A Probabilistic Re-View on Felsenthal and Machover's "The Measurement of Voting Power"

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1 The Book

Dan Felsenthal and Moshé Machover's (1998) monograph on *The Measurement of Voting Power* served a double purpose, of concisely presenting the state of the art of the theory of weighted voting systems, and of initiating novel strains of research in the area. The authors achieved these goals by a careful use of the mathematical tools, game theory and probability theory. The mathematical frame was developed not in an ivory tower seclusion, but along pertinent applications such as US-American court cases, or the Council of Ministers of the European Union. The interplay of ideal theory and concrete applications proved most fertile.

In Augsburg we repeatedly worked through the book in the course of seminars for our students who have a strong background in probability theory and statistics. Therefore we paid particular attention to the book's probabilistic language, and experimented with the technical vocabulary, in order to optimize communication with non-mathematical contemporaries. An instant stumbling stone was felt to be the phonetic closeness of two central notions of the subject, *voting weight* and *voting power*. In German they translate into *Stimmgewicht* and *Stimmkraft*. Since the German language puts a strong emphasis on the first syllable of a compound word, a negligent speaker may offer the audience an audible *Stimm*. . ., followed by a murmured . . .*something*, thus completely missing the point. For this reason we tried to separate the notions more clearly. We kept *voting weight*, but replaced *absolute voting power* by *influence probability*, and *relative voting power* by *power share*. The term *share* indicates that the ensemble of these indices totals unity, whence they form a *power distribution*.

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In the present paper we explicate the probabilistic approach, in as far as we found it telling and conducive. The approach is by no means new. It dates back at least to Straffin (1978, 1988), and the Felsenthal and Machover (1998) monograph makes excellent use of it. Nevertheless we believe that a 're-view' on its role may prove useful.

In their final Chap. 8, Felsenthal and Machover (1998) make a point to take abstentions seriously. We maintain that ternary voting profiles provide a sufficiently general reference set supporting both, ternary decision rules that permit abstentions, and binary decision rules that are restricted to Yea-Nay voting (Sect. 2). The ensuing development depends on the probability distribution adopted. A model is truly ternary when it assigns positive weights to voting profiles with at least one abstention. The case of abstention probability zero leads back to the binary setting. The Penrose/Banzhaf models (Sect. 3) and the Shapley/Shubik models (Sect. 4) come with abstention probabilities $t \in [0, 1)$ that afford a smooth transition between ternary and binary settings. We conclude with an outlook on bloc decision rules, a prime example being provided by the Council of Ministers of the European Union (Sect. 5).

2 Ternary Voting Profiles

Let N denote an *assembly* consisting of finitely many *agents* j. When a proposal is tabled and a vote is taken, the results are recorded as a vector $a = (a_j)_{j \in N}$, a *voting profile*. The vote of agent j is reported as $a_j = \textit{yea}$ when j votes Yea, $a_j = \textit{nay}$ when j votes Nay, or $a_j = \textit{abstain}$ when j abstains. The natural ordering among these values is $\textit{nay} \leq \textit{abstain} \leq \textit{yea}$. Felsenthal and Machover (1998, p. 282) use the coding $\textit{nay} = -1$, $\textit{abstain} = 0$, and $\textit{yea} = 1$.

The ensemble of all voting profiles constitutes the *ternary profile space*

$$\Omega_N = \left\{ (a_j)_{j \in N} \;\middle|\; a_j \in \{\textit{nay}, \textit{abstain}, \textit{yea}\}, \text{ for all } j \in N \right\}.$$

Every profile $a \in \Omega_N$ induces a *region of growing acceptance* consisting of those profiles b that express at least as much acceptance as is reported in a,

$$[a, \textit{yea}] = \left\{ (b_j)_{j \in N} \in \Omega_N \;\middle|\; a_j \leq b_j, \text{ for all } j \in N \right\}.$$

