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A C^0 interior penalty discontinuous Galerkin Method for fourth-order total variation flow. II: Existence and uniqueness

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We prove the existence and uniqueness of a solution of a C^0 Interior Penalty Discontinuous Galerkin (C^0 IPDG) method for the numerical solution of a fourth-order total variation flow problem that has been developed in part I of the paper. The proof relies on a nonlinear version of the Lax-Milgram Lemma. It requires to establish that the nonlinear operator associated with the C^0 IPDG approximation is Lipschitz continuous and strongly monotone on bounded sets of the underlying finite element space.

KEYWORDS
C^0 interior penalty discontinuous Galerkin method, existence and uniqueness, fourth-order total variation flow

1 | INTRODUCTION

We consider the following fourth-order total variation flow (TVF) problem:

$$\frac{\hat{\partial} w}{\hat{\partial} \hat{t}} + \beta \hat{\Delta} \hat{\nabla} \cdot \frac{\hat{\nabla} w}{|\hat{\nabla} w|} = 0 \qquad \text{in } \hat{Q} := \hat{\Omega} \times (0, \hat{T}), \tag{1.1a}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta \frac{\hat{\nabla} w}{|\hat{\nabla} w|} = \mathbf{n}_{\hat{\Gamma}} \cdot \hat{\nabla} \hat{\nabla} \left(\hat{\nabla} \cdot \frac{\hat{\nabla} w}{|\hat{\nabla} w|} \right) = 0 \qquad \text{on } \hat{\Sigma} := \hat{\Gamma} \times (0, \hat{T}), \tag{1.1b}$$

$$w(\cdot, 0) = w^0 \quad \text{in } \hat{\Omega}. \tag{1.1c}$$

where $\hat{\Omega} \subset \mathbb{R}^2$ is a bounded domain with boundary $\hat{\Gamma} = \partial \hat{\Omega}$, $\hat{T} > 0$ is the final time, $\beta > 0$ is some constant, $\mathbf{n}_{\hat{\Gamma}}$ stands for the exterior unit normal at $\hat{\Gamma}$, and $w^0 \in L^2(\hat{\Omega})$ is some given initial data. The fourth order Equation (1.1a) has to be understood as the flow problem

$$-\frac{\hat{\partial} w}{\hat{\partial} \hat{t}} \in \partial E_{H^{-1}}(w)$$

associated with the total variation-H^{-1} (TV-H^{-1}) minimization of the energy functional

$$E(w) = \beta \int_{\hat{\Omega}} |\hat{\nabla} w| \ dx, \qquad \beta > 0, \tag{1.2}$$

where $\partial_{H^{-1}} E(w)$ is the H^{-1} subdifferential of E. In fact, if we introduce an inner product on $H^{-1}(\hat{\Omega}$ according to

$$(w, z)_{-1,\hat{\Omega}} := (\hat{\nabla}(-\hat{\Delta}^{-1} w), \hat{\nabla}(-\hat{\Delta}^{-1} z))_{0,\hat{\Omega}},$$

the subdifferential

$$\partial_{H^{-1}} E(w) = \{v \in H^{-1}(\hat{\Omega}) \ | \ (v, z - w)_{-1,\Omega} \leq E(z) - E(w) \ \text{ for all } \ z \in H^{-1}(\hat{\Omega})\}$$

reads as follows (cf., eg, [1]):

$$\partial_{H^{-1}} E(w) = \{\hat{\Delta}\hat{\nabla} \cdot \boldsymbol{\xi} \ | \ \boldsymbol{\xi}(\hat{x}) \in \partial\Phi(\hat{\nabla} w(\hat{x}))\}.$$

Here, $\Phi(|\boldsymbol{\eta}|)$ and $\partial\Phi(|\boldsymbol{\eta}|)$ are given by

$$\Phi(\boldsymbol{\eta}) = \beta \ | \ \boldsymbol{\eta} \ |, \qquad \partial\Phi(\boldsymbol{\eta}) = \begin{cases} \beta\boldsymbol{\eta}/ \ | \ \boldsymbol{\eta} \ | & \text{if } \ \boldsymbol{\eta} \neq \mathbf{0} \\ \{\boldsymbol{\tau} \in \mathbb{R}^2 \ | \ |\boldsymbol{\tau}| \leq \beta\} & \text{if } \ \boldsymbol{\eta} = \mathbf{0} \end{cases}. \tag{1.3}$$

The fourth-order TVF problem (Equations (1.1a) to (1.1c) describes surface relaxation below the roughening temperature. We note that similar fourth-order TVF problems occur in image recovery. For more details we refer to Bhandari and coworkers [2] and the references therein.

In the sequel, we consider the regularized fourth-order TVF problem

$$\frac{\hat{\partial} w}{\hat{\partial} \hat{t}} + \beta\hat{\Delta}\hat{\nabla} \cdot ((\delta^2 + |\hat{\nabla} w|^2)^{-1/2} \hat{\nabla} w) = 0 \quad \text{ in } \ \hat{Q}, \tag{1.4a}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta(\delta^2 + |\hat{\nabla} w|^2)^{-1/2} \hat{\nabla} w = 0 \quad \text{ on } \ \hat{\Sigma}, \tag{1.4b}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta\hat{\nabla}(\hat{\nabla} \cdot (\delta^2 + |\hat{\nabla} w|^2)^{-1/2} \hat{\nabla} w) = 0 \quad \text{ on } \ \hat{\Sigma},$$

$$w(\cdot, 0) = w^0 \quad \text{ in } \ \hat{\Omega}, \tag{1.4c}$$

where $\delta > 0$ is a regularization parameter. We further consider a scaling in both the time variable and the spatial variables according to

$$t = \delta\hat{t}, \qquad x_i = \delta\hat{x}_i, \ \ 1 \leq i \leq 2. \tag{1.5}$$

Setting $T := \delta\hat{T}, \Omega := \delta\hat{\Omega}, \Gamma := \partial\Omega, Q := \Omega \times (0, T), \Sigma := \Gamma \times (0, T)$, and $u^0(x) = w^0(\delta^{-1} x)$, as well as

$$\omega(\nabla u) := 1 + |\nabla u|^2, \tag{1.6}$$

the scaled and regularized fourth order TVF problem reads as follows

$$\frac{\partial u}{\partial t} + \beta\delta^2 \Delta\nabla \cdot (\omega(\nabla u)^{-1/2} \nabla u) = 0 \quad \text{ in } \ Q, \tag{1.7a}$$

$$\mathbf{n}_\Gamma \cdot \beta\delta^2 (\omega(\nabla u)^{-1/2} \nabla u) = \mathbf{n}_\Gamma \cdot \beta\delta^2 \nabla(\nabla \cdot ((\omega(\nabla u)^{-1/2} \nabla u)) = 0 \quad \text{ on } \ \Sigma, \tag{1.7b}$$

$$u(\cdot, 0) = u^0 \quad \text{ in } \ \Omega. \tag{1.7c}$$

The numerical solution of the regularized fourth-order TVF problem with periodic boundary conditions has been considered in Kohn and Versieux [3] based on a mixed formulation of the implicitly in time discretized problem. At each time-step, this amounts to the solution of 2 second-order elliptic PDEs by standard Lagrangian finite elements with respect to a triangulation of the computational domain Ω. On the other hand, a C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) method has been developed and implemented in Bhandari and coworkers [2]. The advantage of the C^0IPDG approach is that it directly applies to the fourth-order problem and thus only requires the numerical solution of one equation by using the same Lagrangian finite elements as in the mixed method.

The paper is organized as follows: After some basic notations from matrix analysis and Lebesgue and Sobolev spaces presented in Section 2, in Section 3 we recall the C^0IPDG approximation of the implicity in time discretized, regularized, and scaled fourth-order TVF problem from Bhandari and coworkers [2]. Section 4 is devoted to a proof of the existence and uniqueness of a solution of the C^0IPDG approximation by an application of the nonlinear version of the Lax-Milgram Lemma. In particular, this requires to show that the nonlinear operator associated with the C^0IPDG approximation is Lipschitz continuous and strongly monotone on bounded subsets of the underlying function space.

