

Dirac Fermions on a Two-Dimensional Lattice and the Intermediate Metallic Phase

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Abstract

Consequences of different discretizations of the two-dimensional Dirac operator on low energy properties (e.g., the number of nodes) and their relations to gauge properties are discussed. Breaking of the gauge invariance was suggested in a recent work by M. Bocquet, D. Serban, and M.R. Zirnbauer [cond-mat/9910480] in order to destroy an intermediate metallic phase of lattice Dirac fermions with random mass. It is explained that such a procedure is inconsistent with the underlying lattice physics. Previous results point out that the logarithmic growth of the slope of the average density of states with the system size, obtained in the field-theoretical calculation of M. Bocquet et al., could be a precursor for the appearance of an intermediate metallic phase.

I. INTRODUCTION

Two-dimensional (2D) Dirac (Majorana) fermions can be derived in statistical physics (2D Ising model [1]) and in models of condensed matter systems (quasi-particles in d -wave superconductors [2,3], in the resonant valence bond state of the two-dimensional Heisenberg model [4], or in the quantum Hall effect [5,6,7,8]) always from lattice models. The relation of all these lattice models and 2D Dirac fermions is based on the agreement of their low-energy properties on certain length scales. Using continuous Dirac fermions, which have a linear dispersion, their $k = 0$ node is identified with one of the nodes of the original lattice model. However, there are technical reasons to study a discrete (lattice) version of the Dirac operator because of problems related to the unrestricted spectrum of the continuum model. In the presence of randomness there is also the particular problem of correlations on a characteristic length scale which restricts the use of the unrealistic white noise randomness. In the renormalization group approach [1] or in the saddle-point approximation [9] it is sufficient to specify the lattice by a cutoff of the wavevector, assuming that only the $k = 0$ node is important. For a more detailed discussion of the model one must define the lattice structure explicitly. The simplest case would be a nearest neighbor (NN) discretization of the lattice difference operator

$$\nabla_j f(r) = \frac{1}{2}[f(r + e_j) - f(r - e_j)], \quad (1)$$

where e_j is the lattice unit vector in direction j on a square lattice. This choice, which was used in several papers by the author, exhibits an “intermediate phase”, characterized by a non-vanishing density of states (DOS), in the presence of a random mass for $|\langle m \rangle| \leq m_c$. It turned out that this phase is metallic with finite conductivity [12].

The discretization (1) was criticized in a recent work by Bocquet et al. [13] because of the associated sublattice symmetry. It was argued that the non-vanishing average DOS of this model [12] is a sole consequence of the fact that it belongs to a special universality class, and that a different discretization would have a vanishing average DOS. Their argument is based on the renormalization group calculation of Ref. [1] which gives a *linearly vanishing* DOS at $E = 0$. (However, problems with this calculation exist because the RG has a strange behavior in which the slope of the DOS *diverges* logarithmically with the system size [13].) Since the non-vanishing DOS represents an intermediate metallic phase, observed and discussed for several models in Refs. [9,12,14,15,16,17], the change of the discretization would result in a destruction of this phase.

In the following, an interpretation of the 2D Dirac fermions with discretization (1) is given in terms of a tight-binding model with flux π . It will be shown that the additional sublattice symmetry is a special gauge symmetry. The destruction of the gauge symmetry by an additional next-nearest neighbor term in (1) leads to a lattice model with an inconsistent gauge field representation. More general, a change of the discretization affects physical quantities like the Hall conductivity which depends on the number of nodes. Finally, it will be shown, using a result from a previous calculation, that breaking of the sublattice symmetry does not necessarily result in a vanishing average DOS at $E = 0$. This indicates that even though the properties of the intermediate metallic phase vary with a change of the discretization, its very existence is rather stable.

