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Two-Dimensional Electrons in a Strong Magnetic Field with Disorder: Divergence of the Localization Length.

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Abstract. – Electrons on a square lattice with half a flux quantum per plaquette are considered. An effective description for the current loops is given by a two-dimensional Dirac theory with random mass. It is shown that the conductivity and the localization length can be calculated from a product of Dirac Green's functions with the *same* frequency. This implies that the delocalization of electrons in a magnetic field is due to a critical point in a phase with a spontaneously broken *discrete* symmetry. The estimation of the localization length is performed for a generalized model with N fermion levels using the Schwarz inequality and a $1/N$-expansion. An argument for the existence of two Hall transition points is given in terms of percolation theory.

Lattice models for the two-dimensional electron gas in a strong magnetic field are of increasing interest because of recent experiments with GaAs heterostructures [1]. Moreover, lattice models offer new ways to the study of the quantum Hall transition. The purpose of this article is to discuss a concept for the investigation of the localization length of the lattice model.

The tight-binding Hamiltonian for non-interacting electrons on a square lattice with magnetic flux ϕ reads, in Landau gauge,

$$H = -t\sum_r \left[\exp\left[2i\pi y\phi/\phi_0\right]c^\dagger(r+e_x)\,c(r) + c^\dagger(r+e_y)\,c(r) + \text{h.c.}\right] + \sum_r V(r)\,c^\dagger(r)\,c(r)\,. \qquad (1)$$

$e_{x,y}$ are lattice unit vectors, and $c^\dagger$ and c are fermion creation and annihilation operators, respectively. $V(r)$ is a potential which represents an additional structure or disorder on the lattice. The dispersion for $V = 0$ and half a flux quantum per plaquette ($\phi = \phi_0/2$) is $E = \pm 2t\sqrt{\cos^2 k_1 + \cos^2 k_2}$. A linear (relativistic) approximation for $E \sim 0$ around the four nodes $k_j = \pm\pi/2$ is possible. For H this is still very complicated [2], and a further simplification is useful to lift the degeneracy of the four nodes. This can be achieved by the introduction of a next-nearest-neighbour hopping term and a staggered potential in H [3, 4].

In this case only one gap opens at the quantum Hall transition. Expansion of $k = (\pi/2, \pi/2) + p$ for small p vectors leads to the large-scale approximation by the Dirac Hamiltonian $H' = \sigma \cdot p + \sigma_3 M$ with Pauli matrices σ_j. Disorder, originally introduced in H by a staggered random potential V, is described in H' by a random mass M [3]. The random mass model is probably not in the universality class of the network model of Chalker and Coddington [5] because the latter takes a random phase (*i.e.* a random vector potential in the Dirac representation) into account. According to a recent study [4] the large-scale approximation by a Dirac model is valid if only one gap opens at the quantum Hall transition. This is in agreement with our model because the degeneracy of the nodes was lifted by the next-nearest-neighbour hopping term and the staggered chemical potential. Physical quantities can be obtained from the Green's function

$$G(\omega) \equiv \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} i\omega + M & h \\ h^\dagger & i\omega - M \end{pmatrix}^{-1}, \tag{2}$$

where the Fourier component of h is $p_1 - ip_2$. For instance, the frequency-dependent conductivity is given by Kubo's formula $\sigma_{xx}(\omega) = (e^2/h)\,\omega^2 \sum_r r^2 \langle G_{jj'}(r, 0; i\omega) G_{j'j}(0, r; -i\omega) \rangle$, where $G_{jj'}(r, 0; i\omega) G_{j'j}(0, r; -i\omega) = |G_{jj'}(r, 0; i\omega)|^2$. The localization length ξ_1 is defined as the decay of the function $C_{jj'}(r, \omega) \equiv \langle |G_{jj'}(r, 0; i\omega)|^2 \rangle$ in space. There exists a relation of the above expression, which is composed of Green's functions at frequencies with *opposite sign* (retarded and advanced Green's functions), with a product of Green's functions at the *same* frequency. This follows from the block matrices in (2),

$$G_{jj}(r, r'; -i\omega) = -G_{j'j'}(r', r; i\omega), \quad G_{jj'}(r, r'; -i\omega) = -G_{jj'}(r', r; i\omega) \quad (j \neq j'). \tag{3}$$