2 | BASIC NOTATIONS

For vectors $\underline{\mathbf{x}} = (x_1, \cdots, x_n)^T, \underline{\mathbf{y}} = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$ and for matrices $\underline{\underline{\mathbf{A}}} = (a_{ij})_{i,j=1}^n, \underline{\underline{\mathbf{B}}} = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ we denote by $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$ and $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}}$ the Euclidean inner product $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}} = \sum_{i=1}^n x_i y_i$ and the Frobenius inner product $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = \sum_{i,j=1}^n a_{ij} b_{ij}$. In particular, $\mid \mathbf{x} \mid := (\mathbf{x} \cdot \mathbf{x})^{1/2}$ and $\mid \underline{\underline{\mathbf{A}}} \mid := (\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{A}}})^{1/2}$ refer to the Euclidean norm and the Frobenius norm, respectively.

We will further use standard notation from Lebesgue and Sobolev space theory (cf., eg, [4]). In particular, for a bounded domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we refer to $L^p(D)$, $1 \le p < \infty$, as the Banach space of pth power Lebesgue integrable functions on D with norm $\|\cdot\|_{0,p,D}$ and to $L^\infty(D)$ as the Banach space of essentially bounded functions on D with norm $\|\cdot\|_{0,\infty,D}$. Moreover, we denote by $W^{s,p}(D)$, $s \in \mathbb{R}_+$, $1 \le p \le \infty$, the Sobolev spaces with norms $\|\cdot\|_{s,p,D}$. We note that for $p=2$ the spaces $L^2(D)$ and $W^{s,2}(D) = H^s(D)$ are Hilbert spaces with inner products $(\cdot,\cdot)_{0,2,D}$ and $(\cdot,\cdot)_{s,2,D}$. In the sequel, we will suppress the subindex 2 and write $(\cdot,\cdot)_{0,D}$, $(\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{0,D}$, $\|\cdot\|_{s,D}$ instead of $(\cdot,\cdot)_{0,2,D}$, $(\cdot,\cdot)_{s,2,D}$ and $\|\cdot\|_{0,2,D}$, $\|\cdot\|_{s,2,D}$. The space $W_0^{s,p}(D)$ is the closure of C_0^∞ with respect to the $\|\cdot\|_{s,p,D}$-norm. We refer to $W^{-s,p}(D)$, $s \in \mathbb{R}_+$, $1 \le p \le \infty$, as the dual of $W_0^{s,q}(D)$, where $1/p + 1/q = 1$. In particular, $H^{-s}(D) = (H_0^s(D))^*$.

3 | C^0 IPDG APPROXIMATION

We perform a discretization in time of Equation (1.7) with respect to a partition of the time interval $[0, T]$ into subintervals $[t_{m-1}, t_m]$, $1 \le m \le M$, $M \in \mathbb{N}$, of length $\Delta t := t_m - t_{m-1} = T/M$. Denoting by u^m some approximation of u at time t_m, for $1 \le m \le M$ we have to solve the problems

$$u^m - u^{m-1} + \Delta t \beta \delta^2 \Delta \nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m) = 0 \;\text{ in }\; \Omega, \tag{3.1a}$$

$$\mathbf{n}_\Gamma \cdot \beta \delta^2 (\omega(\nabla u^m)^{-1/2} \nabla u^m) = 0 \;\text{ on }\; \Gamma, \tag{3.1b}$$

$$\mathbf{n}_\Gamma \cdot \beta \delta^2 \nabla (\nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m)) = 0 \;\text{ on }\; \Gamma. \tag{3.1c}$$

We reformulate the second term on the left-hand side of Equation (3.1a) according to

$$\Delta \nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m) = \nabla \cdot \nabla (\nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m)) = \nabla \cdot \nabla \cdot \nabla (\omega(\nabla u^m)^{-1/2} \nabla u^m). \tag{3.2}$$

As has been shown in Bhandari and coworkers [2], we have

$$\nabla(\omega(\nabla u^m)^{-1/2}\nabla u^m) = \omega(\nabla u^m)^{-3/2}\underline{\underline{\mathbf{M}}}(u^m)D^2u^m, \tag{3.3}$$

where D^2u^m is the 2×2 matrix of second partial derivatives of u^m and the matrix $\underline{\underline{\mathbf{M}}}(u^m)$ is given by

$$\underline{\underline{\mathbf{M}}}(u^m) := \begin{pmatrix} 1+\left(\frac{\partial u^m}{\partial x_2}\right)^2 & -\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2} \\ -\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2} & 1+\left(\frac{\partial u^m}{\partial x_2}\right)^2 \end{pmatrix}. \tag{3.4}$$

We note that the matrix $\underline{\underline{\mathbf{M}}}(u^m)$ is symmetric positive definite with the eigenvalues

$$\lambda_{\min}(\underline{\underline{\mathbf{M}}}(u^m)) = 1, \qquad \lambda_{\max}(\underline{\underline{\mathbf{M}}}(u^m)) = 1 + |\nabla u^m|^2. \tag{3.5}$$

Setting

$$\underline{\underline{\mathbf{A}}}_1(v) := \omega(\nabla v)^{-3/2}\underline{\underline{\mathbf{M}}}(v), \tag{3.6}$$

the weak formulation of the implicitly in time discretized regularized fourth order TVF problem (3.1a) to (3.1c) reads: Find

$$u^m \in V := \{v \in H^2(\Omega) \ | \ \mathbf{n}_\Gamma \cdot \beta\delta^2\omega(\nabla v)^{-1/2}\nabla v = 0 \ \text{ on } \ \Gamma\}$$

such that for all $v \in V$ it holds

$$(u^m - u^{m-1}, v)_{0,\Omega} + \Delta t\beta\delta^2 \int_\Omega (\underline{\underline{\mathbf{A}}}_1(u^m)D^2u^m) : D^2v \ dx = 0. \tag{3.7}$$

For the discretization in space we assume $\mathcal{T}_h$ to be a geometrically conforming, simplicial triangulation of Ω. We denote by $\mathcal{E}_h(\Omega)$ and $\mathcal{E}_h(\Gamma)$ the set of edges of $\mathcal{T}_h$ in the interior of Ω and on the boundary Γ, respectively, and set $\mathcal{E}_h := \mathcal{E}_h(\Omega)\cup\mathcal{E}_h(\Gamma)$. For $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we denote by h_K and h_E the diameter of K and the length of E, and we set $h := \max(h_K \ | \ K \in \mathcal{T}_h)$. Due to the assumptions on $\mathcal{T}_h$ there exist constants $0 < c_R \le C_R$, $0 < c_Q \le C_Q$, and $0 < c_S \le C_S$ such that for all $K \in \mathcal{T}_h$ it holds

$$c_R h_K \le h_E \le C_R h_K, \qquad E \in \mathcal{E}_h(\partial K), \tag{3.8a}$$

$$c_Q h \le h_K \le C_Q h, \tag{3.8b}$$

$$c_S h_K^2 \le \text{meas}(K) \le C_S h_K^2. \tag{3.8c}$$

Denoting by $P_k(T)$, $k \in \mathbb{N}$, the linear space of polynomials of degree $\le k$ on T, for $k \in \mathbb{N}$ we define

$$V_h := \{v_h \in C^0(\overline{\Omega}) \ | \ v_h|_T \in P_k(T), \ \ T \in \mathcal{T}_h\}, \tag{3.9}$$

and note that $V_h \subset H^1(\Omega)$, but $V_h \not\subset H^2(\Omega)$. Further, we introduce

$$\underline{\underline{\mathbf{M}}}_h := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega)^{2\times 2} \ | \ \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2\times 2}, \ \ K \in \mathcal{T}_h\} \tag{3.10}$$

as the space of element-wise polynomial moment tensors. For interior edges $E \in \mathcal{E}_h(\Omega)$ such that $E = K_+ \cap K_-, K_\pm \in \mathcal{T}_h$ and boundary edges on Γ we introduce the average and jump of ∇v_h according to

$$\{\nabla v_h\}_E := \begin{cases} \frac{1}{2}(\nabla v_h|_{E\cap K_+} + \nabla v_h|_{E\cap K_-}) & E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E & E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{3.11a}$$

$$[\nabla v_h]_E := \begin{cases} \nabla v_h|_{E\cap K_+} - \nabla v_h|_{E\cap K_-} & E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E & E \in \mathcal{E}_h(\Gamma) \end{cases}. \tag{3.11b}$$

