2/4 + U^2}$ for g and $E'_n = \Delta$ for h. This affects the matrix elements of (3), especially for strong coupling U and for a small number of photons. A similar observation was made for the two-site Bose–Hubbard model, where the Hartree (Gross–Pitaevskii) approximation has a much more regular spectrum and dynamics than the original quantum model [7, 8].

With the increasing accuracy of experimentally obtained spectral and dynamical data of trapped atoms [13–15] and photons in microcavities [16], it may become important to go beyond simplifying approximations. A possible way is to apply recursive methods such as the CF discussed in this paper. Moreover, quantum models with recursive solutions may present a bridge between models with simple spectra (e.g., the harmonic oscillator or the JC model) and models with random-matrix spectra, as proposed for nuclei [17, 18].

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