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A posteriori error estimates for adaptive finite element discretizations of boundary control problems

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Abstract — We are concerned with an *a posteriori* error analysis of adaptive finite element approximations of boundary control problems for second order elliptic boundary value problems under bilateral bound constraints on the control which acts through a Neumann type boundary condition. In particular, the analysis of the errors in the state, the co-state, the control, and the co-control invokes an efficient and reliable residual-type *a posteriori* error estimator as well as data oscillations. The proof of the efficiency and reliability is done without any regularity assumption. Adaptive mesh refinement is realized on the basis of a bulk criterion. The performance of the adaptive finite element approximation is illustrated by a detailed documentation of numerical results for selected test problems.

Keywords: *a posteriori* error analysis, boundary control problems, control constraints, adaptive finite element methods, residual-type *a posteriori* error estimators, data oscillations

1. INTRODUCTION

Adaptive finite element methods for the efficient numerical solution of boundary and initial-boundary value problems for partial differential equations have reached some state of maturity as documented by a series of monographs on this subject published during the past decade (cf., e.g., [1,3,4,14,27,28]). The concepts for an *a posteriori* error analysis include residual-type estimators [2,3,28], hierarchical type estimators [5,19,20], error estimators that are based on local averaging [9,29], the goal oriented dual weighted approach [4,14], and functional type error majorants [27].

However, considerably less work has been done with regard to an error analysis of finite element approximations of optimal control problems for partial differential equations. In the unconstrained case, we refer to [4,6], whereas residual-type *a posteriori* error estimators in the control constrained case have been developed, analyzed and implemented in [17,21,23] for distributed controls and in [24] for boundary controls.

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The paper is organized as follows: in Section 2, we consider a boundary control problem for a two-dimensional, second order elliptic PDE with a quadratic objective functional and bilateral constraints on the control. The optimality conditions are given in terms of the state, the co-state, the control, and the co-control (Lagrangian multiplier for the control). Section 3 deals with the discretization of the control problem using continuous, piecewise linear finite elements for the state and the co-state and elementwise constant approximations of the control and the co-control with respect to shape regular simplicial triangulations of the computational domain. In Section 4, we present the residual-type *a posteriori* error estimator for the global discretization errors in the state, the co-state, the control, and the co-control which consists of edge and element residuals. As opposed to [24], the error analysis includes data oscillations, since we do not assume any regularity of the data. Section 5 is devoted to the reliability of the error estimator, i.e., we show that, up to data oscillations, the estimator provides an upper bound for the global discretization errors. In Section 6, we prove the efficiency of the estimator. In particular, it is shown that, modulo data oscillations, the error estimator also gives rise to a lower bound. Section 7 focuses on the adaptive mesh refinement. Here, we use a bulk criterion which is realized by a greedy algorithm. In Section 8, we provide a documentation of numerical results in order to illustrate the performance of the error estimator.

2. THE BOUNDARY CONTROL PROBLEM

Adopting standard notation from Lebesgue and Sobolev space theory, we consider the following boundary control problem for a linear second order elliptic boundary value problem with constrained controls on part of the boundary

$$\text{minimize } J(y,u) := \frac{1}{2}\|y-y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\|u-u^d\|^2_{0,\Gamma_1} \tag{2.1a}$$

$$\text{over } (y,u) \in H^1_{0,\Gamma_2}(\Omega) \times K$$

$$\text{subject to } \quad -\Delta y + cy = f \quad \text{in } \Omega \tag{2.1b}$$

$$n \cdot \nabla y = u \quad \text{on } \Gamma_1. \tag{2.1c}$$

Here, Ω stands for a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \varnothing$. As far as the data are concerned we assume $f, y^d \in L^2(\Omega)$, $u^d \in L^2(\Gamma_1)$ and $c, \alpha \in \mathbb{R}_+$. The set K of constraints is given by

$$K := \left\{ v \in L^2(\Gamma_1) | \psi^{(1)}(x) \leqslant v(x) \leqslant \psi^{(2)}(x) \text{ f.a.a. } x \in \Gamma_1 \right\} \tag{2.2}$$

where $\psi^{(\nu)} \in L^\infty(\Gamma_1)$, $1 \leqslant \nu \leqslant 2$, with $\psi^{(1)}(x) < \psi^{(2)}(x)$ f.a.a. $x \in \Gamma_1$.

It is well-known that under the above assumptions the boundary control problem (2.1a)–(2.1c) admits a unique solution $(y,u) \in V \times K$, where $V := H_{0,\Gamma_2}(\Omega)$ (cf., e.g., [15,22,23]). The solution is characterized by the existence of a co-state $p \in V$ and a

Lagrange multiplier for the inequality constraints (co-control) $\sigma \in L^2(\Gamma_1)$ such that

$$a(y,v) = \ell_1(v), \quad v \in V \tag{2.3a}$$

$$a(p,v) = \ell_2(v), \quad v \in V \tag{2.3b}$$

$$u = u^d + \frac{1}{\alpha}\,(p-\sigma) \tag{2.3c}$$

$$\sigma \in \partial I_K(u). \tag{2.3d}$$

Here, $a(\cdot,\cdot)$ stands for the bilinear form

$$a(u,v) := \int_\Omega (\nabla u \cdot \nabla v + cuv)\,\mathrm{d}x, \quad u,v \in V.$$

The functionals $\ell_\nu : V \to \mathbb{R}$, $1 \leqslant \nu \leqslant 2$, are given by

$$\ell_1(v) := \int_\Omega fv\,\mathrm{d}x + \int_{\Gamma_1} uv\,\mathrm{d}s, \qquad \ell_2(v) := \int_\Omega \left(y^d - y\right) v\,\mathrm{d}x$$

and $\partial I_K : L^2(\Omega) \to 2^{L^2(\Omega)}$ denotes the subdifferential of the indicator function I_K of the constraint set K (cf., e.g., [18]).

We note that the inclusion (2.3d) can be written as the variational inequality

$$(\sigma, v-u)_{0,\Gamma_1} \geqslant 0, \quad v \in K. \tag{2.4}$$

We define the active control sets $\mathscr{A}^{(\nu)}(u)$, $1 \leqslant \nu \leqslant 2$, as the maximal open sets $A^{(\nu)} \subset \Gamma_1$ such that $u(x) = \psi^{(\nu)}(x)$ f.a.a. $x \in A^{(\nu)}$ and the inactive control set $\mathscr{I}(u)$ according to $\mathscr{I}(u) := \bigcup_{\varepsilon>0} B_\varepsilon$, where B_ε is the maximal open set $B \subset \Gamma_1$ such that $\psi^{(1)}(x) + \varepsilon \leqslant u(x) \leqslant \psi^{(2)}(x) - \varepsilon$ for almost all $x \in B$. Then, the variational inequality (2.4) can be equivalently stated by means of the complementarity conditions:

$$\sigma(x) \leqslant 0 \text{ a.e. in } \mathscr{A}^{(1)}(u), \quad \sigma(x) = 0 \text{ a.e. in } \mathscr{I}(u), \quad \sigma(x) \geqslant 0 \text{ a.e. in } \mathscr{A}^{(2)}(u). \tag{2.5}$$

In particular, we may split the adjoint control σ according to

$$\sigma \;=\; \sigma^{(1)} + \sigma^{(2)} \tag{2.6}$$

where $\sigma^{(1)} \in L^2_-(\Gamma_1)$ with $\mathrm{supp}(\sigma^{(1)}) = \mathrm{cl}(\mathscr{A}^{(1)}(u))$ and $\sigma^{(2)} \in L^2_+(\Gamma_1)$ with $\mathrm{supp}(\sigma^{(2)}) = \mathrm{cl}(\mathscr{A}^{(2)}(u))$. It follows from (2.5) that $\sigma^{(\nu)}$, $1 \leqslant \nu \leqslant 2$, satisfy the complementarity conditions

$$(\sigma^{(1)}, u - \psi^{(1)})_{0,\Gamma_1} = 0, \qquad (\sigma^{(2)}, \psi^{(2)} - u)_{0,\Gamma_1} = 0. \tag{2.7}$$

3. FINITE ELEMENT APPROXIMATION

We assume that $\mathscr{T}_h(\Omega)$ is a shape-regular simplicial triangulation of Ω such that the subsets $\Gamma_\nu \subset \Gamma$, $1 \leqslant \nu \leqslant 2$, inherit geometrically conforming triangulations $\mathscr{T}_h(\Gamma_\nu)$. We refer to $\mathscr{N}_h(D)$ and $\mathscr{E}_h(D)$, $D \subseteq \overline{\Omega}$, as the sets of vertices and edges of $\mathscr{T}_h(\Omega)$ in D. We denote by h_T the diameter of an element $T \in \mathscr{T}_h(\Omega)$ and by h_E the length of an edge $E \in \mathscr{E}_h(D)$. Denoting by $P_0(D), D \subset \bar{\Omega}$, the set of constant functions on D, we define f_h, y_h^d as the elementwise constant functions $f_h|_T := f_T$, $y_h^d|_T := y_T^d \in P_0(T)$, $T \in \mathscr{T}_h(\Omega)$, where f_T is given by the integral mean with respect to T, i.e.,

$$f_T \ := \ |T|^{-1} \int_T f \ \mathrm{d}x$$

and $y_T^d \in P_0(T)$ is given analogously.

We refer to $V_h \subset V$ as the finite element space of continuous, piecewise linear finite elements with respect to $\mathscr{T}_h(\Omega)$

$$V_h \ := \ \left\{ v_h \in C(\overline{\Omega}) \mid v_h|_T \in P_1(T), \ v_h|_{\partial T \cap \Gamma_2} = 0, \ T \in \mathscr{T}_h(\Omega) \right\}$$

and define $W_h \subset L^2(\Gamma_1)$ as the linear space of piecewise constants with respect to $\mathscr{E}_h(\Gamma_1)$

$$W_h \ := \ \left\{ w_h \in L^2(\Gamma_1) \mid w_h|_E \in P_0(E), \ E \in \mathscr{E}_h(\Gamma_1) \right\}.$$

We denote by $u_E^d \in P_0(E)$, $E \in \mathscr{E}_h(\Gamma_1)$ the integral mean of u^d with respect to $E \in \mathscr{E}_h(\Gamma_1)$, i.e.,

$$u_E^d \ := \ h_E^{-1} \int_E u^d \ \mathrm{d}s.$$

We define $u_h^d \in W_h$ by $u_h^d|_E \in P_0(E), E \in \mathscr{E}_h(\Gamma_1)$.

The lower bound $\psi^{(1)}$ and the upper bound $\psi^{(2)}$ for the boundary controls are approximated by $\psi_h^{(\nu)} \in W_h$, $1 \leqslant \nu \leqslant 2$, such that $\psi_E^{(1)} := \psi_h^{(1)}|_E < \psi_h^{(2)}|_E =: \psi_E^{(2)}$, $E \in \mathscr{E}_h(\Gamma_1)$.