The average $\{\Delta v_h\}_E$ and jump $[\Delta v_h]_E$ are defined analogously. We further denote by $\mathbf{n}_E$ the unit normal vector on E pointing in the direction from K_+ to K_-. In the sequel, for $E \in \mathcal{E}_h$ we will frequently use

$$| \{v_h w_h\}_E | \le 2\{|v_h|\}_E\{|w_h|\}_E, \tag{3.12a}$$

$$| [v_h w_h]_E | \le 4\{|v_h|\}_E\{|w_h|\}_E. \tag{3.12b}$$

In fact, for $E \in \mathcal{E}_h(\Omega)$ Equations (3.12a) and (3.12b) follow from

$$| \{v_h w_h\}_E | \le \frac{1}{2}(|v_h|_{E_+}|w_h|_{E_+} + |v_h|_{E_-}|w_h|_{E_-}) \le 2\{|v_h|\}_E\{|w_h|\}_E,$$
$$| [v_h w_h]_E | \le (|v_h|_{E_+}|w_h|_{E_+} + |v_h|_{E_-}|w_h|_{E_-}) \le 4\{|v_h|\}_E\{|w_h|\}_E,$$

whereas it is obvious for $E \in \mathcal{E}_h(\Gamma)$. We will also use

$$\sum_{E\in\mathcal{E}_h} [v_h w_h]_E = \sum_{E\in\mathcal{E}_h} \{v_h\}_E[w_h]_E + \sum_{E\in\mathcal{E}_h(\Omega)} [v_h]_E\{w_h\}_E. \tag{3.13}$$

Following the general approach [5] for DG approximations of second-order elliptic boundary value problems, in Bhandari and coworkers [2] we have derived the following C^{0}IPDG approximation of Equation (3.7): Find $u_h^m \in V_h$ such that for all $v_h \in V_h$ it holds

$$(u_h^m, v_h)_{0,\Omega} + \Delta t\beta\delta^2 a_h^{IP}(u_h^m, v_h; u_h^m) = (u_h^{m-1}, v_h)_{0,\Omega}, \qquad v_h \in V_h. \tag{3.14}$$

Here, for $z_h \in V_h$ the mesh-dependent semilinear C^0IPDG form $a_h^{IP}(\cdot,\cdot;z_h) : V_h \times V_h \to \mathbb{R}$ is given by

$$\begin{aligned} a_h^{IP}(u_h, v_h; z_h) := &\sum_{K\in\mathcal{T}_h} (\underline{\underline{\mathbf{A}}}_1(z_h)D^2u_h, D^2v_h)_{0,K} \\ &- \sum_{E\in\mathcal{E}_h} (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(z_h)D^2u_h\}_E\mathbf{n}_E, \mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4}\nabla v_h]_E)_{0,E} \\ &- \sum_{E\in\mathcal{E}_h} (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(z_h)D^2v_h\}_E\mathbf{n}_E, \mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4}\nabla u_h]_E)_{0,E} \\ &+ \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1}(\mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4}\nabla u_h]_E, \mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4}\nabla v_h]_E)_{0,E}, \end{aligned} \tag{3.15}$$

where $\alpha > 0$ is a penalty parameter and

$$\underline{\underline{\mathbf{A}}}_2(z_h) := \omega(\nabla z_h)^{-5/4}\underline{\underline{\mathbf{M}}}(z_h). \tag{3.16}$$

4 | EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE C^{0}IPDG APPROXIMATION

The existence and uniqueness of a solution of the C^{0}IPDG approximation Equation (3.14) can be shown using the following nonlinear analogue of the Lax-Milgram Lemma.

Theorem 4.1 *Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$ and associated norm $\|\cdot\|_V$ and let V^* be the dual space with norm $\|\cdot\|_{V^*}$. We denote by $\langle\cdot,\cdot\rangle_{V^*,V}$ the dual pairing between V^* and V. Let $A : V \to V^*$ be a nonlinear operator with $A(0) = 0$ that is Lipschitz continuous on $B(0, R) := \{v \in V \mid \|v\|_V \le R\}$, $R > 0$, that is, there exists a constant $\Gamma(R) > 0$ such that for all $v, w \in V$ it holds*

$$\|A(v) - A(w)\|_{V_h^*} \le \Gamma(R)\ \|v - w\|_V. \tag{4.1}$$

Moreover, assume that $A: V \to V^*$ is strongly monotone on B(0, R), that is, there exists a constant $\gamma(R) > 0$ such that for all v, w ∈ B(0, R) it holds

$$\langle A(v) - A(w), v - w\rangle_{V^*,V} \geq \gamma(R) \ \ \|v - w\|_V^2. \tag{4.2}$$

Then, for any $\ell \in V^*$ with

$$\|\ell\|_{V^*} \leq \frac{\Gamma(R)^2}{\gamma(R)} \left(1 - \sqrt{1 - \frac{\gamma(R)^2}{\Gamma(R)^2}}\right) R, \tag{4.3}$$

the nonlinear equation

$$Au = \ell \tag{4.4}$$

has a unique solution u ∈ B(0, R).

Proof. We refer to $\tau : V^* \to V$ as the Riesz mapping, that is,

$$\langle \ell, v\rangle_{V^*,V} = (\tau\ell, v)_V, \qquad \ell \in V^*, \ \ v \in V. \tag{4.5}$$

Then, $u \in B(0, R)$ is a solution of Equation (4.4) if and only if u is a fixed point of the nonlinear map $T : V \to V$ defined by means of

$$T(v) := v - \rho(\tau A(v) - \tau\ell), \qquad v \in V, \ \ \rho > 0.$$

Due to Equation (4.5) we have

$$\|T(v) - T(w)\|_V^2 = \|v - w\|_V^2 - 2\rho\langle A(v) - A(w), v - w\rangle_{V^*,V} + \rho^2\|A(v) - A(w)\|_{V^*}^2. \tag{4.6}$$

Now, using Equations (4.1) and (4.2) it follows that

$$\|T(v) - T(w)\|_V^2 \leq q\|v - w\|_V^2, \qquad q := 1 - 2\rho\gamma(R) + \rho^2\Gamma(R)^2.$$

For $\rho \in (0, 2\gamma(R)/\Gamma(R)^2)$ we have $q < 1$ and hence, T is a contraction on $B(0, R)$. We note that q attains its minimum $q_{min} = 1 - \gamma(R)^2/\Gamma(R)^2$ for $\rho_{min} = \gamma(R)/\Gamma(R)^2$. Moreover, choosing $w = 0$ in Equation (4.6) and observing $A(0) = 0$, we have

$$\|T(v) - T(0)\|_V^2 \leq q_{\min}\|v\|_V^2,$$

and hence, for $v \in B(0, R)$ it holds

$$\|T(v)\|_V \leq \|T(v) - T(0)\|_V + \|T(0)\|_V \leq \sqrt{q_{\min}}R + \rho\|\ell\|_{V^*}.$$

■

Consequently, we have

$$\|T(v)\|_V \leq R, \tag{4.7}$$

if $\ell \in V^*$ satisfies Equation (4.3). We deduce from Equation (4.7) that $T(B(0, R)) \subset B(0, R)$. The Banach fixed point theorem asserts the existence and uniqueness of a fixed point in $B(0, R)$.