This identity reflects the Lorentz covariance of the Dirac theory. It implies $|G_{jj}(r, r'; i\omega)|^2 = -G_{jj}(r, r'; i\omega) G_{j'j'}(r, r'; i\omega)$ and $|G_{jj'}(r, r'; i\omega)|^2 = -G_{j'j}(r, r'; i\omega) G_{jj'}(r, r'; i\omega)$. That means only the Green's function with one frequency is required for the evaluation of transport or localization properties in the present model.

The physics of the electrons can be understood as the statistics of current loops, created by the magnetic field. Depending on the potential, there are local current loops with two different directions. This observation is central for the understanding of the Hall transition discussed subsequently. The current loops are composed from the Dirac fermions by the creation and annihilation of particle-hole pairs. The direction can be reversed globally by a time-reversal transformation. In terms of the Green's function this means $G(\omega) \rightarrow -\sigma_3 G(\omega) \sigma_3$. In general the system is not invariant under this transformation because one direction of the currents is favoured. The favoured direction characterizes the electronic state of a Hall plateau. The transition between the Hall plateaux corresponds to a symmetric point with $\omega = M = 0$ which is also a critical point with a divergent localization length. The discrete symmetry at this critical point plays a fundamental role for the lattice electrons in a magnetic field. Taking now a random mass, which is symmetrically distributed around zero, this symmetry is spontaneously broken for the average Green's function [6]. It will be discussed in the following that the localization length also diverges at two critical points in the presence of disorder where the symmetry is broken. This is an extension of a previous work [7] where the Hall conductivity was calculated. Critical points with a divergent localization length in a symmetry-broken regime are known from Anderson localization in systems without magnetic field and dimensionality $d > 2$. However, the physics of the latter is different because there are no current loops. As a manifestation of this difference a *continuous* symmetry appears [8] instead of discrete symmetry found for localization in the presence of a magnetic field. The large-scale approximation for the continuous symmetry

leads to a non-linear sigma-model contrary to the Dirac theory in the presence of the magnetic field.

The Schwarz inequality can be applied to get a lower bound for C:

$$|\langle G_{jj''}(r, 0; i\omega) G_{j'j'''}(0, r; i\omega)\rangle|^2 \leq \langle |G_{jj''}(r, 0; i\omega)|^2\rangle \langle |G_{j'j'''}(0, r; i\omega)|^2\rangle \qquad (4)$$

with $j'' = j$, $j''' = j'$ or $j'' = j'$, $j''' = j$. Writing $C'_{jj}(r, \omega) \equiv \langle G_{jj}(r, 0; i\omega) G_{j'j'}(0, r, i\omega)\rangle$ and $C'_{jj'}(r, \omega) \equiv \langle G_{jj'}(r, 0; i\omega) G_{j'j}(0, r, i\omega)\rangle$ one has $|C'_{jj'}(r, \omega)| \leq |C_{jj'}(r, \omega)|$. C' will be used subsequently because it is easier to calculate than C. The average correlation functions are translational invariant. Therefore, the corresponding Fourier components $\tilde{C}(k, \omega)$ can be used to calculate the localization length

$$\xi_1 \propto \sqrt{-\frac{\nabla_k^2 \tilde{C}(k, \omega)}{\tilde{C}(k, \omega)}}\Bigg|_{k=\omega=0} \qquad (5)$$

