Long-range correlations in disordered graphene

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Angaben zur Veröffentlichung / Publication details:

Ziegler, Klaus G. 2008. "Long-range correlations in disordered graphene." *Physical Review B* 78 (12): 125401. https://doi.org/10.1103/physrevb.78.125401.

PHYSICAL REVIEW B **78**, 125401 (2008)

Long-range correlations in disordered graphene

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(Received 31 May 2008; revised manuscript received 15 August 2008; published 2 September 2008)

The appearance of long-range correlations near the Dirac point of a Dirac-type spinor model with random vector potential is studied. These correlations originate from a spontaneously broken symmetry and their corresponding Goldstone modes. Using a strong-disorder expansion, correlation functions and matrix elements are analyzed and compared with results from a weak-disorder expansion. The local density of states correlation and the overlap between states above and below the Dirac point are characterized by a long-range behavior. The correlation range decreases with the distance from the Dirac point. Transport is diffusive and the diffusion coefficient is proportional to the one-particle scattering time for any strength of disorder. A consequence of the special properties of particle-hole scattering is a constant microwave conductivity for weak, as well as for strong disorder, describing a deviation from conventional Drude-type transport. Some properties of the model can be linked to a kind of Kondo scale, which is generated by disorder. Finally, the properties of the wave functions at the Dirac point are characterized by their participation ratios, indicating a critical state at the Dirac point.

DOI: 10.1103/PhysRevB.78.125401 PACS number(s): 81.05.Uw, 71.55.Ak, 72.10.Bg, 73.20.Jc

I. INTRODUCTION

The electronic properties of graphene are closely related to the existence of a spinor wave function and the two Dirac nodes in the band structure.[1–3] This implies an unconventional behavior which is associated with the Klein paradox.[4] An important effect is the scattering between the hole and the particle sector of the Dirac cones, leading, for instance, to the zitterbewegung.[5–10] A constant contribution to the microwave conductivity is another consequence of this effect, causing a deviation from conventional Drude-type transport.[11–13] The latter has been studied only for weak disorder. It would be interesting to analyze these effects, their modification or even their destruction in the presence of strong disorder.

The presence of disorder in graphene has been discussed in a number of recent papers, considering different physical conditions. It seems that disorder should appear in the effective Dirac-type Hamiltonian in the form of a random vector potential due to instability of the translational order in two dimensions (2D).[14,3,15–17] Scattering by a random vector potential affects only the phase of the wave functions. Its influence on transport is such that no localization has been observed, even for very strong disorder.[18] However, it is believed that disorder has a substantial effect on the magnetoresistance, observed as a suppression of the weak-localization peak.[14,19] Another effect, which might be caused by disorder, are fluctuations of the charge distribution near the Dirac point. Recent experiments have revealed that in graphene long-range charge fluctuations exist.[20]

The Dirac point is very special because it is associated with zero-energy modes and a vanishing density of states (DOS). Consequently, it might be very sensitive to disorder. It is known that, depending on the type of disorder, the properties of Dirac fermions are strongly affected by randomness.[21,22] An example is the prediction of a dramatic change of the density of states due to disorder. On the other hand, some quantities, such as conductivity, are remarkably robust.[23,21,24,25] Of particular interest is the question how disorder affects the interband scattering. Weak disorder has no effect on particle-hole scattering, only on the interparticle scattering (cf. Ref. 11). All this indicates that the Dirac point is an interesting object to study in more detail, especially when disorder is strong.

To understand the effect of a random vector potential in a Dirac-type Hamiltonian, it has to be noticed that the Dirac point is characterized by a chiral symmetry. We discuss this symmetry and its spontaneous breaking by the random vector potential. It is known that the spontaneously broken symmetry implies a robust diffusion mode for weak disorder[24] and for strong disorder.[18] In this paper we will demonstrate that the local density of states correlations have a long-ranged behavior, although the average density of states is finite. Some properties can be linked to a kind of Kondo scale, generated by disorder.[26,27] The analysis of the scattering between different parts of the Dirac cone leads to correlations between different energy states, whose correlation are expressed in terms of transition matrix elements. A special case is the corresponding matrix elements of the position operator with respect to $\pm E$ energy states, which appear as important contributions to the microwave conductivity. Finally, the second moment of the local density of states can be associated with the inverse participation ratio. This quantity characterizes the properties of the wave functions at the Dirac point.

In order to study the effect of symmetry breaking in the presence of disorder, we apply a perturbation theory for strong disorder. Strong-disorder expansions have been quite successful in different fields of physics. In the case of Dirac fermions it has been observed that the expansion has a number of advantages in comparison with a weak-disorder expansion.[18] The main reason is that the leading order of the expansion is quite simple. Nevertheless, the strong-disorder expansion cannot be applied directly to one-particle Green's function of the Dirac-type Hamiltonian because it has, in contrast to a Schrödinger-type Hamiltonian, divergent contributions from self-crossing loops. (The strong-disorder ex-

pansion can be formulated as an expansion in terms of the hopping elements of the Hamiltonian.) These problems can be avoided by applying the expansion to the saddle-point integration. Then the integration is restricted to the most important contributions, namely the saddle-point manifold, which takes into account the underlying symmetries of the model. In this case the strong-coupling expansion can be controlled and leads in leading order to a model with diffusion propagator. It describes the Goldstone modes which are generated by spontaneous symmetry breaking (SSB). The results of this expansion can be compared to those of the more common weak-disorder expansion. The latter gives in leading order a factorization of average two-particle Green's function into a product of two average one-particle Green's functions, which can be treated within the self-consistent Born approximation. Both approaches are compared to study similarities and differences of both regimes.

The paper is organized as follows. In Sec. II we introduce the generalized Dirac model and discuss spontaneous symmetry breaking. This includes the definition of some physical quantities, which are studied in the rest of the paper. The framework of functional integration for our model is presented in Sec. III, and the corresponding symmetries are analyzed in Sec. IV. Then the averaging with respect to the random vector potential and the related saddle-point integration are performed (Sec. V). The results of the calculations are discussed and compared with weak-disorder approximations in Sec. VI.

II. MODEL AND SPONTANEOUS SYMMETRY BREAKING

In this paper we study the generalized Dirac Hamiltonian for a spinor-1/2 state in a random vector potential

$$H = H_0 + v\sigma_1, \quad H_0 = h_1\sigma_1 + h_2\sigma_2. \tag{1}$$

σ_j $(j=0,1,2,3)$ are Pauli matrices, h_j is an operator that acts in space, and v is space diagonal with random variables v_r. The latter are independently Gaussian distributed with zero mean and variance g. Special cases for H_0 are 2D Dirac fermions, whose Fourier components are

$$h_j = k_j$$

and the tight-binding model on a honeycomb lattice with

$$h_1 = -t\sum_{j=1}^{3}\cos(\vec{a}_j\cdot\vec{k}), h_2 = -t\sum_{j=1}^{3}\sin(\vec{a}_j\cdot\vec{k}),$$

with the lattice vectors of the honeycomb lattice

$$\vec{a}_1 = (-\sqrt{3}/2, 1/2), \quad \vec{a}_2 = (0,-1), \quad \vec{a}_3 = (\sqrt{3}/2, 1/2).$$

H_0 can be diagonalized as $H_0 = \mathrm{dia}\, g(e_k, -e_k)$ with $e_k = \sqrt{h_1^2+h_2^2}$. The Hamiltonian H is invariant under the continuous transformation $H \to e^{i\alpha\sigma_3} H e^{i\alpha\sigma_3}$. This symmetry can be broken spontaneously. To measure SSB, we consider the one-particle Green's function

$$G(z) = (H+z)^{-1}$$

and calculate the difference of this expression before and after the symmetry transformation. For the special case with parameter value $\alpha = i\pi/2$ this reads

$$G(z) - i\sigma_3 G(z) i\sigma_3 = (H+z)^{-1} - (H-z)^{-1}.$$

Since z breaks the symmetry, we send it to zero. If we choose $z = i\epsilon$, the results for the diagonal elements of G are proportional to the local density of states at the Dirac point:

$$\rho_r(0) = \delta_{rr}(H) \propto \lim_{\epsilon\to 0}[(H+i\epsilon)^{-1}_{rr} - (H-i\epsilon)^{-1}_{rr}].$$

Thus, a nonzero $\rho_r(0)$ indicates SSB. This is similar to spontaneous symmetry breaking in classical systems, e.g., in the case of the magnetic phase transition of a classical Heisenberg model.[28] However, there is a remarkable difference: In the Heisenberg model there is SSB only for systems with dimensionality d higher than 2, whereas this restriction does not hold in a disordered system. Even for $d=0$, where $H = v\sigma_1$, we have SSB, since the DOS reads

$$\langle\rho(0)\rangle = \lim_{\epsilon\to 0}\int \frac{\epsilon}{\epsilon^2+v^2}P(v)dv = \lim_{\epsilon\to 0}\int \frac{1}{1+x^2}P(\epsilon x)dx > 0$$

for a continuous distribution function $P(v)$, e.g., a Gaussian. Another interesting point is that the second moment of the density of states diverges like ϵ^{-1}:

$$\langle\rho(0)^2\rangle = \int \frac{\epsilon^2}{(\epsilon^2+v^2)^2}P(v)dv = \epsilon^{-1}\int \frac{1}{(1+x^2)^2}P(\epsilon x)dx. \tag{2}$$

This simply reflects the fact that the distribution of the density of states has a Lorenzian form.

