Integer Quantum Hall Effect for Lattice Fermions.

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Abstract. – A two-dimensional lattice model for non-interacting fermions in a magnetic field with half a flux quantum per plaquette and N levels per site is considered. This is a model which exhibits the integer quantum Hall effect (IQHE) in the presence of disorder. It presents an alternative to the continuous picture for the IQHE with Landau levels. The large-N limit can be solved: two Hall transitions appear and there is an interpolating behaviour between the two Hall plateaux. Although this approach to the IQHE is different from the traditional one with Landau levels because of different symmetries (continuous for Landau levels and discrete here), some characteristic features are reproduced. For instance, the slope of the Hall conductivity is infinite at the transition points and the electronic states are delocalized only at the transitions.

We consider in two dimensions non-interacting lattice fermions in a homogeneous magnetic field with half a flux quantum per plaquette. This problem was originally studied some time ago by Fisher and Fradkin [1]. It was shown by these authors that the large-scale limit is equivalent to a two-dimensional Dirac theory. In a recent article Ludwig *et al.* [2] extended this model by introducing a staggered chemical potential. In the large-scale limit this parameter appears as the mass M of the Dirac fermions. Therefore, the purpose of this parameter is to create a gap of the effective Dirac theory between the particle and the hole band. Thus, two parameters control this system of non-interacting fermions: the staggered chemical potential and the energy of the particle E. In order to observe the integer quantum Hall effect (IQHE) one must introduce disorder. In the traditional approach to the IQHE, which is based on a continuum model with Landau levels [3, 4], this is achieved by using random fluctuations of the chemical potential. However, for the model under consideration it was argued by Ludwig *et al.* [2] that randomness of the chemical potential alone cannot create a non-vanishing density of states (DOS) for $M = E = 0$. Their argument is based on a perturbation theory with respect to disorder. Consequently, disorder in M would not lead to a physical Hall transition because of the absence of states at the transition point $M = E = 0$. In order to get a non-zero DOS for the physical Hall system, Ludwig *et al.* introduced another random quantity, an additional potential V which corresponds to energy fluctuation. The need of this potential is somehow surprising because a random chemical potential was sufficient to get the physical Hall transition in the famous approach to the IQHE by Pruisken *et al.* [3,4]. It will be argued subsequentially that, in contrast to the claim by Ludwig *et al.*, a random mass is also sufficient for a non-zero DOS in the lattice model of refs. [1,2].

First of all, we notice that the result of Ludwig *et al.* is in contradiction to an earlier work [5] where it was shown that the DOS is *non-zero* at $M = E = 0$. This result is based on a rigorous proof. Therefore, it would be interesting to understand why the perturbative approach of Ludwig *et al.* does not reproduce the correct behaviour of the DOS.

We will explain in this article that disorder in the Dirac mass M can lead to fluctuations in V (around $V = 0$). This is due to the fact that the same average Green's function can be created by different types of external fluctuations. Using N-level fermions we obtain the fluctuations in V from a suitable representation and a non-zero DOS in the large-N limit.

To introduce the details of the model, the pure system of non-interacting fermions on a two-dimensional lattice is considered along the lines of refs. [1,2]. This describes lattice fermions with nearest-neighbour hopping rate $t = 1$ and next–nearest-neighbour hopping rate $t'/4$ in a staggered potential $\mu(-1)^{x+y}$. If we identify fermions with the four corners of the unit cell, the related Hamiltonian reads in Fourier representation

$$H(k) =$$

$$= \begin{pmatrix} \mu & 1+e^{-ik_x} & \tau(1-e^{-ik_y})(1-e^{-ik_x}) & 1+e^{-ik_y} \\ 1+e^{ik_x} & -\mu & 1+e^{-ik_y} & -\tau(1-e^{-ik_y})(1-e^{ik_x}) \\ -\tau(1-e^{ik_y})(1-e^{ik_x}) & 1+e^{ik_y} & \mu & -1-e^{ik_x} \\ 1+e^{ik_y} & \tau(1-e^{ik_y})(1-e^{-ik_x}) & -1-e^{-ik_x} & -\mu \end{pmatrix} \quad (1)$$

with $\tau = it'/4$. Expansion of $k = (\pi, \pi) + ap$ for small p vectors leads to the large-scale approximation which breaks up the Hamiltonian (1) into two independent Dirac Hamiltonians $H_\pm = \sigma\cdot p + \sigma_3(m \pm T')$, with Pauli matrices σ_j. The lattice constant a is scaled out with $T' = t'/a$ and $m = \mu/a$. The two Dirac theories describe particles with different masses $m \pm T'$, respectively. For large-scale properties like the Hall transition it is sufficient to consider only the light particle with $M = m - T'$. The Hamiltonians obey individually a discrete symmetry $H_\pm \to -\sigma_3 H_\pm \sigma_3$ for zero mass [2,5]. This is an important observation because the effective theory for the Landau-level system obeys a continuous symmetry [3,4]. That means the two approaches to the IQHE are fundamentally different. Nevertheless, we will see that certain physical properties are similar in both cases.

