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Angaben zur Veröffentlichung / Publication details:

Carstensen, Carsten, and Ronald H. W. Hoppe. 2005. "Convergence analysis of an adaptive edge finite element method for the 2D eddy current equations." *Journal of Numerical Mathematics* 13 (1): 19–32. https://doi.org/10.1515/1569395054069017.

J. Numer. Math., Vol. 13, No. 1, pp. 19–32 (2005)

Convergence analysis of an adaptive edge finite element method for the 2D eddy current equations

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Received September 14, 2004

Abstract — For the 2D eddy currents equations, we design an adaptive edge finite element method (AEFEM) that guarantees an error reduction of the global discretization error in the $H(\mathrm{curl})$-norm and thus establishes convergence of the adaptive scheme. The error reduction property relies on a residual-type *a posteriori* error estimator and is proved for discretizations based on the lowest order edge elements of Nédélec's first family. The main ingredients of the proof are the reliability and the strict discrete local efficiency of the estimator as well as the Galerkin orthogonality of the edge element approximation.

Keywords: Adaptive edge finite element method, convergence analysis, guaranteed error reduction

1. INTRODUCTION

Given a bounded, simply connected domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, consider the 2D stationary eddy currents equations

$$\mathbf{curl}\,\chi\,\mathrm{curl}\,\mathbf{j} + \sigma\mathbf{j} = \mathbf{f} \quad \text{in } \Omega \tag{1.1}$$

$$\mathbf{t}\cdot\mathbf{j} = 0 \quad \text{on } \Gamma \tag{1.2}$$

where χ, σ are positive constants, $\mathbf{f} \in \mathbf{H}(\mathrm{div};\Omega)$, and $\mathbf{t}$ refers to the unit tangential vector on Γ. The variational formulation of (1.1), (1.2) amounts to the computation of $\mathbf{j} \in \mathbf{H}_0(\mathrm{curl};\Omega) := \{\mathbf{q} \in \mathbf{H}(\mathrm{curl};\Omega) \mid \mathbf{t}\cdot\mathbf{q} = 0 \text{ on } \Gamma\}$ such that

$$a(\mathbf{j},\mathbf{q}) = \ell(\mathbf{q}), \quad \mathbf{q} \in \mathbf{H}_0(\mathrm{curl};\Omega). \tag{1.3}$$

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Here, the bilinear form $a(\cdot,\cdot) : \mathbf{H}_0(\mathrm{curl};\Omega) \times \mathbf{H}_0(\mathrm{curl};\Omega) \to \mathbb{R}$ and the functional $\ell(\cdot) : \mathbf{H}_0(\mathrm{curl};\Omega) \to \mathbb{R}$ are given by

$$a(\mathbf{j},\mathbf{q}) := \int_\Omega [\chi \mathrm{curl}\,\mathbf{j}\,\mathrm{curl}\,\mathbf{q} + \sigma \mathbf{j}\cdot\mathbf{q}]\,\mathrm{d}x$$

$$\ell(\mathbf{q}) := \int_\Omega \mathbf{f}\cdot\mathbf{q}\,\mathrm{d}x.$$

In the sequel, we denote by $\|\cdot\|_{\mathrm{curl};\Omega}$ the standard graph norm

$$\|\mathbf{q}\|_{\mathrm{curl};\Omega} := \left(\|\mathrm{curl}\,\mathbf{q}\|_{0,\Omega}^2 + \|\mathbf{q}\|_{0,\Omega}^2\right)^{1/2}, \quad \mathbf{q} \in \mathbf{H}(\mathrm{curl};\Omega)$$

and by $\|\cdot\|_a$ the energy norm

$$\|\mathbf{q}\|_a := a(\mathbf{q},\mathbf{q})^{1/2}, \quad \mathbf{q} \in \mathbf{H}(\mathrm{curl};\Omega)$$

which are equivalent, i.e., $\|\cdot\|_{\mathrm{curl};\Omega} \approx \|\cdot\|_a$.

We assume that $\mathscr{T}_H(\Omega)$ is a shape-regular simplicial triangulation of Ω and refer to $\mathscr{N}_H(\omega)$ and $\mathscr{E}_H(\omega), \omega \subset \Omega$, as the sets of vertices and edges of $\mathscr{T}_H$ in $\omega \subset \Omega$. We denote by h_T the diameter of an element $T \in \mathscr{T}_H(\Omega)$ and by h_E the length of an edge $E \in \mathscr{E}_H(\Omega)$. Further, we refer to $\omega_E = T_+ \cup T_-$ as the union of the triangles $T_\pm \in \mathscr{T}_H(\Omega)$ sharing the common edge $E \in \mathscr{E}_H(\Omega)$.

The variational equation (1.3) is discretized by the lowest order edge elements of Nédélec's first family

$$\mathbf{Nd}_1(T) := \left\{ \exists \alpha \in \mathbb{R}^2, \exists \beta \in \mathbb{R} \;\; \forall x = (x_1,x_2) \in T : \mathbf{q}(x) = \alpha + \beta \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right\}$$

with the degrees of freedom

$$\int_E \mathbf{t}_E \cdot \mathbf{q}\,\mathrm{d}s, \quad E \in \mathscr{E}_H(T).$$

The associated curl-conforming edge element space $\mathbf{Nd}_1(\Omega;\mathscr{T}_H(\Omega)) \subset \mathbf{H}(\mathrm{curl};\Omega)$ is given by

$$\mathbf{Nd}_1(\Omega;\mathscr{T}_H(\Omega)) := \{\mathbf{q}_H \in \mathbf{H}(\mathrm{curl};\Omega) \mid \;\; \mathbf{q}_H|_T \in \mathbf{Nd}_1(T), \quad T \in \mathscr{T}_H(\Omega)\}$$

and we refer to $\mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ as the subspace

$$\mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)) := \{\mathbf{q}_H \in \mathbf{Nd}_1(\Omega;\mathscr{T}_H(\Omega)) \mid \;\; \mathbf{t}\cdot\mathbf{q}_H = 0 \;\text{ on } \Gamma\}.$$