Then, the finite element approximation of the boundary control problem (2.1a)–(2.1c) amounts to the computation of $(y_h, u_h) \in V_h \times K_h$ as the solution of the constrained finite dimensional minimization problem

$$\text{minimize } J_h(y_h,u_h) := \frac{1}{2} \, \|y_h - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|u_h - u^d\|_{0,\Omega}^2 \tag{3.1a}$$

$$\text{over } (y_h,u_h) \in V_h \times K_h \tag{3.1b}$$

$$\text{subject to } a(y_h,v_h) = \ell_{h,1}(v_h), \quad v_h \in V_h. \tag{3.1c}$$

Here, K_h denotes the discrete constraint set for the boundary controls

$$K_h := \left\{ \ w_h \in W_h \,|\, \psi_h^{(1)} \leqslant w_h \leqslant \psi_h^{(2)} \ \right\} \tag{3.2}$$

whereas the functional $\ell_{h,1} : V_h \to \mathbb{R}$ is given by

$$\ell_{h,1}(v_h) \;:=\; \int_\Omega f v_h \;\mathrm{d}x \;+\; \int_{\Gamma_1} u_h v_h \;\mathrm{d}s, \quad v_h \in V_h.$$

The optimality conditions for (3.1a)–(3.1c) involve the existence of a discrete co-state $p_h \in V_h$ and of a discrete co-control $\sigma_h \in W_h$ such that

$$a(y_h, v_h) = \ell_{h,1}(v_h), \quad v_h \in V_h \tag{3.3a}$$

$$a(p_h, v_h) = \ell_{h,2}(v_h), \quad v_h \in V_h \tag{3.3b}$$

$$u_h = u_h^d + \frac{1}{\alpha}(M_h p_h - \sigma_h) \tag{3.3c}$$

$$\sigma_h \in \partial I_{K_h}(u_h)\;. \tag{3.3d}$$

Here, $\ell_{h,2} : V_h \to \mathbb{R}$ stands for the functional

$$\ell_{h,2}(v_h) := \int_\Omega (y^d - y_h) v_h \,\mathrm{d}x, \quad v_h \in V_h$$

and $M_h : V_h \to W_h$ is the averaging operator given by

$$M_h v_h|_E := h_E^{-1} \int_E v_h \,\mathrm{d}s, \quad E \in \mathscr{E}_h(\Gamma_1). \tag{3.4}$$

We define $\mathscr{A}^{(\nu)}(u_h)$, $1 \leqslant \nu \leqslant 2$, and $\mathscr{I}(u_h)$ as the discrete active and inactive control sets according to

$$\mathscr{A}^{(1)}(u_h) \;:=\; \bigcup \left\{ E \in \mathscr{E}_h(\Gamma_1) \mid u_h|_E = \psi_h^{(1)}|_E \right\} \tag{3.5a}$$

$$\mathscr{A}^{(2)}(u_h) \;:=\; \bigcup \left\{ E \in \mathscr{E}_h(\Gamma_1) \mid u_h|_E = \psi_h^{(2)}|_E \right\} \tag{3.5b}$$

$$\mathscr{I}(u_h) \;:=\; \bigcup \left\{ T \in \mathscr{T}_h(\Omega) \mid u_h|_T < \psi_h|_T \right\}. \tag{3.5c}$$

As in the continuous regime, we split the adjoint control σ_h by means of

$$\sigma_h \;=\; \sigma_h^{(1)} + \sigma_h^{(2)} \tag{3.6}$$

with $\mathrm{supp}(\sigma_h^{(1)}) = \mathscr{A}^{(1)}(u_h)$ and $\mathrm{supp}(\sigma_h^{(2)}) = \mathscr{A}^{(2)}(u_h)$ such that

$$\sigma_h^{(1)}|_E \leqslant 0, \quad E \in \mathscr{A}^{(1)}(u_h), \qquad \sigma_h^{(2)}|_E \geqslant 0, \quad E \in \mathscr{A}^{(2)}(u_h). \tag{3.7}$$

The inclusion (3.3d) implies that $\sigma_h^{(\nu)}$, $1 \leqslant \nu \leqslant 2$, satisfy the complementarity conditions

$$(\sigma_h^{(1)}, u_h - \psi_h^{(1)})_{0,\Gamma_1} = 0, \qquad (\sigma_h^{(2)}, \psi_h^{(2)} - u_h)_{0,\Gamma_1} = 0. \tag{3.8}$$

4. THE RESIDUAL TYPE *A POSTERIORI* ERROR ESTIMATOR

The residual type error estimator consists of easily computable element and edge residuals with respect to the finite element approximations $y_h \in V_h$ and $p_h \in V_h$ of the state $y \in V$ and the co-state $p \in V$ as well as of data oscillations.

In particular, we define

$$\eta_y := \left(\sum_{T\in\mathscr{T}_h(\Omega)} \eta_{y,T}^2 + \sum_{E\in\mathscr{E}_h(\Omega)} (\eta_{y,E}^{\text{int}})^2 + \sum_{E\in\mathscr{E}_h(\Gamma_1)} (\eta_{y,E}^{\Gamma_1})^2 \right)^{1/2} \tag{4.1a}$$

$$\eta_p := \left(\sum_{T\in\mathscr{T}_h(\Omega)} \eta_{p,T}^2 + \sum_{E\in\mathscr{E}_h(\Omega)} (\eta_{p,E}^{\text{int}})^2 + \sum_{E\in\mathscr{E}_h(\Gamma_1)} \sum_{\nu=1}^{2} (\eta_{p,E}^{\Gamma_1,\nu})^2 \right)^{1/2} . \tag{4.1b}$$

The element residuals $\eta_{y,T}$, $\eta_{p,T}$ and the edge residuals $\eta_{y,E}^{\text{int}}$, $\eta_{p,E}^{\text{int}}$ associated with the interior edges $E \in \mathscr{E}_h(\Omega)$ as well as the edge residuals $\eta_{y,E}^{\Gamma_1}$, $\eta_{p,E}^{\Gamma_1,\nu}$, $1 \leqslant \nu \leqslant 2$, with respect to the boundary edges $E \in \mathscr{E}_h(\Gamma_1)$ are given by

$$\eta_{y,T} := h_T \, \|f - cy_h\|_{0,T}, \quad T \in \mathscr{T}_h(\Omega) \tag{4.2a}$$

$$\eta_{p,T} := h_T \, \|y^d - y_h - cp_h\|_{0,T}, \quad T \in \mathscr{T}_h(\Omega) \tag{4.2b}$$

$$\eta_{y,E}^{\text{int}} := h_E^{1/2} \, \|\nu_E \cdot [\nabla y_h]\|_{0,E}, \quad E \in \mathscr{E}_h(\Omega) \tag{4.2c}$$

$$\eta_{p,E}^{\text{int}} := h_E^{1/2} \, \|\nu_E \cdot [\nabla p_h]\|_{0,E}, \quad E \in \mathscr{E}_h(\Omega) \tag{4.2d}$$

$$\eta_{y,E}^{\Gamma_1} := h_E^{1/2} \, \|u_h - \nu_E \cdot \nabla y_h\|_{0,E}, \quad E \in \mathscr{E}_h(\Gamma_1) \tag{4.2e}$$

$$\eta_{p,E}^{\Gamma_1,1} := h_E^{1/2} \, \|\nu_E \cdot \nabla p_h\|_{0,E}, \quad E \in \mathscr{E}_h(\Gamma_1) \tag{4.2f}$$

$$\eta_{p,E}^{\Gamma_1,2} := \|M_h p_h - p_h\|_{0,E}, \quad E \in \mathscr{E}_h(\Gamma_1). \tag{4.2g}$$

Here, for an interior edge

$$E = T_1 \cap T_2, \quad T_\nu \in \mathscr{T}_h(\Omega), \quad 1 \leqslant \nu \leqslant 2$$

we refer to ν_E as the exterior unit normal vector on E directed towards T_2, whereas $[\nabla y_h]$ and $[\nabla p_h]$ denote the jumps of $\nabla y_h, \nabla p_h$ across E.

The residual type error estimator η for the finite element approximation of the boundary control problem (2.1a)–(2.1c) is then given by

$$\eta := \left(\eta_y^2 + \eta_p^2\right)^{1/2}. \tag{4.3}$$

The error analysis further invokes the data oscillations

$$\mu_h(u^d) := \Big(\sum_{E\in\mathscr{E}_h(\Gamma_1)} \mu_E(u^d)^2 \Big)^{1/2}, \quad \mu_E(u^d) := \|u^d - u^d_E\|_{0,E} \tag{4.4a}$$

$$\mu_h(\psi^{(\nu)}) := \Big(\sum_{E\in\mathscr{E}_h(\Gamma_1)} \sum_{\nu=1}^{2} \mu_E(\psi^{(\nu)})^2 \Big)^{1/2}, \quad \mu_E(\psi^{(\nu)}) := \|\psi^{(\nu)} - \psi^{(\nu)}_E\|_{0,E} \tag{4.4b}$$

$$osc_h(f) := \Big(\sum_{T\in\mathscr{T}_h(\Omega)} osc_T(f)^2 \Big)^{1/2}, \quad osc_T(f) := h_T\|f - f_T\|_{0,T} \tag{4.4c}$$

$$osc_h(y^d) := \Big(\sum_{T\in\mathscr{T}_h(\Omega)} osc_T(y^d)^2 \Big)^{1/2}, \quad osc_T(y^d) := h_T\|y^d - y^d_T\|_{0,T}. \tag{4.4d}$$

Compared to the element residuals $\eta_{y,T}$, $\eta_{p,T}$ and the edge residuals $\eta^{\text{int}}_{y,E}$, $\eta^{\text{int}}_{p,E}$ as well as $\eta^{\Gamma_1}_{y,E}$, $\eta^{\Gamma_1}_{p,E}$, the data oscillations $osc_h(f)$, $osc_h(y^d)$ are of the same order for non smooth f, y^d and of higher order for smooth f, y^d, e.g., $f, y^d \in H^1(\Omega)$.

5. RELIABILITY OF THE ERROR ESTIMATOR

In this section, we show that, up to data oscillations, the discretization errors in the state, the co-state, the control, and the co-control

$$|||z - z_h||| := \|y - y_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} + \|u - u_h\|_{0,\Gamma_1} + \|\sigma - \sigma_h\|_{0,\Gamma_1} \tag{5.1}$$

can be bounded by the residual-type error estimator η as given by (4.3).