Physical quantities

A characteristic property of the Hamiltonian H in Eq. (1) is that its spectrum is symmetric with respect to the energy $E=0$ (the Dirac point), consisting of states $\Psi_{\pm E}$ in the upper and the lower band (particle and hole parts of the Dirac cones). These states are connected by the transformation $\Psi_{-E} = \sigma_3\Psi_E$ because the eigenvalue equation

$$H\Psi_E = E\Psi_E$$

gives

$$\sigma_3 H\sigma_3\sigma_3\Psi_E = E\sigma_3\Psi_E,$$

which becomes with $\sigma_3 H\sigma_3 = -H$

$$H\sigma_3\Psi_E = -E\sigma_3\Psi_E.$$

Scattering between Ψ_E and Ψ_{-E} leads to interesting physical phenomena, such as the zitterbewegung[5,6] or a constant contribution to the microwave conductivity.[11,29,13] The latter can be seen by considering a typical contribution to the frequency-dependent conductivity at frequency ω (cf. Appendix A) as

$$\sigma^{II}(\omega) = -\frac{e^2}{4h}\omega\int_{-\omega/2}^{\omega/2}\sum_r r_k^2 \mathrm{Tr}_2[G_{r0}(\omega/2-E-i\epsilon)G_{0r}(-\omega/2 - E+i\epsilon) + G_{r0}(-\omega/2+E-i\epsilon)G_{0r}(\omega/2+E+i\epsilon)]dE, \tag{3}$$

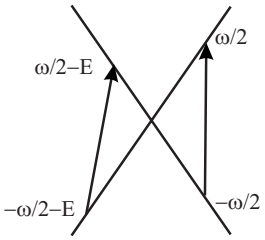

FIG. 1. Scattering on the Dirac cone: schematic scattering process between energy states $\Phi_{E-\omega/2}$ and $\Phi_{E+\omega/2}$, as well as between $\Phi_{-\omega/2}$ and $\Phi_{\omega/2}$. This type of scattering is relevant for the microwave conductivity $\sigma^{II}(\omega)$ in Eq. (5).

where Tr_2 is the trace with respect to Pauli matrices. This can also be expressed in terms of the matrix elements of r_k^2 between the states $\Phi_{\pm\omega/2-E}$ (cf. Appendix B) as

$$\sigma^{II}(\omega) = -\frac{e^2}{4h}\omega\int_{-\omega/2}^{\omega/2}(\langle\Phi_{-\omega/2+E}|r_k^2|\Phi_{\omega/2+E}\rangle + \langle\Phi_{\omega/2-E}|r_k^2|\Phi_{-\omega/2-E}\rangle)dE. \tag{4}$$

For a system at the Dirac point $E=0$, we expect that the main contribution to the conductivity comes from the scattering between $\Phi_{-\omega/2}$ and $\Phi_{\omega/2}$, where the scattering does not change momentum (cf. Fig. 1). Therefore, we consider in the following only:

$$\sigma_0(\omega) = -\frac{e^2}{4h}\omega^2\langle\Phi_{-\omega/2}|r_k^2|\Phi_{\omega/2}\rangle + \langle\Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle) = -\frac{e^2}{2h}\omega^2\langle\Phi_{-\omega/2}|r_k^2|\Phi_{\omega/2}\rangle, \tag{5}$$

where the last equation is a consequence of the symmetry of the matrix element. Then a constant conductivity, found theoretically[11,12] and experimentally,[13] requires a matrix element that diverges for $\omega\sim 0$ as

$$\langle\Phi_{-\omega/2}|r_k^2|\Phi_{\omega/2}\rangle \sim \omega^{-2}.$$

There are two other correlation functions that are of interest for the characterization of the disordered system of quasiparticles in graphene. One is

$$C_{rr'} = \langle \mathrm{Tr}_2[\delta_{rr'}(H-E)\delta_{r'r}(H-E)]\rangle, \tag{6}$$

which is related to the dc conductivity $\sigma_{\mu\mu}$ at $T=\omega=0$ through the expression[25]

$$\sigma_{\mu\mu} = 2\pi\frac{e^2}{\hbar}\lim_{\epsilon\to 0}\epsilon^2\sum_{r'}(r_\mu - r_\mu)^2 C_{rr'}. \tag{7}$$

The other correlation function is

$$D_{rr'} = \langle \mathrm{Tr}_2[\delta_{rr}(H-E)]\mathrm{Tr}_2[\delta_{r'r'}(H-E)]\rangle, \tag{8}$$

which is the correlation of the local density of states at sites r and r'. For $r'=r$, D_{rr} is related to the participation ratio and the inverse participation ratio (cf. Appendix C). These are quantities that measure the statistical properties of the wave functions and their localization behavior.[30] The participation ratio

$$p^{(2)} = \frac{\langle\rho_r\rangle^2}{\langle\rho_r^2\rangle}$$

vanishes for localized states because the local density of states ρ_r has a broad distribution with divergent second moments in this case, similar to the single-site density of states of Eq. (2). Using D_{rr} the participation ratio reads

$$p^{(2)} = \lim_{\epsilon\to 0}\frac{\langle\rho_r\rangle^2}{D_{rr}}. \tag{9}$$

Moreover, the inverse participation ratio is also related to the second moment of the local density of states (cf. Appendix C):

$$P^{(2)} = \lim_{\epsilon\to 0}\epsilon D_{rr}. \tag{10}$$

It vanishes as one approaches the regime of extended states, coming from the localized regime.

The density operator $\delta(H)$ can be expressed by the one-particle Green's function

$$G^{\pm} = (H \pm i\epsilon)^{-1}$$

as

$$\delta_{rr'}(H) = \frac{1}{2i\pi}(G^-_{rr'} - G^+_{rr'}).$$

With the relation $\sigma_3 G^{\pm}\sigma_3 = -G^{\mp}$ the delta function also reads

$$\delta_{rr'}(H) = \frac{i}{2\pi}[G^+_{rr'} + \sigma_3 G^+_{rr'}\sigma_3] = \frac{i}{\pi}\begin{pmatrix} G^+_{rr',11} & 0 \\ 0 & G^+_{rr',22}\end{pmatrix}.$$

This relation implies for the correlation functions at the Dirac point $E=0$

$$C_{rr'} = -\frac{1}{\pi^2}\sum_{j=1,2}\langle G^+_{rr',jj}G^+_{r'r,jj}\rangle, \tag{11}$$

$$D_{rr'} = -\frac{1}{\pi^2}\langle \mathrm{Tr}_2[G^+_{rr}]\mathrm{Tr}_2[G^+_{r'r'}]\rangle. \tag{12}$$

Moreover, away from the Dirac point at energies $E=\pm\omega/2$ it gives

$$A_{rr'}(\omega) = -\langle \mathrm{Tr}_2[\sigma_3 G_{rr'}(\omega/2+i\epsilon)\sigma_3 G_{r'r}(\omega/2+i\epsilon)]\rangle, \tag{13}$$

which can be used to write for the conductivity of Eq. (5)

$$\sigma_0(\omega) = -\frac{e^2}{2h}\omega^2\sum_r r_k^2 A_{r0}(\omega) = -\frac{e^2}{2h}\omega^2\langle\Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle. \tag{14}$$