II. DISCUSSION OF THE MODEL

The 2D Dirac Hamiltonian is given as

$$H = H_0 + m\sigma_3, \quad H_0 = i\nabla_1\sigma_1 + i\nabla_2\sigma_2$$

with Pauli matrices σ_j and a random mass with $\langle m_r \rangle = 0$. ∇_j can be any discrete lattice difference operator. It must be antisymmetric in order to have a Hermitean matrix H . For instance, it could be a NN difference operator (1) or a combination of a NN and a NNN difference operator. The former will be called “ π flux discretization” and the latter “flux breaking (FB) discretization” for reasons which are explained subsequently. The additional NNN difference operator adds new nodes to the dispersion of the pure system. To understand the difference of the discretizations one can consider the pure model, i.e. the Hamiltonian without the random mass. Then the dispersion for the π flux discretization is

$$E_\pi = \pm\sqrt{\sin^2 k_1 + \sin^2 k_2}$$

which has nodes at $k_j = 0, \pm\pi$. For the FB discretization the dispersion is

$$E_{FB} = \pm\sqrt{[\sin k_1 + \sin(2k_1)]^2 + [\sin k_2 + \sin(2k_2)]^2}.$$

with nodes at $k_j = 0, \pm\pi, \arccos(-1/2)$. Additional nodes of the pure system can increase the DOS of the disordered system. This gives rise to the question whether all nodes contribute to the DOS at low energy or only some. On the other hand, the nodes with non-zero k can be removed by adding a diagonal hopping term to the Hamiltonian. As it was shown in a previous study [11] (and will be discussed in Sect. III), the $k = 0$ node is sufficient for a non-zero average DOS at $E = 0$. Thus the states with wavevector $k \approx 0$ appear with non-vanishing density.

Two sublattices can be constructed by considering two layers of the original square lattice, where the layers refer to the Dirac index $\alpha = 1, 2$. Then the sublattices are defined through the condition that for the site (r, α) either $r_1 + r_2 + \alpha$ is odd (sublattice Λ_1) or even (sublattice Λ_2). The Hamiltonian with π flux discretization acts on these two sublattices separately:

$$H = P_1 H P_1 + P_2 H P_2, \quad (2)$$

where P_j is the projector on Λ_j . Adding next-nearest neighbor (NNN) terms to (1) results in the Hamiltonian H_{FB} which couples both sublattices.

The transformation $H \rightarrow H S \sigma_3$ can be applied to the Hamiltonian with

$$S_{rr'} = (-1)^{r_1+r_2} \delta_{rr'}.$$

H commutes with $S \sigma_3$ [11,13]. It changes the sign of $P_2 H P_2$ but leaves $P_1 H P_1$ unchanged. This means it is a gauge transformation on Λ_2 .

It is important to notice that property (2) does not mean that one can project on one of these sublattices and ignore the other projection because both sublattices are still statistically coupled through the same random mass. This can be seen in the product of the Green's function with its complex conjugate $|(H + i\epsilon)_{r\alpha, r'\alpha'}^{-1}|^2$. It was shown [12] that for $\alpha' = \alpha + 1 \pmod{2}$ the symmetry of H implies

$$|(H + i\epsilon)_{r\alpha, r'\alpha'}^{-1}|^2 = -(H + i\epsilon)_{r\alpha, r'\alpha}^{-1} (H + i\epsilon)_{r\alpha', r'\alpha'}^{-1}$$

and

$$|(H + i\epsilon)_{r\alpha, r'\alpha'}^{-1}|^2 = -(H + i\epsilon)_{r\alpha, r'\alpha'}^{-1} (H + i\epsilon)_{r\alpha', r'\alpha}^{-1}.$$

Thus $|(H + i\epsilon)_{r\alpha, r'\alpha'}^{-1}|^2$ on one sublattice is the product of Green's functions from *both* sublattices. The non-trivial properties of the two-particle Green's function are a consequence of the Green's functions on the two sublattices which share a common random mass.

