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A probabilistic synopsis of binary decision rules

Olga Ruff · Friedrich Pukelsheim

Abstract Decision rules for Yes–No voting systems are placed in a probabilistic framework. Selfdual and permutationally invariant distributions are introduced. Under such distributions, the mean success margin of the majority rule and of the unanimity rule are shown to bound the mean success margin of all other decision rules. For bloc decision rules in the Penrose/Banzhaf model, a product formula for the voters' influence probabilities is derived. Other indices and the Shapley/Shubik model are also discussed.

1 Introduction

Decision rules and the measurement of voting power originates from, and usually is oriented toward, game theory. Felsenthal and Machover (1998) present a detailed overview of the subject, including a critical assessment of concepts and methods. Von Neumann and Morgenstern (1944) laid the foundation for the game-theoretic approach, and Shapley and Shubik (1954), Shapley (1962), Coleman (1971), Owen (1971), and Dubey (1975) followed their lead. An alternative approach based on probability models was put forward by Straffin (1977a,b, 1988) for its appeal of modelling the voters' behavior. This article focusses on the probabilistic view, as does Straffin, but more for its analytical potential and the decision-theoretic outlook.

With our statistical background we occasionally feel some irritation that, when authors make probabilistic statements, it remains unclear to which probability space they refer to. Is it the set of voters? Or the permutations into which the voters may be aligned? Is it the space of voting profiles? Is it the set of all agenda proposals to be treated during the voting process? Our article grew out of an attempt to extract just

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one reference space and see how much of the current theory can be developed within the space chosen.

Our reference space Ω_N consists of vectors a, called voting profiles. The components may take two values, $a_j = 1$ or $a_j = -1$, for every voter j in a finite assembly N. A voting profile a records whether participant j votes Yea ($a_j = 1$) or Nay ($a_j = -1$). This space of all voting profiles figures prominently also in the Felsenthal and Machover (1998) monograph, and in current research literature such as Laruelle and Valenciano (2004, 2005).

Section 2 introduces a decision rule W_N as a set of voting profiles that forms a monotone, nonempty, and proper subset of the space Ω_N. Important events are $C_j(W_N)$, consisting of the voting profiles where voter j may exert critical decisiveness. An important function is the success margin $\sigma_{W_N}(a)$, the difference between the number of voters for whom the voting profile a is a success and the number of those for whom it is a failure.

Section 3 turns to general probability assumptions. Two properties become vital, selfduality of a distribution, and permutational invariance. For such distributions, Theorem 2 proves that the mean success margin of a decision rule W_N is bounded from above by the mean success margin of the straight majority rule, while it is bounded from below by the mean success margin of the unanimity rule. The theorem shows that the mean success margin has extreme bounds depending on general probabilistic properties, rather than being owed to the structure of a particular model.

Section 4 turns to the Penrose/Banzhaf model, due to Penrose (1946) and Banzhaf (1965). In this model it is easy to see that the sensitivity of a decision rule coincides with its mean success margin (Theorem 3). In Sect. 5, we overview other power indices by identifying them as conditional probabilities or conditional expectations in the Penrose/Banzhaf model.

In Sect. 6, we investigate bloc decision rules. Theorem 4 derives a product formula for the influence probability of voter j, in the presence of a prespecified partitioning into blocs. The formula splits into two factors, the impact of voter j in his or her bloc B, and the impact of bloc B relative to the other blocs of the partitioning. This generalizes and compactifies a result due to Felsenthal and Machover (2002). Partitionings of the assembly N into blocs are also employed by Straffin (1978), Laruelle and Valenciano (2004) and, for the investigation of list apportionments in proportional representation systems, by Leutgäb and Pukelsheim (2009).

Section 7 merges the Shapley/Shubik indices into the present approach. As pointed out by Straffin (1977a) and Dubey and Shapley (1979), the Shapley/Shubik model may be based on a two-stage usage of uniform distributions. Section 8 concludes the article with some final remarks.

2 Voting profiles and decision rules

Let N be an *assembly*, a finite set, of *n voters*. A *voting profile* is a vector $a = (a_j)_{j\in N}$ with binary components, $a_j := 1$ in case voter $j \in N$ is a *Yea-voter*, or $a_j := -1$ in case j is a *Nay-voter*. Altogether the voting profiles form the *space of voting profiles*

$$\Omega_N := \{-1, 1\}^N.$$

Let $1_N := (1, \ldots, 1)$ denote the *unity vector* in the set Ω_N. For a given voting profile $a \in \Omega_N$ the component-wise partial ordering $\leq$ of vectors induces the interval region $[a, 1_N] := \{b \in \Omega_N : a \leq b \leq 1_N\}$. A binary *decision rule* (also known as a simple voting game) is a subset $W_N \subseteq \Omega_N$ enjoying three properties:

(1) $[a, 1_N] \subseteq W_N$ for all $a \in W_N$,
(2) $1_N \in W_N$,
(3) $-1_N \notin W_N$.

The monotonicity property (1) is central: if a voting profile a is in W_N and b is above a in the component-wise partial ordering, $b \geq a$, then b is also in W_N. Given (1), properties (2) and (3) simply mean that the decision rule W_N, as a subset of voting profiles, is nonempty and proper, $\emptyset \neq W_N \neq \Omega_N$. The voting profiles in W_N are termed *positive*, for the reason that they are taken to represent the positive outcomes of the voting procedure; they are also known as winning coalitions, or winning configurations. The voting profiles in the complement $W_N^c := \Omega_N \setminus W_N$ are called *negative*.

A family of decision rules that is of particular relevance for practical committee work is formed by weighted decision rules. Let $w = (w_j)_{j \in N} \in (0, \infty)^N$ be a vector of *voting weights*. The sum of all components of w is denoted by $w_+ := \sum_{j \in N} w_j$. For a given voting profile a the sum of the voting weights of its Yea-voters is $\sum_{j:\, a_j = 1} w_j$, and defines the *profile weight* of a.