A subset $W_N \subseteq \Omega_N$ is called a *decision rule* when it satisfies the three properties

$$[a, \textit{yea}] \subseteq W_N, \text{ for all } a \in W_N, \tag{1}$$

$$(\textit{yea}, \ldots, \textit{yea}) \in W_N, \tag{2}$$

$$(\textit{abstain}, \ldots, \textit{abstain}) \notin W_N. \tag{3}$$

The profiles a that constitute the subset W_N are called *winning*, in the sense that a proposal is carried if and only if a belongs to W_N. We do not consider systems in which the final outcome might be a tie. Therefore the complement $W_N^C = \Omega_N \setminus W_N$ comprises the profiles that are *loosing*. Thus a subset W_N is a decision rule if and only if (1) it is *acceptance-monotonic*: if a is winning and b reports at least as much acceptance as does a, then b is also winning, (2) unanimous acceptance is winning, and (3) unanimous abstention is loosing.

Now we fix some decision rule W_N, and investigate its merits from the point of view of agent j. Two events transpire to be of particular interest. First, there is the set $A_j(W_N)$ of *agreeable profiles*, when j agrees with the final outcome. Second, there is the set $C_j(W_N)$ of *critical profiles*, when the vote of j is decisive to turn the profile winning or loosing. Let the notation $(a_i)_{i \neq j}$ & $(yea)_j$ represent the profile where the votes of the other agents $i \neq j$ are concatenated with a Yea from agent j. Similarly $(a_i)_{i \neq j}$ & $(nay)_j$ is to indicate that the votes of the others is completed with j's Nay. The two events mentioned may then be described as follows:

$$A_j(W_N) = \left\{ a \in W_N \;\middle|\; a_j = yea \right\} \cup \left\{ a \in W_N^C \;\middle|\; a_j = nay \right\},$$

$$C_j(W_N) = \left\{ a \in \Omega_N \;\middle|\; (a_i)_{i \neq j} \,\&\, (yea)_j \in W_N \text{ and } (a_i)_{i \neq j} \,\&\, (nay)_j \in W_N^C \right\}.$$

So far the exposition is descriptive and qualitative. It is only now that we consider quantitative indices. All of them originate from a probability measure P given on the ternary profile space Ω_N, with some of them being peculiar to an agent j:

$P[W_N]$	the *efficiency* of the decision rule W_N,
$P\big[A_j(W_N)\big]$	the *success probability* of agent j,
$P\big[C_j(W_N)\big]$	the *influence probability* of agent j,
$P\big[C_j(W_N)\big] / \Sigma_P(W_N)$	the *power share* of agent j, utilizing
$\Sigma_P(W_N) = \sum_{i \in N} P\big[C_i(W_N)\big]$	the *influence sensitivity* of the decision rule W_N.

The indices coincide with those in the monograph (Felsenthal and Machover 1998), except that in The Book they are related to specific distributions, namely the Penrose/Banzhaf and Shapley/Shubik distributions in their variants with abstention probability equal to zero (see below). In particular, our notion of Penrose/Banzhaf influence probability of agent j is the same as their *Banzhaf power* (or absolute Banzhaf index) of j, and our power share of agent j coincides with their *Banzhaf index* of voting power (or relative Banzhaf index) of j. Our motivation for not specializing the probabilistic assumptions too early is that there are results like Theorem 1 below which hold quite generally. To this end we need to introduce some notation.

The *dual profile* dual(a) of a ternary voting profile $a \in \Omega_N$ is defined by reversing the votes of all agents $j \in N$,

$$\big(\mathrm{dual}(a)\big)_j = \begin{cases} nay & \text{in case } a_j = yea, \\ abstain & \text{in case} a_j = abstain, \\ yea & \text{in case} a_j = nay. \end{cases}$$

A distribution P is said to be *selfdual* when a voting profile and its dual are assigned identical probabilities, $P[\{a\}] = P\big[\{\mathrm{dual}(a)\}\big]$.

A distribution P is said to be *exchangeable* when it remains invariant under all permutations of the assembly N. In the presence of exchangeability, a maximal invariant statistic tallies the yeas, nays, and abstentions of a voting profile $a \in \Omega_N$ into the three counts $\mathrm{Yea}(a)$, $\mathrm{Nay}(a)$, and $\mathrm{Abst}(a)$, respectively.

