Anderson localization of two-dimensional Dirac fermions: a perturbative approach

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Anderson localization is studied for two-dimensional Dirac fermions in the presence of strong random scattering. Averaging with respect to the latter leads to a graphical representation of the correlation function with entangled random walks and three-vertices which connect three different types of propagators. This approach indicates Anderson localization along a semi-infinite line, where the localization length is inversely proportional to the scattering rate.

I. INTRODUCTION

The physics of diffusion and Anderson localization is based on the picture that quantum particles scatter on impurities or defects of the underlying lattice structure. This represents a complex dynamical system which can be treated in practice only by some simplifying assumptions. First, we consider only independent particle of the system and average over all possible scattering effects. For the latter we introduce a static distribution by assuming that the relevant scattering processes happen only on time scales that are large in comparison to the tunneling process of the quasiparticle in the lattice.

For the diffusive regime of such a disordered system exist powerful methods, such as the nonlinear sigma model [1–3] and the weak-localization approach [4–6]. The latter is based on a perturbation series in powers of $1/k_F l$, where k_F is the Fermi wavevector and l the mean-free path. This approach, however, is not directly applicable if $k_F \sim 0$, which is, for instance, the case for Dirac fermions at the spectral nodes [7–10]. The special transport properties at these nodes have attracted great interest in the context of graphene [11, 12] and topological insulators [13]. Therefore, it is important to develop a flexible approach which allows us to study the related physics.

An alternative perturbative approach to the above mentioned methods was suggested recently, based on the idea that E_b/η is a small parameter (E_b is the band width, η the scattering rate) [14]. The scattering rate is related to the scattering time τ by $\eta = \hbar/\tau$ and to the mean-free path l by $\eta = \hbar v_F/l$ (v_F is the Fermi velocity). Thus, in contrast to the weak-localization approach the expansion parameter $E_b l/\hbar v_F$ depends on the bandwidth rather than on the Fermi wavevector k_F. This approach enables us to study the regime with short mean-free path l, where we expect Anderson localization. The latter phenomenon is connected with a special type of symmetry breaking: While diffusion breaks the time-reversal invariance of the underlying microscopic system, Anderson localization breaks the scaling invariance of diffusion by creating a finite scale, the localization length. This is similar to the Kosterlitz-Thouless transition in the XY model, where thermal fluctuations create vortex pairs whose correlation decays exponentially [15, 16]. This can be understood in a more formal way: In the presence of weak disorder we have diffusion, characterized by the diffusion propagator $K_q \propto 1/(i\omega + Dq^2)$ with diffusion coefficient D, which has two poles $q_\pm = \pm\sqrt{-i\omega/D}$ for the wavevector q. For Anderson localization, on the other hand, one would expect the appearence of poles away from the real axis, where the distance from the latter is proportional to the inverse localization length. However, this would imply that $K_{q=0}$ is finite in the limit $\omega \to 0$, which violates the general property $K_{q=0} \propto 1/i\omega$ [17, 18]. It was found within the strong-scattering expansion that there is only one pole that has a distance from the real axis inversely proportional to the localization length, whereas the other pole approaches the real axis with $\omega \to 0$ [14]. This result shall be used in this paper to study the localization properties of 2D Dirac fermions in the presence of strong scattering.

The article is organized as follows. Fundamental quantities and the model are defined in Sect. II. Then in Sect. III A we briefly summarize the perturbation theory of Ref. [14]. The three progagators of the theory are discussed for 2D Dirac fermions in Sects. III B and III C. These results for the propagators are used in Sect. III D to show that the linked cluster expansion of the correlation function is convergent for strong scattering. Finally, the results are summarized in Sect. IV.

II. TRANSITION PROBABILITY

At weak scattering we expect diffusion, a behavior known from classical physics, where the mean-square displacement of a particle position grows linearly with time. This behavior is also valid for quantum systems [19]. It provides our basic understanding for a large number of transport phenomena, such as the metallic behavior in electronic systems. Starting point is the transition probability for a particle, governed by the random Hamiltonian H, to move from the site r' on a lattice to another lattice site r within the time t:

$$P_{rr'}(t) = \sum_{j,j'} \langle |\langle r,j|e^{-iHt}|r',j'\rangle|^2\rangle_d \ , \tag{1}$$

where $\langle ...\rangle_d$ is the average with respect to randomly distributed disorder. The indices j, j' refer to different bands of the system. In the following we will focus on the specific case of the 2D Dirac Hamiltonian $H = v_F \vec{p}\cdot\vec{\sigma} + H_1$ where H_1 is a random term with mean zero, and where v_F is the Fermi velocity. The components of the vector $\vec{\sigma} = (\sigma_1, \sigma_2)$ are Pauli matrices. Assuming a cut-off λ for the momentum, there is an effective bandwidth $E_b = 2v_F\lambda^2$. In this case $j, j' = 1, 2$ are spinor indices.