III. FUNCTIONAL INTEGRAL

All three correlation functions in Eqs. (11)–(13) are written as products of two Green's functions. We can express these products, before averaging, as a Gaussian functional

integral with two independent Gaussian fields—a boson (complex) field χ_{rk} and a fermion (Grassmann) field Ψ_{rk} ($k=1,2$) and their conjugate counterparts $\bar{\chi}_{rk}$ and $\bar{\Psi}_{rk}$:

$$G_{rr',jj'}(i\epsilon)G_{r'r,k'k}(i\epsilon) = \int \chi_{r'j'}\bar{\chi}_{rj}\Psi_{rk}\bar{\Psi}_{r'k'} \times \exp(-S_0(E))\mathcal{D}\Psi\mathcal{D}\chi, \quad (15)$$

where $S_0(E)$ is a quadratic form of the four-component field $\phi_r=(\chi_{r1},\chi_{r1},\Psi_{r2},\Psi_{r2})$

$$S_0(E) = -i\sum_{r,r'} \phi_r \cdot (\mathbf{H} + i\epsilon + E)_{r,r'}\bar{\phi}_{r'} (\epsilon > 0). \quad (16)$$

The use of the mixed field ϕ_r has the advantage that an extra normalization factor for the integral is avoided. The extended Hamiltonian $\mathbf{H}=\mathrm{dia}\, g(H,H)$ of S_0 acts in the boson and in the fermion sector separately. Using Eq. (15), the correlation functions now read

$$C_{rr'} = -\frac{1}{\pi^2}\sum_j \int \chi_{r'j}\bar{\chi}_{rj}\Psi_{rj}\bar{\Psi}_{r'j}\langle\exp[-S_0(0)]\rangle\mathcal{D}\Psi\mathcal{D}\chi,$$

$$A_{rr'}(\omega) = -\sum_{j,j'}(-1)^{j+j'}\int \chi_{r'j'}\bar{\chi}_{rj}\Psi_{rj}\bar{\Psi}_{r'j'}\langle\exp[- S_0(\omega/2)]\rangle\mathcal{D}\Psi\mathcal{D}\chi,$$

and

$$D_{rr'} = -\frac{1}{\pi^2}\sum_{j,j'}\int \chi_{rj}\bar{\chi}_{rj}\Psi_{r'j'}\bar{\Psi}_{r'j'} \times\langle\exp[-S_0(0)]\rangle\mathcal{D}\Psi\mathcal{D}\chi.$$

In the case of $C_{rr'}$ and $A_{rr'}(\omega)$ we arrange the products of the fields such that pairs at the same site are neighboring factors:

$$C_{rr'} = \frac{1}{\pi^2}\sum_j \int \chi_{r'j}\bar{\Psi}_{r'j}\Psi_{rj}\bar{\chi}_{rj}\langle\exp[-S_0(0)]\rangle\mathcal{D}\Psi\mathcal{D}\chi,$$

$$A_{rr'}(\omega) = \sum_{j,j'}(-1)^{j+j'}\int \chi_{r'j'}\bar{\Psi}_{r'j'}\Psi_{rj}\bar{\chi}_{rj} \times\langle\exp[-S_0(\omega/2)]\rangle\mathcal{D}\Psi\mathcal{D}\chi.$$

IV. SUPERSYMMETRY

The Hamiltonian $\mathbf{H}$ in S_0 is invariant under the transformation

$$\mathbf{U} = U_0(q,p)\mathbf{U}_S U_0(q',p') = \begin{pmatrix} e^{i(q+q')\sigma_3}(1+\psi\bar{\psi}/2) & e^{i(q+p')\sigma_3}\psi\sigma_3 \\ e^{i(q'+p)\sigma_3}\bar{\psi}\sigma_3 & e^{i(p+p')\sigma_3}(1-\psi\bar{\psi}/2)\end{pmatrix}, \quad (17)$$

with

$$\mathbf{U}_S = \exp\begin{pmatrix} 0 & \psi\sigma_3 \\ \bar{\psi}\sigma_3 & 0\end{pmatrix}, \quad U_0(q,p) = \begin{pmatrix} e^{iq\sigma_3} & 0 \\ 0 & e^{ip\sigma_3}\end{pmatrix}.$$

For $\gamma_j=\mathrm{dia}\ g(\sigma_j,\sigma_j)$ there is the symmetry transformation

$$\mathbf{U}\gamma_j\mathbf{U}' = \mathbf{U}\mathbf{U}^{-1}\gamma_j = \gamma_j,$$

with

$$\mathbf{U}' = U_0(q',p')\mathbf{U}_S U_0(q,p) = \begin{pmatrix} e^{i(q+q')\sigma_3}(1+\psi\bar{\psi}/2) & e^{i(q'+p)\sigma_3}\psi\sigma_3 \\ e^{i(q+p')\sigma_3}\bar{\psi}\sigma_3 & e^{i(p+p')\sigma_3}(1-\psi\bar{\psi}/2)\end{pmatrix}. \quad (18)$$

This implies the invariance

$$\mathbf{H} \to \mathbf{U}\mathbf{H}\mathbf{U}' = \mathbf{H}. \quad (19)$$

To simplify the notation one can introduce the phases φ_{ij} with

$$\varphi_{11} = q+q', \quad \varphi_{12} = q+p', \quad \varphi_{21} = q'+p, \quad \varphi_{22} = p+p'$$

and write

$$\mathbf{U}\mathbf{U}' = \begin{pmatrix} e^{2i\varphi_{11}\sigma_3}(1+\psi\bar{\psi}) + e^{2i\varphi_{12}\sigma_3}\psi\bar{\psi} & [e^{i(\varphi_{12}+\varphi_{22})\sigma_3} + e^{i(\varphi_{11}+\varphi_{21})\sigma_3}]\psi\sigma_3 \\ [e^{i(\varphi_{11}+\varphi_{21})\sigma_3} + e^{i(\varphi_{12}+\varphi_{22})\sigma_3}]\bar{\psi}\sigma_3 & e^{2i\varphi_{22}\sigma_3}(1-\psi\bar{\psi}) - e^{2i\varphi_{21}\sigma_3}\psi\bar{\psi}\end{pmatrix}. \quad (20)$$

V. AVERAGED CORRELATION FUNCTIONS

Averaging Eq. (15) over the Gaussian distribution of v_r means replacing $\exp(-S_0)$ by $\langle\exp(-S_0)\rangle$ on the right-hand side of the equation. The latter can be written again as an exponential function $\langle\exp(-S_0)\rangle=\exp(-S_1)$, where the new function S_1 contains also quartic terms of the field ϕ:

$$S_1 = -i\sum_{r,r'}\phi_r\cdot(\mathbf{H}_0 + i\epsilon + E)_{r,r'}\bar{\phi}_{r'} + g\sum_r(\phi_r\cdot\gamma_1\bar{\phi}_r)^2. \quad (21)$$

Then it is convenient to transform the integration variables (Hubbard-Stratonovich transformation) as

$$\begin{pmatrix} \chi_r\bar{\chi}_r & \chi_r\bar{\Psi}_r \\ \Psi_r\bar{\chi}_r & \Psi_r\bar{\Psi}_r \end{pmatrix} \to \mathbf{Q}_r = \begin{pmatrix} Q_r & \Theta_r \\ \bar{\Theta}_r & -iP_r \end{pmatrix}, \tag{22}$$

where Q_r, P_r are symmetric 2×2 matrices and Θ_r, $\bar{\Theta}_r$ are 2×2 matrices whose elements are independent Grassmann variables. Now the correlation functions can be rewritten as correlation functions in the new field $\mathbf{Q}_r$:

$$C_{rr'} = \frac{1}{g^2\pi^2}\sum_j \int (\sigma_1\Theta)_{r',jj}(\sigma_1\bar{\Theta})_{r,jj}\exp(-S_2)\mathcal{D}[\mathbf{Q}], \tag{23}$$

$$A_{rr'}(\omega) = \frac{1}{g^2}\int \mathrm{Tr}_2(\sigma_3\sigma_1\Theta_{r'})\mathrm{Tr}_2(\sigma_3\sigma_1\bar{\Theta}_r)\exp(-S_2)\mathcal{D}[\mathbf{Q}], \tag{24}$$

and

$$D_{rr'} = -\frac{1}{g^2\pi^2}\int \mathrm{Tr}_2(-i\sigma_1 P_{r'})\mathrm{Tr}_2(\sigma_1 Q_r)\exp(-S_2)\mathcal{D}[\mathbf{Q}], \tag{25}$$

with

$$S_2 = \sum_{r,r'}\frac{1}{g}\mathrm{Tr}\, g(\mathbf{Q}_r^2) + \ln[\det g[\mathbf{H}_0 + i\epsilon + E - 2\gamma_1\mathbf{Q}]]. \tag{26}$$