The dispersion of H_- is $E = \pm\sqrt{M^2 + p^2}$. This is a Dirac theory with a particle and a hole band which are separated by a gap with width $2M$. We can vary the number of particles/holes by varying the width of the gap. Suppose the energy is in the lower (hole) band. Varying the gap means we add/remove particles and holes to/from the edges of the gap. This has no effect on the current because those states are not occupied. On the contrary, we can have the energy in the upper (particle) band. Varying the gap means that we add or remove the same number of particles and holes. Consequently, there is again no net current change because both contributions cancel each other. The situation is different if the energy is inside the gap: only the lower (hole) band is completely filled whereas the upper (particle) band is empty. Varying the gap means adding/removing holes to/from the system if the energy passes a band edge. This implies an additional current. Finally, holes and particles can be exchanged by the transformation $M \to -M$. This gives a particle (hole) current for $M > 0$ ($M < 0$), respectively. To make this more explicit, the Hall conductivity σ_{xy} is calculated as the response to an external static field q_y

$$\sigma_{xy} = j_x/E_y = \frac{i}{q_y}\int \sum_{r,r'}{}' \mathrm{Tr}[\sigma_x(H - i\omega + E)^{-1}_{r,r'}(H - i\omega + E + q_y\sigma_y)^{-1}_{r',r}]\frac{\mathrm{d}\omega}{2\pi}, \qquad (2)$$

where H is either H_- or H_+ and $\sum'$ is the sum normalized with the number of lattice sites. Using the Green's function $G(E - i\omega) = (H - i\omega + E)^{-1}$, we obtain for $q_y \sim 0$

$$\sim i \sum_{r,r',r''}{}' \int \mathrm{Tr}[\sigma_x G(E - i\omega; r, r')G(E - i\omega; r', r'')\sigma_y G(E - i\omega; r'', r)]\frac{\mathrm{d}\omega}{2\pi}.$$

For infinite cut-off of the k-integration (*i.e.* infinite bands) we find in units of $e^2/\hbar$ [2]

$$\sigma_{xy} = \frac{M}{4\pi|M|}\Theta(|M| - |E|), \qquad (3)$$

where M can be the light or the heavy mass. This result reflects correctly the qualitative interpretation of the Hall conductivity. The Hall conductivity for the original lattice fermion problem is the sum of the Hall conductivities from the light and the heavy mass, such that the total σ_{xy} has a jump from 0 to 1.

Now we introduce disorder through a random Dirac mass M. Disorder creates a DOS inside the gap of the pure system. This means that the bands are broadened such that their inner tails can overlap. The gap is closed if the fluctuations δM are larger than M [5]. That means we have a compact region in M around $M = 0$ for which the DOS $\rho(E)$ is non-zero. We cannot distinguish particle and hole contributions to the current as in the pure Dirac theory because the particle-hole symmetry is spontaneously broken in this case [5]: due to the fluctuations in M new complicated (mixed) states are created inside the gap of the pure system. They lead to a new Hall conductivity which interpolates between the two Hall plateaux. Once there is a gap again (if $|M|$ is larger than the maximal fluctuations $|\delta M|$), then the Hall plateaux appear (fig. 1). In the following we will generalize the model of ref. [2] by introducing N levels for the fermion Hamiltonian $H_\pm$. Then it will be briefly discussed that the large-N Hall conductivity describes indeed such a behaviour. Here we can use results obtained in previous studies of the Dirac Hamiltonian H_- [6, 7].

As a generalization of $H_\pm$ to N levels of fermions, we introduce $H^{\alpha\alpha'} = H_0^{\alpha\alpha'} - \delta M_r^{\alpha\alpha'}\sigma_3$ ($\alpha, \alpha' = 1, 2, \ldots, N$), with $H_0^{\alpha\alpha'} = H_\pm\delta^{\alpha\alpha'}$ and for the Hermitian random matrix δM, we assume $\langle \delta M_r^{\alpha\alpha'}\delta M_r^{\alpha''\alpha'''}\rangle = (g/N)\delta^{\alpha\alpha'''}\delta^{\alpha'\alpha''}$. That means only random fluctuations couple the N different Dirac systems. Physical quantities like the DOS per site or the conductivities are