By definition, the *weighted decision rule* $W_N(q; w)$ contains the voting profiles for which the weight exceeds $q w_+$, for some pre-specified (*relative*) *quota* $q \in [0, 1)$:

$$W_N(q; w) := \left\{ a \in \Omega_N : \sum_{j : a_j = 1} w_j > q w_+ \right\}.$$

Equivalently we might refer to the *absolute quota* $Q := q w_+$. However, for the discussion of ternary decision rules in Käufl et al. (2010) we find the relative quota q the more natural quantity to work with, whence we prefer q over Q.

In the symmetric case all voters possess the same voting weight, turning the weight vector into $w = \lambda 1_N$ for some $\lambda > 0$. The most prominent examples are the *unanimity rule* U_N and the *straight majority rule* M_N:

$$\begin{aligned} U_N &:= W_N(1 - 1/n; 1_N) = \{1_N\}, \\ M_N &:= W_N(1/2; 1_N) = \{a \in \Omega_N : a_+ > 0\}. \end{aligned}$$

Similarly to the weight total w_+, the component sum of a voting profile a is denoted by $a_+ := \sum_{j \in N} a_j$. If positive, it is the margin by which the Yea-voters outnumber the Nay-voters. If zero, there is a tie. If negative, it indicates more Nay-voters than Yea-voters. In general, the number of Yea-voters is given by $(1/2)(1_N + a)_+ = (n + a_+)/2$, while the number of Nay-voters is obtained from $(1/2)(1_N - a)_+ = (n - a_+)/2$. This yields the obvious identity $a_+ = (n + a_+)/2 - (n - a_+)/2$, which is just another manifestation of a_+ signifying the margin between Yea- and Nay-voters.

Let $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ denote the j-th Euclidean unit vector. The voting profiles in which the vote of j becomes *critical* (decisive) are assembled in the event

$$C_j(W_N) := \left\{a \in \Omega_N\colon \left(a \in W_N^c, a + 2e_j \in W_N\right) \quad \text{or} \quad \left(a \in W_N, a - 2e_j \in W_N^c\right)\right\}.$$

That is, a voter is either *entry-critical* (critical outside a voting profile), when leaving the Nay-voters and joining the Yea-voters turns a negative voting profile into the positive. Or the voter is *exit-critical* (critical in a voting profile), when switching from the Yea-voters to the Nay-voters turns a positive voting profile into the negative.

The critical event $C_j(W_N)$ is understood better by concentrating on the set of competitors of voter j, i.e., $N \setminus \{j\}$. To this end let $\Pi_{N\setminus\{j\}}$ be the projection of the space $\Omega_N = \{-1, 1\}^N$ onto $\Omega_{N\setminus\{j\}} = \{-1, 1\}^{N\setminus\{j\}}$, the $(n-1)$-dimensional marginal space omitting voter j. This is one instance—out of more to follow—where it becomes instrumental to use sets as subscripts, such as the assembly N or a subset $N \setminus \{j\}$, rather than their cardinalities.

Given a marginal voting profile $b \in \Omega_{N\setminus\{j\}}$ without voter j, we denote by $(b; 1)$ the full voting profile when j concurs with a Yea, and by $(b; -1)$ when j joins in with a Nay. The set of voting profiles where the vote of j is critical may then be rewritten as:

$$C_j(W_N) = \left\{a \in \Omega_N\colon \left(\Pi_{N\setminus\{j\}}(a); 1\right) \in W_N, \quad \left(\Pi_{N\setminus\{j\}}(a); -1\right) \in W_N^c\right\}.$$

Let $D_j(W_N) := \Pi_{N\setminus\{j\}}\left(C_j(W_N)\right)$ denote the image under the projection $\Pi_{N\setminus\{j\}}$ of the critical event $C_j(W_N)$.

Theorem 1 *Let W_N be a decision rule for an assembly N. Then $C_j(W_N)$ is the pre-image of $D_j(W_N)$:*

$$C_j(W_N) = \Pi^{-1}_{N\setminus\{j\}}\left(D_j(W_N)\right).$$

Proof A vector $a \in \Omega_N$ is mapped to the image $b := \Pi_{N\setminus\{j\}}(a) \in \Omega_{N\setminus\{j\}}$, with components $b_i = a_i$ for all $i \neq j$. The vector $b \in \Omega_{N\setminus\{j\}}$ has two pre-images, $(b; -1)$ and $(b; 1)$. Hence, we obtain $D_j(W_N) = \left\{b \in \Omega_{N\setminus\{j\}}\colon (b; -1) \in W_N^c, (b; 1) \in W_N\right\}$. Evidently the pre-image of $D_j(W_N)$ reproduces the event $C_j(W_N)$. □

A voting profile $a \in \Omega_N$ is said to be a *success* for voter j provided it is positive and j is a Yea-voter ($a \in W_N$, $a_j = 1$), or it is negative and j is a Nay-voter ($a \in W_N^c$, $a_j = -1$). A positive voting profile is taken to be a *failure* for a Nay-voter, as is a negative voting profile for a Yea-voter. The notion of success is emphasized by Laruelle and Valenciano (2005) as a property capturing an aspect somewhat complementary to criticality.

The difference between the number of voters for which a voting profile $a \in \Omega_N$ is a success, and the number of the voters for which it appears to be a failure, defines the *success margin* $\sigma_{W_N}(a)$ of the voting profile a in the decision rule W_N:

$$\sigma_{W_N}(a) := \begin{cases} a_+ & \text{in case } a \in W_N, \\ -a_+ & \text{in case } a \in W_N^c. \end{cases}$$

A particular emphasis is placed on those positive voting profiles $a \in W_N$ appearing to be a failure to a majority of voters. For such voting profiles the success margin is negative, whence its negative part represents the *majority deficit*, $\delta_{W_N}(a) := \sigma^-_{W_N}(a) = \left(|\sigma_{W_N}(a)| - \sigma_{W_N}(a)\right)/2$.