Tr g is the graded trace, Tr_2 the trace with respect to Pauli matrices, and det g the graded determinant.[24]

Saddle-point manifold

The integration in Eqs. (23)–(25) can be performed in saddle-point approximation. The saddle point (SP) is obtained as the solution of $\delta S_2=0$. Assuming a solution of the form

$$\mathbf{Q}_0 = -i\frac{\eta}{2}\gamma_1,$$

we obtain the parameter η from the SP equation

$$\eta = ig\ \mathrm{Tr}_2 G_{rr}(E + i\epsilon + i\eta). \tag{27}$$

For $\epsilon=E=0$ the SP equation is invariant under the global symmetry transformation $\gamma_1\mathbf{Q}_0 \to \mathbf{U}\gamma_1\mathbf{Q}_0\mathbf{U}'$ of Eq. (19). This transformation leads to the SP manifold

$$\mathbf{Q}_r' = -i\frac{\eta}{2}\gamma_1\mathbf{U}_r\mathbf{U}_r' = -i\frac{\eta}{2}\mathbf{U}_r'^{-1}\gamma_1\mathbf{U}_r', \tag{28}$$

where $\mathbf{U}_r$ and $\mathbf{U}_r'$ are obtained from Eqs. (17) and (18) by replacing the transformation parameters ψ, … by space-dependent variables ψ_r, …. The form of $\mathbf{Q}_r'$, which is dictated by the symmetry of Sec. IV, implies for the action on the SP manifold that (i) the quadratic term in S_2 vanishes and (ii) the remaining term becomes

$$S' = \ln\det g(\mathbf{H}_0 + i\epsilon + E + i\eta\mathbf{U}\mathbf{U}') = \ln\det g[\mathbf{U}^{-1}(\mathbf{H}_0 + i\epsilon + E)\mathbf{U}'^{-1} + i\eta]. \tag{29}$$

This action contains the symmetry breaking field $i\epsilon+E$. Expansion in powers of η^{-1} yields, after renaming $\mathbf{U}^{-1}\to\mathbf{U}$,

$$S' = \frac{\epsilon - iE}{\eta}\mathrm{Tr}\, g(\mathbf{U}\mathbf{U}') + \frac{1}{\eta^2}\mathrm{Tr}\, g(\mathbf{U}'\mathbf{U}\mathbf{H}_0\mathbf{U}'\mathbf{U}\mathbf{H}_0) + O(\eta^{-3}).$$

Rescaling $\varphi\to\eta^{-1}\varphi$ and $\psi\to\eta^{-1}\psi$ does not change the integration measure and allows us to perform an expansion of $\mathbf{U}\mathbf{U}'$ in powers of η^{-1} up to $o(\eta^{-3})$:

$$\mathbf{U}\mathbf{U}' = \begin{pmatrix} \sigma_0 + 2i\varphi_{11}\sigma_3/\eta - 2\varphi_{11}^2\sigma_0/\eta^2 + 2\psi\bar{\psi}\sigma_0/\eta^2 & 2\psi\sigma_3/\eta + 2i(\varphi_{11}+\varphi_{22})\psi\sigma_0/\eta^2 \\ 2\bar{\psi}\sigma_3/\eta + 2i(\varphi_{11}+\varphi_{22})\bar{\psi}\sigma_0/\eta^2 & \sigma_0 + 2i\varphi_{22}\sigma_3/\eta - 2\varphi_{22}^2\sigma_0/\eta^2 - 2\psi\bar{\psi}\sigma_0/\eta^2 \end{pmatrix}, \tag{30}$$

where we have used $\varphi_{12}+\varphi_{21}=q+p'+q'+p=\varphi_{11}+\varphi_{22}$. This provides an expansion of the action, where the leading order is a quadratic form in terms of the fields ψ and φ:

$$S' = \sum_{r,r'} K_{rr'}(\epsilon - iE)(-\varphi_{11,r}\varphi_{11,r'} + \varphi_{22,r}\varphi_{22,r'} + 2\psi_r\bar{\psi}_{r'}) + \frac{2}{\eta^2}\sum_{r,r'} K_{rr'}(0)(\varphi_{11,r}+\varphi_{22,r})(\varphi_{11,r'}+\varphi_{22,r'})\psi_r\bar{\psi}_{r'} + O(\eta^{-7}), \tag{31}$$

with

$$K_{rr'}(\epsilon - iE) = 4\eta^{-4}\left[\left[\sum_{j=1,2}\sum_{\bar{r}} h_{j,r\bar{r}}h_{j,\bar{r}r} + 2(\epsilon - iE)\eta\right]\delta_{rr'} - \sum_{j=1,2} h_{j,rr'}h_{j,r'r}\right]. \tag{32}$$

There is only one term that couples the Grassmann field ψ with the field φ. It turns out (cf. Appendix D) that this term drops out after the integration over φ. Moreover, the Jacobian J of the transformation should be $J=2^{-N}+o(\eta^{-2})$ (N is the number of lattice sites) in order to satisfy the condition

$$\int e^{-S'} J\mathcal{D}[\psi,\varphi] = 1. \tag{33}$$

The symmetry-breaking term $\epsilon - iE$ appears as a prefactor of a diagonal term. This action can now be used to calculate the correlation functions in Eqs. (23)–(25). An expansion of the components of the matrix field yields

$$\sigma_1 Q = -\frac{i}{2}\left[\eta\sigma_0 + 2i\varphi_{11}\sigma_3 + \frac{2}{\eta}(-\varphi_{11}^2 + \psi\bar{\psi})\sigma_0\right] - i\sigma_1 P$$
$$= -\frac{i}{2}\left[\eta\sigma_0 + 2i\varphi_{22}\sigma_3 - \frac{2}{\eta}(\varphi_{22}^2 + \psi\bar{\psi})\sigma_0\right],$$

$$\sigma_1\Theta = -i\left[\sigma_3 + \frac{i}{\eta}(\varphi_{11}+\varphi_{22})\sigma_0\right)\psi,$$

$$\sigma_1\bar{\Theta} = -i\left[\sigma_3 + \frac{i}{\eta}(\varphi_{11}+\varphi_{22})\sigma_0\right)\bar{\psi}.$$

Integration over the fields ψ and φ with respect to the quadratic action S' in

$$\langle\ldots\rangle = \int \cdots e^{-S'}\mathcal{D}[\psi,\varphi] \tag{34}$$

leads to [up to $o(\eta^{-1})$]

$$\langle\sigma_1 Q\rangle = i\sigma_0\eta/2, -i\langle\sigma_1 P\rangle = i\sigma_0\eta/2, \tag{35}$$

since the integration rules imply

$$\langle-\varphi_{11}^2\rangle + \langle\psi\bar{\psi}\rangle = \langle\varphi_{22}^2\rangle + \langle\psi\bar{\psi}\rangle = 0.$$

It is also a consequence of Eq. (33). Moreover, we need for the evaluation of the correlation functions in Eqs. (23)–(25) the following expressions:

$$\langle(\sigma_1\Theta_{r'})_{jj}(\sigma_1\bar{\Theta}_r)_{jj}\rangle = -\langle\psi_{r'}\bar{\psi}_r\rangle + o(\eta^{-1}),$$

$$\langle(\mathrm{Tr}_2(\sigma_3\sigma_1\Theta_{r'})\mathrm{Tr}_2(\sigma_3\sigma_1\bar{\Theta}_r)\ \rangle = -4\langle\psi_{r'}\bar{\psi}_r\rangle + o(\eta^{-1}),$$

and

$$\langle\mathrm{Tr}_2(-i\sigma_1 P_{r'})\mathrm{Tr}_2(\sigma_1 Q_r)\rangle = -\eta^2 - 4\langle\psi_r\bar{\psi}_{r'}\rangle\langle\psi_{r'}\bar{\psi}_r\rangle.$$

This leads to

$$C_{rr'} = -\frac{2}{g^2\pi^2}\langle\psi_{r'}\bar{\psi}_r\rangle = \frac{1}{g^2\pi^2}K_{rr'}^{-1} \tag{36}$$

and

$$A_{rr'}(\omega) = -\frac{4}{g^2}\langle\psi_{r'}\bar{\psi}_r\rangle = \frac{2}{g^2}K_{rr'}^{-1}. \tag{37}$$

Finally, the correlation function of the local density of states reads

$$D_{rr'} = \frac{1}{g^2\pi^2}(\eta^2 + 4\langle\psi_r\bar{\psi}_{r'}\rangle\langle\psi_{r'}\bar{\psi}_r\rangle) = \frac{\eta^2}{g^2\pi^2} + \frac{1}{g^2\pi^2}K_{rr'}^{-1}K_{r'r}^{-1}. \tag{38}$$

In summary, all the correlations are expressed in terms of the inverse of the matrix K of Eq. (32). Interesting is that only correlations of the Grassmann field ψ appear, whereas the real fields $\varphi_{jj'}$ do not contribute, at least in the approximation up to $o(\eta^{-1})$.