With the expression (1) we obtain, for instance, the mean-square displacement as

$$\langle (r_k - r'_k)^2\rangle = \sum_r (r_k - r'_k)^2 P_{rr'}(t)$$

and the diffusion coefficient as

$$D = \lim_{\epsilon\to 0} \epsilon^2 \sum_r (r_k - r'_k)^2 \int_0^\infty P_{rr'}(t)e^{-\epsilon t}dt \ .$$

The time integral in the last expression can also be written in terms of the Green's function as

$$\int_0^\infty P_{rr'}(t)e^{-\epsilon t}dt = \int Tr_2\left\{G_{r,r'}(E+i\epsilon)\left[G_{r',r}(E-i\epsilon) - G_{r',r}(E+i\epsilon)\right]\right\}dE \ , \tag{2}$$

where Tr_2 is the trace with respect to the spinor index. The one-particle Green's function is defined as the resolvent $G(z) = (H-z)^{-1}$ of the Hamiltonian H, and $G_{r0}(E+i\epsilon)$ describes the propagation of a particle with energy E from the origin to a site r.

The correlation function of the Green's functions with poles on different half planes is dominant, whereas the correlation function of the Green's functions with poles only on one half plane is the derivative of $Tr_2 G_{r,r}(E+i\epsilon)$ with respect to E:

$$\frac{\partial}{\partial E} Tr_2\left[G_{r,r}(E+i\epsilon)\right] = \sum_{r'} Tr_2\left[G_{r,r'}(E+i\epsilon)G_{r',r}(E+i\epsilon)\right] \ .$$

Therefore, more important is the other term in (2):

$$Tr_2\left[\langle G_{r,r'}(E+i\epsilon)G_{r',r}(E-i\epsilon)\rangle_d\right] \ . \tag{3}$$

A direct application of a perturbation theory for strong scattering would be an expansion of this expression in powers of the off-diagonal terms $v_F\vec{p}\cdot\vec{\sigma}$ (hopping expansion). However, such an expansion fails for small ϵ because the expansion terms diverge with $\epsilon \to 0$. This is a consequence of the fact that we have poles in the upper and in the lower half plane that move to the real axis for $\epsilon \to 0$. It has been shown in Ref. [14] that this correlation of the Green's functions agrees for large distances $|r - r'|$ with the correlation function of a random-phase model, described by the expression

$$K_{rr'} = \frac{\langle C^{-1}_{rr'} \det C\rangle_a}{\langle \det C\rangle_a} \tag{4}$$

with

$$C_{rr'} = 2\delta_{rr'} - \sum_{j,j'} e^{i\alpha_{rj}} h_{rj,r'j'} \sum_{j'',r''} h^\dagger_{r'j',r''j''} e^{-i\alpha_{r''j''}} \ . \tag{5}$$

The brackets $\langle ... \rangle_a$ mean integration with respect to the angular variables $\{0 \leq \alpha_{rj} < 2\pi\}$, normalized by 2π. These angles represent the relevant part of the disorder fluctuations, which are subject to long-range correlations of the Green's functions. Here it should be noticed that there is an invariance of C with respect to a global phase change $\alpha_{rj} \to \alpha_{rj} + \phi$. Moreover, we have

$$h_{rr'} = \sigma_0 \delta_{rr'} + 2i\eta(v_F \vec{p} \cdot \vec{\sigma} - i\bar{\eta})^{-1}_{rr'} \quad \text{with } \bar{\eta} = \eta + \epsilon \ , \tag{6}$$

where $\eta \geq 0$ is the scattering rate in units of $v_F \lambda^2 = E_b$. In the limit $\epsilon \to 0$ the propagator h is unitary:

$$hh^\dagger = \mathbf{1} - 4\epsilon(1-\epsilon)\bar{\eta}(p^2 + \bar{\eta}^2)^{-1} \ . \tag{7}$$