The Hamiltonian $H_j = P_j H P_j$ describes a tight-binding model on the sublattice Λ_j with hopping elements

$$H_{r1,r\pm e_1 2} = H_{r2,r\pm e_1 1} = \pm i$$

$$H_{r1,r\pm e_2 2} = H_{r2,r\mp e_2 1} = \pm 1$$

which represents a Hamiltonian with flux π . Adding NNN terms to ∇_j changes this flux because the new terms have the same phase factors as the NN terms but on links which are twice as large as those of the NN hopping terms. This leads to an inconsistency in the model because particles would experience a magnetic field whose strength depends on the length of the hops. Moreover, the invariance property $[H, S\sigma_3]_- = 0$, which is the gauge invariance on sublattice Λ_2 , is violated by the NNN hopping term.

It is possible to add NNN hopping terms to H which are consistent with the node structure of the π flux discretization. The simplest is

$$h_{nnn} = H_0^2 = (i\nabla_1\sigma_1 + i\nabla_2\sigma_2)^2.$$

It preserves the gauge field of the NN term. Another one is

$$h'_{nnn} = H_0^2\sigma_3 = (i\nabla_1\sigma_1 + i\nabla_2\sigma_2)^2\sigma_3$$

which does not preserve the gauge field because it contributes a positive hopping element for $\alpha = 1$ but a negative hopping element for $\alpha = 2$. Nevertheless, both NNN terms obey the condition

$$[h_{nnn}, S\sigma_3]_- = [h'_{nnn}, S\sigma_3]_- = 0, \quad (3)$$

but only h'_{nnn} satisfies the defining symmetry of class D of Altland and Zirnbauer [19]

$$h'_{nnn} = -\sigma_1 h'_{nnn}{}^T \sigma_1.$$

Property (3) can be used to estimate the non-vanishing average DOS at $E = 0$, as demonstrated in Refs. [10,11].

The discussion of the DOS requires only the diagonal part of the one-particle Green's function $G_{rr,aa} = (H \pm i\epsilon\sigma_0)_{rr,aa}^{-1}$, which can be represented by a sum over closed random walks [18] beginning at r and returning to it with the same Dirac index. Formally, the random walk representation can be obtained from the hopping expansion of the Green's function

$$(H_0 + m\sigma_3 + i\epsilon\sigma_0)^{-1} = (m\sigma_3 + i\epsilon\sigma_0)^{-1} \sum_{l \geq 0} [H_0(m\sigma_3 + i\epsilon\sigma_0)^{-1}]^l.$$

This expansion is convergent for sufficiently large ϵ . After averaging over the random mass one can perform an analytic continuation to arbitrarily small $\epsilon > 0$. The loops experience an effective gauge field because of complex hopping elements. An example is a simple plaquette (Fig. 1). In general the flux per plaquette is $\phi = \pi$, in units of the flux quantum ϕ_0 . An important property of the random walks follows from

$$G(-m, i\epsilon) = \sigma_2 G^T(m, i\epsilon) \sigma_2$$

because this means that the random walks are reversed by the change of the sign of the mass. An interpretation is that the currents in the model can be reversed by reversing the sign of the mass, which is related to the change of the sign of the Hall conductivity σ_{xy} with m [5].

A. Hall Conductivity

A quantity which is sensitive to the type of discretization but robust against disorder is the Hall conductivity σ_{xy} . It can be measured as the linear response to an external gauge field [5]. In a pure system ($m = \text{const.}$) with Hamiltonian $H = m\sigma_3 + h_1\sigma_1 + h_2\sigma_2$ it reads [8]

$$\sigma_{xy} = \frac{m}{2} \int \frac{1}{(m^2 + h_1^2 + h_2^2)^{3/2}} \frac{d^2k}{(2\pi)^2}$$

in units of e^2/\hbar . For $m \sim 0$ only the nodes contribute significantly to the intergral:

$$\sigma_{xy}(m) = \frac{m}{|m|} f(|m|),$$

where $f(0) \neq 0$ is proportional to the number of nodes. In the case of the π flux Dirac operator this gives just a Hall step for *each* sublattice which means a single Hall step for the corresponding π flux tight-binding model. For the modified Hamiltonian $H_0 + m\sigma_3 + h'_{nnn}$ the Hall conductivity is

$$\frac{1}{2} \int \frac{m + \sin^2 k_1 + \sin^2 k_2}{[(m + \sin^2 k_1 + \sin^2 k_2)^2 + \sin^2 k_1 + \sin^2 k_2]^{3/2}} \frac{d^2k}{(2\pi)^2}$$

which gives

$$\sigma_{xy} \sim \text{const.} + \frac{1}{2\pi} \frac{m}{|m|} \quad (m \sim 0).$$