We now turn to evaluating the events of interest by means of appropriate probability distributions.

3 Selfdual and permutationally invariant distributions

The aim is to equip the space of voting profiles Ω_N with probability distributions P permitting a meaningful a priori analysis of decision rules W_N.

The share of all positive voting profiles $P(W_N)$ is called the *P-efficiency* and serves as an indicator for the decision-making ability of the decision rule W_N under P. The *influence probability* of voter j in the decision rule W_N (also known as swing probability) is defined to be the probability of j being critical, $P\left(C_j(W_N)\right)$. The sum of all influence probabilities, $\Sigma_P(W_N) := \sum_{j \in N} P\left(C_j(W_N)\right)$, is termed the *P-sensitivity* of the decision rule W_N.

The critical events $C_j(W_N)$, $j \in N$, generally neither cover the space of voting profiles Ω_N, nor are pairwise disjoint. Hence, there is no reason for the P-sensitivity to be equal to unity. However, the P-sensitivity can be used to normalize the influence probabilities into $P\left(C_j(W_N)\right)/\Sigma_P(W_N)$, the *power share* of voter j under P. The power shares form a probability distribution on the set of voters N, preserving for any two voters $i \neq j$ the ratio of their influence probabilities, $P\left(C_i(W_N)\right)/P\left(C_j(W_N)\right)$.

Two structural properties of P become essential. A distribution P is said to be *selfdual* when $P(\{a\}) = P(\{-a\})$ holds for all voting profiles $a \in \Omega_N$. Selfduality means that the probability of a positive profile a for a proposal is just the same as the probability for the dual voting profile $-a$ for the negation of that proposal.

A distribution P is called *permutationally invariant* when $P \circ \pi^{-1} = P$ holds for all permutations $\pi\colon N \to N$. To see the effect of permutational invariance, we write the space Ω_N as the disjoint union of the sets of voting profiles with a fixed number k of Yea-voters:

$$\Omega_N = \biguplus_{k=0}^{n} \left\{ {N \atop k} \right\}, \quad \left\{ {N \atop k} \right\} := \{a \in \Omega_N\colon (n + a_+)/2 = k\}.$$

The subset $\left\{ {N \atop k} \right\}$ has cardinality $\binom{n}{k}$. Within such a subset, a permutationally invariant distribution behaves like a uniform distribution:

$$P(\{a\}) = \frac{1}{\binom{n}{k}} P\left(\left\{ {N \atop k} \right\}\right) \quad \text{for all} \quad a \in \left\{ {N \atop k} \right\}.$$

Theorem 2 *Let W_N be a decision rule for an assembly N.*

(i) *The success margin σ_{W_N} and the majority deficit δ_{W_N} satisfy $\sigma_{W_N} = \sigma_{M_N} - 2\delta_{W_N} \leq \sigma_{M_N}$, where σ_{M_N} is the success margin of the straight majority rule M_N. In particular, every distribution P fulfills*

$$\mathrm{E}_P[\sigma_{W_N}] \leq \mathrm{E}_P[\sigma_{M_N}].$$

(ii) *Every selfdual distribution P fulfills*

$$\mathrm{E}_P[\sigma_{W_N}] = 2 \sum_{a \in W_N} a_+ P(\{a\}) = 2 \sum_{k=1}^{n} (2k-n) P\left(W_N \cap \left\{ {N \atop k} \right\}\right).$$

(iii) *Every selfdual and permutationally invariant distribution P fulfills*

$$\mathrm{E}_P[\sigma_{W_N}] \geq \mathrm{E}_P[\sigma_{U_N}] = 2nP(\{1_N\}).$$

Proof

(i) The absolute value of any success margin is equal to the success margin of the straight majority rule, since $|\sigma_{W_N}(a)| = |a_+| = \sigma_{M_N}(a)$. The assertions follow from $\delta_{W_N} = (\sigma_{M_N} - \sigma_{W_N})/2$, and $\sigma_{W_N} \leq |\sigma_{W_N}| = \sigma_{M_N}$.

(ii) For $a \in \Omega_N$, we define the indicator function

$$\mathbb{1}\{a \in W_N\} = \begin{cases} 1 \text{ in case } a \in W_N, \\ 0 \text{ in case } a \in W_N^c. \end{cases}$$

Thus, the success margin turns into $\sigma_{W_N}(a) = (2 \cdot \mathbb{1}\{a \in W_N\} - 1)a_+$. It follows that $\mathrm{E}_P[\sigma_{W_N}] = 2\sum_{a \in W_N} a_+ P(\{a\}) - \sum_{a \in \Omega_N} a_+ P(\{a\})$. The last sum vanishes, due to $a_+ = -(-a_+)$ and the selfduality of P:

$$2 \sum_{a \in \Omega_N} a_+ P(\{a\}) = \sum_{a \in \Omega_N} a_+ P(\{a\}) - \sum_{a \in \Omega_N} (-a)_+ P(\{-a\}) = 0.$$

In the assertion, the second equality is a rearrangement according to the number of Yea-voters, $k = (n + a_+)/2$. Since $-1_N \notin W_N$, we have $k \geq 1$. The last sum has at most n terms, while the penultimate sum may have up to $2^n - 1$ terms.