It is convenient to introduce the generating functional $\log(\langle \det(C + a) \rangle_a)$ with the $N \times N$ matrix a (N is the number of lattice sites). Then we obtain from (4)

$$K_{rr'} = \frac{\langle C^{-1}_{rr'} \det C \rangle_a}{\langle \det C \rangle_a} = \frac{\partial}{\partial \alpha_{r'r}} \log(\langle \det(C + a) \rangle_a) \Big|_{a=0} \ . \tag{8}$$

In the remainder of this paper we will show that for strong scattering

$$K_{rr'} \sim \frac{1}{2} g_{r-r'} \ , \tag{9}$$

where g_r decays exponentially, except for a line where it is constant.

III. THREE-VERTEX EXPANSION

A. General idea

We briefly recapitulate the perturbative expansion of Ref. [14], which relies on the idea that the expression (8) can be treated within a linked cluster expansion of $\langle \det(C + a) \rangle_a$. The latter is generated by the expansion of the determinant in Eq. (4)

$$\det(C + \alpha) = \frac{2^N}{\det g} e^A \quad \text{with} \quad A = Tr[\log(\mathbf{1} + \frac{1}{2} g(-C' + \bar{C}))] \quad \text{and } g = \left(\mathbf{1} + \frac{1}{2} a - \frac{1}{2} \bar{C} \right)^{-1} \tag{10}$$

in powers of $\delta C = C' - \bar{C}$ around $\bar{C}$, where we use

$$C'_{rr'} = 2\delta_{rr'} - C_{rr'} = \sum_{j,j'} e^{i\alpha_{rj}} h_{rj,r'j'} \sum_{r'',j''} h^\dagger_{r'j',r'',j''} e^{-i\alpha_{r'',j''}} \tag{11}$$

and

$$\bar{C}_{rr'} = -\sum_{j,j'} h_{rj,r'j'} e^{i(\phi_j - \phi_{j'})} \ . \tag{12}$$

For the last expression we have chosen fixed phases $\alpha_{rj} = \phi_j$ which are uniform in r. Then a Taylor expansion in the exponent of (10) yields

$$A = Tr[\log(\mathbf{1} - \frac{1}{2} g\delta C)] = -\sum_{l \geq 1} \frac{1}{2^l l} Tr\left[(g\delta C)^l \right] \ . \tag{13}$$

This can be used to expand e^A in powers of δC and perform the angular integration for each expansion term. The angular integration is easy to perform, since the expansion yields products of the phase factors $e^{\pm i\alpha_{rj}}$, whose integration vanishes unless the phases compensate each other in the product. Then the result of the angular integration has a graphical representation in terms of random walks, whose sites are connected pairwise by the propagator $h^\dagger$. An equivalent representation consists of random walk whose steps are given by alternating propagators h and $h^\dagger$. The sites of these walks are connected pairwise by the propagator g. This gives us eventually graphs that consist of three types of propagators, namely h,

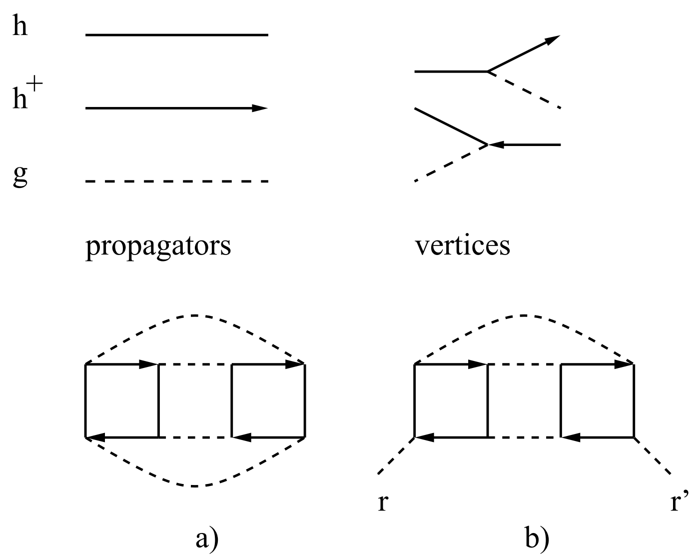

FIG. 1: Propagators, vertices and two typical graphs of the linked cluster expansion: a) is a graph of the generating function and b) is a graph of the correlation function $K_{rr'}$ of Eq. (8), generated from graph a) by differentiation.