In order to apply the previous result to the C^0IPDG method Equation (3.14), we introduce a mesh-dependent semi-norm $|\cdot|_{2,h,\Omega}$ and weighted norm $\|\cdot\|_{2,h,\Omega}$ on V_h according to

$$|v_h|_{2,h,\Omega} := \left(\sum_{K\in\mathcal{T}_h} \int_K D^2v_h : D^2v_h \ \ dx + \ \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_{E\in\mathcal{E}_h} |\mathbf{n}_E \cdot [\nabla v_h]_E|^2 \ \ ds\right)^{1/2}, \tag{4.8a}$$

$$\|v_h\|_{2,h,\Omega} := (\|v_h\|_{0,\Omega}^2 + |v_h|_{2,h,\Omega}^2)^{1/2}. \tag{4.8b}$$

We further note that (3.14) can be written as the nonlinear equation

$$A_h^{DG} u_h^m = \ell_h, \tag{4.9}$$

where the nonlinear operator $A_h^{DG} : V_h \to V_h^*$ and the linear functional $\ell_h \in V_h^*$ are given by

$$\langle A_h^{DG} v_h, w_h \rangle_{V_h^*, V_h} := (v_h, w_h)_{0,\Omega} + \Delta t \beta \delta^2 \ a_h^{DG}(v_h, w_h; v_h), \qquad v_h, w_h \in V_h, \tag{4.10}$$

$$\ell_h(v_h) := (u_h^{m-1}, v_h)_{0,\Omega}, \qquad v_h \in V_h. \tag{4.11}$$

For the proof of Lipschitz continuity on bounded sets and strong monotonicity of the nonlinear operator A_h^{DG} we need the inverse estimates (cf., eg, [6, 7]): For $p \in [1, \infty]$ and $\ell, m \in \mathbb{N}_0$ it holds

$$\|v_h\|_{m,p,K} \le \frac{C_{inv}}{\text{meas}(K)^{max\left(0, \frac{1}{2} - \frac{1}{p}\right)} \ h_K^{m-\ell}} \|v_h\|_{\ell,K}, \qquad v_h \in V_h, \tag{4.12}$$

where C_{inv} is a positive constant that only depends on k, ℓ, m, p, and the shape regularity of the triangulation. We further need the trace inequalities (cf., eg, [8, 9]): For $p \in [1, \infty]$, $m \in \mathbb{N}_0$, and $K \in \mathcal{T}_h$ it holds

$$\|\nabla v_h\|_{m,p,\partial K} \le C_T h_K^{-1/p} \|\nabla v_h\|_{m,p,K}, \qquad v_h \in V_h, \tag{4.13a}$$

$$\|D^2 v_h\|_{m,p,\partial K} \le C_T h_K^{-1/p} \|D^2 v_h\|_{m,p,K}, \qquad v_h \in V_h, \tag{4.13b}$$

where C_T is a positive constant that only depends on k, m, p, and the shape regularity of the triangulation. Moreover, we will frequently use the following Poincaré-Friedrichs inequality for piecewise H^2-functions (cf., eg, [10])

$$\|\nabla v_h\|_{0,\Omega} \le C_{PF} |v_h|_{2,h,\Omega}, \qquad v_h \in V_h, \tag{4.14}$$

where $C_{PF} > 0$ is a constant that only depends on Ω and the shape regularity of the triangulation.

In the sequel, we will frequently use some basic estimates for the weight function $\omega(\boldsymbol{\nabla} v_h)$. In particular, for $\beta > 0$ and $v \in V_h$ it holds

$$\omega(\nabla v)^{-\beta} = (1 + |\nabla v|^2)^{-\beta} \le 1, \tag{4.15a}$$

$$\begin{aligned} \omega(\nabla v)^{-(\beta+1)} \mid \nabla v \mid &\le \omega(\nabla v)^{-(\beta+1)}(1 + |\nabla v|^2)^{1/2} \\ &\le \omega(\nabla v)^{-(\beta+1/2)} \le 1. \end{aligned} \tag{4.15b}$$

Moreover, for $v, w \in V_h$ and $\xi(s) := w + s(v - w)$, $s \in [0, 1]$, it holds

$$\omega(\nabla v)^{-\beta} - \omega(\nabla w)^{-\beta} = -2\beta \int_0^1 \omega(\nabla \xi(s))^{-\beta-1} \nabla \xi(s) \cdot \nabla(v - w) \ ds, \tag{4.16a}$$

$$\begin{aligned} \omega(\nabla v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\nabla w)^{-\beta} \underline{\underline{\mathbf{M}}}(w) &= \int_0^1 \omega(\nabla \xi(s))^{-\beta} \underline{\underline{\mathbf{F}}}(\xi(s); v - w) \ ds \\ &-2\beta \int_0^1 \omega(\nabla \xi(s))^{-\beta-1} \nabla \xi(s) \cdot \nabla(v - w) \underline{\underline{\mathbf{M}}}(\xi(s)) \ ds, \end{aligned} \tag{4.16b}$$

where the matrix $\underline{\underline{\mathbf{F}}}(v; w)$, $v, w \in V_h$ is given by

$$\underline{\underline{\mathbf{F}}}(v; w) := \begin{pmatrix} 2 \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_2} & \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} \\ \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} & 2 \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1} \end{pmatrix}, \qquad v, w \in V_h. \tag{4.17}$$

An easy computation yields

$$|\underline{\underline{\mathbf{F}}}(v; w)|^2 \le 5 \ |\nabla v|^2 \ |\nabla w|^2. \tag{4.18}$$

It follows from Equations (4.15b) and (4.16a) that

$$| \omega(\nabla v)^{-\beta} - \omega(\nabla w)^{-\beta} | \leq 2\beta \, | \nabla(v-w) |, \tag{4.19a}$$

whereas in view of Equations (3.5), (4.15b), (4.16b), and (4.18) we have

$$| \omega(\nabla v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\nabla w)^{-\beta} \underline{\underline{\mathbf{M}}}(w) | \leq (2\beta + \sqrt{5}) \, | \nabla(v-w) |, \tag{4.19b}$$

We will first show that the nonlinear operator A_h^{DG} is Lipschitz continuous on the ball

$$B_h(0,R) := \{ v_h \in V_h \ | \ \|v_h\|_{2,h,\Omega} \leq R \}. \tag{4.20}$$

Theorem 4.2 *The nonlinear operator A_h^{DG} is Lipschitz continuous on the ball $B_h(0, R)$. In particular, there exists $\Gamma(h, R) > 0$ such that*

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} \leq \Gamma(h,R) \ \|v_h - w_h\|_{2,h,\Omega}, \qquad v_h, w_h \in B_h(0,R). \tag{4.21}$$

Proof. For $v_h, w_h \in B_h(0, R)$ we set $\xi_h := v_h - w_h$. In view of the definition (4.10) of the nonlinear operator A_h^{DG} we have

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} = \sup_{\|z_h\|_{2,h,\Omega} \leq 1} | \langle A_h^{DG} v_h - A_h^{DG} w_h, z_h \rangle_{V_h^*, V_h} | =$$

$$\sup_{\|z_h\|_{2,h,\Omega} \leq 1} | (\xi_h, z_h)_{0,\Omega} + \Delta t \beta \delta^2 \ (a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h)) | . \tag{4.22}$$

According to the definition (3.15) of the semilinear form $a_h^{DG}(\cdot, \cdot; \cdot)$ we find

$$a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h) = \tag{4.23}$$

$$\sum_{K \in \mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h) D^2 w_h) : D^2 z_h \ dx$$

$$- \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E$$

$$- \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \ ds$$

$$- \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 z_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E$$

$$- \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 z_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E) \ ds$$

$$+ \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E$$

$$- \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \ ds.$$

We will estimate the four terms on the right-hand side of Equation (4.23) separately.
(1) For the first term on the right-hand side of Equation (4.23) we obtain

$$\sum_{K \in \mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h) D^2 w_h) : D^2 z_h \ dx$$

$$= \underbrace{\sum_{K \in \mathcal{T}_h} \int_K \underline{\underline{\mathbf{A}}}_1(v_h) D^2 \xi_h : D^2 z_h \ dx}_{=I_1} + \underbrace{\sum_{K \in \mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)) D^2 w_h : D^2 z_h \ dx}_{=I_2}.$$

In view of Equations (3.5), (3.6), and (4.15a) and using Hölder's inequality as well as the Cauchy-Schwarz inequality, we get the following upper bound for I_1:

$$\begin{aligned} \mid I_1 \mid &\leq \sum_{K\in\mathcal{T}_h} \int_K \mid D^2\xi_h \| D^2 z_h \mid \;\; dx \\ &\leq \sum_{K\in\mathcal{T}_h} \left(\int_K |D^2\xi_h|^2 \;\; dx \right)^{1/2} \left(\int_K |D^2 z_h|^2 \;\; dx \right)^{1/2} \\ &\leq \left(\sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2 \;\; dx \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \;\; dx \right)^{1/2}. \end{aligned} \tag{4.24}$$