A simple calculation for the pure system gives $\xi_1 = |M|^{-1}$. To evaluate the correlation function $C'(r, \omega)$ a generalization of the Hamiltonian H' is introduced which describes N levels of fermions per site [7]: $H^{\alpha\alpha'} = H_0^{\alpha\alpha'} - \delta M_r^{\alpha\alpha'} \sigma_3$ ($\alpha, \alpha' = 1, 2, \ldots, N$) with $H_0^{\alpha\alpha'} = (\sigma \cdot p + \langle M\rangle \sigma_3) \delta^{\alpha\alpha'}$. The distribution of the Hermitian random matrix (a Gaussian unitary ensemble [9]) δM is given by $\langle \delta M_r^{\alpha\alpha'} \delta M_{r'}^{\alpha''\alpha'''}\rangle = (g/N) \delta^{\alpha\alpha'''} \delta^{\alpha'\alpha''} \delta_{r, r'}$. That means only the random fluctuations couple the N different Dirac systems. This model has the advantage that a finite band of states is created due to disorder in the limit $N \to \infty$ and in the expansion around this limit. Thus the limit $N \to \infty$ mimics a system with finite fluctuations. The Hall transition is then due to the opening of a gap for sufficiently large values of $|\langle M\rangle|$ (cf. [7]). This behaviour is similar to the Hall transition on a pure lattice where the transition is also due to the opening of a gap [4, 10]. An alternative picture is related to strong disorder with unrestricted fluctuations (*e.g.*, a Gaussian distribution) and finite N. In this case the average density of states is non-zero for any value of $\langle M\rangle$. Then the correlation function C' decays exponentially.

The product of two Green's functions with the same frequency can be expressed as a functional integral [11],

$$G_{j,j''}^{\alpha\alpha'}(r, r'; i\omega) G_{j'',j'''}^{\alpha''\alpha'''}(r'', r'''; i\omega) = -\int \bar{\Psi}_{r',j''}^{\alpha'} \Psi_{r,j}^{\alpha} \bar{\chi}_{r''',j'''}^{\alpha'''} \chi_{r'',j'}^{\alpha''} \exp[-S_1] \prod_r \mathrm{d}\Phi_r \, \mathrm{d}\bar{\Phi}_r \qquad (6)$$

with the action (sum convention for the level index α)

$$S_1 = i\sigma_\omega \Big[-(\Phi, (H_0 + i\omega\sigma_0)\, \bar{\Phi}) + \sum_r \delta M_r^{\alpha\alpha'} (\Phi_r^{\alpha'} \cdot \sigma_3 \bar{\Phi}_r^{\alpha}) \Big]. \qquad (7)$$

$\sigma_\omega = \mathrm{sign}(\omega)$ and the field is $\Phi_{r,j}^{\alpha} = (\Psi_{r,j}^{\alpha}, \chi_{r,j}^{\alpha})$. The first component is Grassmann and the second is complex. The complex component is added to normalize the functional integral in (6). It also provides a transparent representation of the product of Green's functions with the same frequency as required for C and C'. Averaging with Gaussian-distributed fluctuations yields $\langle \exp[-S_1]\rangle = \exp[-S_2]$. S_2 is obtained from S_1 by replacing the second term by $(g/N) \sum_r (\Phi_r^{\alpha} \cdot \sigma_3 \bar{\Phi}_r^{\alpha'})(\Phi_r^{\alpha'} \cdot \sigma_3 \bar{\Phi}_r^{\alpha})$. This interaction term can also be created by complex (2×2)-matrix fields Q, P and a complex Grassmann field ψ which couple to the composite field $\sum_{\alpha=1}^{N} \Phi_r^{\alpha} \bar{\Phi}_r^{\alpha}$. The new field does not depend on α, the level index of the N-level fermions. The level degree of freedom can be eliminated by integrating out the field Φ in the functional

integral. This leads to $\exp[-NS(Q, P, \psi)]$ with the action [12]

$$S(Q, P, \psi) = (1/g) \sum_r [\mathrm{Tr}_2(Q_r\sigma_3)^2 + \mathrm{Tr}_2(P_r\sigma_3)^2] + \log\det(H_0 - 2Q + i\omega\sigma_0) -$$

$$- \log\det(H_0 + 2iP + i\omega\sigma_0) + (1/N) \sum_{\mu,\mu'=1}^{4} \int (\boldsymbol{I})_{\mu,\mu'}(k)\, \psi_{k,\mu}\, \overline{\psi}_{-k,\mu'}\, \mathrm{d}^2k + O(N^{-2}). \quad (8)$$