(iii) Since $1_N \in W_N$, part (ii) yields

$$\mathrm{E}_P[\sigma_{W_N}] = 2nP(\{1_N\}) + 2 \sum_{k=1}^{n-1} (2k-n) P\left(W_N \cap \left\{ {N \atop k} \right\}\right).$$

It remains to show that the sum is nonnegative. If n is even, its term for $k = n/2$ vanishes. Therefore, we may quite generally subdivide the range of summation into two regions of equal cardinality, $1 \leq k < n/2$ and $n/2 < k \leq n-1$. Applying permutational invariance and selfduality, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-1}(2k-n)P\left(W_N \cap \begin{Bmatrix} N \\ k \end{Bmatrix}\right) \\
&\quad = \sum_{1 \leq k < n/2} (n-2k)\left(P\left(W_N \cap \begin{Bmatrix} N \\ n-k \end{Bmatrix}\right) - P\left(W_N \cap \begin{Bmatrix} N \\ k \end{Bmatrix}\right)\right) \\
&\quad = \sum_{1 \leq k < n/2} \frac{n-2k}{\binom{n}{k}}\, P\left(\begin{Bmatrix} N \\ k \end{Bmatrix}\right)\left(\#\left(W_N \cap \begin{Bmatrix} N \\ n-k \end{Bmatrix}\right) - \#\left(W_N \cap \begin{Bmatrix} N \\ k \end{Bmatrix}\right)\right).
\end{aligned}$$

Since a decision rule W_N is monotone, by its defining property (1), any set of positive voting profiles with many Yea-voters $(n-k)$ outnumbers any set of positive voting profiles with only a few Yea-voters (k), for all $1 \leq k < n/2$. Hence the final sum is nonnegative. □

4 The Penrose/Banzhaf model

The simplest distributional model is the *Penrose/Banzhaf distribution*, P_N, the uniform distribution on the space Ω_N:

$$P_N(\{a\}) := \frac{1}{\#\Omega_N} = \frac{1}{2^n} \quad \text{for all} \quad a \in \Omega_N.$$

The Penrose/Banzhaf distribution makes the voting behavior of all voters $j \in N$ stochastically independent, $P_N = \bigotimes_{j \in N} P_{\{j\}}$, with identical marginals $P_{\{j\}} =$ Bernoulli$(1/2)$. The distribution P_N is selfdual and permutationally invariant. The Penrose/Banzhaf efficiency (also known as Coleman's *power of a collectivity to act*) is $P_N(W_N) = \omega/2^n$, where $\omega := \#W_N$ is the number of positive voting profiles.

For a given decision rule W_N, the *Penrose/Banzhaf influence probability of the voter* $j \in N$ evaluates to

$$P_N\left(C_j(W_N)\right) = P_N \circ \Pi^{-1}_{N \setminus \{j\}}\left(D_j(W_N)\right) = P_{N\setminus\{j\}}\left(D_j(W_N)\right) = \frac{\eta_j}{2^{n-1}},$$

where $\eta_j := \#D_j(W_N)$ denotes the *swing score* (also known as Banzhaf score) of voter j. The following result is well-known (see Felsenthal and Machover 1998, Theorem 3.3.5).

Theorem 3 *For every decision rule W_N its Penrose/Banzhaf sensitivity coincides with its Penrose/Banzhaf mean success margin:*

$$\Sigma_{P_N}(W_N) = \frac{1}{2^{n-1}} \sum_{a \in W_N} a_+ = \mathrm{E}_{P_N}[\sigma_{W_N}].$$

Proof By definition, $\Sigma_{P_N}(W_N) = \eta_+/2^{n-1}$. We set $\omega_j := \#\{a \in W_N : a_j = 1\}$. If voter j changes camps, then the positive voting profiles for which j is critical become negative and drop out: $\#\{a \in W_N \colon a_j = -1\} = \omega_j - \eta_j$. We obtain $\omega = 2\omega_j - \eta_j$, i.e., $\eta_j = 2\omega_j - \omega$. Substituting $\omega_j = \sum_{a \in W_N} \mathbb{1}\{a_j = 1\}$ and summing over j, we get $\eta_+ = 2\left(\sum_{a \in W_N} \sum_{j \in N} \mathbb{1}\{a_j = 1\}\right) - n\omega = 2\left(\sum_{a \in W_N} (n + a_+)/2\right) - n\omega = \sum_{a \in W_N} a_+$. This proves the first equality. The second equation follows from Theorem 2. $\square$

Theorems 2 and 3 imply that the unanimity rule U_N and the straight majority rule M_N provide bounds for the Penrose/Banzhaf sensitivity of any decision rule W_N:

$$\frac{n}{2^{n-1}} = \Sigma_{P_N}(U_N) \ \leq\ \Sigma_{P_N}(W_N) \ \leq\ \Sigma_{P_N}(M_N) = \frac{n}{2^{n-1}} \binom{n-1}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2n}{\pi}}.$$

5 Conditional power indices in the Penrose/Banzhaf model

Other power indices emerge from the Penrose/Banzhaf model as conditional probabilities, or as conditional expectations. For an appraisal of these indices see Felsenthal and Machover (1998).

5.1 Conditioning on the set of positive voting profiles

A first group of power indices conditions on the decision rule W_N itself, thereby emphasizing its interpretation as the set of all positive voting profiles.

The Penrose/Banzhaf probability of the critical event $C_j(W_N)$ given the positive voting profiles W_N is better known as *Coleman's power to prevent action*:

$$\begin{aligned} \mathrm{E}_{P_N}\left[\mathbb{1}\left\{a \in C_j(W_N)\right\} \mid W_N\right] = P_N\left(C_j(W_N) \mid W_N\right) &= \frac{P_N\left(C_j(W_N) \cap W_N\right)}{P_N(W_N)} \\ &= \frac{\eta_j/2^n}{\omega/2^n} = \frac{\eta_j}{\omega}. \end{aligned}$$

The penultimate equality uses the fact that the number of positive voting profiles for which voter j is exit-critical amounts to $\#\left(C_j(W_N) \cap W_N\right) = \#\left(C_j(W_N) \cap W_N^c\right) = (1/2)\,\#C_j\,(W_N) = \eta_j$.

Coleman's power to initiate action is given by $P_N\left(C_j(W_N) \mid W_N^c\right) = \eta_j/(2^n - \omega)$. Here, the conditioning event is the set of negative voting profiles W_N^c, and η_j represents the number of negative voting profiles where voter j is entry-critical. The harmonic mean of the two Coleman indices reproduces the Penrose/Banzhaf influence probability $P_N\left(C_j(W_N)\right)$. This reproduction property applies in every probability space (Ω_N, P) where the relation $P\left(C_j(W_N) \cap W_N\right) = P\left(C_j(W_N) \cap W_N^c\right)$ holds true.