Thus, the additional term h'_{nnn} contributes a constant Hall conductivity at small values of m . These results indicate that approximations of the original model should not change the number of nodes nor violate the structure of the Hamiltonian by adding new terms of the type h'_{nnn} in order to get the correct Hall conductivity.

A general rule is that all nodes of the lattice model should be included in the calculation of the low energy properties. This has severe consequence for the renormalization group (RG) calculation which usually deals only with one large length scale. The various nodes (different length scales) will create additional couplings under the RG transformation, leading eventually to a strong coupling behavior.

III. MODEL WITH SINGLE NODE

To study the contribution of the $k = 0$ node to the average DOS, all nodes of the π flux discretization must be removed except for that at $k = 0$. This can be achieved by adding a diagonal term to H

$$H_1 = H + (\Delta - 2)\sigma_3$$

with $2\Delta f(r) = f(r+e_1)+f(r-e_1)+f(r+e_2)+f(r-e_2)$. Obviously, this Hamiltonian violates the sublattice gauge-invariance condition:

$$\Delta\sigma_3 S = -\sigma_3 S \Delta.$$

Without randomness (i.e. $m = 0$) the dispersion is

$$E_1(k_1, k_2) = \pm\sqrt{(\cos k_1 + \cos k_2 - 2)^2 + \sin^2 k_1 + \sin^2 k_2}$$

which is non-zero except for the node at $k_j = 0$. Now this Hamiltonian can be coupled by a random field m' to the Hamiltonian with nodes at $\pm\pi$

$$H_2 = H - (\Delta + 2)\sigma_3$$

as

$$\hat{H} = \begin{pmatrix} H_1 & m'\sigma_3 \\ m'\sigma_3 & H_2 \end{pmatrix}.$$

Both Hamiltonians H_1 and H_2 violate the sublattice gauge-invariance condition (3) but preserve the symmetry of class D

$$\hat{H} = -\sigma_1 \hat{H}^T \sigma_1.$$

The Green's function can be projected on the subspace of H_1

$$[H_1 + i\epsilon - m'\sigma_3(H_2 + i\epsilon)^{-1}\sigma_3 m']^{-1}$$

It was shown that this expression leads to a non-vanishing average DOS at $E = 0$ [11], indicating that the projection on one node produces already a non-vanishing average DOS.

As a remark it should be noted that the pure 2D Ising model at the critical point is governed by the Hamiltonian [9]

$$H_{2DIM} = a(1 + a^2 + 2a \cos k_1 - 2 \cos k_2)\sigma_3 + 2a(\sin k_1 \sigma_1 + \sin k_2 \sigma_2)$$

with $a = \sqrt{2} - 1$. This Hamiltonian is similar to H_1 as it has only one node at $k = 0$.

IV. CONCLUSIONS

The previous discussion is based on the one-particle Green's function without referring to an effective field theory. This approach is motivated by the aim to avoid additional symmetries which appear by the introduction of new fields and which are not related to the original Hamiltonian or the one-particle Green's function. In the case of the Hamiltonian H , regardless of the discretization of ∇_j , there are two *discrete* symmetries:

$$H \rightarrow -\sigma_3 H \sigma_3 \quad (4)$$

for the ensemble and, therefore, for the *average* one-particle Green's function. Moreover, there is a symmetry under the discrete transformation

$$H \rightarrow -\sigma_1 H^T \sigma_1, \quad (5)$$

which defines class D of Ref. [19]. It holds for each realization of the model with Hamiltonian H . h_{nnn} and h'_{nnn} , both break symmetry (4) whereas only h_{nnn} breaks symmetry (5).