VI. DISCUSSION

All three quantities in Eqs. (36)–(38) are related to the same correlation function, namely to $K_{rr'}^{-1} = -2\langle\psi_r\bar{\psi}_{r'}\rangle$ of Eq. (32). The latter, or its Fourier components $1/K(q)$ with

$$K(q) = \frac{8}{\eta^3}[\epsilon' + c(q)], \quad \epsilon' = \epsilon - iE, \tag{39}$$

can be considered as the propagator of the average two-particle Green's function $\langle G_{rr'}G_{r'r}\rangle$. It describes the motion of two particles, created at the same time at site r' (cf. Appendix B). Although the two particles are independent (i.e., we ignore their Coulomb interaction here), the averaging over the random vector potential creates an *effective interaction* between them. This interaction is presented by the quartic term in S_1 of Eq. (21), whose strength is g, the variance of the Gaussian distributed vector potential. A consequence of the interaction is that the two-particle propagator describes diffusion, when we consider $\epsilon' = i\omega/2$ and study $q \sim 0$:

$$\frac{1}{K(q)} \sim \frac{\eta^3/4}{i\omega + Dq^2},$$

with the diffusion coefficient

$$D = \left.\frac{1}{2}\frac{\partial^2 c(q)}{\partial q_l^2}\right|_{q=0} = \frac{1}{2\eta}\int_k \sum_j \left(\frac{\partial h_j}{\partial k_l}\frac{\partial h_j}{\partial k_l} - \frac{\partial^2 h_j}{\partial k_l^2}h_j\right). \tag{40}$$

Here it has been assumed that D is isotropic.

Assuming weak disorder, the action S' in Eq. (29) can be expanded in powers of η. (This weak-disorder approach is valid even for $\epsilon = \omega = 0$, in contrast to the factorization approach of Sect VIE) A diffusion propagator was also found for this case, with a the different diffusion coefficient though:[25]

$$D_w = \frac{g}{4\pi\eta}$$

for Dirac fermions and $E=0$. Thus, the physics of the average two-particle Green's function is diffusive, both for weak (i.e., for $\eta \ll 1$) and for strong disorder (i.e., for $\eta \gg 1$). The diffusion coefficient depends on the one-particle scattering rate η and the Hamiltonian H_0, given by its components h_j in Eq. (1). It is remarkable that both diffusion coefficients, the one of the strong-disorder expansion in Eq. (40), as well as D_w of the weak-disorder expansion, are proportional to the

one-particle scattering time $\tau=\eta^{-1}$. The latter can be calculated from the self-consistent Born approximation.[31] However, D always decreases with increasing disorder, whereas D_w is not monotonous with g but has a minimum due to the extra factor g. For realistic values of g, where the variance of the Gaussian distribution is $0<g<1$, the diffusion coefficient D_w decreases with g. In any case, the increasing behavior of D_w is beyond the validity of the weak-disorder approach.

The integration with respect to the SP manifold of Sec. V allows us to identify the relevant transformation parameters for the long-range correlation functions. From the results in Eqs. (36)–(38) it is obvious that only ψ, i.e., the transformation that mixes fermions and bosons, is relevant. On the other hand, the transformation inside the bosonic and inside the fermionic sector, provided by the parameters $\varphi_{jj'}$, is not relevant for the long-range correlations. This is in agreement with previous calculations for the conductivity, where the integration with respect to $\varphi_{jj'}$ has not been taken into account.[32] If we project the symmetry transformation by choosing $\psi=\bar{\psi}=0$ in Eqs. (17) and (18), there would also be a Goldstone mode, which becomes massless as we send the symmetry-breaking term to zero (i.e., $\epsilon\to 0$). This case has two interesting consequences: (i) the average density of states (DOS) would be divergent at $E=0$ due to

$$\langle\sigma_1 Q_r\rangle = -i\eta\sigma_0(\langle\varphi_{11,r}^2\rangle - 1/2),$$

and (ii) the long-range behavior of $C_{rr'}$, $A_{rr'}$, and $D_{rr'}$ would disappear.

For the calculation of the physical quantities of Sec. V we need the following expressions. The diagonal elements of K^{-1} diverge logarithmically with $\epsilon\sim 0$ as

$$K_{rr}^{-1} = \int_q \frac{1}{K(q)} \sim K_0 \ln(\epsilon),$$

which implies a divergence of the second moment of the local density of states D_{rr}. Summation over the lattice sites gives

$$\sum_r K_{r0}^{-1} = \frac{1}{K(q=0)} = \frac{\eta^3}{4\epsilon'}. \tag{41}$$

Finally, for the matrix element of r_k^2 and the conductivity we need

$$\sum_r r_k^2 K_{r0}^{-1} = -\left.\frac{\partial^2}{\partial q_k^2}\frac{1}{K(q)}\right|_{q=0} = \frac{\eta^3}{2\epsilon'^2}D. \tag{42}$$

A. One-particle scattering rate η

In the following the one-particle scattering rate η will be discussed for the specific case of Dirac fermions. Then the SP Eq. (27) reads

$$\eta = \frac{g}{2\pi}(\eta+iE)\ln\left[1+\frac{\lambda^2}{(\eta+iE)^2}\right], \tag{43}$$

where λ is the momentum cutoff of the Dirac fermions. For weak disorder (i.e., $g\ll 1$), the scattering rate is also weak.

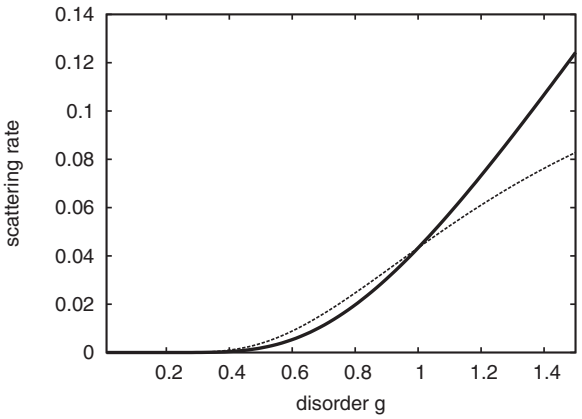

FIG. 2. Scattering rate (full curve) and density of states at the Dirac point (dashed curve) for Dirac fermions. Both quantities practically vanish over a wide range of disorder g.