Then, the edge finite element discretization of (1.3) reads as follows: Find $\mathbf{j}_H \in \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ as the solution of

$$a(\mathbf{j}_H,\mathbf{q}_H) = \ell(\mathbf{q}_H), \quad \mathbf{q}_H \in \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)). \tag{1.4}$$

An adaptive edge finite element method (AEFEM) consists of successive loops of the cycle

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}. \tag{1.5}$$

Here, SOLVE means the numerical solution of the edge element approximation (1.4). We remark that efficient multilevel iterative schemes are available [3,6,13–16,21]. They are based on the Helmholtz decomposition of the edge element space according to

$$\mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)) = \mathbf{Nd}^0_{1,0}(\Omega;\mathscr{T}_H(\Omega)) \oplus \mathbf{Nd}^{\perp}_{1,0}(\Omega;\mathscr{T}_H(\Omega)). \tag{1.6}$$

Here, $\mathbf{Nd}^0_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ refers to the subspace of irrotational vector fields

$$\mathbf{Nd}^0_{1,0}(\Omega;\mathscr{T}_H(\Omega)) := \{\mathbf{q}_h \in \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)) \mid \ \ \mathrm{curl}\,\mathbf{q}_h = 0\} \tag{1.7}$$

and $\mathbf{Nd}^{\perp}_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ is given by

$$\begin{aligned}&\mathbf{Nd}^{\perp}_{1,0}(\Omega;\mathscr{T}_H(\Omega))\\ &\quad := \{\mathbf{q}_h \in \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)) \mid \ (\mathbf{q}_h,\mathbf{q}^0_h)_{0,\Omega} = 0,\ \ \mathbf{q}^0_h \in \mathbf{Nd}^0_{1,0}(\Omega;\mathscr{T}_H(\Omega))\}.\end{aligned} \tag{1.8}$$

Denoting by $S_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ the finite element space of continuous, piecewise linear finite elements with respect to $\mathscr{T}_H(\Omega)$, we have the representation [17]

$$\mathbf{Nd}^0_{1,0}(\Omega;\mathscr{T}_H(\Omega)) = \mathbf{grad}\, S_{1,0}(\Omega;\mathscr{T}_H(\Omega)). \tag{1.9}$$

Consequently, $\mathbf{Nd}^{\perp}_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ can be interpreted as the subspace of weakly solenoidal vector fields.

The following step ESTIMATE invokes the efficient and reliable *a posteriori* error estimation of the global discretization error. This area has reached some state of maturity documented by a bundle of monographs and numerous research articles published during the past decade (cf. [1,4,5,12,23] and the references therein). With regard to the development, analysis, and implementation of a residual-type *a posteriori* error estimator for edge element discretizations of the eddy currents equations in 3D we refer to [7]. Its adaptation to 2D problems results in the estimator

$$\eta_H^2 := \sum_{T\in\mathscr{T}_H(\Omega)} \eta_T^2 + \sum_{E\in\mathscr{E}_H(\Omega)} \eta_E^2. \tag{1.10}$$

The element residuals η_T, $T \in \mathscr{T}_H(\Omega)$, and the edge residuals η_E, $E \in \mathscr{E}_H(\Omega)$, are given by

$$\eta_T^2 \ := \ h_T^2\,\|\mathbf{f} - \sigma\mathbf{j}_H\|^2_{0,T} \tag{1.11}$$

$$\eta_E^2 \ := \ h_E\,\|[\chi\,\mathrm{curl}\,\mathbf{j}_H]\|^2_{0,E} + h_E\,\|[\nu_E\cdot\sigma\mathbf{j}^0_H]\|^2_{0,E} \tag{1.12}$$

where $[\cdot]$ denotes the jump of the respective quantity across $E \in \mathscr{E}_H(\Omega)$ and $\mathbf{j}^0_H \in \mathbf{Nd}^0_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ refers to the irrotational part of $\mathbf{j}_H$.

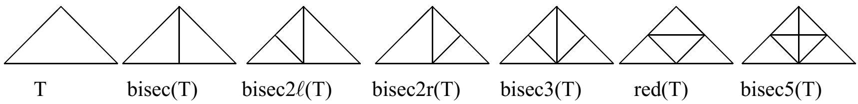

Figure 1. Possible refinements of one triangle T in the step REFINE. Data representations and a MATLAB realization are provided in [9].

However, up to now, the convergence analysis of the full adaptive scheme (1.5) is restricted to conforming and mixed finite element methods [10,18,19].

The bulk criterion in the step MARK was introduced and analyzed in [8,11,18] for displacement-based AFEMs. Here, it leads to a selection of a subset $\mathscr{M}_1$ of elements $T \in \mathscr{T}_H(\Omega)$ and a subset $\mathscr{M}_2$ of edges $E \in \mathscr{E}_H(\Omega)$ such that

$$\vartheta_1 \eta_H^2 \leqslant \sum_{T\in\mathscr{M}_1} \eta_T^2 + \sum_{E\in\mathscr{M}_2} \eta_E^2 \tag{1.13}$$

for some universal constant $0 < \vartheta_1 < 1$.

The convergence analysis further involves the data terms

$$\mu_H^2 := \sum_{E\in\mathscr{E}_H(\Omega)} \mu_E^2, \quad \mu_E^2 := h_T^2 \|\operatorname{div} \mathbf{f}\|_{0,\omega_E}^2 \tag{1.14}$$

which have to be controlled by selecting a subset $\mathscr{M}_3$ of edges $E \in \mathscr{E}_H(\Omega)$ such that for some $0 < \vartheta_2 < 1$

$$\vartheta_2 \mu_H^2 \leqslant \sum_{E\in\mathscr{M}_3} \mu_E^2. \tag{1.15}$$

The final step REFINE involves the refinement of elements and edges selected in MARK. Typical refinements of a triangle $T \in \mathscr{T}_H$ are displayed in Fig. 1.