Reliability results for conforming finite element discretizations of standard elliptic boundary value problems take advantage of Galerkin orthogonality which does not hold true for the boundary control problem under consideration. However, we observe that the discrete state and co-state $y_h, p_h \in V_h$ may also be considered as finite element approximations of the coupled elliptic system: Given $u_h \in W_h$, find $y(u_h), p(u_h) \in V$ such that

$$a(y(u_h), v) = (f, v)_{0,\Omega} + (u_h, v)_{0,\Gamma_1}, \quad v \in V \tag{5.2}$$

$$a(p(u_h), v) = -(y(u_h) - y^d, v)_{0,\Omega}, \quad v \in V. \tag{5.3}$$

Obviously, we have

$$\|y(u_h) - y\|_{1,\Omega} \leqslant c_0^{-1} c(\Gamma_1)\, \|u - u_h\|_{0,\Gamma_1}, \quad \|p(u_h) - p\|_{1,\Omega} \leqslant c_0^{-1} \|y - y(u_h)\|_{0,\Omega} \tag{5.4}$$

where $c_0 := \min(1, c)$ and $c(\Gamma_1)$ is the constant in the trace inequality

$$\|v\|_{0,\Gamma_1} \leqslant c(\Gamma_1)\|v\|_{1,\Omega}, \quad v \in V. \tag{5.5}$$

Moreover, choosing $v = p(u_h) - p$ in (5.2) and $v = y(u_h) - y$ in (5.3), we find

$$(p - p(u_h), u - u_h)_{0,\Gamma_1} = -\|y - y(u_h)\|_{0,\Omega}^2 \leqslant 0. \tag{5.6}$$

In terms of the auxiliary state $y(u_h)$ and the auxiliary co-state $p(u_h)$, we first show that, up to $\eta_{p,E}^{\Gamma_1,2}$ and data oscillations, the error $|||z - z_h|||$ can be bounded from above by the discretization errors $\|y - y(u_h)\|_{1,\Omega}$ and $\|p - p(u_h)\|_{1,\Omega}$.

Lemma 5.1. *Let $y(u_h)$ and $p(u_h)$ be the intermediate state and intermediate adjoint state as given by* (5.2), (5.3) *and let $\mu_h(u^d), \mu_h(\psi^{(\nu)}$, $1 \leqslant \nu \leqslant 2$, be the data oscillations according to* (4.4a) *and* (4.4b)*. Then, there exist positive constants $C_\nu, 1 \leqslant \nu \leqslant 5$, depending only on α and Ω, such that*

$$\begin{aligned} |||z - z_h||| \;\leqslant\; & C_1 \|y_h - y(u_h)\|_{1,\Omega} + C_2 \|p_h - p(u_h)\|_{1,\Omega} \\ & + C_3 \eta_{p,E}^{\Gamma_1,2} + C_4 \sum_{\nu=1}^{2} \mu_h(\psi^{(\nu)}) + C_5 \mu_h(u^d). \end{aligned} \tag{5.7}$$

Proof. Observing (3.3a), (3.3b) and (5.4), we find

$$\|y - y_h\|_{1,\Omega} \leqslant \|y_h - y(u_h)\|_{1,\Omega} + c_0^{-1} c(\Gamma_1) \|u - u_h\|_{0,\Gamma_1} \tag{5.8}$$

$$\begin{aligned} \|p - p_h\|_{1,\Omega} &\leqslant \|p_h - p(u_h)\|_{1,\Omega} + c_0^{-1} \|y - y_h\|_{0,\Omega} \\ &\leqslant \|p_h - p(u_h)\|_{1,\Omega} + c_0^{-1} \|y_h - y(u_h)\|_{1,\Omega} + c_0^{-1} c(\Gamma_1) \|u - u_h\|_{0,\Gamma_1}. \end{aligned} \tag{5.9}$$

Further, in view of (2.3c) and (3.3c)

$$\begin{aligned} \|\sigma - \sigma_h\|_{0,\Gamma_1} &\leqslant \alpha \Big(\|u - u_h\|_{0,\Gamma_1} + \mu_h(u^d) \Big) + \|p - M_h p_h\|_{0,\Gamma_1} \\ &\leqslant \alpha \Big(\|u - u_h\|_{0,\Gamma_1} + \mu_h(u^d \Big) + c(\Gamma_1) \|p - p_h\|_{1,\Omega} + \|p_h - M_h p_h\|_{0,\Gamma_1} \\ &\leqslant (\alpha + c_0^{-1} c(\Gamma_1)^2) \|u - u_h\|_{0,\Gamma_1} + c(\Gamma_1) \|p_h - p(u_h)\|_{1,\Omega} \\ &\quad + c_0^{-1} c(\Gamma_1) \|y_h - y(u_h)\|_{1,\Omega} + \|p_h - M_h p_h\|_{0,\Gamma_1} + \alpha \mu_h(u^d). \end{aligned} \tag{5.10}$$

On the other hand, taking again (2.3c),(3.3c) into account and using Young's inequality

$$\begin{aligned} \alpha \|u - u_h\|_{0,\Gamma_1}^2 \;=\; & (\sigma_h - \sigma, u - u_h)_{0,\Gamma_1} + (p - p_h, u - u_h)_{0,\Gamma_1} \\ & + (p_h - M_h p_h, u - u_h)_{0,\Gamma_1} + \alpha (u^d - u_h^d, u - u_h)_{0,\Gamma_1} \\ \leqslant\; & (\sigma_h - \sigma, u - u_h)_{0,\Gamma_1} + (p - p_h, u - u_h)_{0,\Gamma_1} \\ & + \frac{\alpha}{4} \|u - u_h\|_{0,\Gamma_1}^2 + \frac{2}{\alpha} \|p_h - M_h p_h\|_{0,\Gamma_1}^2 + \frac{2}{\alpha} \mu_h^2(u^d). \end{aligned} \tag{5.11}$$

We split $\sigma_h - \sigma$ by means of (2.6), (3.6). Then, the complementarity conditions (2.7), (3.7) and (3.8) allow to estimate $(\sigma_h^{(1)} - \sigma^{(1)}, u - u_h)_{0,\Gamma_1}$ according to

$$\begin{aligned}(\sigma_h^{(1)} - \sigma^{(1)}, u - u_h)_{0,\Gamma_1} &= \underbrace{(\sigma_h^{(1)}, u - \psi^{(1)})_{0,\Gamma_1}}_{\leqslant 0} + (\sigma_h^{(1)} - \sigma^{(1)}, \psi^{(1)} - \psi_h^{(1)})_{0,\Gamma_1} \\ &\quad + \underbrace{(\sigma_h^{(1)}, \psi_h^{(1)} - u_h)_{0,\Gamma_1}}_{=0} - \underbrace{(\sigma^{(1)}, u - \psi^{(1)})_{0,\Gamma_1}}_{=0} - \underbrace{(\sigma^{(1)}, \psi_h^{(1)} - u_h)_{0,\Gamma_1}}_{\geqslant 0} \\ &\leqslant (\sigma_h^{(1)} - \sigma^{(1)}, \psi^{(1)} - \psi_h^{(1)})_{0,\Gamma_1}.\end{aligned}$$

An application of Young's inequality implies that for some $\varepsilon > 0$

$$(\sigma_h^{(1)} - \sigma^{(1)}, u - u_h)_{0,\Gamma_1} \leqslant \frac{\varepsilon\alpha}{4} \|\sigma^{(1)} - \sigma_h^{(1)}\|_{0,\Gamma_1}^2 + \frac{1}{\varepsilon\alpha} \mu_h^2(\psi^{(1)}). \tag{5.12}$$

Likewise, observing the same complementarity conditions, we can estimate $(\sigma_h^{(2)} - \sigma^{(2)}, u - u_h)_{0,\Gamma_1}$ according to

$$\begin{aligned}(\sigma_h^{(2)} - \sigma^{(2)}, u - u_h)_{0,\Gamma_1} &= \underbrace{(\sigma_h^{(2)}, u - \psi^{(2)})_{0,\Gamma_1}}_{\leqslant 0} + (\sigma_h^{(2)} - \sigma^{(2)}, \psi^{(2)} - \psi_h^{(2)})_{0,\Gamma_1} \\ &\quad + \underbrace{(\sigma_h^{(2)}, \psi_h^{(2)} - u_h)_{0,\Gamma_1}}_{=0} - \underbrace{(\sigma^{(2)}, u - \psi^{(2)})_{0,\Gamma_1}}_{=0} - \underbrace{(\sigma^{(2)}, \psi_h^{(2)} - u_h)_{0,\Gamma_1}}_{\geqslant 0} \\ &\leqslant (\sigma_h^{(2)} - \sigma^{(2)}, \psi^{(2)} - \psi_h^{(2)})_{0,\Gamma_1} \leqslant \frac{\varepsilon\alpha}{4} \|\sigma^{(2)} - \sigma_h^{(2)}\|_{0,\Gamma_1}^2 + \frac{1}{\varepsilon\alpha} \mu_h^2(\psi^{(2)}). \end{aligned} \tag{5.13}$$

Since $\sigma^{(1)} - \sigma_h^{(1)}$ and $\sigma^{(2)} - \sigma_h^{(2)}$ have disjoint supports on Γ_1, we have

$$\|\sigma - \sigma_h\|_{0,\Gamma_1}^2 = \|\sigma^{(1)} - \sigma_h^{(1)}\|_{0,\Gamma_1}^2 + \|\sigma^{(2)} - \sigma_h^{(2)}\|_{0,\Gamma_1}^2.$$

Hence, (5.12) and (5.13) result in

$$(\sigma_h - \sigma, u - u_h)_{0,\Gamma_1} \leqslant \frac{\varepsilon\alpha}{4} \|\sigma - \sigma_h\|_{0,\Gamma_1}^2 + \frac{1}{\varepsilon\alpha} \sum_{\nu=1}^{2} \mu_h^2(\psi^{(\nu)}). \tag{5.14}$$

On the other hand, in view of (5.6), for the second term on the right-hand side in (5.11) we obtain

$$(p - p_h, u - u_h)_{0,\Gamma_1} \leqslant (p(u_h) - p_h, u - u_h)_{0,\Gamma_1}.$$

Using Young's inequality again, the right-hand side can be further estimated according to

$$(p(u_h) - p_h, u - u_h)_{0,\Gamma_1} \leqslant \frac{\alpha}{4} \|u - u_h\|_{0,\Gamma_1}^2 + \frac{c(\Gamma_1)^2}{\alpha} \|p(u_h) - p_h\|_{1,\Omega}^2. \tag{5.15}$$