$h^\dagger$ and g, and two types of three-vertices (cf. Fig. 1). Moreover, the linked cluster expansion is based on the relation $\langle e^A\rangle_a = e^{\langle A\rangle_c}$, where $\langle A\rangle_c$ consists of those diagrams from the expansion of $\langle e^A\rangle_a$ which are connected (or linked) graphs [14, 15]. The latter provides an expansion, where the number of terms increases exponentially with the number of propagators. Then we have a convergent expansion when we can make the contribution of each propagator small. This will be discussed for 2D Dirac lattice fermions with finite momentum cut-off λ in the next section.

B. Propagators for 2D Dirac fermions

Now we have to analyze the three propagators of the three-vertex expansion for 2D Dirac fermions. The three-vertices contribute only a factor 1. The Fourier components of the propagators h and $h^\dagger$ are

$$h_k \sim \frac{-1+\kappa^2}{1+\kappa^2}\sigma_0 + 2i\frac{\vec\kappa\cdot\vec\sigma}{1+\kappa^2}, \quad h_k^\dagger \sim \frac{-1+\kappa^2}{1+\kappa^2}\sigma_0 - 2i\frac{\vec\kappa\cdot\vec\sigma}{1+\kappa^2} \quad (\kappa = k/\eta) \tag{14}$$

for $\bar\eta \sim \eta$. These propagators decay exponentially in real space on the scale η, as explained in App. A 1. This is important for the perturbation theory, since δC contributes at least one factor with an off-diagonal element of h:

$$\delta C_{rr'} = C'_{rr'} - \bar C_{rr'} = \sum_{j,j'} e^{i\alpha_{rj}} h_{rj,r'j'} \sum_{r'',j''} h^\dagger_{r'j',r'',j''} e^{-i\alpha_{r''j''}} + \sum_{j,j'} h_{rj,r'j'} e^{i(\phi_j - \phi_{j'})} \ . \tag{15}$$

In particular, the diagonal element reads

$$\delta C_{rr} \sim -\sum_{j,j''}\sum_{r''\neq r} e^{i\alpha_{rj}} h^\dagger_{rj',r'',j''} e^{-i\alpha_{r''j''}} \tag{16}$$

for $\epsilon \sim 0$ and $\eta \gg v_F\lambda^2$. This expression contains only an off-diagonal propagator $h_{rr''}$ with $r'' \neq r$. Therefore, $\delta C_{rr'}$ decays exponentially on the scale $1/\eta$. Thus, only the propagator g determines the convergence of the perturbation series when we take $\eta \gg v_F\lambda^2$. Its Fourier components are [14]

$$\tilde g_q = \frac{1}{1-\frac{1}{2}\bar C_q} \sim \frac{\eta/2}{\epsilon + 2i\vec q\cdot\vec s + 4q^2/\eta} \ , \tag{17}$$

where $\vec s = (\cos\Delta, \sin\Delta)$ depends on the global phase difference $\Delta = \phi_1 - \phi_2$. This propagator is more subtle than the propagators h and $h^\dagger$.

C. Discussion of the propagator g

The propagator $\tilde{g}_q$ is invariant under a global phase shift but it is sensitive to the difference Δ of two uniform phases ϕ_1 and ϕ_2. In particular, the position of its poles with respect to q_1 depends on $c = \cos\Delta$, $s = \sin\Delta$:

$$q_\pm = -i\frac{\eta c}{4} \pm i\sqrt{\eta^2c^2/16 + \epsilon\eta/4 + q_2^2 + i\eta s q_2/2}\ . \tag{18}$$

The corresponding poles with respect to q_2 are obtained by interchanging c and s. It should be noticed that any function of $\tilde{g}_q$ in which we integrate with respect to q does not depend on Δ. This is because $\vec{s}$ appears only as $\vec{q}\cdot\vec{s} = q\cos\varphi$, where φ is the angle between $\vec{q}$ and $\vec{s}$. An example of such a function is $\det g$. In other words, the special choice of Δ affects only space-dependent quantities, such as the correlation functions. On the other hand, fixing of Δ was only necessary to define a starting point of our perturbation expansion. This indicates that Δ should be fixed by a variational procedure to optimize the leading order. According to the standard procedure in mean-field theories [15, 16], this would require a global quantity, such as a free energy. The corresponding quantity in our case would be $\log(\det g)$, which, however, does not depend on Δ. Therefore, Δ plays a similar role as the phase angle in $U(1)$–symmetric models, such as the XY model [15, 16]. The propagator of the XY model does not depend on the phase angle, though. Thus, our theory, which is defined by the three propagators h, $h^\dagger$ and g, does not belong to any of the standard classes of field theory. The dependence of the propagator g on the choice of Δ is related to the fact that our system, defined by the determinant $\det C$, is translational invariant, whereas Anderson localization breaks scaling invariance by creating a finite length scale. This will be explained in more detail once the we have determined the behavior of g_r.