To study the Dirac point, we rewrite Eq. (43) as

$$\eta = -iE + \frac{\lambda}{\sqrt{\exp(2\pi\eta/g(iE+\eta))-1}}, \tag{44}$$

and take the limit $E=0$

$$\eta_0 = \frac{\lambda}{\sqrt{e^{2\pi/g}-1}}. \tag{45}$$

This is shown in Fig. 2. For $|E|\ll\eta_0$ we can solve the SP Eq. (44) by an expansion in E: $\eta=\eta_0+o(E)$. On the other hand, for $|E|\gg|\eta|$ we can iterate Eq. (43) with the initial value

$$\bar{\eta} = \frac{g}{2\pi}iE\ln[1-\lambda^2/E^2]$$
$$= \frac{gE}{2}\left(\mathrm{sign}(E)+\frac{i}{\pi}\ln[\lambda^2/E^2-1]\right)(E^2<\lambda^2). \tag{46}$$

$\bar{\eta}$ is a reasonable approximation of η if

$$g\left|1+\frac{i}{\pi}\ln[\lambda^2/E^2-1]\right| \ll 1.$$

B. Density of states

The density of states is proportional to the average one-particle Green's function

$$\rho(E) = \mathrm{Tr}_2\langle G_{rr}(E-i\epsilon)\rangle$$

and describes two important features of our model—the spontaneous symmetry breaking and the one-particle scattering rate η in the SP approximation [cf. Eq. (27)]. In terms of the SP integration of the functional integral, Eq. (35) provides a finite average density of states at the Dirac point

$$\rho(E=0) = -i\frac{1}{g}\mathrm{Tr}_2\langle\sigma_1 Q\rangle = -\frac{1}{g}\mathrm{Tr}_2\langle\sigma_1 P\rangle = \eta_0/g,$$

which is practically zero for a larger regime of g (cf. Fig. 2). Moreover, for $|E|\gg|\eta|$ we get a linear behavior from Eq. (46)

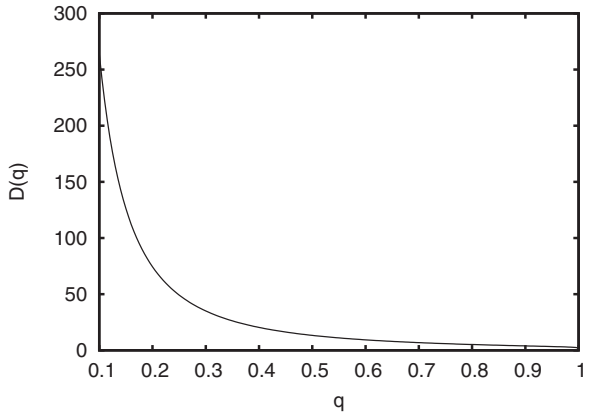

FIG. 3. Fourier transform of the local density of states correlation function, which diverges like q^{-2}.

$$\rho(E) \approx \frac{1}{g}\mathrm{Re}(\bar{\eta}) = \frac{|E|}{2},$$

which reflects the density of states of pure Dirac fermions. Both results are in good agreement with a self-consistent calculation of the average density of states.[27,33]

Correlations of the local density of states $D_{rr'}$ have a long-range behavior. The corresponding Fourier transform

$$D(q) = \int_k \frac{1}{K(k-q/2)K(k+q/2)} \tag{47}$$

is a function of q^2 and diverges like q^{-2} at $q=0$. It has a kink (or a shoulder) at the edge of the one-particle spectrum λ (cf. Fig. 3).

C. Microwave conductivity

According to Eqs. (14), (37), and (42), the matrix element of r_k^2 reads

$$\langle \Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle = \sum_r r_k^2 A_{r0}(\omega) = -\frac{1}{g^2}\frac{\eta^3}{\omega^2}D.$$

Using the expression of the diffusion coefficient in Eq. (40) and taking $\epsilon \to 0$, the matrix element becomes

$$\langle \Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle = -4\frac{\eta^2}{g^2\omega^2}\int_k \sum_j \left(\frac{\partial h_j}{\partial k_l}\frac{\partial h_j}{\partial k_l} - \frac{\partial^2 h_j}{\partial k_l^2}h_j\right). \tag{48}$$

Lower frequencies ω (i.e., lower energies) are more important for scattering than states of higher energies due to their larger matrix elements. Moreover, if we interpret the matrix element as a measure of localization, the states with $\omega > 0$ are localized on a scale $1/\omega$. The matrix element, together with Eq. (14), gives for the microwave conductivity

$$\sigma_0(\omega) = -\frac{e^2}{4\pi\hbar}\omega^2\langle \Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle \sim \frac{e^2}{\pi\hbar}\frac{\eta_0^2}{g^2}\int_k \sum_j \left(\frac{\partial h_j}{\partial k_l}\frac{\partial h_j}{\partial k_l} - \frac{\partial^2 h_j}{\partial k_l^2}h_j\right) (\omega \ll \eta_0). \tag{49}$$

This result indicates a constant microwave conductivity, at least for $\omega \ll \eta_0$, since the prefactor ω^2 is compensated by the ω^{-2} behavior of the matrix element of r_k^2. It should also be noticed that $\lim_{\omega\to 0}\sigma_0(\omega)$ gives the dc conductivity of Eq. (7). This follows immediately from the definitions of the two conductivities in Eqs. (7) and (14) and from Eqs. (36) and (37).

The ω^{-2} behavior of the r_k^2 matrix element in Eq. (48) obviously does not depend on the special form of h_j, as long as the spinor structure of H_0 exists. This implies that also for a parabolic k_j dependence (e.g., in the case of a graphene bilayer[34–36]), the cancellation of the ω^2 terms in the conductivity takes place.

D. Participation ratios

The inverse participation ratio $P^{(2)}$ of Eq. (10) vanishes like

$$P^{(2)} \sim \epsilon(\ln \epsilon)^2, \tag{50}$$

which indicates the existence of delocalized states at the Dirac point. The participation ratio

$$p^{(2)} \sim (\ln \epsilon)^{-2}, \tag{51}$$

on the other hand, vanishes logarithmically. These two results are consistent with a critical point at $E=0$, where there is a transition from localized to extended states.

E. Perturbation theory for weak disorder and factorization

The fact that diffusion is controlled by the one-particle scattering rate η raises the question about the quality of the one-particle approximation. The latter has been used frequently by factorizing the two-particle Green's function.[37,38,31,15] This approximation should be valid for weak disorder. It is based on the assumption that the Green's functions are uncorrelated and the averaged product is approximately the same as the product of the averaged one-particle Green's functions:

$$\langle G^+_{rr',jj}G^+_{r'r,kk}\rangle \approx \langle G^+_{rr',jj}\rangle\langle G^+_{r'r,kk}\rangle.$$

This allows us to treat the average one-particle Green's functions within the self-consistent Born approximation:

$$\langle G^{\pm}\rangle \approx (H_0 \pm i\eta)^{-1} \equiv (H_0 \pm i\eta)^{-1},$$

where η the imaginary part of the self-energy (or inverse scattering time) determined for Dirac fermions in Eq. (43). Consequently, the one-particle Green's function decays exponentially on the scale η^{-1}. The correlation function $D_{rr'}$ is constant and proportional to η^2, and there is no divergence for $r'=r$. This means that we have lost in the factorization the correlation term

$$\frac{1}{g^2\pi^2}K_{rr'}^{-1}K_{r'r}^{-1}$$

of Eq. (38). These are the substantial differences between the strong-disorder expansion and weak-disorder perturbation theory at the Dirac point. If we go away from the Dirac point, we can study the matrix element r_k^2, approximated by the factorization as

$$\langle\Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle = -\sum_r r_k^2 \mathrm{Tr}_2[\sigma_3 G_{r0}(\omega/2+i\eta)\sigma_3 G_{0r}(\omega/2+i\eta)] \sim \begin{cases} -\eta_0^{-2} & \text{for } \eta_0 \gg \omega \\ -4\omega^{-2} & \text{for } \eta_0 \ll \omega \end{cases},$$

where we have assumed that $\omega \ll \lambda$. In contrast to the result of Eq. (48), the matrix element does not diverge now if we approach the Dirac point $\omega=0$, since there is the finite limit η_0^{-2}. This reflects the finite decay length η_0^{-1} of the average one-particle Green's function. Away from the Dirac point the factorization works better. This can be seen if we insert the matrix element of r_k^2 into the conductivity of Eq. (14)

$$\sigma_0(\omega) = -\frac{e^2}{4\pi\hbar}\omega^2\langle\Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle \sim \frac{e^2}{\pi\hbar} = 2\frac{e^2}{h}(\eta_0 \ll \omega) \tag{52}$$

in the perturbative regime. Thus again, the conductivity does not depend on the frequency. Actually, the validity of the perturbative regime for the matrix element of r_k^2 and the conductivity is big in terms of g due to $\eta_0 \sim e^{-\pi/g}$.