Using (5.14) and (5.15) in (5.11), we end up with

$$\begin{aligned}\|u-u_h\|_{0,\Gamma_1}^2 \;\leqslant\; & \frac{\varepsilon}{2}\|\sigma-\sigma_h\|_{0,\Gamma_1}^2 + 2\Big(\frac{c(\Gamma_1)}{\alpha}\Big)^2 \|p(u_h)-p_h\|_{1,\Omega} \\ & + \frac{4}{\alpha^2}\big(\eta_{p,E}^{\Gamma_1,2}\big)^2 + \frac{4}{\alpha^2}\mu_h^2(u^d) + \frac{2}{\varepsilon\alpha^2}\sum_{\nu=1}^{2}\mu_h^2(\psi^{(\nu)}). \end{aligned} \tag{5.16}$$

Combining (5.8)–(5.10), (5.16) and choosing $\varepsilon > 0$ appropriately, gives the assertion. $\square$

Lemma 5.2. *Let* (y_h, p_h) *be the solution of* (3.3a), (3.3b) *and let* $y(u_h), p(u_h)$ *be the auxiliary state and co-state as given by* (5.2), (5.3), *respectively. Further, let* η_y *and* $\eta_{p,T}$, $\eta_{p,E}^{\rm int}$, $\eta_{p,E}^{\Gamma_1}$ *be the parts of the residual error estimator* η *as given by* (4.1a) *and* (4.2b), (4.2d), (4.2f)*. Then, there exist positive constants* C_ν, $6 \leqslant \nu \leqslant 7$, *depending only on the ellipticity constant* c_0 *and the shape regularity of the triangulation* $\mathscr{T}_h(\Omega)$, *such that*

$$\|y(u_h)-y_h\|_{1,\Omega}^2 \leqslant C_6\eta_y^2 \tag{5.17a}$$

$$\|p(u_h)-p_h\|_{1,\Omega}^2 \leqslant C_7\Big(\eta_y^2 + \sum_{T\in\mathscr{T}_h(\Omega}\eta_{p,T}^2 + \sum_{E\in\mathscr{E}_h(\Omega}(\eta_{p,E}^{\rm int})^2 + \sum_{E\in\mathscr{E}_h(\Gamma_1}(\eta_{p,E}^{\Gamma_1})^2\Big). \tag{5.17b}$$

Proof. The upper bounds can be derived by standard means, using for instance Clément's interpolation operator (cf., e.g., [28]). In particular, for the discretization error $\|y(u_h)-y_h\|_{1,\Omega}$ we obtain

$$\begin{aligned}\|y(u_h)-y_h\|_{1,\Omega}^2 \;\leqslant\; C_6 \Big(& \sum_{T\in\mathscr{T}_h(\Omega)} \underbrace{h_T^2\|f-cy_h\|_{0,T}^2}_{=\eta_{y,T}^2} \\ & + \sum_{E\in\mathscr{E}_h(\Omega)} \underbrace{h_E\|\nu_E\cdot[\nabla y_h]\|_{0,E}^2}_{=(\eta_{y,E}^{\rm int})^2} + \sum_{E\in\mathscr{E}_h(\Gamma_1)} \underbrace{h_E\|g-\nu_E\cdot\nabla y_h\|_{0,E}^2}_{=(\eta_{y,E}^{\Gamma_1})^2}\Big)\end{aligned}$$

which is (5.17a). Applying the same techniques to $\|p(u_h)-p_h\|_{1,\Omega}$, it follows that

$$\begin{aligned}\|p(u_h)-p_h\|_{1,\Omega}^2 \leqslant C\Big(& \sum_{T\in\mathscr{T}_h(\Omega)} h_T^2\|y^d-y(u_h)-cp_h\|_{0,T}^2 \\ & + \sum_{E\in\mathscr{E}_h(\Omega)} \underbrace{h_E\|\nu_E\cdot[\nabla p_h]\|_{0,E}^2}_{=(\eta_{p,E}^{\rm int})^2} + \sum_{E\in\mathscr{E}_h(\Omega)} \underbrace{h_E\|\nu_E\cdot\nabla p_h\|_{0,E}^2}_{=(\eta_{p,E}^{\Gamma_1})^2}\Big). \end{aligned}\tag{5.18}$$

Taking advantage of (5.17a), for the first term on the right-hand side in (5.18) we get

$$
\begin{aligned}
&\sum_{T\in\mathscr{T}_h(\Omega)} h_T^2\|y^d-y(u_h)-cp_h\|_{0,T}^2\\
&\quad\leqslant 2\Big(\sum_{T\in\mathscr{T}_h(\Omega)} \underbrace{h_T^2\|y^d-y_h-cp_h\|_{0,T}^2}_{=\eta_{p,T}^2}+\sum_{T\in\mathscr{T}_h(\Omega)} h_T^2\|y(u_h)-y_h\|_{0,T}^2\Big)\\
&\quad\leqslant 2\sum_{T\in\mathscr{T}_h(\Omega)} \eta_{T,p}^2+2h^2\|y(u_h)-y_h\|_{1,\Omega}^2\\
&\quad\leqslant 2\sum_{T\in\mathscr{T}_h(\Omega)} \eta_{p,T}^2+2h^2C_6^2\eta_y^2. \qquad (5.19)
\end{aligned}
$$

If we use (5.19) in (5.18), we achieve at (5.17b). □

Combining Lemma 5.1 and Lemma 5.2 results in the following reliability estimate:

Theorem 5.1. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3b) *and* (3.3a)–(3.3d)*, and let* η *and* $\mu_h(u^d),\mu_h(\psi^{(\nu)})$, $1\leqslant\nu\leqslant 2$*, be the residual error estimator and the data oscillations as given by* (4.3) *and* (4.4a)–(4.4b)*, respectively. Then, there exist positive constants* Γ *and* C*, depending on* α,c_0,Ω *and the shape regularity of the triangulation* $\mathscr{T}_h(\Omega)$*, such that*

$$
|||z-z_h|||\leqslant\Gamma\eta+C\Big(\mu_h(u^d)+\sum_{\nu=1}^{2}\mu_h(\psi^{(\nu)})\Big). \qquad (5.20)
$$

6. EFFICIENCY OF THE ERROR ESTIMATOR

In this section, we will prove that, up to data oscillations, the estimator η also provides a lower bound for the error $|||z-z_h|||$. In particular, we will show that the local contributions of the estimator can be bounded from above by the local constituents of the error and associated data oscillations. As in the case of standard Lagrange finite element approximations of elliptic boundary value problems (cf., e.g., [28]), we use element and edge bubble functions. We denote by λ_i^T, $1\leqslant i\leqslant 3$, the barycentric coordinates of $T\in\mathscr{T}_h(\Omega)$ and refer to $\vartheta_T:=27\prod_{i=1}^3\lambda_i^T$ as the associated element bubble function. Likewise, λ_i^E, $1\leqslant i\leqslant 2$, stand for the barycentric coordinates of $E\in\mathscr{E}_h(\Omega\cup\Gamma_1)$ and $\vartheta_E:=4\prod_{i=1}^2\lambda_i^E$ denotes the associated edge bubble function. We recall from [28] that there exists constants c_i, $1\leqslant i\leqslant 5$, depending only on the shape regularity of the triangulation $\mathscr{T}_h(\Omega)$ such that for $p_T\in P_k(T)$, $k\in\mathbb{N}_0$, and

$p_E \in P_k(E)$, $k \in \mathbb{N}_0$, there holds

$$\|p_T\|_{0,T}^2 \leqslant c_1\,(p_T, p_T \vartheta_T)_{0,T}, \quad T \in \mathscr{T}_h(\Omega) \tag{6.1a}$$

$$\|p_T \vartheta_T\|_{0,T} \leqslant c_2\,\|p_T\|_{0,T}, \qquad T \in \mathscr{T}_h(\Omega) \tag{6.1b}$$

$$|p_T \vartheta_T|_{1,T} \leqslant c_3 h_T^{-1}\|p_T\|_{0,T}, \quad T \in \mathscr{T}_h(\Omega) \tag{6.1c}$$

$$\|p_E\|_{0,E}^2 \leqslant c_4\,(p_E, p_E \vartheta_E)_{0,E}, \quad E \in \mathscr{E}_h(\Omega \cup \Gamma_1) \tag{6.1d}$$

$$\|p_E \vartheta_E\|_{0,E} \leqslant c_5\|p_E\|_{0,E}, \qquad E \in \mathscr{E}_h(\Omega \cup \Gamma_1). \tag{6.1e}$$

For $E \in \mathscr{E}_h(\Omega)$ and $p_E \in P_k(E)$, $k \in \mathbb{N}_0$, we further refer to $\tilde{p}_E$ as the extension of p_E to $\omega_E := T_1 \cup T_2, E = T_1 \cap T_2$, $T_\nu \in \mathscr{T}_h(\Omega)$, $1 \leqslant \nu \leqslant 2$, in the sense that for fixed $E'_\nu \in \mathscr{E}_h(T_\nu) \setminus \{E\}$, for $x \in T_\nu$ we have $\tilde{p}_E(x) := p_E(x_E)$ where $x_E \in E$ is such that $x - x_E$ is parallel to E'_ν. For $E \in \mathscr{E}_h(\Gamma_1)$, the extension to $\omega_E = T_E \in \mathscr{T}_h(\Omega)$ with $E \in \mathscr{E}_h(T_E)$ is done similarly. Again, referring to [28], there exist positive constants c_i, $6 \leqslant i \leqslant 7$, which only depend on the shape regularity of $T \in \mathscr{T}_h(\Omega)$ such that

$$\|\tilde{p}_E \vartheta_E\|_{0,\omega_E} \leqslant c_6 h_E^{1/2}\|p_E\|_{0,E} \tag{6.2a}$$

$$|\tilde{p}_E \vartheta_E|_{1,\omega_E} \leqslant c_7 h_E^{-1/2}\|p_E\|_{0,E} \tag{6.2b}$$

Lemma 6.1. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3d) *and* (3.3a)–(3.3d) *and let* $\eta_{y,T}, osc_T(f)$ *be given by* (4.2a) *and* (4.4c)*, respectively. Then, there exists a positive constant* γ *depending only on the shape regularity of* $\mathscr{T}_h(\Omega)$ *such that for* $T \in \mathscr{T}_h(\Omega)$

$$\eta_{y,T}^2 \leqslant \gamma\Big(\|y-y_h\|_{1,T}^2 + osc_T^2(f)\Big). \tag{6.3}$$

Proof. We have

$$\eta_{y,T}^2 = h_T^2\|f - cy_h\|_{0,T}^2 \leqslant 2h_T^2\|f_h - cy_h\|_{0,T}^2 + 2osc_T^2(f). \tag{6.4}$$