Generally the two Coleman indices do not sum to unity. In either case normalization reproduces the Penrose/Banzhaf influence probabilities.

An alternative idea is that in case of an increasing number of critical Yea-voters they should be assigned decreasing pay-offs. This reasoning originates from a game-theoretic approach, the winning subset of Yea-voters having to share a fixed prize. For a voting profile $a \in \Omega_N$ we define the vector $\gamma(a)$ to indicate whether voter j is exit-critical ($\gamma_j(a) := 1$) or not ($\gamma_j(a) := 0$). Hence, the component sum $\gamma_+(a) := \sum_{j\in N} \gamma_j(a)$ indicates the number of exit-critical Yea-voters. The *Burgin/Shapley index* is defined as

$$\begin{aligned}\mathrm{E}_{P_N}\left[\frac{1}{\gamma_+(a)}\mathbb{1}\left\{a \in C_j(W_N)\right\} \mid W_N\right] &= \sum_{a\in C_j(W_N)} \frac{1}{\gamma_+(a)} P_N\left(\{a\} \mid W_N\right) \\ &= \sum_{a\in C_j(W_N)\cap W_N} \frac{1}{\gamma_+(a)}\,\frac{1/2^n}{\omega/2^n} = \frac{1}{\omega}\sum_{a\in C_j(W_N)\cap W_N}\frac{1}{\gamma_+(a)}.\end{aligned}$$

The normalized versions of the Burgin/Shapley indices are called *Johnston indices*.

5.2 Conditioning on the set of minimal-positive voting profiles

A second group of indices arises when the conditioning event is taken to be the set of *minimal-positive voting profiles*, i.e., voting profiles wherein every Yea-voter is exit-critical:

$$W_N^{\min} := \left\{a \in W_N\colon a - 2e_j \in W_N^c \quad \text{for all} \quad j \in N \text{ with } a_j = 1\right\}.$$

The interval regions that are induced by the minimal-positive voting profiles characterize the decision rule: $W_N = \bigcup_{a\in W_N^{\min}}[a, 1_N]$. Kirsch and Langner (2009) make do with minimal-positive voting profiles to calculate influence probabilities.

The indices corresponding to Coleman's power to prevent action are

$$\mathrm{E}_{P_N}\left[\mathbb{1}\left\{a \in C_j(W_N)\right\} \,\middle|\, W_N^{\min}\right] = \frac{\#\left(C_j(W_N)\cap W_N^{\min}\right)}{\#W_N^{\min}}.$$

Normalization yields the *Holler/Packel public good indices*.

In minimal-positive voting profiles every Yea-voter is exit-critical, $\gamma_+(a) = (n + a_+)/2$. The indices that run parallel to the Burgin/Shapley indices are the *Deegan/Packel indices*

$$\mathrm{E}_{P_N}\left[\frac{1}{\gamma_+(a)}\mathbb{1}\left\{a \in C_j(W_N)\right\} \,\middle|\, W_N^{\min}\right] = \frac{1}{\#W_N^{\min}} \sum_{a\in C_j(W_N)\cap W_N^{\min}} \frac{1}{\gamma_+(a)}.$$

Their total happens to be equal to unity, since for all $a\in W_N^{\min}$ we get $\sum_{j\in N} \mathbb{1}\left\{a\in C_j(W_N)\right\} = \gamma_+(a)$, and $\sum_{j\in N} \mathrm{E}_{P_N}\left[\frac{1}{\gamma_+(a)}\mathbb{1}\left\{a\in C_j(W_N)\right\} \,\middle|\, W_N^{\min}\right] = 1$.

Although the Penrose/Banzhaf uniform distribution is the most prominent model, bloc decision rules give rise to other interesting distributions.

6 Bloc decision rules

We assume an assembly N and its set of voting profiles Ω_N to be given, together with some decision rule W_N. A *partitioning* $\mathcal{P}$ of the assembly N is a decomposition of N into pairwise disjoint subsets. Its subsets $B \in \mathcal{P}$ are called *blocs*.

The smallest partitioning is $\{N\}$, embracing just the single bloc N. The largest partitioning is $\{\{j\}: j \in N\}$, featuring only *trivial*, i.e., one-element, blocs $\{j\}$. These two configurations are extreme and only of theoretical interest. Practical examples use partitionings $\mathcal{P}$ consisting of more than one and less than n blocs. In Exhibit 1, we partition the former EEC into four blocs, $\mathcal{P} = \{\{\text{DE}\}, \{\text{IT}\}, \{\text{FR}\}, \{\text{NL, BE, LU}\}\}$. The big Member States stay alone, while the Benelux states join into a bloc of three.

The assembly N is a disjoint union of the blocs $B \in \mathcal{P}$, and its voting profile space is a Cartesian product of the profile spaces of the blocs:

$$N = \biguplus_{B \in \mathcal{P}} B, \quad \Omega_N = \prod_{B \in \mathcal{P}} \Omega_B.$$

Now a voting profile in Ω_N is a block vector $a = (a_B)_{B \in \mathcal{P}}$, with components $a_B := (a_j)_{j \in B}$.

Given a bloc $B \in \mathcal{P}$, we consider B as an assembly in its own right, with associated space of voting profiles Ω_B. We assume that every bloc is given an *internal* decision rule W_B. The final decision, in the grand assembly N, is preceded by internal bloc decisions. If in bloc B the internal voting profile a_B is positive, then all members of the bloc vote Yea in the final voting profile. If a_B is negative, all of them vote Nay. Therefore, the contribution of bloc B to the final voting profile is given by $(2 \cdot \mathbb{1}\{a_B \in W_B\} - 1)1_B$, namely 1_B in case $a_B \in W_B$, and -1_B otherwise. This leads to the definition of the *bloc decision rule*,

$$W_N|(W_B)_{B \in \mathcal{P}} := \left\{(a_B)_{B \in \mathcal{P}} \in \Omega_N\colon\ ((2 \cdot \mathbb{1}\{a_B \in W_B\} - 1)1_B)_{B \in \mathcal{P}} \in W_N\right\} \subseteq \Omega_N.$$

Theorem 4 treats the partitioning $\mathcal{P}$ as yet another assembly, as in Straffin (1978), Felsenthal and Machover (2002), and Laruelle and Valenciano (2004). Its space of voting profiles $\Omega_{\mathcal{P}}$ is equipped with a decision rule $W_{\mathcal{P}}$ that is induced by the decision rule W_N.