Since the number of fermion levels N appears in front of the new action, the limit $N \to \infty$ corresponds to a saddle-point integration for the fields Q and P. $\boldsymbol{I}(k)$ is the matrix of the Gaussian fluctuations around the saddle-point solution $Q = Q_0 + \delta Q$ and $P = iQ_0 + \delta P$, where Q_0 is the $N \to \infty$ solution $Q_0 = -(1/2)[i\eta\sigma_0 + M_s\sigma_3]$. M_s is a shift of the average Dirac mass and η shifts the frequency in the Green's function. Introducing $m = \langle M \rangle + M_s$ as the effective (renormalized) Dirac mass the imaginary shift is $\eta = \sigma_\omega \sqrt{\exp[-2\pi/g] - m^2}$ for $|m| \leqslant \exp[-\pi/g] \equiv m_c$ and zero otherwise [12].

The details of the derivation of $S(Q, P, \psi)$ from S_2 and the evaluation of the saddle-point integration can be found in ref. [12]. However, expression (8) can also be reconstructed using the fact that $\int \exp[-S_1]\, d\ldots = 1$. The $N \to \infty$-limit gives $Q_0 = -iP_0$ which implies that the large N-terms cancel each other in the action (8). First-order fluctuations do not contribute at the saddle-point. And second-order fluctuations appear as quadratic forms of δQ and δP with the matrix $\boldsymbol{I}$. Therefore, the condition $\int \exp[-NS(Q, P, \psi)]\, d\ldots = 1$ is satisfied if a quadratic form of $\boldsymbol{I}$ appears with a complex Grassmann field. This result reflects the supersymmetric construction of the functional integral.

For the asymptotic behaviour of a large correlation length it is sufficient to consider $k \sim 0$. If $k = 0$ the matrix $\boldsymbol{I}$ reads

$$\boldsymbol{I}(k=0) = \begin{pmatrix} 1/g - 2\alpha\mu^2 & 0 & 0 & 2\beta \\ 0 & 2/g - 4\alpha|\mu|^2 & 0 & 0 \\ 0 & 0 & 2/g - 4\alpha|\mu|^2 & 0 \\ 2\beta & 0 & 0 & 1/g - 2\alpha\mu^{*2} \end{pmatrix}, \quad (9)$$

where $\mu = m + i\eta$, $\alpha = \int (|\mu|^2 + k^2)^{-2}\, \mathrm{d}^2k/4\pi^2$ and $\beta = \int k^2(|\mu|^2 + k^2)^{-2}\, \mathrm{d}^2k/4\pi^2$. The inner (2×2)-matrix of $\boldsymbol{I}(k=0)$ is diagonal with positive matrix elements $2/g - 4\alpha|\mu|^2$. It leads always to a finite correlation length. However, this part of $\boldsymbol{I}$ does not contribute to C' because $C'_{jj'} = \boldsymbol{I}^{-1}_{j+2(j-1),\, j'+2(j'-1)}$ [12]. Therefore, only the projected submatrix (P_{14} is the projector on the indices 1, 4)

$$P_{14}\boldsymbol{I}^{-1}P_{14} = \begin{pmatrix} 1/g - 2\mu^2\alpha + \mu^2 A_1 k^2/|\mu|^2 & 2\beta + B_1 k^2/|\mu|^2 \\ 2\beta + B_1 k^2/|\mu|^2 & 1/g - 2\mu^{*2}\alpha + \mu^{*2} A_1 k^2/|\mu|^2 \end{pmatrix}^{-1} \quad (10)$$

with positive constants A_1, B_1 is of interest here. It describes the critical behaviour of the model due to one vanishing eigenvalue for $k = 0$ at two critical points $m = \pm m_c$. The critical points are also characterized by a vanishing imaginary part of the saddle-point Q_0, *i.e.* at the points of a vanishing density of states.