From the symmetry point of view alone it is not entirely clear under which conditions a vanishing average DOS at $E = 0$ exist. The competition of different nodes (i.e. different length scales) cannot be described only by global symmetries but requires more detailed knowledge. However, there is a simple argument in terms of the hopping expansion which indicates that the number of *independent* random terms in the Dirac Hamiltonian H is crucial for the behavior of the average DOS around $E = 0$. Taking the zero-dimensional limit of H , i.e. the leading order of the hopping expansion, there is a power law

$$\langle \rho(E) \rangle \sim \rho_0 |E|^\alpha,$$

where $\alpha = 0$ (random mass) and $\alpha = 1$ (two-component random vector potential). The latter, of course, violates also the symmetry (5).

The non-vanishing DOS for the class D model in $d = 2$ with a diffusive behavior was recently discussed by Senthil and Fisher [16] and Read and Green [17] in terms of the RG flow of a non-linear sigma model. Although this approach was criticized by Bocquet et al. [13], its result is in agreement with that obtained with a different approach [12]. The appearance of the intermediate phase is quite natural for the 2D Dirac fermions, since the pure model has a singular “metallic” phase at $E = m = 0$ with conductivity $e^2/h\pi$. This phase is robust against a random vector potential, where the value of the conductivity remains unchanged [8]. A random mass has apparently a stronger effect because it reduces the conductivity at $m = 0$ by a factor $1/(1 + g/\pi)$, where g is the variance of the random mass [12]. Moreover, it broadens the singular phase at $m = 0$ to an interval $-m_c < m < m_c$ with non-vanishing DOS. In terms of the random bond Ising model the non-vanishing DOS reflects the existence of the Griffiths-McCoy-Wu phase, which cannot be seen in the perturbative RG approach [20].

The occurrence of vortex-like excitations might be an important effect, as suggested in Refs. [13,16,17]. Using the π flux discretization, these vortices can be created by local edge currents in areas where the sign of the random mass changes: An area with a positive mass has a positive Hall conductivity whereas the surrounding area with negative mass has a Hall conductivity with opposite sign. The resulting edge currents can have a long-range interaction. An effective model for this behavior can be found in terms of the Q matrix field theory of Ref. [12], in which the random mass is replaced by a matrix field. Details will be published separately.

In conclusion, the change of the discretization has a strong effect on the node structure of the Dirac Hamiltonian. The correct discretization is determined by

the effective gauge field which the Dirac fermions experience. In the case of Dirac fermions with random mass, however, the average DOS at low energies is relatively robust against the change of the discretization. In particular, the $k \approx 0$ modes have a substantial contribution to the average DOS. This indicates that also the intermediate metallic phase of the Dirac fermions with random mass and π flux discretization should be robust under a change of the discretization.

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FIGURES

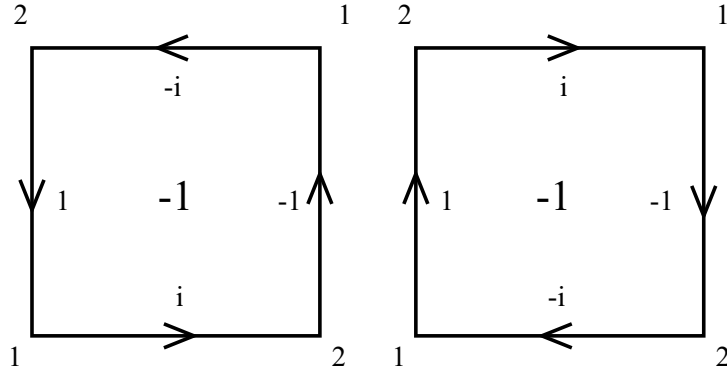


FIG. 1. Simple plaquettes are created from the Hamiltonian H_0 with nearest neighbor discretization. They enclose a flux $\phi = \pi$ which gives $e^{i\phi} = -1$. The sublattice gauge transformation $H \rightarrow HS\sigma_3$ does not change this property.