Theorem 4 *Let $\mathcal{P}$ be a partitioning of the assembly N. With decision rule W_N for N, and internal decision rules W_B for the blocs $B \in \mathcal{P}$, we introduce*

$$W_{\mathcal{P}} := \{c \in \Omega_{\mathcal{P}}\colon (c_B 1_B)_{B \in \mathcal{P}} \in W_N\}, \quad Q_{\mathcal{P}} := \bigotimes_{B \in \mathcal{P}} \text{Bernoulli}(P_B(W_B)).$$

Then we have, for every bloc $A \in \mathcal{P}$ and for all voters $j \in A$:

$$P_N\left(C_j\left(W_N|(W_B)_{B\in\mathcal{P}}\right)\right) = P_A\left(C_j(W_A)\right)\; Q_{\mathcal{P}}\big(C_A(W_{\mathcal{P}})\big).$$

Proof For every voting profile a in Ω_N, the function $c_B(a_B) := (2 \cdot \mathbb{1}\{a_B \in W_B\} - 1)1_B$ induces the voting profile $c(a) := (c_B(a_B))_{B\in\mathcal{P}}$ in $\Omega_{\mathcal{P}}$. A voter $j \in A$ is critical in Ω_N, with respect to the bloc decision rule $W_N|(W_B)_{B\in\mathcal{P}}$, if and only if j is critical in W_A and the bloc A is critical in $W_{\mathcal{P}}$:

$$C_j\big(W_N|(W_B)_{B\in\mathcal{P}}\big) = \big\{a \in \Omega_N\colon a_A \in C_j(W_A),\;\; c(a) \in C_A(W_{\mathcal{P}})\big\}.$$

By Theorem 1, the event $C_A(W_{\mathcal{P}}) = \Pi^{-1}_{\mathcal{P}\setminus\{A\}}(D_A(W_{\mathcal{P}}))$ depends on the blocs in $\mathcal{P} \setminus \{A\}$, only. Since the distribution P_N is a product, $P_N = \bigotimes_{B\in\mathcal{P}} P_B$, we obtain

$$P_N\left(C_j\left(W_N\,|(W_B)_{B\in\mathcal{P}}\right)\right) = P_A\left(C_j(W_A)\right)\; P_{N\setminus A}\left(\left\{(c_B)_{B\in\mathcal{P}\setminus\{A\}} \in D_A(W_{\mathcal{P}})\right\}\right).$$

In the last factor we re-introduce the marginal space Ω_A:

$$P_{N\setminus A}\left(\left\{(c_B)_{B\in\mathcal{P}\setminus\{A\}} \in D_A(W_{\mathcal{P}})\right\}\right) = P_N\left(\{b \in C_A(W_{\mathcal{P}})\}\right) = P_N \circ b^{-1}\left(C_A(W_{\mathcal{P}})\right).$$

The distribution of the random vector $b = (c_B)_{B\in\mathcal{P}}$ under P_N turns out to be

$$P_N \circ b^{-1} = \bigotimes_{B\in\mathcal{P}} P_B \circ c_B^{-1} = \bigotimes_{B\in\mathcal{P}} \text{Bernoulli}(p_B),$$

with $p_B := P_B(\{c_B = 1\}) = P_B(\{a_B \in \Omega_B\colon a_B \in W_B\}) = P_B(W_B)$. This yields the distribution $Q_{\mathcal{P}}$ as claimed in the assertion. □

Trivial blocs $B = \{j\}$ do not contribute anything novel to the product formula. Indeed, the sole decision rule for them is $W_{\{j\}} = \{1\}$. Hence, the first factor in the product formula equals unity, $P_{\{j\}}\left(C_j(W_{\{j\}})\right) = 1$. Moreover, trivial blocs enter into the distribution $Q_{\mathcal{P}}$ as a Bernoulli$(1/2)$ component, since $P_{\{j\}}(W_{\{j\}}) = 1/2$. For this reason trivial blocs are often omitted when listing a partitioning.

In other words, if in a partitioning $\mathcal{P}$ a voter j stands alone, then the behavior of the trivial bloc $\{j\}$ in the partitioning assembly $\mathcal{P}$ is identical with the behavior of the voter j in the original assembly N, with probability $1/2$ of being a Yea-voter. Thus, a voter who stays back as a one-element bloc remains passive, and falls victim to the nontrivial blocs of the partitioning $\mathcal{P}$.

In Theorem 4, the distribution $Q_{\mathcal{P}}$ is a product of Bernoulli distributions. In general, it is no longer the case that a Yea must emerge with probability $1/2$. A Yea in bloc B occurs with probability equal to the Penrose/Banzhaf efficiency, $P_B(W_B)$, and a Nay has probability $1 - P_B(W_B)$. In Exhibit 1, the Benelux bloc votes Yea under the unanimity rule with probability $1/8$.

Other instances may give rise to distributions with correlated components. A prominent example is the Shapley/Shubik distribution.