The *success margin* $\sigma(W_N)(a)$ is defined to be the difference between the number of those who vote in favor of the final outcome, and those who vote against it,

$$\sigma(W_N)(a) = \begin{cases} \mathrm{Yea}(a) - \mathrm{Nay}(a) & \text{in case } a \in W_N, \\ \mathrm{Nay}(a) - \mathrm{Yea}(a) & \text{in case } a \in W_N^C. \end{cases}$$

Two decision rules deserve special attention. The first is the *unanimity rule* U_N, signaling acceptance when nobody is objecting, and the second is the *straight majority rule* M_N, requiring the Yeas to outnumber the Nays,

$$U_N = \Big\{(a_j)_{j\in N} \in \Omega_N \;\Big|\; \mathrm{Yea}(a) > 0 = \mathrm{Nay}(a)\Big\},$$

$$M_N = \Big\{(a_j)_{j\in N} \in \Omega_N \;\Big|\; \mathrm{Yea}(a) > \mathrm{Nay}(a)\Big\}.$$

Theorem 1. *Let the ternary profile space Ω_N be equipped with be a selfdual and exchangeable probability distribution P.*

Then every decision rule W_N has its expected success margin lying between the expected success margins of the unanimity rule and of the straight majority rule,

$$\mathrm{E}_P\big[\sigma(U_N)\big] \le \mathrm{E}_P\big[\sigma(W_N)\big] \le \mathrm{E}_P\big[\sigma(M_N)\big].$$

Proof. See Proposition 4.1 in Birkmeier et al. (2011). □

The unanimity rule and the straight majority rule are two instances of the wider class of *weighted decision rules* $W_N[q; (w_j)_{j\in N}]$. Such a rule is determined by a *quota* $q \in [0, 1)$, and *voting weights* $w_j > 0$ for agents $j \in N$. For a given voting profile a let $\mathrm{YCW}(a) = \sum_{j\in N: a_j = yea} w_j$ designate the *Yea-voters' cumulative weight*, and $\mathrm{NCW}(a) = \sum_{j\in N: a_j = nay} w_j$ the *Nay-voters' cumulative weight*. A ternary voting profile a is defined to be winning, $a \in W_N[q; (w_j)_{j\in N}]$, when the Yea-voters' cumulative weight exceeds the fraction q of the cumulative weight of all non-abstainers, $\mathrm{YCW}(a) > q \cdot \big(\mathrm{YCW}(a) + \mathrm{NCW}(a)\big)$.

3 The Penrose/Banzhaf Model

The *Penrose/Banzhaf distribution* P_N^t assumes that all agents act independently, abstain with a common *abstention probability* $t \in [0, 1)$, and divide the remaining likelihood $1 - t$ equally between a Yea and a Nay. In this model, a ternary voting profile $a \in \Omega_N$ carries the probability

$$P_N^t[\{a\}] = \frac{1}{2^{\text{Yea}(a)+\text{Nay}(a)}} (1-t)^{\text{Yea}(a)+\text{Nay}(a)}\, t^{\text{Abst}(a)}.$$

When the ternary parameter t vanishes, voting profiles that contain an abstention are assigned zero probability. Thus a profile carries positive mass only when every agent votes Yea or Nay. That is, with $t = 0$ the ternary Penrose/Banzhaf model reduces to the familiar binary Penrose/Banzhaf model. The ternary Penrose/Banzhaf model thus embraces the binary Penrose/Banzhaf model as a degenerate case. A sample result is provided by Theorem 2.