The main result of this paper to prove the following error reduction property.

Theorem 1.1 (error reduction property). *Let $\mathbf{j}_h$ and $\mathbf{j}_H$ be the adaptive edge element approximations to $\mathbf{j}$ with respect to $\mathscr{T}_h(\Omega)$ and $\mathscr{T}_H(\Omega)$. Then, there exist positive constants $\rho_\nu < 1$, $1 \leqslant \nu \leqslant 2$, and C depending only on ϑ_ν, $1 \leqslant \nu \leqslant 2$, in* (1.13), (1.15) *and on the shape regularity of the triangulations such that*

$$\|\mathbf{j} - \mathbf{j}_h\|_a^2 \leqslant \rho_1 \|\mathbf{j} - \mathbf{j}_H\|_a^2 + C\mu_H^2 \tag{1.16}$$

$$\mu_h^2 \leqslant \rho_2 \mu_H^2. \tag{1.17}$$

The paper is organized as follows. Section 2 is devoted to the reliability of the error estimator η_H. The main ingredient in the proof is the strict discrete local efficiency which is addressed in Section 3. Together with the reliability of the estimator, the bulk criteria (1.13), (1.15), and the orthogonality of the edge element approximation, it allows the proof of the error reduction (1.16), (1.17) which will be done in Section 4.

2. RELIABILITY OF THE ERROR ESTIMATOR

Throughout this paper, $A \lesssim B$ abbreviates $A \leqslant CB$ with a mesh-size independent, generic constant $C > 0$. Finally, $A \approx B$ abbreviates $A \lesssim B \lesssim A$. The paper adopts standard notation for Lebesgue and Sobolev spaces and norms.

Denoting by $\mathbf{e_j} := \mathbf{j} - \mathbf{j}_H$ the global discretization error, the main result of this section establishes the reliability of the error estimator η.

Theorem 2.1. *There holds*

$$\|\mathbf{e_j}\|^2_{\mathrm{curl};\Omega} \lesssim \eta_H^2 + \mu_H^2 . \tag{2.1}$$

The proof of (2.1) uses the decomposition of the error into its irrotational and its weakly solenoidal part which allows to estimate both parts separately. It relies on the Helmholtz decomposition of $\mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega)$ according to

$$\mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega) = \mathbf{H}^0_{0,\Gamma}(\mathrm{curl};\Omega) \oplus \mathbf{H}^{\perp}_{0,\Gamma}(\mathrm{curl};\Omega) \tag{2.2}$$

into the subspace of irrotational vector fields

$$\mathbf{H}^0_{0,\Gamma}(\mathrm{curl};\Omega) := \{\mathbf{q} \in \mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega) \mid \ \mathrm{curl}\,\mathbf{q} = 0\}$$

and the subspace of weakly solenoidal vector fields

$$\mathbf{H}^{\perp}_{0,\Gamma}(\mathrm{curl};\Omega) \\ := \{\mathbf{q} \in \mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega) \mid \ (\mathbf{q},\mathbf{q}^0)_{0,\Omega} = 0,\, \mathbf{q}^0 \in \mathbf{H}^0_{0,\Gamma}(\mathrm{curl};\Omega)\}.$$

It is easy to see that $\mathbf{e_j} \in \mathbf{H}_{0,\Gamma}(\Omega)$ satisfies the error equation

$$a(\mathbf{e},\mathbf{q}) = r(\mathbf{q}), \qquad \mathbf{q} \in \mathbf{H}_{0,\Gamma}(\Omega) \tag{2.3}$$

where the residual r is given by

$$r(\mathbf{q}) := (\mathbf{f},\mathbf{q})_{0,\Omega} - a(\mathbf{j}_H,\mathbf{q}). \tag{2.4}$$

In view of the Helmholtz decomposition (2.2), we decompose the error $\mathbf{e_j} \in \mathbf{H}_{0,\Gamma}(\Omega)$ according to

$$\mathbf{e_j} := \mathbf{e_j}^0 + \mathbf{e_j}^{\perp}, \qquad \mathbf{e_j}^0 \in \mathbf{H}^0_{0,\Gamma}(\mathrm{curl};\Omega), \quad \mathbf{e_j}^{\perp} \in \mathbf{H}^{\perp}_{0,\Gamma}(\mathrm{curl};\Omega). \tag{2.5}$$

Then, it follows readily from (2.3) that the irrotational part $\mathbf{e_j}^0$ and the weakly solenoidal part $\mathbf{e_j}^{\perp}$ satisfy the error equations

$$a(\mathbf{e_j}^0,\mathbf{q}^0) \ = \ r(\mathbf{q}^0), \quad \mathbf{q}^0 \in \mathbf{H}^0_{0,\Gamma}(\Omega) \tag{2.6}$$

$$a(\mathbf{e_j}^{\perp},\mathbf{q}^{\perp}) \ = \ r(\mathbf{q}^{\perp}), \quad \mathbf{q}^{\perp} \in \mathbf{H}^{\perp}_{0,\Gamma}(\Omega). \tag{2.7}$$