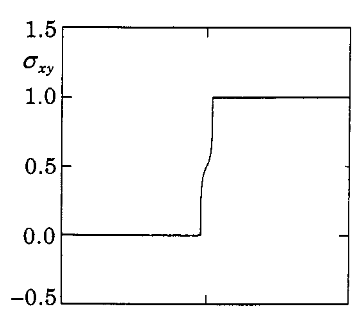

Fig. 1. – Hall conductivity σ_{xy} in units of e^2/h as a function of the staggered chemical potential M for disorder strength $g = 2.2$.

proportional to the number of levels N. It is natural to normalize these quantities by N. The purpose of the $N \to \infty$ limit is to obtain in a simple way a compact distribution to the DOS $\rho(E, M) = (1/2)\sum_\mu G_{\mu,\mu}(E + i0^+)$ from δM which fills the gap. This is a consequence of the well-known fact that $N \to \infty$ gives a semicircular density $\rho(M) \equiv \rho(E = 0, M)$ [8]. However, for a non-compact distribution (*e.g.* Gaussian) and finite N there is no gap at all; the non-compact fluctuations δM extend the DOS over the whole real axis. The trick is now to consider the large-N limit and $1/N$ expansion (a finite order of these terms does not create particles outside the $N \to \infty$ DOS) as a technical device in order to maintain a density $\rho(M)$ with compact support. There are Hall transitions at the edges of this «band» which was created by the mass fluctuations. This result indicates that the Hall plateaux are only due to the opening of a gap in $\rho(E, M)$. If we take $N < \infty$ there should be still a qualitative change in σ_{xy} due to a cross-over analogous to the transition with the compact DOS because the DOS in the gap is exponentially small for $|M| > M_c$. Therefore, σ_{xy} reaches almost the plateau value whereas we find a smooth change between the two plateau values by varying M between $-\infty$ and ∞. Nevertheless, in a real system it is more natural to truncate the large fluctuations.

The Green's function can be expressed formally as a functional integral for non-interacting fermions [9]

$$G^{\alpha\alpha'}_{\mu,\mu'}(z; r, r') = [(H_0 - \delta M\sigma_3 + z\sigma_0)^{-1}]^{\alpha\alpha'}_{r,\mu; r',\mu'} = -i\int \bar{\Psi}^{\alpha'}_{r',\mu'} \Psi^{\alpha}_{r,\mu} \exp[-S_1] \prod_r d\Phi_r \, d\bar{\Phi}_r, \quad (4)$$

with the action (sum convention for α)

$$S_1 = is_z[-(\Phi, (H_0 + z\sigma_0)\bar{\Phi}) + \sum_r \delta M_r^{\alpha\alpha'}(\Phi_r^{\alpha'} \cdot \sigma_3 \bar{\Phi}_r^{\alpha})], \quad (5)$$

with $s_z = \mathrm{sign}(\mathrm{Im}\, z)$ and the field $\Phi^{\alpha}_{r,\mu} = (\Psi^{\alpha}_{r,\mu}, \chi^{\alpha}_{r,\mu})$. The first component is Grassmann and the second complex. The complex component is added to normalize the functional integral in (4). Averaging with Gaussian-distributed fluctuations yields

$$S_2 = -is_z(\Phi, (H_0 + z\sigma_0)\bar{\Phi}) + \frac{g}{N}\sum_r (\Phi_r^{\alpha} \cdot \sigma_3 \bar{\Phi}_r^{\alpha})^2 . \quad (6)$$

Thus we have derived an effective field theory for Φ which serves as a generating functional for the average Green's function. It is important to notice that *not only* δM creates the fermion-fermion interaction in (6) but also other types of disorder. For instance, the interaction can also be created by a term which couples to a matrix field ($\mu = 1, \ldots, 4$ includes the complex and Grassmann components): $(N/g) Q_{r;\mu,\mu'}(\sigma_3)_{\mu'} Q_{r;\mu',\mu}(\sigma_3)_{\mu} - iQ_{r;\mu,\mu'}\Phi^{\alpha}_{r,\mu'}\bar{\Phi}^{\alpha}_{r,\mu}$. This implies that the distribution δM can be transformed into another distribution with a new «random variable» Q (which does not have a probability measure but some generalized distribution including Grassmann variables). In other words, we can write

$$\langle [(H_0 - \delta M\sigma_3 + z\sigma_0)^{-1}]^{\alpha\alpha} \rangle_{\delta M} = \langle [(H_0 - Q + z\sigma_0)^{-1}]^{\alpha\alpha} \rangle_Q . \quad (7)$$