Likewise, using Equations (3.8b), (3.8c), (4.16b), the inverse inequality Equation (4.12), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), and observing $\|D^2 w_h\|_{0,K} \leq \|w_h\|_{2,h,\Omega} \leq R, K \in \mathcal{T}_h$, we can estimate I_2 from above as follows:

$$\begin{aligned} \mid I_2 \mid &\leq \sum_{K\in\mathcal{T}_h} \int_K \mid \underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h) \| D^2 w_h \| D^2 z_h \mid \;\; dx \\ &\leq (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \int_K \mid \nabla\xi_h \| D^2 w_h \| D^2 z_h \mid \;\; dx \\ &\leq (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \|\nabla\xi_h\|_{0,\infty,K} \left(\int_K |D^2 w_h|^2 \;\; dx \right)^{1/2} \left(\int_K |D^2 z_h|^2 \;\; dx \right)^{1/2} \\ &\leq (3+\sqrt{5}) c_S^{-1/2} C_{inv} R \sum_{K\in\mathcal{T}_h} h_K^{-1} \|\nabla\xi_h\|_{0,K} \|D^2 z_h\|_{0,K} \\ &\leq (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} R h^{-1} \left(\sum_{K\in\mathcal{T}_h} \|\nabla\xi_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \right)^{1/2} \\ &\leq (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \right)^{1/2}. \end{aligned}$$

Hence, setting $C_A^{(1)} := (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R$, we thus have

$$\mid I_2 \mid \leq C_A^{(1)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \right)^{1/2}. \tag{4.25}$$

(2) Setting $\tilde{\omega}(\nabla v_h, \nabla w_h) := \omega(\nabla v_h)^{-1/4} - \omega(\nabla w_h)^{-1/4}$, the second term on the right-hand side of Equation (4.23) can be written as

$$\begin{aligned} &\sum_{E\in\mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \;\; \mathbf{n}_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \\ &\qquad - \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \;\; \mathbf{n}_E \;\; \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \;\; ds \\ &\quad = \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 \xi_h\}_E \;\; \mathbf{n}_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \;\; ds}_{=II_1} \end{aligned}$$

$$+ \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{(\underline{\underline{\mathbf{A}}}_2(v_h) - \underline{\underline{\mathbf{A}}}_2(w_h))D^2 w_h\}_E \;\; \mathbf{n}_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \;\; ds}_{=II_2}$$

$$+ \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \;\; \mathbf{n}_E \;\; \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla z_h]_E \;\; ds.}_{=II_3}$$

Setting $E_1 := E_+$, $E_2 := E_-$, for $E \in \mathcal{E}_h(\Omega)$, and using Equations (3.5), (3.8a), (3.16), (4.15a), and the trace inequality Equation (4.13b), for the first term II_1 we find

$$\begin{aligned}
| II_1 | &\le \sum_{E\in\mathcal{E}_h} \int_E | \{D^2 \xi_h\}_E || \mathbf{n}_E \cdot [\nabla z_h]_E | \;\; ds \\
&\le \frac{1}{2} \sum_{E\in\mathcal{E}_h(\Omega)} \int_E \sum_{i=1}^{2} |D^2 \xi_h|_{E_i}|| \mathbf{n}_E \cdot [\nabla z_h]_E | \;\; ds + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E | D^2 \xi_h | \, | \mathbf{n}_E \cdot [\nabla z_h]_E | \;\; ds \\
&\le \sum_{E\in\mathcal{E}_h(\Omega)} \sum_{i=1}^{2} h_E^{1/2} \Big(\int_E |D^2 \xi_h|_{E_i}|^2 \;\; ds \Big)^{1/2} h_E^{-1/2} \Big(\int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 \;\; ds \Big)^{1/2} \\
&+ \sum_{E\in\mathcal{E}_h(\Gamma)} \Big(h_E^{1/2} \int_E |D^2 \xi_h|^2 \;\; ds \Big)^{1/2} h_E^{-1/2} \Big(\int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 \;\; ds \Big)^{1/2} \\
&\le C_R^{1/2} \Big(\sum_{K\in\mathcal{T}_h} h_K \|D^2 \xi_h\|_{0,\partial K}^2 \Big)^{1/2} \Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 \;\; ds \Big)^{1/2} \\
&\le C_R^{1/2} C_T \Big(\sum_{K\in\mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 \;\; ds \Big)^{1/2} .
\end{aligned}$$

We thus have

$$| II_1 | \le C_A^{(2)} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 \;\; ds \Big)^{1/2}, \tag{4.26}$$

where $C_A^{(2)} := C_R^{1/2} C_T$. In a similar way, using (3.5),(3.8a)-(3.8c),(3.16),(4.16b), the inverse inequality (4.12), the trace inequality (4.13a), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), and observing $\|D^2 w_h\|_{0,K} \le \|w_h\|_{2,h,\Omega} \le R, K \in \mathcal{T}_h$, the second term II_2 can be estimated from above according to

$$\begin{aligned}
| II_2 | &\le \Big(\frac{5}{2} + \sqrt{5} \Big) C_R^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \|\nabla \xi_h\|_{0,\infty,K}^2 h_K \int_{\partial K} |D^2 w_h|^2 \;\; ds \Big)^{1/2} \\
&\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 \;\; ds \Big)^{1/2} \\
&\le \Big(\frac{5}{2} + \sqrt{5} \Big) C_R^{1/2} C_T \Big(\sum_{K\in\mathcal{T}_h} \|\nabla \xi_h\|_{0,\infty,K}^2 \int_K |D^2 w_h|^2 \;\; ds \Big)^{1/2}
\end{aligned}$$

$$\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla z_h]_E|^2\ ds\right)^{1/2}$$

$$\leq \left(\frac{5}{2}+\sqrt{5}\right) c_Q^{-1}c_S^{-1}C_{inv}C_R^{1/2}C_T R h^{-1}\left(\sum_{K\in\mathcal{T}_h}\|\nabla\xi_h\|_{0,K}^2\right)^{1/2}$$

$$\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla z_h]_E|^2\ ds\right)^{1/2}$$

$$\leq C_A^{(3)}h^{-1}|\xi_h|_{2,h,\Omega}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla z_h]_E|^2\ ds\right)^{1/2}, \tag{4.27}$$

where $C_A^{(3)} := \left(\frac{5}{2}+\sqrt{5}\right) c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_T R$. In a similar way, for II_3 we obtain

$$|\ II_3\ | \leq C_R^{1/2}\left(\sum_{K\in\mathcal{T}_h}\|\nabla z_h\|_{0,\infty,K}^2 h_K\int_{\partial K}|D^2 w_h|^2 ds\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2 ds\right)^{1/2}$$

$$\leq C_R^{1/2}C_T\left(\sum_{K\in\mathcal{T}_h}\|\nabla z_h\|_{0,\infty,K}^2\int_K|D^2 w_h|^2 dx\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2 ds\right)^{1/2}$$

$$\leq c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_T R h^{-1}|z_h|_{2,h,\Omega}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2\ ds\right)^{1/2}.$$

and hence,

$$|\ II_3\ | \leq C_A^{(4)}h^{-1}|z_h|_{2,h,\Omega}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2\ ds\right)^{1/2}, \tag{4.28}$$

where $C_A^{(4)} := c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_T R$.

(3) For the third term on the right-hand side of Equation (4.23) we have

$$\sum_{E\in\mathcal{E}_h}\int_E (\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2 z_h\}_E\ \mathbf{n}_E\ \ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla v_h]_E$$

$$-\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 z_h\}_E\ \mathbf{n}_E\ \ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla w_h]_E)\ ds$$

$$=\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{(\underline{\underline{\mathbf{A}}}_2(v_h)-\underline{\underline{\mathbf{A}}}_2(w_h))D^2 z_h\}_E\mathbf{n}_E\ \ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla v_h]_E\ ds}_{=III_1}$$

$$+\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 z_h\}_E\mathbf{n}_E\ \ \mathbf{n}_E\cdot[\tilde{\omega}(\nabla v_h,\nabla w_h)\nabla v_h]_E\ ds}_{=III_2}$$

$$+\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 z_h\}_E\mathbf{n}_E\ \ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E\ ds.}_{=III_3}$$

The terms III_1, III_2, and III_3 can be estimated from above in much the same way as the corresponding terms for II. We obtain

$$| III_1 | \leq C_A^{(5)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 \ ds \right)^{1/2}, \tag{4.29}$$

where $C_A^{(5)} := \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$, and

$$| III_2 | \leq C_A^{(6)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 \ ds \right)^{1/2}, \tag{4.30a}$$

$$| III_3 | \leq C_A^{(7)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 \ ds \right)^{1/2}, \tag{4.30b}$$

where $C_A^{(6)} = C_A^{(7)} := c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$.