Theorem 2. *Let the ternary profile space Ω_N be equipped with the Penrose/Banzhaf distribution P_N^t, with abstention probability $t \in [0, 1)$, and let W_N be an arbitrary decision rule.*

(i) *For all agents $j \in N$, success and influence probabilities are related through*

$$P_N^t\big[A_j(W_N)\big] = \frac{1-t}{2} + \frac{1-t}{2} P_N^t\big[C_j(W_N)\big].$$

(ii) *The influence sensitivity and the expected success margin of W_N fulfill*

$$\Sigma_{P_N^t}(W_N) = \frac{1}{1-t} \mathrm{E}_{P_N^t}\big[\sigma(W_N)\big].$$

Proof. See Propositions 5.1 and 5.2 in Birkmeier et al. (2011). □

With $t = 0$, this coincides with the results in Theorems 3.2.16 and 3.3.5 in Felsenthal and Machover (1998), see also Ruff and Pukelsheim (2010). Birkmeier (2011, Satz 2.3.4) presents a version of part (i) dealing with a slightly larger set of profiles that are considered a success for agent j, namely those that are agreeable to agent j combined with those wherein j abstains (and which might be considered "weakly agreeable").

4 The Shapley/Shubik Model

The *Shapley/Shubik distribution* S_N^t on Ω_N is built up in three stages. The first stage, dealing with abstentions, is new. We propose to assume all agents to abstain independently, with a common abstention probability $t \in [0, 1)$. Under this assumption the number ℓ of those who abstain follows a binomial distribution, $\frac{n!}{\ell!(n-\ell)!}t^{\ell}(1-t)^{n-\ell}$. The second and third stages are standard. The number of Yea-voters k is taken to attain each of its possible values $0, \ldots, n-\ell$ with the same probability, $1/\big(n-\ell+1\big)$. Third, each of the $\frac{n!}{k!\ell!(n-k-\ell)!}$ profiles with k Yeas, ℓ abstentions, and $n-k-\ell$ Nays is considered equally likely. Thus, with some of the factorial terms canceling out and after re-substituting Yea(a) for k and Yea(a) + Nay(a) for $n-\ell$, the total probability of a ternary voting profile $a \in \Omega_N$ becomes

$$S_N^t[\{a\}] = \frac{\text{Yea}(a)!\ \text{Nay}(a)!}{\big(\text{Yea}(a)+\text{Nay}(a)+1\big)!}\ (1-t)^{\text{Yea}(a)+\text{Nay}(a)}\ t^{\text{Abst}(a)}.$$

In binary models, it is well-known that every decision rule W_N has Shapley/Shubik influence sensitivity equal to unity. This entails two intriguing consequences, that the Shapley/Shubik sensitivity is insensitive to the specific decision rule W_N, and that the Shapley/Shubik influence probability of an agent j coincides with her or his power share. In ternary Shapley/Shubik models, the first conclusion persists, the second does not.

Theorem 3. *Let the ternary profile space Ω_N be equipped with the Shapley/Shubik distribution S_N^t, with abstention probability $t \in [0,1)$, and let n be the cardinality of the assembly N.*

Then all decision rules W_N share an identical influence sensitivity,

$$\Sigma_{S_N^t}(W_N) = \frac{1-t^n}{1-t}.$$

Proof. See Satz 2.3.9 in Birkmeier (2011). □

The right hand side is the same as $1+t+\cdots+t^{n-1}$. Hence its limit equals n, the number of agents, as the abstention probability t tends to unity. This is quite plausible since, with the likelihood of abstention growing, each of the n agents is getting to be more and more critical when casting a clear Yea- or Nay-vote, and in the end acquires an influence probability equal to unity.

5 The EU Council of Ministers

In some applications the grand assembly N is partitioned into disjoint subsets, called *blocs*. The associated two-tier voting system is composed of internal decision rules within blocs, and a second-level decision rule among bloc delegates. An example is the Council of Ministers of the European Union, where the entirety of the Union citizens, N, is partitioned into the 27 blocs of its Member States' citizenries that are represented by their Ministers.

In the binary Penrose/Banzhaf bloc model, the influence probability of citizen $j \in N$ in a two-tier system typically factorizes into the product of the internal influence probability of j in her or his bloc B, times the second-level influence probability of bloc B relative to the specified bloc partitioning, see Straffin (1978), Felsenthal and Machover (2002), Laruelle and Valenciano (2004), or Ruff and Pukelsheim (2010). These product formulas generalize to carry over to ternary Penrose/Banzhaf bloc models, see Birkmeier (2011) for a *bottom up* construction (Satz 5.1.3) as well as for a *top down* construction (Satz 5.1.6).