Before we Fourier transform $\tilde{g}_q$ of Eq. (17) we consider the asymptotic case $\eta\sim\infty$, where we neglect the quadratic term $4q^2/\eta$ in the denominator of $\tilde{g}_q$. Then the remaining linear term gives only one pole $q_+ = i\epsilon/2c - (s/c)q_2$, and the Cauchy integration yields

$$\frac{\eta}{2}\int_q \frac{e^{-iq\cdot r}}{\epsilon + 2i\vec{q}\cdot\vec{s}} = i\pi sgn(c)\eta\Theta(-cr_1)e^{\epsilon r_1/2c}\delta_{r_2,sr_1/c} \tag{19}$$

with the Heaviside step function Θ. In the limit $\epsilon\to 0$ this describes a propagator that vanishes everywhere except for the line $r_2 = \tan\Delta\Theta(-cr_1)r_1$. On this line its value is constant and proportional to the scattering rate.

The Fourier transform of the full propagator $\tilde{g}_q$ yields a very similar result, except for a softening of the sharp line $r_2 = \tan\Delta\Theta(-cr_1)r_1$:

$$\tilde{g}_q \to g_r = \frac{\eta}{2}\int_q \frac{e^{-iq\cdot r}}{\epsilon + 2i\vec{q}\cdot\vec{s} + 4q^2/\eta} \tag{20}$$

gives, according to App. A 2, an exponential decay of g_r on the scale $1/\eta$ off the line $r_2 = \tan\Delta\Theta(-cr_1)r_1$:

$$g_r \sim -\frac{\eta\pi}{8c}Ce^{-\eta|cr_1|(s+|c|r_2/|r_1|)^2/8}\times\begin{cases}1 & \text{for } cr_1<0\\ e^{-\eta cr_1/2} & \text{for } cr_1>0\end{cases}\ , \tag{21}$$

where $\eta \gg v_F\lambda^2$, $\epsilon\sim 0$, and the coefficient C is an integral given in Eq. (A8). This propagator is depicted in Fig. 2. Its behavior can be understood as Anderson localization away from this line.

D. Convergent linked cluster expansion for strong scattering

The exponentially decaying behavior of δC and of g on the scale $1/\eta$ implies that the linked cluster expansion of the generating function $\log(\langle\det(C+a)\rangle_a)\Big|_{a=0}$, whose number of graphs grows exponentially with the number of vertices/propagators, is convergent for sufficient large scattering rate η. The constant line of g does not change this fact because the g appear in a loop, according to the Taylor expansion in Eq. (13): Even if the constant line of g contributes on a certain distance along the loop, this is only possible in one direction. Then there is always a contribution from exponentially decaying terms in the

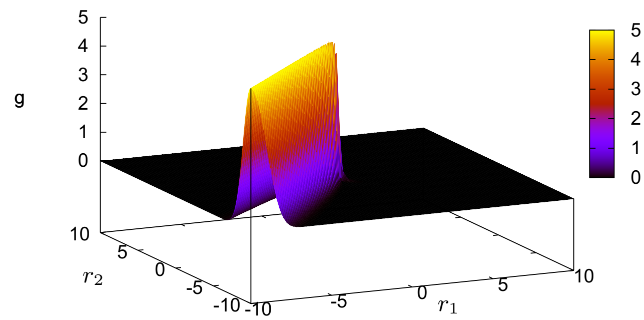

FIG. 2: (Color online) The Propagator g from Eq. (21) with $\Delta = \pi/4$ describes Anderson localization along a semi-infinite line.

opposite direction, in order to close the loop. Therefore, the constant contribution is compensated by an exponentially decaying contribution.