Setting $z_h := (f_h - cy_h)|_T \vartheta_T$, applying (6.1a) and observing $\Delta y_h|_T = 0$, Green's formula and the fact that z_h is an admissible test function in (3.3a) imply

$$\begin{aligned} h_T^2\|f_h - cy_h\|_{0,T}^2 &\leqslant c_1 h_T^2 (f_h + \Delta y_h - cy_h, z_h)_{0,T} \\ &= c_1 h_T^2\Big(-a(y_h,z_h) + (f,z_h)_{0,T} + (f_h - f, z_h)_{0,T}\Big) \\ &= c_1 h_T^2\Big(a(y-y_h,z_h) + ((f_h - f), z_h)_{0,T}\Big) \\ &\leqslant c_1\Big(h_T^2 C_0\|y-y_h\|_{1,T}|z_h|_{1,T} + h_T osc_T(f))\|z_h\|_{0,T}\Big). \end{aligned} \tag{6.5}$$

Now, by (6.1b), (6.1c) and Young's inequality, (6.5) gives rise to

$$h_T^2\|f_h - cy_h\|_{0,T}^2 \leqslant c_1\Big(c_3|y-y_h|_{1,T}^2 + c_2 h_T^2 osc_T^2(f)\Big) + \frac{1}{2}h_T^2\|f_h - cy_h\|_{0,T}^2. \tag{6.6}$$

Combining (6.4) and (6.6), readily gives the assertion. □

Lemma 6.2. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3d) *and* (3.3a)–(3.3d) *and let* $\eta_{p,T}, osc_T(y^d)$ *be given by* (4.2b) *and* (4.4d)*, respectively. Then, there exists a positive constant* γ *depending only on the shape regularity of* $\mathscr{T}_h(\Omega)$ *such that for* $T \in \mathscr{T}_h(\Omega)$

$$\eta_{p,T}^2 \leqslant \gamma \big(\|p-p_h\|_{1,T}^2 + h_T^2 \|y-y_h\|_{0,T}^2 + osc_T^2(y^d) \big). \tag{6.7}$$

Proof. The assertion (6.7) can be verified by using the same arguments as in the proof of Lemma 6.1. □

Lemma 6.3. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3d) *and* (3.3a)–(3.3d) *and let* $\eta_{y,E}^{\text{int}}$ *and* $\eta_{p,E}^{\text{int}}$ *be given by* (4.2c) *and* (4.2d)*, respectively. Then, there exists a positive constant* γ *depending only on the shape regularity of* $\mathscr{T}_h(\Omega)$ *such that for* $E \in \mathscr{E}_h(\Omega)$

$$(\eta_{y,E}^{\text{int}})^2 \;\leqslant\; \gamma \left(\|y-y_h\|_{1,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{y,T_\nu}^2 \right) \tag{6.8}$$

$$(\eta_{p,E}^{\text{int}})^2 \;\leqslant\; \gamma \left(\|p-p_h\|_{1,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{p,T_\nu}^2 \right). \tag{6.9}$$

Proof. We set $p_E := (\nu_E \cdot [\nabla y_h])|_E$ and $z_h := \tilde{p}_E \vartheta_E$. We use (6.1d), apply Green's formula, observe that $\tilde{z}_h$ is an admissible test function in (3.3a), and take advantage of (6.2a), (6.2b) to obtain

$$\begin{aligned}
(\eta_{y,E}^{\text{int}})^2 \;&=\; h_E \|\nu_E \cdot [\nabla y_h]\|_{0,E}^2 \\
&\leqslant\; c_4 h_E (\nu_E \cdot [\nabla y_h], p_E \vartheta_E)_{0,E} = c_4 h_E \sum_{\nu=1}^{2} (\nu_{\partial T_\nu} \cdot [\nabla y_h], z_h)_{0,\partial T_\nu} \\
&=\; c_4 h_E \big(a(y_h - y, z_h) + (f - c y_h, z_h)_{0,\omega_E} \big) \\
&\leqslant\; c_4 h_E^{1/2} \|\nu_E \cdot [\nabla y_h]\|_{0,E} \left(c_7 \max(1,c) \|y-y_h\|_{1,\omega_E} + c_6 \left(\sum_{\nu=1}^{2} \eta_{y,T_\nu}^2 \right)^{1/2} \right).
\end{aligned}$$

An application of Young's inequality results in (6.8). The proof of (6.9) is done in exactly the same way. □

Lemma 6.4. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3d) *and* (3.3a)–(3.3d) *and let* $\eta_{y,E}^{\Gamma_1}$ *and* $\eta_{p,E}^{\Gamma_1,1}$ *be given by* (4.2e) *and* (4.2f)*, respectively. Then, there exists a positive constant* γ *depending only on the shape regularity of* $\mathscr{T}_h(\Omega)$ *such that for* $E \in \mathscr{E}_h(\Gamma_1)$

$$(\eta_{y,E}^{\Gamma_1})^2 \leqslant \gamma \big(\|y-y_h\|_{1,T_E}^2 + \eta_{y,T_E}^2 + \|u-u_h\|_{0,E}^2 \big) \tag{6.10}$$

$$(\eta_{p,E}^{\Gamma_1,1})^2 \leqslant \gamma \big(\|p-p_h\|_{1,T_E}^2 + \eta_{p,T_E}^2 \big). \tag{6.11}$$

Proof. We choose $p_E := (\nu_E \cdot \nabla y_h - u_h)|_E$ and $z_h := \tilde{p}_E \vartheta_E$. As in the proof of the previous lemma, we observe (6.1d) and apply Green's formula. We further make use of the fact that z_h is an admissible test function in (3.3a), and we apply (6.1e) and (6.2b). This results in

$$\begin{aligned}
(\eta_{y,E}^{\Gamma_1})^2 &= h_E \|u_h - \nu_E \cdot \nabla y_h\|_{0,E}^2 \\
&\leqslant c_4 h_E (\nu_E \cdot \nabla y_h - u_h, p_E \vartheta_E)_{0,E} \\
&= c_4 h_E \big((\nu_{\partial T_E} \cdot \nabla y_h, z_h)_{0,\partial T_E} + (u - u_h, p_E \vartheta_E)_{0,E} \big) \\
&= c_4 h_E \big(a(y_h - y, z_h) + (f - c y_h, z_h)_{0,T_E} + (u - u_h, p_E \vartheta_E)_{0,E} \big) \\
&\leqslant c_4 h_E^{1/2} \|u_h - \nu_E \cdot \nabla y_h\|_{0,E} \big(c_7 \max(1,c) \|y - y_h\|_{1,T_E} \\
&\quad + c_5 \|u - u_h\|_{0,E}^2 + c_6 \eta_{y,T_E}^2 \big). \qquad (6.12)
\end{aligned}$$

Applying Young's inequality in (6.12) gives (6.10). The proof of (6.11) is carried out in the same way as (6.12) has been established. □

Lemma 6.5. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3d) *and* (3.3a)–(3.3d) *and let* $\eta_{p,E}^{\Gamma_1,2}$ *and* $\mu_T(u^d)$ *be given by* (4.2g) *and* (4.4a)*, respectively. Then, for* $E \in \mathscr{E}_h(\Gamma_1)$ *there holds*

$$\eta_{p,E}^{\Gamma_1,2} \leqslant \|p - p_h\|_{1,T_E} + \|\sigma - \sigma_h\|_{0,E} + \alpha\big(\|u - u_h\|_{0,E} + \mu_E(u^d)\big). \qquad (6.13)$$

Proof. We have

$$\eta_{p,E}^{\Gamma_1,2} = \|M_h p_h - p_h\|_{0,E} \leqslant \|p - p_h\|_{0,E} + \|M_h p_h - p\|_{0,E} .$$

Observing (2.3c) and (3.3c), for the second term on the right-hand side we find

$$\|M_h p_h - p\|_{0,E} \leqslant \|\sigma - \sigma_h\|_{0,E} + \alpha\big(\|u - u_h\|_{0,E} + \mu_E(u^d)\big)$$

which readily gives (6.13). □

Summarizing the results of Lemmas 6.1 – 6.5, we finally obtain:

Theorem 6.1. *Let* (y,p,u,σ) *and* (y_h,p_h,u_h,σ_h) *be the solutions of* (2.3a)–(2.3d) *and* (3.3a)–(3.3d) *and let* $\eta, \mu_h(u^d)$ *and* $osc_h(y^d), osc_h(f)$ *be given by* (4.3), (4.4a)–(4.4d)*, respectively. Then, there exist positive constants* γ *and* $\varkappa$ *depending only on* c *and the shape regularity of* $\mathscr{T}_h(\Omega)$ *such that*

$$|||z - z_h|||^2 \geqslant \gamma \eta^2 - \varkappa \big(\mu_h^2(u^d) + osc_h^2(f) + osc_h^2(y^d) \big). \qquad (6.14)$$

7. ALGORITHMIC REALIZATION OF THE BULK CRITERION

The refinement of the triangulation $\mathscr{T}_h(\Omega)$ is based on a bulk criterion that has been previously used in the convergence analysis of adaptive finite element for nodal finite element methods [8,13,26] and for nonconforming, mixed and edge element methods [10–12]. Here, we adopt the bulk criterion for the finite element approximation of the Neumann type boundary control problem under consideration: Given the universal constants Θ_i, $1 \leqslant i \leqslant 4$, with $0 < \Theta_i < 1$, the outcome are sets of edges $\mathscr{M}_E^{\rm int} \subset \mathscr{E}_h(\Omega), \mathscr{M}_E^{\Gamma_1,1}, \mathscr{M}_E^{\Gamma_1,2} \subset \mathscr{E}_h(\Gamma_1)$ and sets of elements $\mathscr{M}_{\eta,T}, \mathscr{M}_{osc,T} \subset \mathscr{T}_h(\Omega)$ such that

$$\Theta_1 \sum_{E\in\mathscr{E}_h(\Omega)} \big((\eta_{y,E}^{\rm int})^2 + (\eta_{p,E}^{\rm int})^2\big) \leqslant \sum_{E\in\mathscr{M}_E^{\rm int}} \big((\eta_{y,E}^{\rm int})^2 + (\eta_{p,E}^{\rm int})^2\big) \tag{7.1}$$

$$\Theta_2 \sum_{E\in\mathscr{E}_h(\Gamma_1)} \big((\eta_{y,E}^{\Gamma_1})^2 + (\eta_{p,E}^{\Gamma_1,1})^2 + (\eta_{p,E}^{\Gamma_1,2})^2\big) \leqslant \sum_{E\in\mathscr{M}_E^{\Gamma_1,1}} \big((\eta_{y,E}^{\Gamma_1})^2 + (\eta_{p,E}^{\Gamma_1,1})^2 + (\eta_{p,E}^{\Gamma_1,2})^2\big) \tag{7.2}$$

$$\Theta_3 \Big(\sum_{E\in\mathscr{E}_h(\Gamma_1)} \mu_E^2(u^d) + \sum_{E\in\mathscr{A}_{u_h}} \sum_{\nu=1}^{2} \mu_E^2(\psi^{(\nu)})\Big) \leqslant \sum_{E\in\mathscr{M}_E^{\Gamma_1,2}} \big(\mu_E^2(u^d) + \sum_{\nu=1}^{2} \mu_E^2(\psi^{(\nu)})\big) \tag{7.3}$$

$$\Theta_4 \sum_{T\in\mathscr{T}_h(\Omega)} \big(\eta_{y,T}^2 + (\eta_{p,T})^2\big) \leqslant \sum_{T\in\mathscr{M}_{\eta,T}} \big(\eta_{y,T}^2 + \eta_{p,T}^2\big) \tag{7.4}$$

$$\Theta_5 \sum_{T\in\mathscr{T}_h(\Omega)} \big(osc_T^2(y^d) + osc_T^2(f)\big) \leqslant \sum_{T\in\mathscr{M}_{osc,T}} \big(osc_T^2(y^d) + osc_T^2(f)\big). \tag{7.5}$$