The analysis may be employed to design optimal decision rules that permit abstentions. The underlying notion of optimality builds on a weighted average of the diplomatic *one state, one vote* principle that underlies international relations among Member States, and of the democratic *one person, one vote* principle that would apply to the Union citizens, see Laruelle and Widgrén (1998) and Satz 5.3.3 in Birkmeier (2011). However, it is by no means evident whether the Treaty of Lisbon (2010) would support the two equality principles and, if so, whether they may be mixed into a single optimality criterion.

Nevertheless, a statistical evaluation of previous decision rules used in the EU Council of Ministers leads to the estimates reported in Sect. 1 of Birkmeier (2011). They suggest that, in the past, the Union functioned with a mixture that puts a weight of 10 % on the diplomatic equality principle, and a complementary 90 % weight on the democratic equality principle. With these weightings, the optimal quota is found to be 60.98 %, see Birkmeier (2011, p. 117). This is slightly below the quota of 61.6 % proposed in the Jagiellonian Compromise of Słomczyński and Życzkowski (2010).

The mixture criterion is roughly in line with the composition of the European Parliament where each Member State is guaranteed six seats out of a total of 751 seats. That is, 20 % of the seats are preassigned to the Member States obeying the diplomatic equality principle of *one state, one vote*. The remaining 80 % then might be allocated via a proportional representation apportionment method to honor the democratic equality principle of *one person, one vote*, as proposed in the Cambridge Compromise of Grimmett et al. (2011).

References

Birkmeier, O. (2011). *Machtindizes und Fairness-Kriterien in gewichteten Abstimmungssytemen mit Enthaltungen*. Augsburger Schriften zur Mathematik, Physik und Informatik, Band 18. Berlin: Logos.

Birkmeier, O., Käufl, A., & Pukelsheim, F. (2011). Abstentions in the German Bundesrat and ternary decision rules in weighted voting systems. *Statistics & Decisions*, *28*, 1–16.

Felsenthal, D.S., & Machover, M. (1998). *The Measurement of voting power – theory and practice, problems and paradoxes*. Cheltenham: Edward Elgar.

Felsenthal, D.S., & Machover, M. (2002). Annexations and alliances: When are blocs advantageous a priori? *Social Choice and Welfare*, *19*, 295–312.

Grimmett, G., Laslier, J. -F., Pukelsheim, F., Ramírez-González, V., Rose, R., Słomczyński, W., Zachariasen, M., Zyczkowski, K. (2011). *The allocation between the EU Member States of the seats in the European Parliament*. European Parliament, Directorate-General for Internal Policies, Policy Department C: Citizen's Rights and Constitutional Affairs, Note 23.03.2011 (PE 432.760).

Laruelle, A., & Valenciano, F. (2004). On the meaning of Owen-Banzhaf coalitional value in voting situation. *Theory and Decision*, *56*, 113–123.

Laruelle, A., & Widgrén, M. (1998). Is the allocation of voting power among the EU States fair? *Public Choice*, *94*, 317–339.

Ruff, O., & Pukelsheim, F. (2010). A probabilistic synopsis of binary decision rules. *Social Choice and Welfare*, *35*, 501–516.

Słomczyński, W., & Życzkowski, K. (2010). Jagiellonian compromise – an alternative voting system for the Council of the European Union. In M. A. Cichocki, & Życzkowski, K. (Eds.), *Distribution of power and voting procedures in the European Union* (pp. 43–58). London: Ashgate.

Straffin, P. D. (1978). Probability models for power indices. In P. C. Ordeshook (Ed.), *Game theory and political science* (pp. 477–510). New York: New York University Press.

Straffin, P. D. (1988). The Shapley-Shubik and Banzhaf power indices as probabilities. In A. E. Roth (Ed.), *The Shapley value – essays in honor of Lloyd S. Shapley* (pp. 71–81). Cambridge: Cambridge University Press.

Treaty of Lisbon. (2010). Treaty on European Union, Articles 1–19. *Official Journal of the European Union*, *C83*, (30.2.2010) 13–27.