Now we return to the right-hand side of Eq. (8) to calculate the correlation function $K_{rr'}$. There is the leading term from the prefactor in Eq. (10)

$$-\frac{\partial \log(\det g)}{\partial \alpha_{r'r}}\Big|_{a=0} = \frac{1}{2} g_{rr'} \ , \tag{22}$$

and for the expansion terms of the graphical representation the differentiation means breaking up a propagator g into two propagators g, since only g depends on a:

$$\frac{\partial g_{\bar{r}\bar{r}'}}{\partial \alpha_{r'r}}\Big|_{a=0} = -\frac{1}{2} g_{\bar{r}r'} g_{r\bar{r}'} \ . \tag{23}$$

Thus, the correlation function is graphically a sum of random walks from r to r', as depicted in Fig. 1b). These walks are estimated as $O(\eta^4 e^{-2\eta})$, where η^2 is the estimate of the two external g propagators and $\eta^2 e^{-2\eta}$ from the remaining loop. The correlation function (8) then is

$$K_{rr'} = \frac{1}{2} g_{r-r'} + O(\eta^4 e^{-2\eta}) \ , \tag{24}$$

which agrees with (9).

IV. CONCLUSIONS

Using the linked cluster expansion of Ref. [14], we have studied the exponentially decaying correlation function $K_{rr'}$ in (4) for strong scattering. This expansion is constructed from three different propagators, where two of them decay exponentially in all directions and one that decays exponentially only away from a semi-infinite line. Along this semi-infinite line it is constant. The three propagators are connected by two types of vertices, as depicted in Fig. 1. The leading term of the convergent linked cluster expansion is given by the anisotropic propagator g through (24). Therefore, g characterizes Anderson localization of 2D Dirac fermions at strong scattering. Here it should be noticed that we have considered a system on an infinite torus (i.e., for periodic boundary conditions in both directions). Other boundary conditions may change the location of the semi-infinite line, from which Anderson localization appears. In particular, this may also fix the direction of the semi-infinite localization line, which is arbitrary on the infinite system.

Appendix A: Calculation of Propagators

1. Propagator h

Fourier transformation of h_k gives the h_r that decays exponentially on the scaled $1/\eta$.

$$h_q \to h_r = \int_q e^{-iq\cdot r} h_q = \eta^2 \int_q \frac{e^{-iq\cdot r}}{\eta^2+q^2}\left[\left(-1+\frac{q^2}{\eta^2}\right)\sigma_0 + \frac{2i}{\eta}\vec{q}\cdot\vec{\sigma}\right]$$

$$= \sigma_0 \int_q e^{-iq\cdot r} + 2\eta^2 \int_q \frac{e^{-iq\cdot r}}{\eta^2+q^2}\left(-\sigma_0 + \frac{i}{\eta}\vec{q}\cdot\vec{\sigma}\right) = \sigma_0\delta_{r,0} + 2\eta^2 \int_q \frac{e^{-iq\cdot r}}{\eta^2+q^2}\left(-\sigma_0 + \frac{i}{\eta}\vec{q}\cdot\vec{\sigma}\right) \ . \quad \text{(A1)}$$

The integral on the right-hand side is finite for an infinite cut-off. Therefore, we can perform an integration over the entire $\mathbf{R}^2$. This enables us to employ a Cauchy integration. Without restricting the generality we choose $r_j \neq 0$ and obtain for $k \neq j$

$$I_r = \int_{-\lambda}^{\lambda}\int_{-\infty}^{\infty} \frac{e^{-iq_1r_1-iq_2r_2}}{\eta^2+q_1^2+q_2^2} dq_j dq_k = \frac{\pi}{\eta} e^{-\eta|r_j|}\chi_k, \quad \chi_k = \int_{-\lambda}^{\lambda} \frac{e^{-|r_j|\eta(\sqrt{1+q_k^2/\eta^2}-1)+iq_kr_k}}{\sqrt{1+q_k^2/\eta^2}} dq_k \ , \quad \text{(A2)}$$

with $|\chi_k| < \infty$. This yields

$$h_r = \sigma_0\delta_{r,0} - 2\eta^2 I_r\sigma_0 - 2\eta\frac{\partial I_r}{\partial r_1}\sigma_1 - 2\eta\frac{\partial I_r}{\partial r_2}\sigma_2 \ , \quad \text{(A3)}$$

whose off-diagonal terms decay exponentially on the scale η according to (A2). The diagonal term reads

$$h_0 = \sigma_0\left(1 - 2\eta^2\int_q \frac{1}{\eta^2+q^2}\right) \sim -\sigma_0\delta_{r,0} \ ,$$

where the asymptotic result is for $\eta \gg v_F\lambda^2$.