(4) Finally, for the fourth term on the right-hand side of Equation (4.23) we get

$$\begin{aligned}
&\alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \\
&\qquad - \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \ ds \\
&= \underbrace{\alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \ ds}_{=IV_1} \\
&+ \underbrace{\alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \ \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla z_h]_E \ ds}_{=IV_2} \\
&+ \underbrace{\alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla w_h]_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E \ ds.}_{=IV_3}
\end{aligned} \tag{4.31}$$

Using Equation (3.8a), (4.15a), the trace inequality (4.13a), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), for IV_1 we obtain

$$\begin{aligned}
| IV_1 | &\leq \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E | \mathbf{n}_E \cdot [\nabla \xi_h]_E || \mathbf{n}_E \cdot [\nabla z_h]_E | \ ds \\
&\leq \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
&\leq C_A^{(8)} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} \int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2},
\end{aligned}$$

where $C_A^{(8)} := \alpha$. Setting $K_1 := K_+$ and $K_2 := K_-$ for $E \in \mathcal{E}_h(\Omega)$, $E = K_+ \cap K_-$, and $K_1 = K_2 = K$ for $E \in \mathcal{E}_h(\Gamma)$, $E \in \mathcal{E}_h(K \cap \Gamma)$, the term IV_2 can be estimated from above as follows:

$$| IV_2 | \leq \alpha \sum_{E\in\mathcal{E}_h} \sum_{i=1}^{2} \|\nabla \xi_h\|_{0,\infty,K_i} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2}$$
$$\left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}.$$

Using Equations (3.8b), (3.8c), the inverse inequality (4.12), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), for IV_1, we have

$$\sum_{i=1}^{2} \|\nabla \xi_h\|_{0,\infty,K_i} \leq c_R^{-1} c_S^{-1} C_{inv} h^{-1} \sum_{i=1}^{2} \|\nabla \xi_h\|_{0,K_i}$$
$$\leq 2c_R^{-1} c_S^{-1} C_{inv} h^{-1} \|\nabla \xi_h\|_{0,\Omega} \leq 2c_R^{-1} c_S^{-1} C_{inv} C_{PF} h^{-1} |\xi_h|_{2,h,\Omega}.$$

Hence, observing $\left(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2} \leq \|w_h\|_{2,h,\Omega} \leq R$, we obtain

$$| IV_2 | \leq C_A^{(9)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}. \tag{4.32}$$

where $C_A^{(9)} := 2\alpha c_R^{-1} c_S^{-1} C_{inv} C_{PF} R$. In the same way we get

$$| IV_3 | \leq C_A^{(10)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}. \tag{4.33}$$

where $C_A^{(10)} := C_A^{(9)}$.

Setting $C_A := \sum_{i=1}^{10} C_A^{(i)}$, it follows from Equations (4.22) and (4.33) that

$$| \langle A_h^{DG} v_h - A_h^{DG} w_h, z_h \rangle_{V_h^*, V_h} | \leq \max(1, \beta \Delta t \delta^2 C_A h^{-1}) \ \ \|\xi_h\|_{2,h,\Omega} \|z_h\|_{2,h,\Omega},$$

which implies Equation (4.21) with $\Gamma(h, R) := \max(1, \beta \Delta t \delta^2 C_A h^{-1})$. ∎

Theorem 4.3 *Under the assumption that there exist constants $0 < \kappa \ll 1$ and $C_\Delta > 0$ such that*

$$\beta \Delta t \delta^2 \leq C_\Delta h^{4+\kappa}, \tag{4.34}$$

for sufficiently small $0 < h < 1$ there exists $\gamma(h, R) > 0$ such that for v_h, $w_h \in B_h(0, R)$ it holds

$$\langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h \rangle_{V_h^*, V_h} \geq \gamma(h, R) \|v_h - w_h\|_{2,h,\Omega}^2. \tag{4.35}$$

Proof. For $v_h, w_h \in B_h(0, R)$ we set $\xi_h := v_h - w_h$. Taking the definition (4.10) of the nonlinear operator A_h^{DG} into account, we have

$$\langle A_H^{DG} v_h - A_h^{DG} w_h, \xi_h \rangle_{V_h^*, V_h} = \|\xi_h\|_{0,\Omega}^2 + \beta \Delta t \delta^2 (a_h^{DG}(v_h, \xi_h; v_h) - a_h^{DG}(w_h, \xi_h; w_h)). \tag{4.36}$$

Recalling the definitions (3.6), (3.16) of $\underline{\underline{\mathbf{A}}}_1$ and $\underline{\underline{\mathbf{A}}}_2$, for the second term on the right-hand side of Equation (4.36) it follows that

$$a_h^{DG}(v_h, \xi_h; v_h) - a_h^{DG}(w_h, \xi_h; w_h) = \sum_{K\in\mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h)) D^2 w_h) : D^2 \xi_h \ \ dx$$
$$- \sum_{E\in\mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \ \ \mathbf{n}_E \ \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E$$

$$- \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 w_h\}_E \ \ \mathbf{n}_E \ \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4}\nabla \xi_h]_E) \ \ ds$$

$$- \sum_{E\in\mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h)D^2 \xi_h\}_E \ \ \mathbf{n}_E \ \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}\nabla v_h]_E$$

$$- \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 \xi_h\}_E \ \ \mathbf{n}_E \ \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4}\nabla w_h]_E) \ \ ds$$

$$+ \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}\nabla v_h]_E \ \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}\nabla \xi_h]_E$$

$$- \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4}\nabla w_h]_E \ \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4}\nabla \xi_h]_E) \ \ ds. \tag{4.37}$$

As in the previous theorem, we will estimate the four terms on the right-hand side of Equation (4.37) separately.

(1) For the first term we obtain

$$\sum_{K\in\mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h)D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h)D^2 w_h) \ : \ D^2\xi_h \ \ dx$$

$$= \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h \ : \ D^2\xi_h \ \ dx}_{=I_1} + \underbrace{\sum_{K\in\mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h))D^2 w_h \ : \ D^2\xi_h \ \ dx.}_{=I_2}$$

As far as I_1 is concerned, due to Equations (3.5) and (3.6) we have

$$\int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h \ : \ D^2\xi_h \ \ dx \geq (1 + \|\nabla v_h\|_{0,\infty,K}^2)^{-3/2}\|D^2\xi_h\|_{0,K}^2.$$

Using Equations (3.8b), (3.8c), the inverse inequality (4.12), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), and observing $\|v_h\|_{2,\,h,\,\Omega} \leq R$, we get

$$\begin{aligned}\|\nabla v_h\|_{0,\infty,K}^2 &\leq c_S^{-2}C_{inv}^2 h_K^{-2}\|\nabla v_h\|_{0,K}^2 \leq c_Q^{-2}c_S^{-2}C_{inv}^2 h^{-2}\|\nabla v_h\|_{0,\Omega}^2\\ &\leq c_Q^{-2}c_S^{-2}C_{inv}^2 C_{PF}^2 h^{-2}\|v_h\|_{2,h,\Omega}^2 \leq \gamma_M^{(0)} h^{-2},\end{aligned}$$

where $\gamma_M^{(0)} := c_Q^{-2}c_S^{-2}C_{inv}^2 C_{PF}^2 R^2$. Observing $h \leq 1$, it follows that

$$(1 + \|\nabla v_h\|_{0,\infty,K}^2)^{-3/2} \geq h^3(h^2 + \gamma_M^{(0)})^{-3/2} \geq h^3(1 + \gamma_M^{(0)})^{-3/2} = \gamma_M^{(1)} h^3,$$

where $\gamma_M^{(1)} := (1 + \gamma_M^{(0)})^{-3/2}$. Hence, we obtain the following lower bound for I_1:

$$I_1 \geq \gamma_M^{(1)} h^3 \sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2. \tag{4.38}$$