We set

$$\mathscr{M}_E := \mathscr{M}_E^{\rm int} \cup \bigcup_{\nu=1}^{2} \mathscr{M}_E^{\Gamma_1,\nu}, \quad \mathscr{M}_T := \mathscr{M}_{\eta,T} \cup \mathscr{M}_{osc,T}$$

and refine an element $T \in \mathscr{T}_h(\Omega)$ regularly (i.e., subdividing it into four congruent subtriangles by joining the midpoints of the edges), if

- $T \in \mathscr{M}_T$ or
- $T \cap \Gamma_1 = \varnothing$ and at least two edges $E \in \mathscr{E}_h(T)$ belong to $\mathscr{M}_E$ or
- $T \cap \Gamma_1 \neq \varnothing$ and $E \in \mathscr{E}_h(T) \cap \mathscr{E}_h(\Gamma_1)$ belongs to $\mathscr{M}_E$.

Denoting by $\mathscr{N}_E := \{E' \in \mathscr{E}_h(\Gamma_1) | E' \cap E \neq \varnothing\}$ the set of all neighboring edges of $E \in \mathscr{E}_h(\Gamma_1)$, we define the set

$$\mathscr{F}_h(u_h) := \partial\mathscr{A}(u_h) \cup \partial\mathscr{I}(u_h)$$

where

$$\begin{aligned} \partial\mathscr{A}(u_h) &:= \bigcup \{E \subset \mathscr{A}(u_h) \mid \mathscr{N}_E \cap \mathscr{I}(u_h) \neq \varnothing\} \\ \partial\mathscr{I}(u_h) &:= \bigcup \{E \subset \mathscr{I}(u_h) \mid \mathscr{N}_E \cap \mathscr{A}(u_h) \neq \varnothing\}. \end{aligned}$$

The set $\mathscr{F}_h(u_h)$ represents a neighborhood of the discrete free boundary between the discrete active and inactive sets $\mathscr{A}(u_h)$ and $\mathscr{I}(u_h)$. In order to guarantee a sufficient resolution of the continuous free boundary, at each refinement step, the elements $E \in \mathscr{F}_h(u_h)$ are bisected.

Further irregular refinements by bisection are only performed in order to guarantee that the refined triangulation is geometrically conforming.

The bulk criterion (7.1)–(7.5) is realized by the following greedy algorithm.

Algorithm (bulk criterion):

Step 1. Initialization:
Set

$$\mathscr{M}_E^{\text{int}} := \varnothing, \quad \mathscr{M}_E^{\Gamma_1} := \mathscr{F}_h(u_h), \quad \mathscr{M}_{osc,T,0} := \varnothing, \quad k = 0.$$

Step 2. Iteration loop:

Step 2a. Check edge residuals (interior edges):
If

$$\Theta_1 \sum_{E \in \mathscr{E}_h(\Omega)} \left((\eta_{y,E}^{\text{int}})^2 + (\eta_{p,E}^{\text{int}})^2\right) \leqslant \sum_{E \in \mathscr{M}_E^{\text{int}}} \left((\eta_{y,E}^{\text{int}})^2 + (\eta_{p,E}^{\text{int}})^2\right)$$

set $\mathscr{M}_E^{\text{int}} := \mathscr{M}_{E,k}^{\text{int}}$, $k = 0$, and go to Step 2b, else select some

$$F \in \mathscr{E}_h(\Omega) \setminus \mathscr{M}_{E,k}^{\text{int}}$$

such that

$$\eta_F = \max_{G \in \mathscr{E}_h(\Omega) \setminus \mathscr{M}_{E,k}^{\text{int}}} \left(\eta_{y,E}^{\text{int}}, \eta_{p,E}^{\text{int}}\right)$$

and set

$$\mathscr{M}_{E,k+1}^{\text{int}} := \mathscr{M}_{E,k}^{\text{int}} \cup \{F\}, \quad k := k+1.$$

Step 2b. Check edge residuals (edges on the Neumann boundary):
If

$$\Theta_2 \sum_{E \in \mathscr{E}_h(\Gamma_1)} \left((\eta_{y,E}^{\Gamma_1})^2 + (\eta_{p,E}^{\Gamma_1,1})^2 + (\eta_{p,E}^{\Gamma_1,2})^2\right) \leqslant \sum_{E \in \mathscr{M}_E^{\Gamma_1,1}} \left((\eta_{y,E}^{\Gamma_1})^2 + (\eta_{p,E}^{\Gamma_1,1})^2 + (\eta_{p,E}^{\Gamma_1,2})^2\right)$$

set $\mathscr{M}_E^{\Gamma_1,1} := \mathscr{M}_{E,k}^{\Gamma_1,1}$ and go to Step 2c, else select some

$$F \in \mathscr{E}_h(\Gamma_1) \setminus \mathscr{M}_{E,k}^{\Gamma_1,1}$$

such that

$$\eta_F = \max_{G \in \mathscr{E}_h(\Gamma_1) \setminus \mathscr{M}_{E,k}^{\Gamma_1,1}} \left(\eta_{y,E}^{\Gamma_1}, \eta_{p,E}^{\Gamma_1,1}, \eta_{p,E}^{\Gamma_1,2}\right)$$

and set

$$\mathscr{M}_{E,k+1}^{\Gamma_1,1} := \mathscr{M}_{E,k}^{\Gamma_1,1} \cup \{F\}, \quad k := k+1.$$

Step 2c. Check edge related data terms:
Set

$$\mathscr{M}_{E,k}^{\Gamma_1,2} := \mathscr{M}_{E,k}^{\Gamma_1,1}.$$

If

$$\Theta_3\Big(\sum_{E\in\mathscr{E}_h(\Gamma_1)} \mu_E^2(u^d) + \sum_{E\in\mathscr{A}_{u_h}} \sum_{\nu=1}^{2} \mu_E^2(\psi^{(\nu)})\Big) \leqslant \sum_{E\in\mathscr{M}_{E,k}^{\Gamma_1,2}} \Big(\mu_E^2(u^d) + \sum_{\nu=1}^{2} \mu_E^2(\psi^{(\nu)})\Big)$$

set $\mathscr{M}_E^{\Gamma_1,2} := \mathscr{M}_{E,k}^{\Gamma_1,2}$, $k=0$, and go to Step 2d, else select some

$$\mu_F := \max_{E\in\mathscr{E}_h(\Gamma_1)\setminus\mathscr{M}_{E,k}^{\Gamma_1,2}} \big(\mu_E(u^d), \mu_E(\psi^{(1)}), \mu_E^2(\psi^{(2)})\big)$$

and set

$$\mathscr{M}_{E,k+1}^{\Gamma_1,2} := \mathscr{M}_{E,k}^{\Gamma_1,2} \cup \{F\}, \quad k := k+1.$$

Step 2d. Check element residuals:
If

$$\Theta_4 \sum_{T\in\mathscr{T}_h(\Omega)} \big(\eta_{y,T}^2 + (\eta_{p,T})^2\big) \leqslant \sum_{T\in\mathscr{M}_{\eta,T}} (\eta_{y,T}^2 + \eta_{p,T}^2)$$

set $\mathscr{M}_{\eta,T} := \mathscr{M}_{\eta,T,k}$ and go to Step 2e, else select some

$$\eta_S := \max\Big(\max_{T\in\mathscr{T}_h(\Omega)\setminus\mathscr{M}_{\eta,T,k}} \big(\eta_{y,T}, \eta_{p,T}\big)\Big)$$

and set

$$\mathscr{M}_{\eta,T,k+1} := \mathscr{M}_{\eta,T,k} \cup \{S\}, \quad k := k+1.$$

Step 2e. Check remaining data oscillations:
Set

$$\mathscr{M}_{osc,T,k}^{osc,T} := \mathscr{M}_{\eta,T,k}.$$

If

$$\Theta_5 \sum_{T\in\mathscr{T}_h(\Omega)} \big(osc_T^2(y^d) + osc_T^2(f)\big) \leqslant \sum_{T\in\mathscr{M}_{osc,T}} \big(osc_T^2(y^d) + osc_T^2(f)\big).$$

go to Step 3, else select some

$$\eta_S := \max_{T\in\mathscr{T}_h(\Omega)} \setminus \mathscr{M}_{osc,T,k}\big(osc_T(y^d), osc_T(f)\big)$$

and set

$$\mathscr{M}_{osc,T,k+1} := \mathscr{M}_{osc,T,k} \cup \{S\}, \quad k := k+1.$$

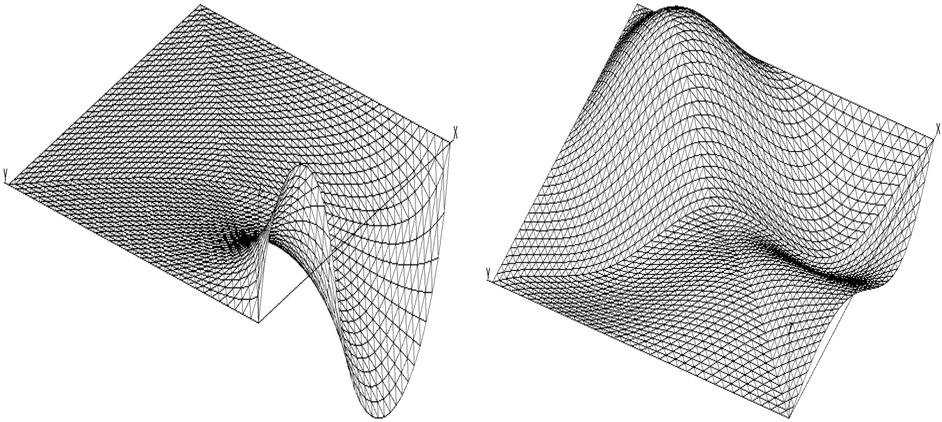

Figure 1. Example 1: Visualization of the optimal state y (left) and of the optimal co-state p (right).