In order to establish an upper bound for the L^2-norm of $\mathbf{e}_\mathbf{j}^0$, we make use of the representation

$$\mathbf{H}^0_{0,\Gamma}(\mathrm{curl};\omega) = \mathbf{grad}\,\mathbf{H}^1_0(\Omega)$$

and introduce the Scott–Zhang interpolation operator $P_H : H^1_{0,\Gamma}(\Omega) \to S_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ which has the following properties [22]:

$$\begin{aligned}
P_H\varphi &= \varphi, \qquad \varphi \in S_{1,0}(\Omega;\mathscr{T}_H(\Omega)) && (2.8)\\
\|\mathbf{grad}\,P_H\varphi\|_{0,T} &\lesssim \|\mathbf{grad}\,\varphi\|_{0,D_T}, \qquad T \in \mathscr{T}_H(\Omega) && (2.9)\\
\|\varphi - P_H\varphi\|_{0,T} &\lesssim h_T\,\|\mathbf{grad}\,\varphi\|_{0,D_T}, \qquad T \in \mathscr{T}_H(\Omega) && (2.10)\\
\|\varphi - P_H\varphi\|_{0,E} &\lesssim h_E^{1/2}\,\|\mathbf{grad}\,\varphi\|_{0,D_E}, \qquad E \in \mathscr{E}_H(\Omega) && (2.11)
\end{aligned}$$

with D_T and D_E being given by

$$\begin{aligned}
D_T &:= \bigcup\{T' \in \mathscr{T}_H(\Omega) \mid \ \mathscr{N}_H(T') \cap \mathscr{N}_H(T) \neq \emptyset\} && (2.12)\\
D_E &:= \bigcup\{T' \in \mathscr{T}_H(\Omega) \mid \ \mathscr{N}_H(T') \cap \mathscr{N}_H(E) \neq \emptyset\}. && (2.13)
\end{aligned}$$

Lemma 2.1. *There holds*

$$\|\mathbf{e}_\mathbf{j}^0\|^2_{0;\Omega} \lesssim \sum_{E\in\mathscr{E}_H(\Omega)} h_E\|[\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]\|^2_{0,E} + \mu_H^2. \tag{2.14}$$

Proof. We have $\mathbf{e}_\mathbf{j}^0 = \mathbf{grad}\,\psi$ for some $\psi \in H^1_0(\Omega)$. Then if we choose $\mathbf{q}^0 = \mathbf{grad}\,\psi$ in (2.6) and observe $r(\mathbf{grad}(P_H\psi)) = 0$, we obtain

$$\begin{aligned}
\|\mathbf{e}_\mathbf{j}^0\|^2_{0;\Omega} &= \|\mathbf{grad}\,\psi\|^2_{0;\Omega} \lesssim (\sigma\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\psi)_{0,\Omega}\\
&= r(\mathbf{grad}\,\psi) = r(\mathbf{grad}(\psi - P_H\psi)). && (2.15)
\end{aligned}$$

If we apply Green's formula, observing $\mathrm{div}\,\mathbf{j}_H|_T = 0$, $T \in \mathscr{T}_H$, and take (2.10), (2.11) into account, we find

$$\begin{aligned}
&|r(\mathbf{grad}(\psi - P_H\psi))| && (2.16)\\
&\quad = |(\mathbf{f},\mathbf{grad}(\psi - P_H\psi))_{0,\Omega} - a(\mathbf{j}_H,\mathbf{grad}(\psi - P_H\psi))|\\
&\quad = \Big| \sum_{T\in\mathscr{T}_H(\Omega)} (\mathrm{div}\,\mathbf{f}, \psi - P_H\psi)_{0,T}\\
&\qquad + \sum_{E\in\mathscr{E}_H(\Omega)} ([\boldsymbol{\nu}_E\cdot\sigma\mathbf{j}_H](\psi - P_H\psi))_{0,E}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{T\in\mathscr{T}_H(\Omega)} \|\operatorname{div}\mathbf{f}\|_{0,T}\,\|\psi - P_H\psi)\|_{0,T} \\
&\quad + \sum_{E\in\mathscr{E}_H(\Omega)} \|[\boldsymbol{\nu}_E\cdot\sigma\mathbf{j}_H]\|_{0,E}\|\psi - P_H\psi\|_{0,E} \\
&\lesssim \sum_{T\in\mathscr{T}_H(\Omega)} h_T\|\operatorname{div}\mathbf{f}\|_{0,T}\,\|\mathbf{grad}\,\psi\|_{0,D_T} \\
&\quad + \sum_{E\in\mathscr{E}_H(\Omega)} h_E^{1/2}\|[\boldsymbol{\nu}_E\cdot\sigma\mathbf{j}_H]\|_{0,E}\|\mathbf{grad}\,\psi\|_{0,D_E} \\
&\lesssim \Big(\sum_{T\in\mathscr{T}_H(\Omega)} h_T^2\|\operatorname{div}\mathbf{f}\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathscr{T}_H(\Omega)} \|\mathbf{grad}\,\psi\|_{0,D_T}^2\Big)^{1/2} \\
&\quad + \Big(\sum_{E\in\mathscr{E}_H(\Omega)} h_E\|[\boldsymbol{\nu}_E\cdot\sigma\mathbf{j}_H]\|_{0,E}^2\Big)^{1/2}\Big(\sum_{E\in\mathscr{E}_H(\Omega)} \|\mathbf{grad}\,\psi\|_{0,D_E}^2\Big)^{1/2} \\
&\lesssim \Big(\mu_H + \Big(\sum_{E\in\mathscr{E}_H(\Omega)} h_E\|[\boldsymbol{\nu}_E\cdot\sigma\mathbf{j}_H]\|_{0,E}^2\Big)^{1/2}\Big)\|\mathbf{e}_{\mathbf{j}}^0\|_{0,\Omega}.
\end{aligned}
$$