In order to estimate I_2 from above, we use Equations (3.8b), (3.8c), (4.19b), Hölder's inequality, the inverse inequality (4.12), the Cauchy-Schwarz inequality, and observe $\|D^2 w_h\|_{0,\,K} \leq \|w_h\|_{2,h,\Omega} \leq R, K \in \mathcal{T}_h$:

$$\begin{aligned}| \ I_2 \ | &\leq (3 + \sqrt{5}) \sum_{K\in\mathcal{T}_h} \int_K | \ \nabla\xi_h \| D^2 w_h \| D^2 \xi_h \ | \ \ \ dx\\ &\leq (3 + \sqrt{5}) \sum_{K\in\mathcal{T}_h} \|\nabla\xi_h\|_{0,\infty,K} \left(\int_K |D^2 w_h|^2 \ \ dx \right)^{1/2} \left(\int_K |D^2\xi_h|^2 \ \ dx \right)^{1/2}\\ &\leq (3 + \sqrt{5}) c_S^{-1} C_{inv} \sum_{K\in\mathcal{T}_h} h_K^{-1} \|\xi_h\|_{0,K} \|D^2 w_h\|_{0,K} \|D^2\xi_h\|_{0,K}\end{aligned}$$

$$\leq (3+\sqrt{5})c_S^{-1}C_{inv}^2 R \sum_{K\in\mathcal{T}_h} h_K^{-1}\|\xi_h\|_{0,K} h_K^{-2}\|\xi_h\|_{0,K}$$
$$\leq (3+\sqrt{5})c_Q^{-3}c_S^{-1}C_{inv}^2 R h^{-3}\sum_{K\in\mathcal{T}_h}\|\xi_h\|_{0,K}^2.$$

Hence, it follows that

$$|\ I_2\ | \leq C_B^{(1)} h^{-3}\|\xi_h\|_{0,\Omega}^2, \tag{4.39}$$

where $C_B^{(1)} := (3+\sqrt{5})c_Q^{-3}c_S^{-1}C_{inv}^2 R$.

(2) We now deal with the second term on the right-hand side of Equation (4.37) which we rewrite as follows:

$$\sum_{E\in\mathcal{E}_h}\int_E (\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2 v_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E$$
$$-\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 w_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla\xi_h]_E)\ ds$$
$$=\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2\xi_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E\ ds}_{=II_1}$$
$$+\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{(\underline{\underline{\mathbf{A}}}_2(v_h)-\underline{\underline{\mathbf{A}}}_2(w_h))D^2 w_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E\ ds}_{=II_2}$$
$$+\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2 w_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\tilde{\omega}(\nabla v_h,\nabla w_h)\nabla\xi_h]_E\ ds,}_{=II_3}$$

where $\tilde{\omega}(\nabla v_h,\nabla w_h) := \omega(\nabla v_h)^{-1/4}-\omega(\nabla w_h)^{-1/4}$. In view of Equations (3.5), (3.8b), (3.12), (3.16), (4.15), Hölder's inequality, the Cauchy-Schwarz inequality, the inverse inequality (4.12), and the trace inequality (4.13b) we can estimate II_1 from above as follows:

$$|\ II_1\ | \leq 8\sum_{E\in\mathcal{E}_h}\int_E \{|D^2\xi_h|\}_E\{|\nabla\xi_h|\}_E\ ds$$
$$\leq 4\sum_{E\in\mathcal{E}_h}\left(\int_E\{|D^2\xi_h|^2\}_E\ ds\right)^{1/2}\left(\int_E\{|\nabla\xi_h|^2\}_E\ ds\right)^{1/2}$$
$$\leq 4\sum_{K\in\mathcal{T}_h}\left(\int_{\partial K}|D^2\xi_h|^2\ ds\right)^{1/2}\left(\int_{\partial K}|\nabla\xi_h|^2\ ds\right)^{1/2}$$
$$\leq 4c_Q^{-1}h^{-1}\sum_{K\in\mathcal{T}_h} h_K^{1/2}\|D^2\xi_h\|_{0,\partial K} h_K^{1/2}\|\nabla\xi_h\|_{0,\partial K}$$
$$\leq 4c_Q^{-1}C_T^2 h^{-1}\left(\sum_{K\in\mathcal{T}_h}\|D^2\xi_h\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\|\nabla\xi_h\|_{0,K}^2\right)^{1/2}$$
$$\leq 4c_Q^{-4}C_{inv}^2 C_T^2 h^{-4}\sum_{K\in\mathcal{T}_h}\|\xi_h\|_{0,K}^2$$
$$\leq 4c_Q^{-4}C_{inv}^2 C_T^2 h^{-4}\|\xi_h\|_{0,\Omega}^2.$$

Hence, we obtain

$$| II_1 | \le C_B^{(2)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \tag{4.40}$$

where $C_B^{(2)} := 4 c_Q^{-4} C_{inv}^2 C_T^2$. Likewise, for II_2 we have

$$\begin{aligned}
| II_2 | &\le 4 \left(\frac{5}{2} + \sqrt{5}\right) \sum_{E \in \mathcal{E}_h} \int_E \{|\nabla \xi_h|^2\}_E \{|D^2 w_h|\}_E \ ds \\
&\le 2 \left(\frac{5}{2} + \sqrt{5}\right) \sum_{E \in \mathcal{E}_h} \left(\int_E \{|\nabla \xi_h|^4\}_E \ ds\right)^{1/2} \left(\int_E \{|D^2 w_h|^2\}_E \ ds\right)^{1/2} \\
&\le 2 \left(\frac{5}{2} + \sqrt{5}\right) \sum_{K \in \mathcal{T}_h} \left(\int_{\partial K} |\nabla \xi_h|^4 \ ds\right)^{1/2} \left(\int_{\partial K} |D^2 w_h|^2 \ ds\right)^{1/2} \\
&= 2 \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-1} h^{-1} \sum_{K \in \mathcal{T}_h} h_K^{1/2} \|\nabla \xi_h\|_{0,4,\partial K}^2 h_K^{1/2} \|D^2 w_h\|_{0,\partial K} \\
&\le 2 \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-1} C_T^2 h^{-1} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,4,K}^4\right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|D^2 w_h\|_{0,K}^2\right)^{1/2} \\
&\le 2 \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-1} c_S^{-1/2} C_{inv} C_T^2 R h^{-2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,K}^4\right)^{1/2} \\
&\le 2 \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R h^{-4} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2.
\end{aligned}$$

It follows that

$$| II_2 | \le C_B^{(3)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \tag{4.41}$$

where $C_B^{(3)} := 2 \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$. Finally, II_3 can be bounded from above in much the same way as II_2. We get

$$| II_3 | \le C_B^{(4)} h^{-4} \|\xi_h\|^2, \tag{4.42}$$

where $C_B^{(4)} := 2 c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$.

(3) For the third term on the right-hand side of Equation (4.37) we have

$$\begin{aligned}
&\sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 \xi_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \\
&- \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 \xi_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E) \ ds \\
&= \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{(\underline{\underline{\mathbf{A}}}_2(v_h) - \underline{\underline{\mathbf{A}}}_2(w_h)) D^2 \xi_h\}_E \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E ds}_{=III_1} \\
&+ \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 \xi_h\}_E \mathbf{n}_E \ \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla v_h]_E ds}_{=III_2} \\
&+ \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 \xi_h\}_E \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E ds.}_{=III_3}
\end{aligned}$$

The three terms can be estimated from above in a similar way as the corresponding terms in II. We obtain

$$\mid III_1 \mid \leq C_B^{(5)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \qquad \mid III_2 \mid \leq C_B^{(6)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \qquad \mid III_3 \mid \leq C_B^{(7)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \tag{4.43}$$

where $C_B^{(5)} := 2\left(\frac{5}{2}+\sqrt{5}\right) c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$, $C_B^{(6)} := 2 c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$, and $C_B^{(7)} := C_B^{(6)}$.