Exhibit 1 Weighted decision rule EEC 1958–1972, and bloc variants

Row no.	Voting profile						Profile weight		
	DE: 4	IT: 4	FR: 4	NL: 2	BE: 2	LU: 1	W_{EU6}	$W_{\text{EU6}}\|M_{\text{Benelux}}$	$W_{\text{EU6}}\|U_{\text{Benelux}}$
1	1	1	1	1	1	1	17	17	17
2	1	1	1	1	1	−1	16	17	12
3	1	1	1	1	−1	1	15	17	12
4	1	1	1	−1	1	1	15	17	12
5	1	1	1	1	−1	−1	14	12	12
6	1	1	1	−1	1	−1	14	12	12
7	1	1	1	−1	−1	1	13	12	12
8	1	1	−1	1	1	1	13	13	13
9	1	−1	1	1	1	1	13	13	13
10	−1	1	1	1	1	1	13	13	13
11	1	1	1	−1	−1	−1	12	12	12
12	1	1	−1	1	1	−1	12	13	–
13	1	−1	1	1	1	−1	12	13	–
14	−1	1	1	1	1	−1	12	13	–
15	1	1	−1	1	−1	1	–	13	–
16	1	1	−1	−1	1	1	–	13	–
17	1	−1	1	1	−1	1	–	13	–
18	1	−1	1	−1	1	1	–	13	–
19	−1	1	1	1	−1	1	–	13	–
20	−1	1	1	−1	1	1	–	13	–

Decision rule	Penrose/Banzhaf influence probability						P/B sensitivity	P/B mean majority deficit	P/B efficiency
	All values to be divided by 64								
W_{EU6}	20	20	20	12	12	0	84	18	14
$W_{\text{EU6}}\|M_{\text{Benelux}}$	24	24	24	12	12	12	108	6	20
$W_{\text{EU6}}\|U_{\text{Benelux}}$	18	18	18	6	6	6	72	24	11

The actual rule W_{EU6} used weights 4, 4, 4, 2, 2, 1, and absolute quota 12. The two bloc variants let Benelux decide internally by straight majority ($W_{\text{EU6}}|M_{\text{Benelux}}$), or by unanimity ($W_{\text{EU6}}|U_{\text{Benelux}}$). The three rules feature 14, 20, 11 positive profiles, respectively. Penrose/Banzhaf influence probabilities appear to vary unpredictably.

7 The Shapley/Shubik model

The *Shapley/Shubik distribution* S_N combines two uniform distributions, see Dubey and Shapley (1979). Every subset $\left\{ \begin{matrix} N \\ k \end{matrix} \right\}$ with a fixed number k of Yea-voters, $k = 0, \ldots, n$, is assigned the same probability $1/(n+1)$. Conditionally on such a subset, its $\binom{n}{k}$ voting profiles are again assumed to be uniformly distributed:

$$S_N(\{a\}) := \frac{1}{(n+1)\binom{n}{(n+a_+)/2}} \quad \text{for all} \quad a \in \Omega_N.$$

It is easy to verify that the Shapley/Shubik distribution is selfdual and permutationally invariant. Moreover, Theorem 5 shows that the family of Shapley/Shubik distributions is projectively consistent with respect to its marginal distributions.

Theorem 5 *For all voters $j \in N$ we have $S_N \circ \Pi^{-1}_{N\setminus\{j\}} = S_{N\setminus\{j\}}$.*

Proof For $b \in \Omega_{N\setminus\{j\}}$ we have $\Pi^{-1}_{N\setminus\{j\}}\{b\} = \{(b; -1), (b; 1)\}$. Let $k := (n + b_+)/2$. The identities $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$ justify the assertion:

$$\begin{aligned} S_N\left(\{(b;-1),(b;1)\}\right) &= \frac{1}{(n+1)\binom{n}{k}} + \frac{1}{(n+1)\binom{n}{k+1}} = \frac{\binom{n}{k+1} + \binom{n}{k}}{(n+1)\binom{n}{k}\binom{n}{k+1}} \\ &= \frac{1}{(k+1)\binom{n}{k+1}} = \frac{1}{n\binom{n-1}{k}} = S_{N\setminus\{j\}}(\{b\}). \end{aligned}$$

□

The Shapley/Shubik influence probability of voter j becomes

$$S_N\left(C_j(W_N)\right) = S_{N\setminus\{j\}}\left(D_j(W_N)\right) = \sum_{k=0}^{n-1} S_{N\setminus\{j\}}\left(\left\{ {N\setminus\{j\} \atop k} \right\} \cap D_j(W_N)\right).$$

Since in the probability space $(\Omega_{N\setminus\{j\}}, S_{N\setminus\{j\}})$ a uniform distribution rules on the subsets $\left\{ {N\setminus\{j\} \atop k} \right\}$, we introduce the counting variables

$$\eta_j(k) := \#\left(\left\{ {N\setminus\{j\} \atop k} \right\} \cap D_j(W_N)\right) =: s_{k+1,j},$$

for all $k \in \{0, \ldots, n-1\}$ and $j \in N$. The number s_{ij} counts the voting profiles consisting of i Yea-voters (including j) and featuring voter j as exit-critical. Altogether they form the $\{1, \ldots, n\} \times N$ *swing matrix* s, with the entries s_{ij} in row i and column j. We obtain

$$S_N\left(C_j(W_N)\right) = \sum_{k=0}^{n-1} \frac{\eta_j(k)}{n\binom{n-1}{k}} = \frac{1}{n!}\sum_{k=0}^{n-1} k!(n-1-k)!\,\eta_j(k)$$
$$= \frac{1}{n!}\sum_{i=1}^{n}(i-1)!(n-i)!\,s_{ij}.$$

Theorem 6 states that the Shapley/Shubik influence probabilities always sum to unity. Hence the notion of sensitivity becomes superfluous, in the Shapley/Shubik model.