Using (2.16) in (2.15) gives the assertion. □

For the estimation of the weakly solenoidal part $\mathbf{e}_{\mathbf{j}}^{\perp}$ of the error, we use a vector-valued counterpart of P_H given as follows

$$
\begin{aligned}
&\mathbf{P_H} : \mathbf{H}^1(\Omega)\cap\mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega) \to \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)) \\
&\mathbf{P_H}\mathbf{q} := \sum_{E\in\mathscr{E}_H(\Omega)} (\mathbf{t}_E\cdot\mathbf{q},\varphi_E)_{0,E}\,\mathbf{q}_H^E.
\end{aligned} \tag{2.17}
$$

Here, $\varphi_E \in S_{1,0}(E;\mathscr{T}_{H/2}(E))$ denotes the nodal basis function associated with the nodal point $\mathrm{mid}(E)$ with respect to bisection of E. Moreover, $\mathbf{q}_H^E \in \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ stands for the canonical edge element basis function associated with the edge E, i.e.,

$$
\int_{E'\in\mathscr{E}_H(\Omega)} \mathbf{t}_{E'}\cdot\mathbf{q}_H^E\,\mathrm{d}s = \delta_{E,E'}.
$$

Referring to D_E^2, $E\in\mathscr{E}_H(\Omega)$ and D_T^2, $T\in\mathscr{T}_H(\Omega)$, as the sets

$$
\begin{aligned}
D_E^2 &:= \bigcup\{T\in\mathscr{T}_H(\Omega)\,|\; E\in\mathscr{E}_H(T)\} \\
D_T^2 &:= \bigcup\{D_E\,|\; E\in\mathscr{E}_H(T)\}
\end{aligned}
$$

we have the following properties of $\mathbf{P_H}$.

Lemma 2.2. *Let* $\mathbf{q} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega)$. *Then, there holds*

$$\mathbf{P_H}\mathbf{q}_H = \mathbf{q}_H, \qquad \mathbf{q}_H \in \mathbf{Nd}_{1,0}(\Omega;\mathscr{T}_H(\Omega)) \tag{2.18}$$

$$\|\mathbf{P_H}\mathbf{q}\|_{0,T} \lesssim \|\mathbf{q}\|_{1,D_T^2} \tag{2.19}$$

$$\|\mathrm{curl}\,\mathbf{P_H}\mathbf{q}\|_{0,T} \lesssim \|\mathbf{q}\|_{1,D_T^2} \tag{2.20}$$

$$\|\mathbf{q}-\mathbf{P_H}\mathbf{q}\|_{0,T} \lesssim h_T\|\mathbf{q}\|_{1,D_T^2} \tag{2.21}$$

$$\|\mathbf{q}-\mathbf{P_H}\mathbf{q}\|_{0,E} \lesssim h_E^{1/2}\|\mathbf{q}\|_{1,D_E^2}. \tag{2.22}$$

Proof. The interpolation property (2.18) follows readily by definition of $\mathbf{P_H}$. Using the shape regularity of the triangulation and the trace inequality, for $T \in \mathscr{T}_H(\Omega)$ we get

$$\|\mathbf{P_H}\mathbf{q}\|_{0,T}^2 \lesssim h_T \sum_{E\in\mathscr{E}_H(T)} |(\mathbf{t}_E\cdot\mathbf{q},\varphi_E)_{0,E}|^2$$
$$\leqslant h_T \sum_{E\in\mathscr{E}_H(T)} \|\mathbf{t}_E\cdot\mathbf{q}\|_{0,E}^2\|\varphi_E\|_{0,E}^2 \lesssim \|\mathbf{q}\|_{1,D_T^2}$$

which proves (2.19). The stability property (2.20) can be shown in a similar way, whereas the approximation properties (2.21) and (2.22) can be verified by Bramble–Hilbert type arguments. □

Lemma 2.3. *There holds*

$$\|\mathbf{e_j}^\perp\|_{0;\Omega}^2 \lesssim \sum_{T\in\mathscr{T}_H(\Omega)} h_T^2\|\mathbf{f}-\sigma\mathbf{j}_H\|_{0,T}^2 + \sum_{E\in\mathscr{E}_H(\Omega)} h_E\|[\chi\,\mathrm{curl}\,\mathbf{j}_H]\|_{0,E}^2. \tag{2.23}$$

Proof. We use the fact that $\mathbf{H}_{0,\Gamma}^\perp(\mathrm{curl};\Omega)$ is continuously imbedded in $\mathbf{H}^1(\Omega)\cap\mathbf{H}_{0,\Gamma}(\mathrm{curl};\Omega)$ and there exists a positive constant C depending only on Ω such that

$$\|\mathbf{q}^\perp\|_{1,\Omega} \leqslant C\|\mathrm{curl}\,\mathbf{q}^\perp\|_{0,\Omega}.$$

We choose $\mathbf{q}^\perp := \mathbf{e_j}^\perp$ in (2.7). Taking $r(\mathbf{P_H}\mathbf{e_j}^\perp) = 0$ into account, it follows that

$$\|\mathbf{e_j}^\perp\|_{\mathrm{curl};\Omega}^2 \lesssim a(\mathbf{e_j}^\perp,\mathbf{e_j}^\perp) = r(\mathbf{e_j}^\perp - \mathbf{P_H}\mathbf{e_j}^\perp). \tag{2.24}$$