(4) For the fourth term on the right-hand side of Equation (4.37) we obtain

$$\begin{aligned}
&\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \\
&\qquad\qquad - \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \;\; \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E) \; ds \\
&= \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \; ds}_{=IV_1} \\
&\quad + \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \;\; \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla \xi_h]_E \; ds}_{=IV_2} \\
&\quad + \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla w_h]_E \;\; \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E \; ds.}_{=IV_3}
\end{aligned} \tag{4.44}$$

In view of Equation (3.13), the first term IV_1 can be further split according to

$$\begin{aligned}
IV_1 &= \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E \;\; \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E \; ds}_{=IV_{11}} \\
&+ \underbrace{\alpha \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E \; ds}_{=IV_{12}} \\
&+ \underbrace{\alpha \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E \;\; \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E \; ds}_{=IV_{13}} \\
&+ \underbrace{\alpha \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1} \int_E \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E \;\; \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E \; ds.}_{=IV_{14}}
\end{aligned}$$

For IV_{11}, setting $E_1 := E_+$ and $E_2 := E_-$ for $E \in \mathcal{E}_h(\Omega)$, we have

$$\begin{aligned}
IV_{11} \geq \alpha &\sum_{E\in\mathcal{E}_h(\Omega)} \left(1 + \frac{1}{2}\sum_{i=1}^{2} \|\nabla v_h\|_{0,\infty,E_i}^2\right)^{-1/2} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \\
&+ \alpha \sum_{E\in\mathcal{E}_h(\Gamma)} (1 + \|\nabla v_h\|_{0,\infty,E}^2)^{-1/2} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds.
\end{aligned}$$

Taking advantage of Equations (3.8b), (3.8c), the inverse inequality (4.12), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), it follows that for $E \in \mathcal{E}_h(\partial K)$ it holds

$$\|\nabla v_h\|_{0,\infty,E} \le \|\nabla v_h\|_{0,\infty,K} \le c_S^{-1/2} C_{inv} h_K^{-1} \|\nabla v_h\|_{0,K}$$
$$\le c_Q^{-1} c_S^{-1/2} C_{inv} h^{-1} \|\nabla v_h\|_{0,\Omega} \le c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} h^{-1} \|v_h\|_{2,h,\Omega} \le c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} h^{-1} R,$$

and hence, observing $h < 1$, we get

$$(1 + \|\nabla v_h\|_{0,\infty,E}^2)^{-1/2} \ge (1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2 h^{-2})^{-1/2}$$
$$= (h^2 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2} h \ge (1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2} h.$$

Consequently, we obtain

$$IV_{11} \ge \alpha \gamma_M^{(2)} h \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds, \tag{4.45}$$

where $\gamma_M^{(2)} := \alpha(1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2}$. The remaining terms IV_{1i}, $2 \le i \le 4$, can be estimated from above similarly as the corresponding terms in Theorem 4.2:

$$|\ IV_{12}\ | \le C_B^{(8)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \qquad |\ IV_{13}\ | \le C_B^{(9)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \qquad |\ IV_{14}\ | \le C_B^{(10)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \tag{4.46}$$

where $C_B^{(8)} := 2\alpha c_Q^{-4} c_R^{-1} C_{inv}^2 C_T^2$ and $C_B^{(9)} = C_B^{(10)} := 2 C_B^{(8)}$. The remaining two terms IV_2 and IV_3 can be estimated from above in the same way. Using Equations (3.8a), (3.8b), (4.19a), the inverse inequality (4.12), the trace inequality (4.13a), the Cauchy-Schwarz inequality, and observing

$$\left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 \ ds \right)^{1/2} \le \|w_h\|_{2,h,\Omega} \le R,$$

we obtain

$$|\ IV_2\ | \le 4\alpha c_Q^{-1/2} c_R^{-1/2} h^{-1/2} \sum_{E \in \mathcal{E}_h} h_E^{-1/2} \int_E |\ \mathbf{n}_E \cdot [\nabla w_h]_E \ | \ \{|\nabla \xi_h|\}_E^2 \ ds$$
$$\le 4\alpha c_Q^{-1/2} c_R^{-1/2} h^{-1/2} \sum_{E \in \mathcal{E}_h} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 \ ds \right)^{1/2} \left(\int_E \{|\nabla \xi_h|\}_E^4 \ ds \right)^{1/2}$$
$$\le 2\alpha c_Q^{-3/2} c_R^{-1/2} h^{-3/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 \ ds \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla \xi_h\|_{0,4,\partial K}^4 \right)^{1/2}$$
$$\le 2\alpha c_Q^{-3/2} c_R^{-1/2} C_T R h^{-3/2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,4,K}^4 \right)^{1/2}$$
$$\le 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R h^{-7/2} \left(\sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^4 \right)^{1/2}$$
$$\le 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R h^{-7/2} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2.$$

Hence, it follows that

$$|\ IV_2\ | \le C_B^{(11)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \tag{4.47}$$

where $C_B^{(11)} := 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R$. Moreover, we get

$$| IV_3 | \le C_B^{(12)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \tag{4.48}$$

where $C_B^{(12)} := C_B^{(11)}$. Setting $C_B := \sum_{i=1}^{12} C_B^{(i)}$ and observing (4.34) as well as $h < 1$, it follows from Equations (4.36) to (4.48) that

$$\begin{aligned} &\langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h \rangle_{V_h^*, V_h} \\ &\ge (1 - C_\Delta C_B h^\kappa) \ \|\xi_h\|_{0,\Omega}^2 + \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h^3 \ \ |\xi_h|_{2,h,\Omega}^2. \end{aligned} \tag{4.49}$$

We choose $h_{min} > 0$ such that

$$q := C_\Delta C_B h_{\min}^\kappa < 1 \qquad \text{and} \qquad \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h_{\min}^3 < 1 - q. \tag{4.50}$$

Then, for $h \le h_{min}$ (4.35) follows from Equations (4.49) and (4.50) with

$$\gamma(h, R) := \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h^3. \tag{4.51}$$

■

Corollary 4.1 *Assume that u_h^{m-1} satisfies*

$$\|u_h^{m-1}\|_{0,\Omega} \le \frac{\Gamma(R)^2}{\gamma(R)} \left(1 - \sqrt{1 - \frac{\gamma(R)^2}{\Gamma(R)^2}} \right) R$$

for some $R > 0$ and that (4.34) holds true. Then, for sufficiently small grid size h, the C^0IPDG approximation (3.14) has a unique solution $u_h^m \in B_h(0, R)$.

Proof. Using the Lipschitz continuity (4.22) and the strong monotonicity (4.35) of the nonlinear operator A_h^{DG}, the result follows from the nonlinear analogue of the Lax-Milgram Lemma (Theorem 4.1). ■

Remark 4.1 If we choose $h_{\min} > 0$ such that Equation (4.49) is satisfied as well as $h_{\min} < \beta C_\Delta C_A$, for $h \le h_{min}$ we have $\Gamma(h, R) = \beta C_\Delta C_A h^{-1}$ in Theorem 4.2 and the application of Theorem 4.1 for $V = V_h$ and $A = A_h^{DG}$ implies that the fixed point operator T is a contraction as long as

$$\rho < 2 \ \frac{\gamma(h, R)}{\Gamma(h, R)^2} = 2 \ \frac{\min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)})}{C_\Delta^2 C_A^2} \ h^5. \tag{4.52}$$

In other words, the contraction property degenerates for $h \to 0$. This reflects the very singular character of the fourth-order TVF.

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REFERENCES

[1] Y. Kashima, *A subdifferential formulation of fourth order singular diffusion equations*, Adv. Math. Sci. Appl. vol. 14 (2004) pp. 49–74.

[2] C. Bhandari, R. H. W. Hoppe, R. Kumar, *A C^0 interior penalty discontinuous Galerkin method for fourth-order total variation flow. I: Derivation of the method and numerical results*. Numer. Methods. Partial Differ. Equ. vol. 35 (2019) pp. 1458–1476.

[3] R. V. Kohn, H. M. Versieux, *Numerical analysis of a steepest-descent PDE model for surface relaxation below the roughening temperature*, SIAM J. Numer. Anal. vol. 48 (2010) pp. 1781–1800.

[4] L. Tartar, *Introduction to sobolev spaces and interpolation theory*, Springer, Berlin, 2007.
[5] D. Arnold et al., *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. vol. 39 (2002) pp. 1749–1779.
[6] S. C. Brenner, L. R. Scott, *The mathematical theory of finite element methods*. 3rd ed., Springer, New York, 2008.
[7] P. G. Ciarlet, *The finite element method for elliptic problems*, SIAM, Philadelphia, 2002.
[8] C. Schwab, *P- and hp-finite element methods*, Oxford University Press, Oxford, 1998.
[9] T. Warburton, J. S. Hesthaven, *On the constants in hp-finite element trace inverse inequalities*, Comput. Methods Appl. Mech. Eng. vol. 192 (2003) pp. 2765–2773.
[10] S. C. Brenner, K. Wang, J. Zhao, *Poincaré-Friedrichs inequalities for piecewise H^2 functions*, Numer. Funct. Anal. Optim. vol. 25 (2004) pp. 463–478.