2. Propagator g

We consider the expression in Eq. (21). Without restricting the generality we assume $r_j \neq 0$ and $k \neq j$. Then we perform the q_j integration first:

$$g_r = \frac{\eta^2}{8}\int_{-\lambda}^{\lambda}\int_{-\infty}^{\infty} \frac{e^{-iq_1r_1-iq_2r_2}}{\epsilon\eta/4 + i\eta(q_1c+q_2s) + q_1^2 + q_2^2} dq_j dq_k = \frac{\eta^2}{8}\int_{-\lambda}^{\lambda}\int_{-\infty}^{\infty} \frac{e^{-iq_1r_1-iq_2r_2}}{(q_+-q_j)(q_--q_j)} dq_j dq_k$$

with $q_\pm$ defined in Eq. (18). A Cauchy integration gives

$$g_r = \frac{\eta^2 2i\pi}{8}\begin{cases} \int_{-\lambda}^{\lambda} \frac{e^{-iq_+r_1-iq_2r_2}}{q_--q_+} dq_2 & \text{for } cr_1 < 0 \\ \int_{-\lambda}^{\lambda} \frac{e^{-iq_-r_1-iq_2r_2}}{q_--q_+} dq_2 & \text{for } cr_1 > 0 \end{cases} . \quad \text{(A4)}$$

This leads to

$$g_r = \Gamma_2\begin{cases} 1 & \text{for } cr_1 < 0 \\ e^{-\eta cr_1/2} & \text{for } cr_1 > 0 \end{cases} , \quad \text{(A5)}$$

where

$$\Gamma_2 = -\frac{\eta\pi}{8c}\int_{-\lambda}^{\lambda} \frac{e^{-\eta(\sqrt{1+4\epsilon/\eta c^2+16q_2^2/\eta^2c^2+i8sq_2/\eta c^2}-1)|cr_1|/4-iq_2r_2}}{\sqrt{1+4\epsilon/\eta c^2+16q_2^2/\eta^2c^2+i8sq_2/\eta c^2}} dq_2 \quad \text{(A6)}$$

and Γ_1 after exchanging c and s. Thus, we have $|\Gamma_k| < \infty$. Moreover, we expand the exponent and the denominator for $\eta \gg v_F\lambda^2$. In leading order we obtain

$$\Gamma_2 \sim -\frac{\eta\pi}{8c}\delta_{s|cr_1|/c^2,-r_2} \ , \tag{A7}$$

and when we also include terms with $1/\eta$ in the exponent

$$\Gamma_2 \sim -\frac{\eta\pi}{8c}e^{-\eta|cr_1|(s+|c|r_2/|r_1|)^2/8}\int_{-\lambda}^{\lambda} e^{-|cr_1|q_2^2/\eta c^4}dq_2 \ . \tag{A8}$$

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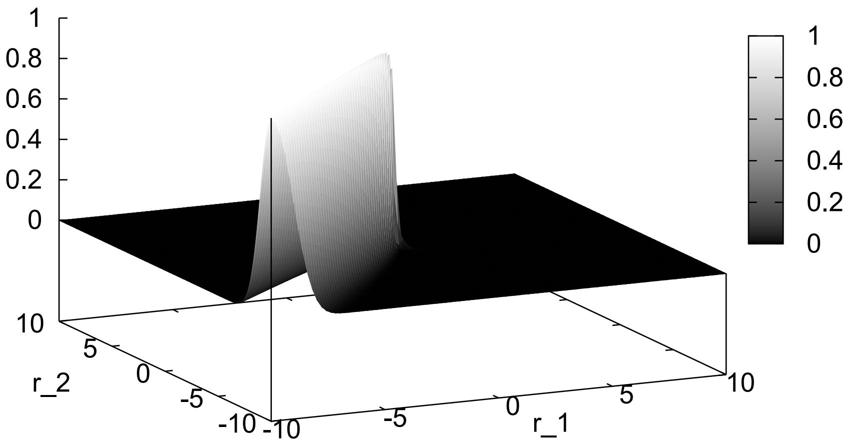

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