8. NUMERICAL RESULTS

We present a documentation of numerical results illustrating the performance of the adaptive finite element approximation for two representative boundary control problems that have been considered in [16] in the framework of primal-dual active set strategies as iterative solvers for such kind of control problems. In particular, we consider the case of constant lower and upper bounds for the control with small α resulting in some type of 'bang-bang' control and an example where the upper bound for the control is highly oscillatory resulting in large values of the associated data oscillation term. The numerical results demonstrate the efficiency and reliability of the estimator and clearly underline the necessity to take data oscillations into account during the adaptive refinement process.

Example 1 (constant bounds).

The data in (2.1a)–(2.1c) have been chosen as follows:

$$\begin{aligned}
\Omega &= (0,1)^2, \quad \Gamma_1 = (0,1)\times\{0\}, \quad c=1\\
y^d &= \sin(2\pi x_1)\sin(2\pi x_2)\exp(2x_1), \quad u^d = \cos(5\pi x_1^2), \quad \alpha = 10^{-3}\\
f &:= 0, \quad \psi^{(1)} = -0.75, \quad \psi^{(2)} = 0.75.
\end{aligned}$$

Figures 1 and 2 show a visualization of the optimal state, the optimal co-state, the optimal control, and the optimal co-control, respectively. The optimal control switches from the lower to the upper bound in a very narrow region which corresponds to the inactive set associated with the optimal control.

The initial simplicial triangulation $\mathscr{T}_{h_0}$ was chosen according to a subdivision of Ω by joining the four vertices resulting in one interior nodal point and four congruent triangles. Since $f=0$ and the lower and upper bounds $\psi^{(\nu)}$, $1 \leqslant \nu \leqslant 2$, are constant, we have $\mu_h(\psi^{(\nu)}) = 0$ and $osc_h(f) = 0$. Figure 3 displays the adaptively generated triangulations after four (left) and eight (right) refinement steps

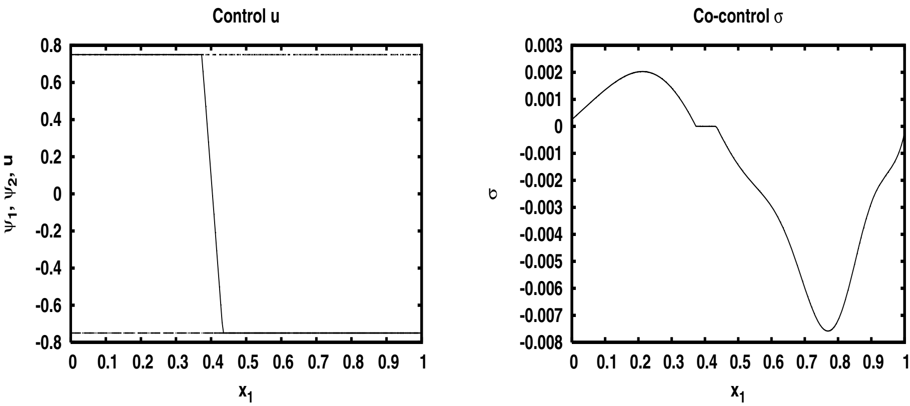

Figure 2. Example 1: Visualization of the optimal control u (left) and of the optimal co-control σ (right). The lower and upper bounds on the control are shown as 'dashed' and 'dotted' lines, respectively.

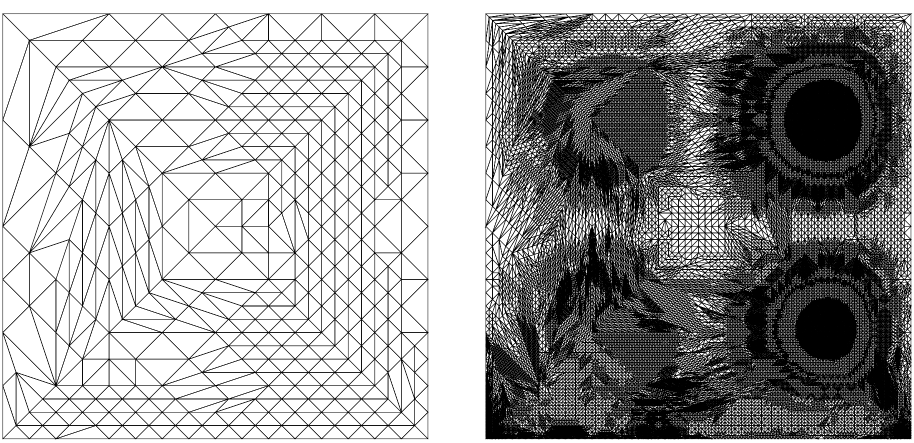

Figure 3. Example 1: Adaptively generated grid after 4 (left) and 8 (right) refinement steps ($\Theta_i = 0.8$, $1 \leqslant i \leqslant 5$, in the bulk criteria).

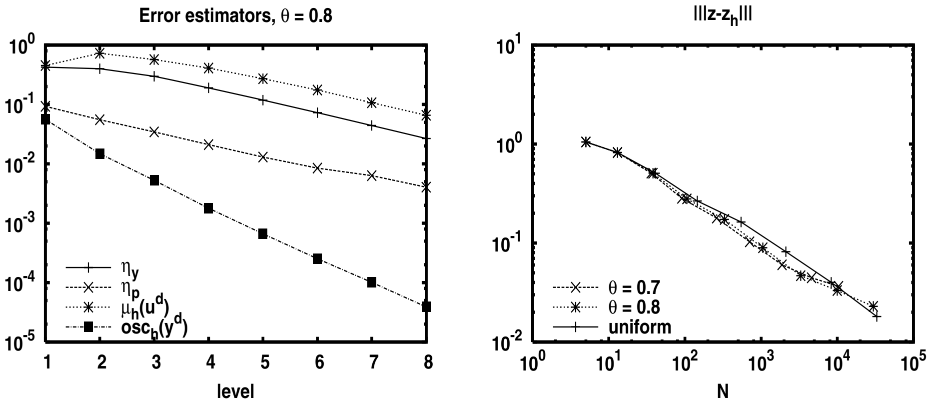

Figure 4. Example 1: Components of the error estimator and data oscillations for $\Theta_i = 0.8$, $1 \leqslant i \leqslant 5$ (left), norm of the total error as a function of the number of grid points for uniform and adaptive refinement (right).

with $\Theta_i = 0.8$ in the bulk criteria. It should be emphasized that we are working with only one grid for both the state and the co-state. Consequently, the grid reflects regions of substantial change in these variables.

Figure 4 (left) displays the actual size of the state and co-state related components of the error estimator as well as the relevant data oscillations as functions of the number of levels in the hierarchy of adaptively generated grids for $\Theta_i = 0.7$ (left) and $\Theta_i = 0.8$ (right). As can be expected, $\mu_h(u^d)$ is dominant due to the highly oscillatory character of u^d.

Tables $1-3$ reflect the history of the refinement process. In particular, Table 1 displays the error reduction in the total error

$$|||z - z_h||| := \left(|y - y_h|^2_{1,\Omega} + |p - p_h|^2_{1,\Omega} + \|u - u_h\|^2_{0,\Gamma_1} + \|\sigma - \sigma_h\|^2_{0,\Gamma_1}\right)^{1/2}$$

and the errors in the state, the co-state, the control, and the co-control, respectively. On the other hand, the values of the components of the residual type *a posteriori* error estimator and of the relevant data oscillations are given in Table 2. Table 3 lists the percentages of elements and edges that have been marked for refinement according to the bulk criteria. Here, $M_{\eta,T}$ and $M_{osc,T}$ stand for the level 1 elements marked for refinement due to the element residuals and the data oscillations, whereas $M_{\eta,E}, M_{\eta,E,\Gamma_1}$ and M_{μ,E,Γ_1} refer to the edges marked for refinement with regard to the interior edge residuals. the edge residuals on Γ_1 and the data oscillations on Γ_1. We note that the sum of the percentages may exceed 100%, since an element or edge may satisfy more than one of the bulk criteria in the adaptive refinement process. We see that the refinement is initially dominated by the data oscillation $\mu_h(u^d)$. whereas at a later stage the element residuals dominate.

Example 2 (highly oscillatory control constraint).

In this example, the upper bound for the control represents a highly oscillatory function on the control boundary. In particular, the data in (2.1a)–(2.1c) have been

Table 1.
Total error, errors in the state, co-state, control, and co-control (Example 1).

l	N_{dof}	$\|\|\|z-z_H\|\|\|$	$\|y-y_H\|_1$	$\|p-p_H\|_1$	$\\|u-u_H\\|_0$	$\\|\sigma-\sigma_H\\|_0$
0	5	1.05e+00	2.80e-01	3.26e-02	7.29e-01	3.38e-03
1	13	8.21e-01	2.66e-01	1.71e-02	5.35e-01	3.14e-03
2	38	5.07e-01	1.58e-01	1.38e-02	3.33e-01	1.74e-03
3	106	2.79e-01	9.75e-02	8.81e-03	1.72e-01	8.45e-04
4	331	1.73e-01	5.59e-02	5.41e-03	1.11e-01	4.27e-04
5	1053	8.97e-02	3.14e-02	3.33e-03	5.46e-02	2.77e-04
6	3311	4.68e-02	1.83e-02	2.05e-03	2.64e-02	1.61e-04
7	9986	3.31e-02	1.07e-02	1.47e-03	2.09e-02	1.10e-04
8	29751	2.29e-02	5.86e-03	8.59e-04	1.61e-02	5.96e-05

Table 2.
Components of the error estimator and data oscillations (Example 1).

l	N_{dof}	η_y	η_p	$\mu_h(u^d)$	$osc_h(y^d)$
0	5	3.14e-02	1.44e-01	7.73e-01	1.42e-01
1	13	4.25e-01	9.28e-02	4.52e-01	5.60e-02
2	38	3.99e-01	5.54e-02	7.26e-01	1.48e-02
3	106	2.98e-01	3.45e-02	5.69e-01	5.28e-03
4	331	1.90e-01	2.11e-02	4.10e-01	1.79e-03
5	1053	1.18e-01	1.30e-02	2.71e-01	6.64e-04
6	3311	7.27e-02	8.43e-03	1.75e-01	2.53e-04
7	9986	4.43e-02	6.33e-03	1.07e-01	1.01e-04
8	29751	2.67e-02	4.05e-03	6.57e-02	3.90e-05

Table 3.
Percentages of elements/edges marked for refinement (Example 1).

l	N_{dof}	$M_{\eta,T}$	$M_{osc,T}$	$M_{\eta,E}$	M_{η,E,Γ_1}	M_{μ,E,Γ_1}
0	5	75.0	75.0	100.0	100.0	100.0
1	13	50.0	50.0	20.0	50.0	100.0
2	38	32.2	44.1	11.1	75.0	75.0
3	106	35.0	34.4	13.4	50.0	62.5
4	331	39.4	21.4	12.5	33.3	53.3
5	1053	44.7	17.6	10.7	36.0	40.0
6	3311	48.6	12.7	9.3	33.3	35.7
7	9986	49.6	8.2	9.6	23.2	33.3
8	29751	43.4	5.7	10.0	21.3	38.0

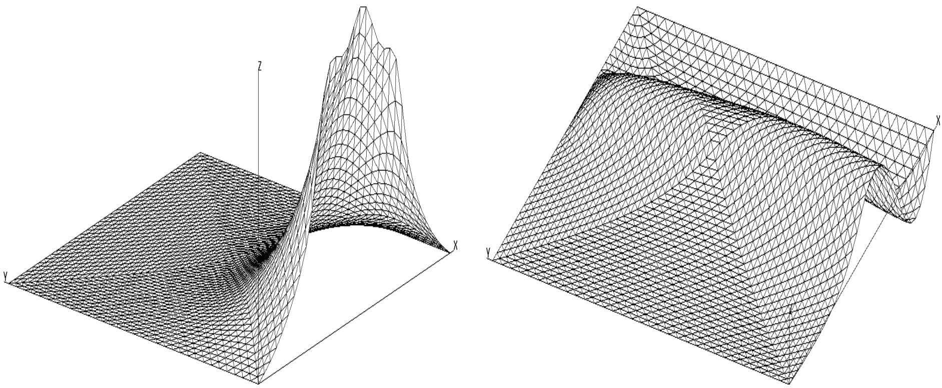

Figure 5. Example 2: Visualization of the optimal state y (left) and of the optimal co-state p (right).