The Fourier components of C' are of the form $\tilde{C}'(k, \omega = 0) = (2\beta + bk^2/|\mu|^2)/(a' + b'k^2/|\mu|^2)$, where b and b' are finite constants and $a' = |1/g - 2\alpha\mu^2|^2 - 4\beta^2$. Equation (5)

yields for the correlation length ξ of C',

$$\xi \sim \xi_0 |m|^{-1}(1/g + (1/\pi) \log |m|)^{-1/2}, \quad \text{for } |m| > m_c \tag{11a}$$

and

$$\xi \sim \xi_0' |\mu|^{-1}(m_c^2 - m^2)^{-1/2}, \quad \text{for } |m| \leq m_c. \tag{11b}$$

For $|m| \geq m_c$ there are two characteristic regimes for the correlation length ξ. One is dominated by $|m| < (1/g + (1/\pi) \log |m|)^{1/2}$, the other by $|m| > (1/g + (1/\pi) \log |m|)^{1/2}$ with the crossover condition $|m| = m_c \exp[\pi m^2]$. The latter gives a very small value of $|m|$ for weak disorder such that $|m| > (1/g + (1/\pi) \log |m|)^{1/2}$ holds for almost the entire range of m. This implies a growth of the correlation length with $|m|^{-1}$ as small values of m are approached. The asymptotic behaviour $\xi \sim (m - m_c)^{-1/2}$ appears only in a very small regime. The correlation length ξ can also be evaluated for $m = m_c$ and $\omega \neq 0$. This leads to a different power law $\xi(\omega) \sim \xi_0 \omega^{-1/3}$ for $\omega \sim 0$.

An alternative approach, based on a different effective field theory with $N = 1$ for the correlation function C, indicates that ξ_1 behaves similarly to ξ [13]. The exponent $\nu \approx 1$ of the localization length, found here as an effective exponent, agrees with recent measurements of the quantum Hall transition in AlGaAs/GaAs heterostructures [1] and in Si MOSFETs [14].

On the other hand, numerical results for network models and lowest Landau level approaches [5, 15-17] suggest that the localization length ξ_1 diverges at the critical point with an exponent $\nu \approx 7/3$. The difference is probably due to strong random phase fluctuations studied in the numerical calculations.

The existence of the two critical points $\pm m_c$, in contrast to the single critical point in the perturbative approach of ref. [3], is a consequence of the spontaneous symmetry breaking due to disorder which creates a non-vanishing density of states between the particle and hole band of the Dirac theory. This non-perturbative phenomenon was discussed in ref. [6, 7, 12]. The single transition point of the other approaches to the integer Hall transition [5, 15-18] is either due to different models (Landau level approach *vs.* lattice fermions) or due to the fact that the two transition points were not seen because of limitations of the numerical resolution. (The distance of the two transitions is $2 \exp[-\pi/g]$ which is extremely small for weak disorder. It can be hidden by finite-size effects.)

Two transition points are also supported by the percolation picture of the lattice model. In this case a space-dependent Dirac mass M_r is considered which is either zero with probability p or $M > 0$ with probability $1 - p$. *I.e.* the average mass $(1 - p)M$ is positive for $p < 1$. The correlation length ξ is infinite for $p = 1$ because this is the symmetric point of the pure system. For $p_c < p < 1$, where p_c is the percolation threshold, the mass is non-zero on some finite clusters of lattice sites according to percolation theory. A large effect on the correlation length is only expected by the creation of an infinite cluster of lattice sites with non-zero mass at the percolation threshold $p = p_c$. If the argument is correct that the quantum Hall transition is connected with classical percolation [19] and corrections due to quantum tunnelling [5], then the transition should occur close to $p = p_c$. Therefore, the critical point is non-zero with $M_c \approx (1 - p_c)M$. Application of the transformation $G \to -\sigma_3 G \sigma_3$ to the Green's function $G(\omega = 0)$ replaces M by $-M$. Thus $-M_c$ is also a critical point.

In conclusion, it was shown that the transport of electrons on a square lattice in a strong magnetic field and a random potential can be described by a product of Dirac Green's functions with the same frequency. Therefore, the physics is characterized by a discrete symmetry. The localization length diverges at two different critical points. The critical exponent was estimated by $\nu \geq 1$ in a wide intermediate regime and by $\nu \geq 1/2$ in a narrow asymptotic regime.

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