VII. CONCLUSIONS

Spinor states, described by a two-dimensional Dirac-type lattice Hamiltonian, were studied in an uncorrelated random vector potential. Our calculation, based on a strong-disorder expansion, has revealed that the quantum states develop long-range correlated fluctuations. In other words, the quantum system transforms the uncorrelated fluctuations of the random vector potential into long-range correlated fluctuations, for instance, of the density of states. The origin of this behavior is spontaneous symmetry breaking (SSB), which develops a massless (long-range) mode. The spontaneous symmetry breaking is measured by the one-particle scattering rate η or the density of states η/g. An important scale for this effect is

$$\eta_0 \sim e^{-\pi/g},$$

which depends on the variance of the Gaussian fluctuations of the random vector potential g. [Although our approach is not valid for very small g (cf. Sec. VI E), this scale is short for all reasonable values $g \approx 1$.] η_0 separates regimes that are controlled by disorder ($\eta \ll \omega$) from that which is controlled by energy (or frequency) ($\eta \ll \omega$). For instance, a central quantity is the one-particle scattering rate has a crossover with respect to ω as

$$\eta \sim \begin{cases} e^{-\pi/g} & \text{for } \eta_0 \gg \omega \\ \omega g & \text{for } \eta_0 \ll \omega \end{cases}.$$

Perturbation theory with respect to disorder can be applied to the regime with $\eta_0 \ll \omega$, at least for quantities such as the conductivity. The correlation length is not affected qualitatively by this cross over, as one can see from the matrix elements of r_k^2

$$\langle\Phi_{\omega/2}|r_k^2|\Phi_{-\omega/2}\rangle = -4\omega^{-2}\begin{cases} (\eta_0^2/g^2)\int_k \sum_j \left(\frac{\partial h_j}{\partial k_l}\frac{\partial h_j}{\partial k_l} - \frac{\partial^2 h_j}{\partial k_l^2}h_j\right) & \text{for } \eta_0 \gg \omega \\ 1 & \text{for } \eta_0 \ll \omega \end{cases},$$

which provides an effective correlation length ω^{-1} for $\langle\Phi_{E,rj}\Phi^* - E, rj\rangle$. As we go away from the Dirac point, the correlation length decreases. On the other hand, the correlation function of the local density of states $D_{rr'}$ clearly distinguishes both regimes, since the corresponding Fourier transform is

$$D(q) \sim \begin{cases} D_0 q^{-2} & \text{for } q \sim 0 \text{ and } \eta_0 \gg \omega \\ D_1\delta(q) & \text{for } \eta_0 \ll \omega \end{cases}.$$

An interesting consequence of the "universal" ω^{-2} behavior of the matrix element of r_k^2 is a constant microwave conductivity in both regimes:

$$\sigma_0(\omega) \sim 2\frac{e^2}{h}\begin{cases} (\eta_0^2/g^2)\int_k \sum_j \left(\frac{\partial h_j}{\partial k_l}\frac{\partial h_j}{\partial k_l} - \frac{\partial^2 h_j}{\partial k_l^2}h_j\right) & \text{for } \eta_0 \gg \omega \\ 1 & \text{for } \eta_0 \ll \omega \end{cases}.$$

This behavior must be seen in contrast to the conventional Drude behavior, where the real part of conductivity decays like ω^{-2}. It is a consequence of the correlated scattering between the upper and the lower part of the Dirac cones (i.e., it is a manifestation of the *Zitterbewegung*). This effect should be experimentally observable, since it survives also in the presence of disorder.

Our results can be summarized by the statement that properties of graphene at the Dirac point are ruled by long-range

correlations. At the distance ω from the Dirac point they decay on the scale ω^{-1}. This has physical consequences, like a constant microwave conductivity, which also has been observed experimentally.[13] Disorder of variance g creates a characteristic scale $\eta_0 = e^{-\pi/g}$, which separates the behavior in a vicinity $\omega \ll \eta_0$ of the Dirac point from another one away from the Dirac point with $\omega \gg \eta_0$. Standard perturbation theories and simple approximations can be applied to the latter because then the behavior is ruled by one-particle properties.

ACKNOWLEDGMENT

I am grateful to B. Dóra for useful discussions.

APPENDIX A: CONDUCTIVITY

The conductivity per site on a lattice with N sites can be evaluated within the Kubo formalism and gives [cf. Eq. (3) of Ref. 25]

$$\begin{aligned}\mathrm{Re}(\sigma_{kk}) &= \frac{e^2}{N\hbar}\pi \int \mathrm{Tr}[[H,r_k]\delta(H-E+\omega) \\ &\quad \times [H,r_k]\delta(H-E)]\frac{f_\beta(E+\omega)-f_\beta(E)}{\omega}dE \\ &= \frac{e^2}{N\hbar}\pi\omega^2 \int \mathrm{Tr}[r_k\delta(H-E+\omega/2)r_k\delta(H-E \\ &\quad -\omega/2)]\frac{f_\beta(E+\omega/2)-f_\beta(E-\omega/2)}{\omega}dE\end{aligned}$$

with the Fermi function at inverse temperature β: $f_\beta(E) = 1/[1+\exp(\beta E)]$. Now we consider

$$\begin{aligned}&\mathrm{Tr}[r_k\delta(H-E+\omega/2)r_k\delta(H-E-\omega/2)] \\ &\quad = \sum_{r,r'} r_k r'_k \mathrm{Tr}_2[\delta_{rr'}(H-E+\omega/2)\delta_{r'r}(H-E-\omega/2)].\end{aligned}$$

We can write

$$2r_k r'_k = -(r_k - r'_k)^2 + r_k^2 + r_k'^2.$$

Moreover, $\delta(H-E+\omega/2)\delta(H-E-\omega/2)=0$ for $\omega \neq 0$. Thus we obtain

$$\begin{aligned}&\sum_{r,r'} r_k r'_k \mathrm{Tr}_2[\delta_{rr'}(H-E+\omega/2)\delta_{r'r}(H-E-\omega/2)] \\ &\quad = -\frac{1}{2}\sum_{r,r'}(r_k - r'_k)^2 \mathrm{Tr}_2[\delta_{rr'}(H-E+\omega/2)\delta_{r'r}(H-E \\ &\quad -\omega/2)].\end{aligned}$$

This gives

$$\begin{aligned}&-\frac{e^2}{2N\hbar}\pi\omega^2 \int \sum_{r,r'}(r_k - r'_k)^2 \mathrm{Tr}_2[\delta_{rr'}(H-E+\omega/2) \\ &\quad \times \delta_{r'r}(H-E-\omega/2)]\frac{f_\beta(E+\omega/2)-f_\beta(E-\omega/2)}{\omega}dE.\end{aligned}$$

Since for low temperatures

$$f_\beta(E) \sim \Theta(-E),$$

we get

$$\begin{aligned}&\sim \frac{e^2}{2\hbar}\pi\omega\sum_r (r_k - r'_k)^2 \int_{-\omega/2}^{\omega/2} \mathrm{Tr}_2[\delta_{rr'}(H-E+\omega/2) \\ &\quad \times \delta_{r'r}(H-E-\omega/2)]dE.\end{aligned}$$

The Dirac delta functions can be expressed by one-particle Green's functions

$$\delta(H+\omega/2) = \frac{1}{2\pi i}[G(\omega/2 - i\epsilon) - G(\omega/2 + i\epsilon)]$$

such that

$$\mathrm{Re}(\sigma_{kk}) \approx \sigma^{\mathrm{I}}(\omega) + \sigma^{\mathrm{II}}(\omega),$$

with

$$\begin{aligned}\sigma^{\mathrm{I}}(\omega) &= \frac{e^2}{4h}\omega \int_{-\omega/2}^{\omega/2} \sum_r (r_k - r'_k)^2 \mathrm{Tr}_2[G_{rr'}(\omega/2 - E - i\epsilon) \\ &\quad \times G_{r'r}(-\omega/2 - E - i\epsilon) + G_{rr'}(\omega/2 - E + i\epsilon) \\ &\quad \times G_{r'r}(-\omega/2 - E + i\epsilon)dE\end{aligned}$$

and

$$\begin{aligned}\sigma^{\mathrm{II}}(\omega) &= -\frac{e^2}{4h}\omega \int_{-\omega/2}^{\omega/2} \sum_r (r_k - r'_k)^2 \mathrm{Tr}_2[G_{rr'}(\omega/2 - E - i\epsilon)G_{r'r}(\\ &\quad -\omega/2 - E + i\epsilon) + G_{rr'}(\omega/2 - E + i\epsilon)G_{r'r}(-\omega/2 - E \\ &\quad - i\epsilon)dE\end{aligned}$$

and because of $G(z) = -\sigma_3 G(-z)\sigma_3$ is

$$\begin{aligned}\sigma^{\mathrm{II}}(\omega) &= -\frac{e^2}{4h}\omega \int_{-\omega/2}^{\omega/2} \sum_r (r_k - r'_k)^2 \mathrm{Tr}_2[G_{rr'}(\omega/2 - E - i\epsilon) \\ &\quad \times G_{r'r}(-\omega/2 - E + i\epsilon) + G_{rr'}(-\omega/2 + E \\ &\quad - i\epsilon)G_{r'r}(\omega/2 + E + i\epsilon)dE.\end{aligned}$$