By Stokes' theorem and the approximation properties (2.21), (2.22) we find

$$r(\mathbf{e_j}^\perp - \mathbf{P_H}\mathbf{e_j}^\perp) = (\mathbf{f},\mathbf{e_j}^\perp - \mathbf{P_H}\mathbf{e_j}^\perp)_{0,\Omega} - a(\mathbf{j}_H,\mathbf{e_j}^\perp - \mathbf{P_H}\mathbf{e_j}^\perp) \tag{2.25}$$
$$\leqslant \sum_{T\in\mathscr{T}_H(\Omega)} \|\mathbf{f}-\sigma\,\mathbf{j}_H\|_{0,T}\|\mathbf{e_j}^\perp - \mathbf{P_H}\mathbf{e_j}^\perp\|_{0,T}$$

$$+ \sum_{E\in\mathscr{E}_H(\Omega)} h_E^{1/2} \|[\chi \operatorname{curl} \mathbf{j}_H]\|_{0,E} \|\mathbf{e}_\mathbf{j}^\perp - \mathbf{P_H}\mathbf{e}_\mathbf{j}^\perp\|_{0,E}$$

$$\lesssim \sum_{T\in\mathscr{T}_H(\Omega)} h_T \|\mathbf{f} - \sigma\,\mathbf{j}_H\|_{0,T}\Big] \|\mathbf{e}_\mathbf{j}^\perp\|_{\mathrm{curl},D_T^2}$$

$$+ \sum_{E\in\mathscr{E}_H(\Omega)} h_E^{1/2} \|[\chi \operatorname{curl} \mathbf{j}_H]\|_{0,E} \|\mathbf{e}_\mathbf{j}^\perp\|_{\mathrm{curl},D_E^2}$$

$$\lesssim \Big[\sum_{E\in\mathscr{E}_H(\Omega)} h_E \|[\chi \operatorname{curl} \mathbf{j}_H]\|_{0,E}^2 + \sum_{T\in\mathscr{T}_H(\Omega)} h_T^2 \|\mathbf{f} - \sigma\,\mathbf{j}_H\|_{0,T}^2 \Big] \|\mathbf{e}_\mathbf{j}^\perp\|_{1,\Omega}$$

$$\lesssim \Big[\sum_{E\in\mathscr{E}_H(\Omega)} h_E \|[\chi \operatorname{curl} \mathbf{j}_H]\|_{0,E}^2 + \sum_{T\in\mathscr{T}_H(\Omega)} h_T^2 \|\mathbf{f} - \sigma\,\mathbf{j}_H\|_{0,T}^2 \Big] \|\mathbf{e}_\mathbf{j}^\perp\|_{\mathrm{curl},\Omega}.$$

Inserting (2.25) into (2.24) gives the assertion. □

Proof of Theorem 2.1. Combining (2.14) from Lemma 2.1 and (2.23) from Lemma 2.3 proves the reliability (2.1) of the estimator. □

3. STRICT DISCRETE LOCAL EFFICIENCY

The strict discrete local efficiency of the error estimator η_H states that up to the local data terms μ_E the $\mathbf{H}$(curl)-norm of the difference $\mathbf{j}_h - \mathbf{j}_H$ in the fine and coarse mesh approximations can be locally bounded from below by the local contributions η_T and η_E of the estimator.

Theorem 3.1. *Let $E \in \mathscr{E}_H(\Omega)$ be a refined edge with $E = T_1 \cap T_2$, $T_\nu \in \mathscr{E}_H(\Omega)$, $1 \leqslant \nu \leqslant 2$, and $\omega_E = T_1 \cup T_2$. Then there holds*

$$\eta_T^2 + \eta_E^2 \lesssim \|\mathbf{j}_h - \mathbf{j}_H\|_{\mathrm{curl},\omega_E}^2 + \mu_E^2. \tag{3.1}$$

The proof of (3.1) is carried out by means of the following results.

Lemma 3.1. *Let $E \in \mathscr{E}_H(\Omega)$ be a refined edge. Then there holds*

$$h_T^2 \|\mathbf{f} - \sigma\,\mathbf{j}_H\|_{0,\omega_E}^2 \lesssim h_T^2\, \|\mathbf{j}_h - \mathbf{j}_H\|_{0,\omega_E}^2. \tag{3.2}$$

Proof. We choose $\varphi_{\omega_E} \in S_{1,0}(\omega_E, \mathscr{T}_h(\omega_E))$ such that

$$(\mathbf{grad}\,\varphi_{\omega_E})|_{\omega_E} = |\omega_E|^{-1} \int_T (\mathbf{f} - \sigma\,\mathbf{j}_H)\,\mathrm{d}\mathbf{x}.$$

It follows that

$$\|\mathbf{grad}\,\varphi_{\omega_E}\|_{0,\omega_E}^2 \leqslant \|\mathbf{f} - \sigma\,\mathbf{j}_H)\|_{0,\omega_E}^2 = \int_{\omega_E} (\mathbf{f} - \sigma\,\mathbf{j}_H) \cdot \mathbf{grad}\,\varphi_{\omega_E}\,\mathrm{d}x.$$

Since $\mathbf{grad}\,\varphi_{\omega_E}$ is an admissible test function for the edge element approximation $\mathbf{j}_h$, we have

$$\int\limits_{\omega_E} \sigma\,\mathbf{j}_h \cdot \mathbf{grad}\,\varphi_{\omega_E}\,\mathrm{d}\mathbf{x} = \int\limits_{\omega_E} \mathbf{f}\cdot \mathbf{grad}\,\varphi_{\omega_E}\,\mathrm{d}x.$$

Consequently, we obtain

$$\begin{aligned} h_T^2\,\|\mathbf{f}-\sigma\,\mathbf{j}_H\|_{0,\omega_E}^2 \;&=\; h_T^2 \int\limits_{\omega_E} \Big(\sigma\,(\mathbf{j}_h-\mathbf{j}_H)\Big)\cdot \mathbf{grad}\,\varphi_{\omega_E}\,\mathrm{d}x \\ &\lesssim\; h_T^2\,\|\mathbf{j}_h-\mathbf{j}_H\|_{0,\omega_E}\,\|\mathbf{f}-\sigma\,\mathbf{j}_H\|_{0,\omega_E}. \end{aligned}$$