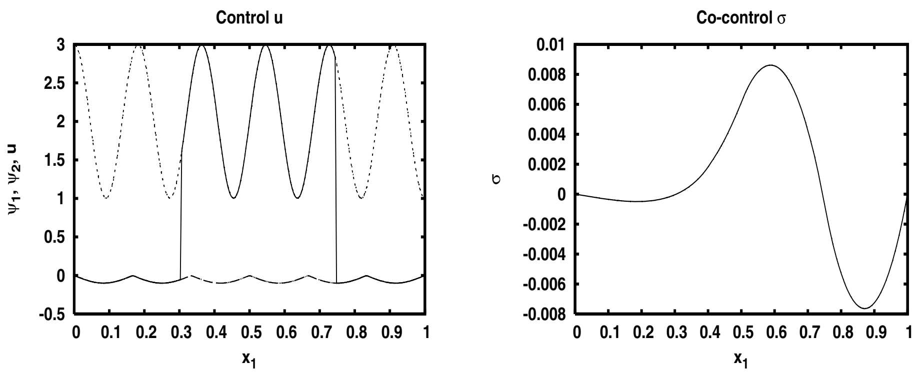

Figure 6. Example 2: Visualization of the optimal control u (left) and of the optimal co-control σ (right). The lower and upper bounds on the control are shown as 'dashed' and 'dotted' lines, respectively.

chosen as follows:

$$\begin{aligned}
\Omega &= (0,1)^2, \quad \Gamma_1 \;=\; (0,1)\times\{0\}, \quad c = 10x_1 + 100x_2 \\
y^d &= \begin{cases} 0, & x_1 \leqslant 0.5 \\ 1, & 0.5 < x_1 \leqslant 0.75, \\ -1, & 0.75 < x_1 \end{cases} \qquad u^d = 0, \quad \alpha = 10^{-5} \\
f &:= 0, \quad \psi^{(1)} = -0.1|\sin(6\pi x_1)|, \quad \psi^{(2)} = 2 + \cos(11\pi x_1).
\end{aligned}$$

Figures 5 and 6 display the optimal state y, the optimal co-state p, the optimal control u, and the optimal co-control σ, respectively. Again, we have a 'bang-bang' type of optimal control which switches from the lower to the upper bound and back again to the lower bound at two points of Γ_1 so that the inactive set at optimality is of measure zero.

The initial simplicial triangulation $\mathcal{T}_{h_0}$ and the parameters Θ_i in the bulk criterion have been chosen as in Example 1. Figure 7 shows the adaptively generated triangulations after four (left) and eight (right) refinement steps.

Table 4.
Total error, errors in the state, co-state, control, and co-control (Example 2).

l	N_{dof}	$\vert\vert\vert z - z_H \vert\vert\vert$	$\vert y - y_H\vert_1$	$\vert p - p_H\vert_1$	$\Vert u - u_H\Vert_0$	$\Vert\sigma - \sigma_H\Vert_0$
0	5	2.02e+00	5.10e-01	3.88e-02	1.47e+00	4.62e-03
1	13	2.08e+00	3.62e-01	3.84e-02	1.67e+00	4.94e-03
2	38	1.04e+00	2.46e-01	2.39e-02	7.61e-01	3.65e-03
3	89	8.66e-01	1.58e-01	1.76e-02	6.89e-01	1.83e-03
4	254	5.15e-01	1.07e-01	1.08e-02	3.97e-01	8.97e-04
5	748	3.42e-01	5.79e-02	7.04e-03	2.76e-01	4.58e-04
6	2368	2.78e-01	3.44e-02	4.51e-03	2.39e-01	3.11e-04
7	7443	2.18e-01	2.11e-02	2.65e-03	1.94e-01	1.77e-04
8	21339	4.00e-02	7.22e-03	1.55e-03	3.11e-02	1.02e-04

Table 5.
Components of the error estimator and data oscillations (Example 2).

l	N_{dof}	η_y	η_p	$\mu_h(\psi^{(1)})$	$\mu_h(\psi^{(2)})$	$osc_h(y^d)$
0	5	0.00e+00	3.30e-01	3.69e-02	1.72e-01	2.71e-01
1	13	1.50e+00	1.69e-01	2.73e-02	6.39e-01	1.05e-01
2	36	6.75e-01	9.85e-02	3.19e-02	6.82e-01	3.56e-02
3	89	3.93e-01	7.38e-02	3.00e-02	6.91e-01	1.61e-02
4	254	2.53e-01	4.68e-02	3.01e-02	6.57e-01	7.00e-03
5	748	1.77e-01	2.99e-02	2.18e-02	4.53e-01	2.90e-03
6	2368	1.10e-01	1.93e-02	1.59e-02	2.69e-01	1.30e-03
7	7443	6.64e-02	1.21e-02	9.96e-03	1.91e-01	6.35e-04
8	21339	4.01e-02	7.66e-03	6.90e-03	1.19e-01	2.33e-04

Table 6.
Percentages of elements/edges marked for refinement (Example 2).

l	N_{dof}	$M_{\eta,T}$	$M_{osc,T}$	$M_{\eta,E}$	M_{η,E,Γ_1}	M_{μ,E,Γ_1}
0	5	50.0	75.0	100.0	100.0	100.0
1	13	43.8	25.0	20.0	100.0	100.0
2	36	35.7	10.7	15.6	75.0	75.0
3	89	44.2	5.8	17.7	50.0	75.0
4	254	50.7	2.6	17.7	26.7	73.3
5	748	56.6	1.3	14.5	24.1	51.7
6	2368	54.7	0.8	13.0	24.5	49.0
7	7443	47.8	0.4	12.6	19.0	36.7
8	21339	40.4	0.3	14.5	21.0	50.4

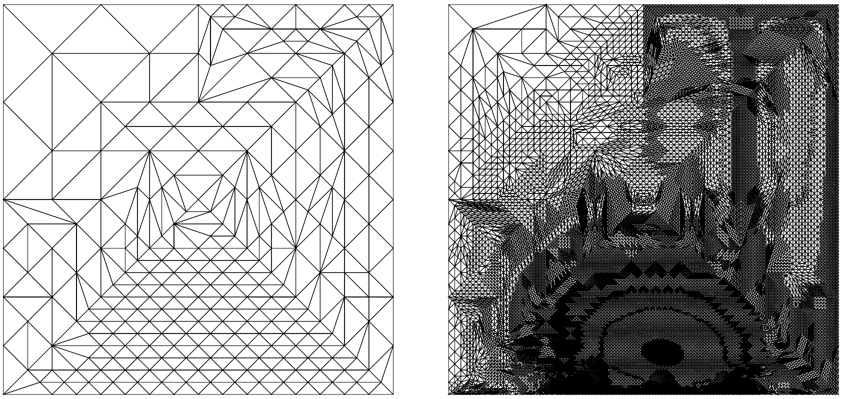

Figure 7. Example 2: Adaptively generated grid after 4 (left) and 8 (right) refinement steps ($\Theta_i = 0.8$, $1 \leqslant i \leqslant 5$, in the bulk criteria).

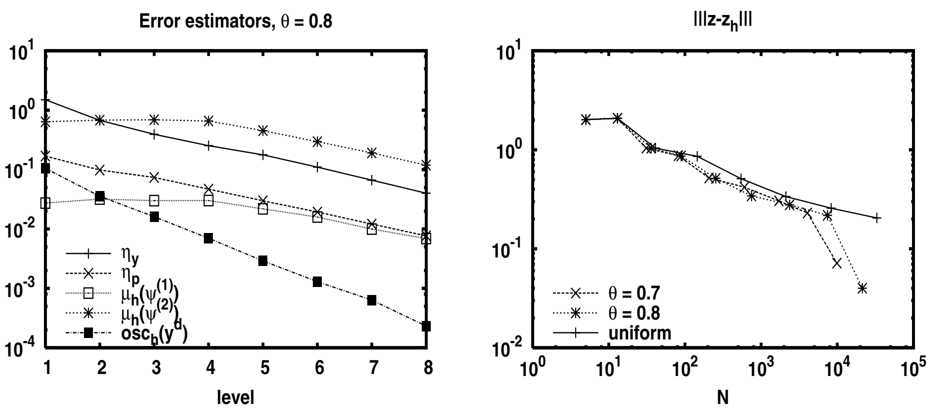

Figure 8. Components of the error estimator and data oscillations for $\Theta_i = 0.8$, $1 \leqslant i \leqslant 5$ (left), adaptive versus uniform refinement (right), Example 2.

Figure 8 (left) shows the components of the error estimator and the relevant data oscillations as functions of the number of levels of the grid hierarchy ($\Theta_i = 0.8$, $1 \leqslant i \leqslant 5$, in the bulk criteria). Furthermore, Fig. 8 (right) displays the total discretization error in the state, co-state, control, and co-control as a function of the total number of degrees of freedom. The benefits of adaptive versus uniform refinement set in, once the highly oscillatory upper bound $\psi^{(2)}$ has been sufficiently resolved. This is also reflected by the results in Tables 4 – 6 which basically give the same information as Tables 1 – 3 for Example 1. The refinement process is at the very beginning clearly dominated by the resolution of $\psi^{(2)}$.

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