APPENDIX B: MATRIX ELEMENTS OF THE ENERGY STATES

Starting from a (localized) state $\Psi(0)$, we can allow the state to evolve in time:

$$\Psi(t) = e^{-iHt}\Psi(0).$$

The question is how the states $\Psi_{\pm E}$ can be reached by this evolution, and how this is influenced by scattering due to disorder. The contribution of the state with energy $\pm E$ to the time evolution is obtained by the Fourier transformation $t \to \pm E$ of $\Psi(t)$ for positive time $t \geq 0$ (since the wave function did not exist for $t < 0$):

$$\Phi_{\pm E} \equiv \int_0^\infty e^{(\pm iE-\epsilon)t}\Psi(t)dt$$

$$= \int_0^\infty e^{(\pm iE-\epsilon)t}e^{-iHt}dt\Psi(0)$$

$$= -i(H \mp E - i\epsilon)^{-1}\Psi(0)$$

$$= -iG(\mp E - i\epsilon)\Psi(0).$$

Since H is Hermitian (i.e., $H^\dagger = H$), the complex conjugate of the wave function is

$$\Phi^*_{\pm E} = iG(\mp E - i\epsilon)^*\Psi^*(0) = iG(\mp E + i\epsilon)^T\Psi^*(0),$$

where G^T is the transposed of G. The matrix element of r_k^2 between $\Phi_{\pm E}$ is

$$\langle\Phi_E|r_k^2|\Phi_{-E}\rangle = \sum_r r_k^2\Phi_{E,rj}\Phi^*_{-E,rj}$$

$$= \sum_{r,j'} r_k^2 G_{r0,jj'}(-E - i\epsilon)G_{0r,j'j}(E + i\epsilon)|\Psi_{j'}(0)|^2 \tag{B1}$$

if we assume that $\Psi(0)$ is localized at the origin of the lattice $r=0$. In the presence of disorder, this expression should be averaged with respect to the latter:

$$\langle\Phi_E|r_k^2|\Phi_{-E}\rangle = \sum_{r,j} r_k^2\langle\Phi_{E,rj}\Phi^*_{-E,rj}\rangle$$

$$= \sum_{r,j,j'} r_k^2\langle G_{r0,jj'}(-E - i\epsilon)G_{0r,j'j}(E + i\epsilon)\rangle$$

$$\times|\Psi_{j'}(0)|^2. \tag{B2}$$

It describes a correlation between the states in the upper and in the lower band, if they evolve from the same initial state $\Psi(0)$. These results can be summarized to the relation

$$\sum_r r_k^2\langle \mathrm{Tr}_2[G_{r0}(-E - i\epsilon)G_{0r}(-E' + i\epsilon)]\rangle = \langle\Phi_E|r_k^2|\Phi_{E'}\rangle$$

if $|\Psi_{j'}(0)|^2 = 1$ for $j' = 1, 2$.

APPENDIX C: INVERSE PARTICIPATION RATIOS

The inverse participation ratio is related to the fourth moment of the normalized eigenfunction Ψ_k with eigenvalue E_k as

$$P^{(2)} = \langle\sum_k |\Psi_{k,r}|^4\delta(E - E_k)\rangle. \tag{C1}$$

This expression vanishes in the delocalized regime. The latter is plausible when we estimate the delocalized wave function by $|\Psi_{k,r}|^2 \sim 1/N$ on a lattice with N sites, taking into account normalization

$$\sum_r |\Psi_{k,r}|^2 = 1.$$

This implies

$$\sum_r\sum_k |\Psi_{k,r}|^2\delta(E - E_k) \sim \sum_k \delta(E - E_k) \sim N\rho,$$

where ρ is the spatially averaged density of states. Moreover, for $P^{(2)}$ we get from Eq. (C1) a vanishing expression for $N \sim \infty$:

$$\sum_k |\Psi_{k,r}|^4\delta(E - E_k) \sim \frac{1}{N^2}\sum_k \delta(E - E_k) \sim \frac{\rho}{N}.$$

$P^{(2)}$ can also be related to D_{rr} by using the relation[30]

$$\lim_{\epsilon\to 0}\epsilon\sum_k |\Psi_{k,r}|^2\delta_\epsilon(E - E_k)\sum_{k'} |\Psi_{k',r}|^2\delta_\epsilon(E - E_{k'})$$

$$= \sum_k |\Psi_{k,r}|^4\delta(E - E_k),$$

which implies

$$P^{(2)} = \lim_{\epsilon\to 0}\epsilon D_{rr}.$$

APPENDIX D: RANDOM-WALK EXPANSION

The presence of the Grassmann field in the action S' of Eq. (29) enables us to write the functional integral $\int\exp(-S')\mathcal{D}[\psi,\varphi]$ in terms of a dense system of self-avoiding random walks (or polymers), whose density is controlled by the symmetry-breaking term. Starting from S' we either use UU' of Eq. (20) or its approximation given in Eq. (30). In both cases the structure of the action with symmetry-breaking parameter ϵ is

$$S' = S_0 + \epsilon\sum_r \psi_r\bar{\psi}_r + \sum_{r,r'} a_{rr'}\psi_r\bar{\psi}_{r'} + \sum_{r,r'} b_{rr'}\psi_r\bar{\psi}_r\psi_{r'}\bar{\psi}_{r'}.$$

S_0 is a term without Grassmann field. The quartic term [which does not appear in the approximated field of Eq. (30)] can be expressed by a quadratic term that couples to a random field. Thus the general structure in terms of the Grassmann field reads

$$S' = S_0 + \epsilon\sum_r \psi_r\bar{\psi}_r + \sum_{r,r'} B_{rr'}\psi_r\bar{\psi}_{r'},$$

where the coefficients $B_{rr'}$ are random and connect only nearest-neighbor sites (this is a consequence of the symmetry). The integration with respect to ψ can be performed first and gives just a determinant

$$\int \exp(-S')\mathcal{D}[\psi] = e^{-S_0}\det(\epsilon + B).$$

Then the determinant can be expanded with respect to B, which creates random walks, whereas ϵ creates an environment of identical points. In other words, a lattice site r is either occupied by an ϵ or it is visited by a random walk with jump rate $B_{rr'}$ between r and a nearest-neighbor site r'. The random walks do not intersect themselves or each other and must be closed. This is a consequence of the integral over the Grassmann field or equivalently it is determined by the structure of the determinant. The contributions of the random

walks depend on the weight of its elements $B_{rr'}$, relative to the value of ϵ. For $\epsilon \sim 0$ (i.e., near the Dirac point) the lattice is completely covered by closed self-avoiding random walks, which can also be considered as the regime of dense polymers. Next, we need to integrate with respect to φ. In general, this will affect the weight of the random walks. In the special case of S' in Eq. (31) we obtain

$$\int \exp(-S')\mathcal{D}[\psi,\varphi] = \int e^{-S_0}\det[K(\epsilon - iE) + zK(0)z]\mathcal{D}[\varphi], \tag{D1}$$

where z is a diagonal matrix with matrix elements $z_{rr} = (\varphi_{11,r} + \varphi_{22,r})/\eta$ and

$$S_0 = \sum_{r,r'} K_{rr'}(\epsilon - iE)(-\varphi_{11,r}\varphi_{11,r'} + \varphi_{22,r}\varphi_{22,r'}).$$

The expansion of the determinant produces terms with products $\Pi_r z_{rr}^{l_r} = \Pi_r[(\varphi_{11,r} + \varphi_{22,r})/\eta)^{l_r}$ $(l_r = 0,1,2)$, where $l_r > 0$ is created by the second term in its argument $z_{rr}K_{rr'}(0)z_{r'r'}$. Integration with the weight factor e^{-S_0}, where at least one site r appears with $l_r > 0$, gives a vanishing Gaussian integral

$$\int e^{-S_0}\prod_r (\varphi_{11,r} + \varphi_{22,r})^{l_r} = 0,$$

since the two quadratic terms in S_0 appear with opposite signs (i.e., φ_{11} must be integrated along the imaginary axis). Therefore, the second term in the determinant of Eq. (D1) does not contribute to the functional integral.

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