Applying Young's inequality gives the assertion. □

Lemma 3.2. *For a refined edge $E \in \mathscr{E}_H(\Omega)$ there holds*

$$\begin{aligned} &h_E\|\,[\chi\,\mathrm{curl}\,\mathbf{j}_H]\,\|_{0,E}^2 \\ &\quad\lesssim \|\mathrm{curl}\,(\mathbf{j}_h-\mathbf{j}_H)\|_{0,\omega_E}^2 + h_T^2\,\|\mathbf{j}_h-\mathbf{j}_H\|_{0,\omega_E}^2 + h_T^2\,\|\mathbf{f}-\sigma\mathbf{j}_H\|_{0,\omega_E}^2. \end{aligned} \tag{3.3}$$

Proof. We choose $\boldsymbol{\psi}_{\omega_E} \in \mathbf{Nd}_{1,0}(\omega_E,\mathscr{T}_h(\omega_E))$ such that

$$\mathbf{t}_{E_\nu}\cdot\boldsymbol{\psi}_{\omega_E}|_E = \begin{cases} [\chi\,\mathrm{curl}\,\mathbf{j}_H]|_{E_\nu}, & 1\leqslant \nu\leqslant 2,\quad E=E_1\cup E_2 \\ 0, & E_\nu \in \mathscr{E}_h(\mathrm{int}(\omega_E)\setminus E). \end{cases}$$

Consequently, we have

$$\|\boldsymbol{\psi}_{\omega_E}\|_{0,\omega_E}^2 \lesssim h_E\,\|[\chi\,\mathrm{curl}\,\mathbf{j}_H]\|_{0,E}^2 = h_E \int\limits_E [\chi\,\mathrm{curl}\,\mathbf{j}_H]\cdot(\mathbf{t}_E\cdot\boldsymbol{\psi}_{\omega_E})\,\mathrm{d}\sigma. \tag{3.4}$$

Since $\boldsymbol{\psi}_{\omega_E}$ is an admissible test function for the edge element approximation $\mathbf{j}_h$, we have

$$\int\limits_{\omega_E} \big(\chi\,\mathrm{curl}\,\mathbf{j}_h\cdot\mathrm{curl}\,\boldsymbol{\psi}_{\omega_E} + \sigma\,\mathbf{j}_h\cdot\boldsymbol{\psi}_{\omega_E}\big)\,\mathrm{d}x = \int\limits_{\omega_E} \mathbf{f}\cdot\boldsymbol{\psi}_{\omega_E}\,\mathrm{d}x. \tag{3.5}$$

On the other hand, taking $\boldsymbol{\psi}_{\omega_E}|_{\partial\omega_E}=0$ and $\mathbf{curl}\,(\chi\,\mathrm{curl}\,\mathbf{j}_H)|_T\equiv 0$, $T\in\mathscr{T}_H(\Omega)$ into account, Stokes' theorem yields

$$\int\limits_E [\chi\,\mathrm{curl}\,\mathbf{j}_H]\,(\mathbf{t}_E\cdot\boldsymbol{\psi}_{\omega_E})\,\mathrm{d}\sigma = -\int\limits_{\omega_E} \chi\,\mathrm{curl}\,\mathbf{j}_H\cdot\mathrm{curl}\,\boldsymbol{\psi}_{\omega_E}\,\mathrm{d}x. \tag{3.6}$$

The combination of (3.4), (3.5) and (3.6) followed by Cauchy inequalities leads to

$$\begin{aligned} h_E\,\|[\chi\,\mathrm{curl}\,\mathbf{j}_H]\|_{0,E}^2 \;&=\; h_E \int\limits_E [\chi\,\mathrm{curl}\,\mathbf{j}_H]\cdot(\mathbf{t}_E\cdot\boldsymbol{\psi}_{\omega_E})\,\mathrm{d}\sigma \\ &\lesssim\; (\|\mathbf{j}_h-\mathbf{j}_H\|_{0,\omega_E} + \|\mathbf{f}-\sigma\mathbf{j}_H\|_{0,\omega_E})\,\|\boldsymbol{\psi}_{\omega_E}\|_{0,\omega_E} \\ &\qquad + h_E\,\|\mathrm{curl}\,\mathbf{j}_h-\mathrm{curl}\,\mathbf{j}_H\|_{0,\omega_E}\,\|\mathrm{curl}\,\boldsymbol{\psi}_{\omega_E}\|_{0,\omega_E}. \end{aligned}$$

This and observing the inverse inequality $\|\mathrm{curl}\boldsymbol{\psi}_{\omega_E}\|_{0,\omega_E} \lesssim h_E^{-1}\,\|\boldsymbol{\psi}_{\omega_E}\|_{0,\omega_E}$ allows to conclude the proof. □

Lemma 3.3. *For a refined edge $E \in \mathscr{E}_H(\Omega)$ we have*

$$h_T^2\|\mathbf{f}-\sigma\,\mathbf{j}_H\|_{0,\omega_E}^2 + h_E\|[\chi\,\mathrm{curl}\,\mathbf{j}_H]\|_{0,E}^2 \lesssim \|\mathbf{j}_h^\perp - \mathbf{j}_H^\perp\|_{\mathrm{curl},\omega_E}^2 + h_T^2\,\|\mathbf{j}_h^0-\mathbf{j}_H^0\|_{0,\omega_E}^2. \tag{3.7}$$

Proof. The proof of (3.7) follows readily by combining Lemma 3.1 and Lemma 3.2. □

Lemma 3.4. *For a refined edge $E \in \mathscr{E}_H(\Omega)$ there holds*

$$h_E\|[\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]\|_{0,E}^2 \lesssim \|\mathbf{j}_h^0-\mathbf{j}_H^0\|_{0,\omega_E}^2 + \mu_E^2. \tag{3.8}$$

Proof. We choose $\varphi_E \in S_{1,0}(\omega_E,\mathscr{T}_h(\omega_E))$ such that

$$\varphi_E(\mathrm{mid}(E)) = [\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]. \tag{3.9}$$