Theorem 6 *Every decision rule W_N has Shapley/Shubik sensitivity equal to unity:*

$$\sum_{j\in N} S_N\left(C_j(W_N)\right) = 1.$$

Proof The assertion is entirely of combinatorial nature: $\sum_{i=1}^{n}(i-1)!(n-i)!s_{i+} = n!$. We show that the left-hand side counts all permutations of n voters, as does the right-hand side. On the left-hand side, the counting is carried out in a way that is dictated by the problem. Without loss of generality we assume that the assembly is enumerated in the form $N = \{1, \ldots, n\}$. Let $\pi(1), \ldots, \pi(n)$ be an arbitrary permutation of the voters. We count the cases where the sole Yea-voters are $\pi(1), \ldots, \pi(i)$, with voter $\pi(i)$ being exit-critical:

$$e_{\pi(1)} + \cdots + e_{\pi(i-1)} + e_{\pi(i)} \in W_N, \qquad e_{\pi(1)} + \cdots + e_{\pi(i-1)} \in W_N^c.$$

Voter $j := \pi(i)$ maintains the exit-critical role in the permutations rearranging the predecessors $\pi(1), \ldots, \pi(i-1)$, or rearranging the successors $\pi(i+1), \ldots, \pi(n)$. This generates $(i-1)!(n-i)!$ permutations. Finally, the number $s_{i+} := \sum_{j\in N} s_{ij}$ is the count of how often voter j takes the position of the exit-critical voter $\pi(i)$. □

Theorem 6 entails the rather strange consequence that, for symmetric decision rules $W_N(q; \lambda 1_N)$ where all voters enjoy the same voting weight λ, the Shapley/Shubik influence probabilities of all voters are equal to $1/n$. They do not depend on the quota q, and therefore the Shapley/Shubik model is incapable of distinguishing the unanimity rule U_N (with quota $q = 1 - 1/n$), from the straight majority rule M_N (with quota $q = 1/2$).

In the Shapley/Shubik model, the mean success margin does not coincide with the sensitivity (which equals unity, by Theorem 6). According to Theorem 2, the lower bound for the mean success margin is given by $\mathrm{E}_{S_N}[\sigma_{U_N}] = 2n/(n+1)$. The upper bound becomes

$$\mathrm{E}_{S_N}[\sigma_{M_N}] = \frac{n+1}{2} - \begin{cases} \frac{1}{2(n+1)} & \text{in case } n \text{ is even,} \\ 0 & \text{in case } n \text{ is odd.} \end{cases} \qquad (*)$$

Indeed, in the straight majority rule M_N a voting profile a is positive if and only if $k := (n + a_+)/2 > n/2$, i.e., $a_+ = 2k - n > 0$. Hence, Theorem 2(ii) gives

$$\mathrm{E}_{S_N}[\sigma_{M_N}] = 2 \sum_{k>n/2} (2k-n) S_N\left(\left\{ \begin{matrix} N \\ k \end{matrix} \right\}\right) = \frac{2}{n+1} \sum_{k>n/2} (2k-n).$$

Now $\sum_{k>n/2}(2k-n) = \lceil n/2 \rceil (\lfloor n/2 \rfloor + 1)$ establishes the two cases in $(*)$.

The Shapley/Shubik model assigns weights to the subsets $\left\{ \begin{matrix} N \\ k \end{matrix} \right\}$ of k Yea-voters that differ from those in the Penrose/Banzhaf model:

$$S_N\left(\left\{ \begin{matrix} N \\ k \end{matrix} \right\}\right) = \frac{1}{n+1} \neq \frac{1}{2^n}\binom{n}{k} = P_N\left(\left\{ \begin{matrix} N \\ k \end{matrix} \right\}\right).$$

Nevertheless, within any such subset the conditional probabilities are the same. For all voting profiles $a \in \left\{ \begin{matrix} N \\ k \end{matrix} \right\}$, we have

$$S_N\left(\{a\} \middle| \left\{ \begin{matrix} N \\ k \end{matrix} \right\}\right) = 1 \Big/ \binom{n}{k} = P_N\left(\{a\} \middle| \left\{ \begin{matrix} N \\ k \end{matrix} \right\}\right).$$

The Shapley/Shubik model has marginal distributions $S_{\{j\}} = \text{Bernoulli}(1/2)$, for all voters $j \in N$, as has the Penrose/Banzhaf model. However, any two voters are stochastically dependent in their behavior, $\mathrm{Cov}_{S_N}[a_i, a_j] = \mathrm{Cov}_{S_{\{i,j\}}}[a_i, a_j] = 1/12$. The correlation coefficient turns out to be $(1/12)/(1/4) = 1/3$.

The positive correlation becomes visible also in the conditional probabilities

$$S_N\left(\{(b;1)\} \middle| \{(b;-1),(b;1)\}\right) = \frac{S_N(\{(b;1)\})}{S_{N\setminus\{j\}}(\{b\})} = \frac{(n+b_+)/2+1}{n+1}.$$

In the Shapley/Shubik model, voter j turns into a Yea-voter with a likelihood that increases with $(n + b_+)/2$, the number of Yea-voters surrounding j. This is reminiscent of the *accessus* procedure in clerical elections. The accession of minority voters to the majority may ease the way to a two-thirds winning configuration (see Colomer and McLean 1998).

8 Conclusion

In this article, we leave the common ground of game theory and favor a probabilistic approach. The set $\Omega_N = \{-1, 1\}^N$ of binary voting profiles in an assembly N allows to treat many prominent power measures known in the literature. This leads to a general theory in which power measures arise from appropriate distributional assumptions.

The approach also yields new results. Section 3 shows that the upper and lower bounds of the expected success margin apply to all selfdual and permutationally invariant distributions, rather than being restricted to special assumptions such as the

Penrose/Banzhaf model. Furthermore bloc decision rules, studied in Section 6, generate a new class of interesting distributions and yield a generalization of the product formula of Felsenthal and Machover (2002). We allow blocs of any sizes and arbitrary internal decision rules. The example of a Benelux bloc in the former EEC illustrates the different power distributions among the six States when the Benelux bloc decides internally by straight majority, or by unanimity.

Finally, we remark that the approach extends to ternary decision rules where abstentions are allowed. With abstention probability $t \in [0, 1)$, Käufl et al. (2010) develop formulas embracing the results of this article as the starting case $t = 0$.

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