The distribution which belongs to $\langle \ldots \rangle_Q$ was investigated in detail in [6, 7]. Here we present just the result for leading order in N: $\langle \ldots \rangle_Q = \int \ldots \exp[-NS(Q, P)] \prod_r dP_r \, dQ_r$ with complex fields Q_r, P_r and

$$S(Q, P) = \frac{1}{g}\sum_r[\mathrm{Tr}_2(Q_r\sigma_3)^2 + \mathrm{Tr}_2(P_r\sigma_3)^2] +$$

$$+ \log\det(H_0 - 2Q + z\sigma_0) - \log\det(H_0 + 2iP + z\sigma_0) + O(N^{-1}). \quad (8)$$

The number of levels N appears in front of the action. Thus, the effect of disorder for $N \to \infty$ can be evaluated in the saddle point (SP) approximation. The SP equation reads

$$\frac{\partial}{\partial Q}\left[\frac{1}{g}\mathrm{Tr}(Q\sigma_3)^2 + \log\det(H_0 - 2Q + z\sigma_0)\right] = 0. \quad (9)$$

A second SP equation appears from the variation of P by replacing $Q \to -iP$. As an ansatz we take a uniform SP solution $Q_0 = -iP_0 = -(1/2)[i\eta\sigma_0 + M_s\sigma_3]$. Then (9) leads to the conditions $\eta = (\eta + \omega - iE)gI$, $M_s = -MgI/(1+gI)$ with the integral $I = \int[(M + M_s)^2 + (\eta + \omega - iE)^2 + k^2]^{-1}\mathrm{d}^2k/2\pi^2$. This means disorder shifts the frequency $\omega \to \omega + \eta$ and the Dirac mass $M \to M' = M + M_s$, where $\eta(M, \omega)$ and $M_s(M, \omega)$ are solutions of the SP equation. The sign of η is fixed by the condition that η must be analytic in ω. This leads to $\mathrm{sign}(\eta) = \mathrm{sign}(\omega)$. Furthermore, $\rho(M)$ is proportional to η. The Hall conductivity per fermion level reads

$$\sigma_{xy} = M'\int(\omega + \eta - iE - i|M'|)^{-1}(\omega + \eta - iE + i|M'|)^{-1}\mathrm{d}\omega/4\pi^2, \quad (10)$$

and with the approximation that M' and η do not depend on ω we get

$$\sigma_{xy} \approx [\mathrm{arctg}((M' + E)/\eta) + \mathrm{arctg}((M' - E)/\eta)]/4\pi^2. \quad (11)$$

The sum of σ_{xy} for both masses is plotted in fig. 1 for $E = 0$. An analogous calculation gives for the longitudinal conductivity per level

$$\sigma_{xx} \approx [\pi/2 - \mathrm{arctg}((\eta^2 + M'^2 - E^2)/2|\eta E|)]/4\pi^2. \quad (12)$$

The transition between the Hall plateaux does not occur at $M = 0$, as suggested in ref. [2], but at $M = \pm M_c$, where $\eta(M)$ vanishes. The distance of these transitions is small for weak disorder and $E = 0$: $\sim \exp[-2\pi/g]$. Thus, it seems to be difficult to resolve the transitions in a real or numerical experiment. The slope of the Hall conductivity $\sigma_{xy}(M)$ is infinite at the transition points in agreement with the Hall transitions for the Landau levels [10]. In the pure limit ($\eta \to 0$) the conductivity $\sigma_{xx}(M)$ is $1/4\pi$ in units of $e^2/\hbar$ outside and zero inside the gap. This is consistent with the behaviour of the Hall conductivity in the pure limit (eq. (3)). In the disordered system $\sigma_{xx}(M)$ vanishes also on the Hall plateaux, where $\eta = 0$, provided $E^2 < M'^2$. $\sigma_{xx}(M)$ vanishes always for $E = 0$. This reflects localization of the states which are created by disorder at the centre of the gap.

Up to now only properties which are obtained in the $N \to \infty$ limit were considered. Of course this is not sufficient to evaluate space-dependent properties like the localization length. The latter requires the fluctuations around the SP. This can be studied on the basis of ref. [7], where a divergent correlation length was found at the transition points $M = \pm M_c$. For $M \neq \pm M_c$ all length scales are finite, which implies localization away from the transition points. A detailed investigation of the localized states must include the evaluation of the localization length exponent ν. This exponent is known for the continuous system with Landau levels from semiclassical approximations [11] and numerical studies to be $\approx 7/3$ [12-15]. However, other values were also found experimentally as well as theoretically [4, 10, 14, 16].

In conclusion, a discussion of the IQHE for lattice fermions with half a flux quantum per plaquette and N levels per site was presented. In the large-N limit two Hall transitions are found with an interpolating behaviour between two Hall plateaux in contrast to the single transition of ref. [2-4]. Although this system is not from the same universality class as the one described in ref. [3, 4] for the Landau level system, some properties are similar, like the infinite slope of the Hall conductivity at the transitions.

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