Since φ_E is piecewise affine along E, it follows that

$$2h_E\int_E [\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]\,\varphi_E\,\mathrm{d}s = h_E\,\|[\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]\|_{0,E}^2. \tag{3.10}$$

Since $\varphi_E|_{\partial\omega_E}=0$ and $\mathrm{div}\,\mathbf{j}_H = 0$ on $T_\pm$, Green's formula applied to $T_\pm$ results in

$$\int_{\omega_E}\sigma\mathbf{j}_H^0\cdot\mathbf{grad}\varphi_E\,\mathrm{d}x = \int_{\omega_E}\sigma\mathbf{j}_H\cdot\mathbf{grad}\varphi_E\,\mathrm{d}x = \int_E[\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]\,\varphi_E\,\mathrm{d}s. \tag{3.11}$$

Observing $\mathbf{grad}\,\varphi_E \in \mathbf{Nd}_{\mathbf{1,0}}^{\mathbf{0}}(\omega_E,\mathscr{T}_h)$, we have

$$\begin{aligned} 0 &= \int_{\omega_E}(\sigma\,\mathbf{j}_h^0-\mathbf{f})\cdot\mathbf{grad}\varphi_E\,\mathrm{d}x \\ &= \int_{\omega_E}\mathrm{div}\,\mathbf{f}\,\varphi_E\,\mathrm{d}x + \int_{\omega_E}\sigma\,\mathbf{j}_h^0\cdot\mathbf{grad}\varphi_E\,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) gives

$$h_E\int_E[\boldsymbol{\nu}_E\cdot\sigma\,\mathbf{j}_H]\,\varphi_E\,\mathrm{d}s \lesssim h_E\,\|\mathrm{div}\,\mathbf{f}\|_{0,\omega_E}\,\|\varphi_E\|_{0,\omega_E} + h_E\,\|\mathbf{j}_h^0-\mathbf{j}_H^0\|_{0,\omega_E}\|\mathbf{grad}\varphi_E\|_{0,\omega_E}. \tag{3.13}$$

Finally, observing that (3.9) implies

$$\|\varphi_E\|_{0,\omega_E} \lesssim h_E^{1/2}\,\|[\nu_E \cdot \sigma\,\mathbf{j}_H]\|_{0,E}$$
$$\|\mathbf{grad}\,\varphi_E\|_{0,\omega_E} \lesssim h_E^{-1/2}\,\|[\nu_E \cdot \sigma\,\mathbf{j}_H]\|_{0,E}$$

(3.10) and (3.13) give the assertion. □

Proof of Theorem 3.1. The proof of (3.1) in Theorem 3.1 now follows readily by combining Lemmas 3.1–3.4. □

4. PROOF OF THE ERROR REDUCTION PROPERTY

We now combine the reliability of the error estimator, the strict discrete local efficiency of the estimator and the Galerkin orthogonality of the edge element approximations to deduce the error reduction property (1.16) of Theorem 1.1.

Proof of Theorem 1.1. Observing $\|\cdot\|_{\mathrm{curl},\Omega} \approx \|\cdot\|_a$, the reliability (2.1) of the error estimator η_H and the bulk criteria (1.13), (1.15) imply

$$\|\mathbf{e_j}\|_a^2 \lesssim \eta_H^2 \lesssim \sum_{T\in\mathcal{M}_1} \eta_T^2 + \sum_{E\in\mathcal{M}_2} \eta_E^2 + \mu_H^2\,.$$

Using the strict discrete local efficiency (3.1), it follows that there exists a constant $C > 0$, depending only on $\vartheta_\nu, 1 \leqslant \nu \leqslant 2$, in the bulk criteria (1.13), (1.15) and on the shape regularity of the triangulations such that

$$\|\mathbf{e_j}\|_a^2 \leqslant C\,\|\mathbf{j}_h - \mathbf{j}_H\|_a^2 + C\mu_H^2\,. \tag{4.1}$$

Due to the Galerkin orthogonality of the edge element approximations and the nestedness of the edge element spaces we have

$$a(\mathbf{j}-\mathbf{j}_h, \mathbf{j}_h - \mathbf{j}_H) = 0$$

which implies

$$\|\mathbf{j}_h - \mathbf{j}_H\|_a^2 = \|\mathbf{j} - \mathbf{j}_H\|_a^2 - \|\mathbf{j} - \mathbf{j}_h\|_a^2\,. \tag{4.2}$$

Combining (4.1) and (4.2) yields

$$C\,\|\mathbf{j} - \mathbf{j}_h\|_a^2 \leqslant (C-1)\,\|\mathbf{j} - \mathbf{j}_H\|_a^2 + C\mu_H^2$$

and hence, (1.16) follows with $\rho_1 = 1 - 1/C$.

For the proof of (1.17) consider $K \in \mathcal{T}_h(\Omega)$ such that $K \subset T \in \mathcal{T}_H(\Omega)$ with $\mathcal{E}_H(T) \cap \mathcal{M}_3 \neq \varnothing$. Then, T is at least halved, and hence,

$$\mu_h^2 \leqslant \frac{1}{2} \sum_{E\in\mathcal{M}_3} \mu_E^2 + \sum_{E\in\mathcal{E}_H(\Omega)\setminus\mathcal{M}_3} \mu_E^2 = \mu_H^2 - \frac{1}{2} \sum_{E\in\mathcal{M}_3} \mu_E^2\,.$$

Observing (1.15), it follows that

$$\frac{\vartheta_2}{2}\,\mu_H^2 \leqslant \frac{1}{2}\sum_{E\in\mathscr{M}_3}\mu_E^2 \leqslant \mu_H^2-\mu_h^2$$

which proves (1.17) with $\rho_2 = 1-\vartheta_2/2$. □

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