Homogeneous isoparametric submanifolds of Hilbert space

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Introduction

The aim of this thesis is to prove rigidity results for homogeneous isoparametric submanifolds of Hilbert space.

A submanifold M of a space form or a Hilbert space V is called isoparametric if its normal bundle is flat and the principal curvatures along parallel normal fields are constant. The beginning of the study of isoparametric hypersurfaces dates back to 1920 and these early investigations culminated in the work of Élie Cartan in the 1930s. In the early 1980s the notion was generalized from isoparametric hypersurfaces to submanifolds of higher codimension in $\mathbb{R}^n$ by Terng ([Ter85]) and others; in a subsequent paper she further generalizes the definition to submanifolds of Hilbert space ([Ter89]).

Homogeneous isoparametric submanifolds are closely related to polar representations, i.e. representations which admit a section, a submanifold that intersects any orbit perpendicularly. Polar representations of compact Lie groups on $\mathbb{R}^n$ were classified by Dadok ([Dad85]). They are orbit-equivalent to s-representations, i.e. isotropy representations of semi-simple symmetric spaces. Thorbergsson proved in 1991 ([Tho91]) that any isoparametric submanifold of $\mathbb{R}^n$ with codimension ≥ 3 is homogeneous. Therefore isoparametric submanifolds are classified except for the case of inhomogeneous ones of codimension two, where 10 cases still remain open cf. [CCJ04].

In infinite dimensions a large class of polar representation is known which arise from finite dimensional hyperpolar actions on compact Lie groups: The so-called $P(G, H)$-actions introduced by Terng ([Ter95]). Many of these (e.g. any with cohomogeneity greater than one) may be seen as an s-representations of an affine Kac-Moody symmetric space, an observation already made though not proven in [HPTT94] and [Ter95]. Gross ([Gro00]) proved on the other hand that s-representation of affine Kac-Moody symmetric spaces are polar and Heintze sketches in [Hei06] a proof for the classification of affine Kac-Moody symmetric spaces.

As in finite dimensions, there is a homogeneity result for isoparametric submanifold of Hilbert space — they are homogeneous if the codimension is greater than one. This result is due to Heintze and Liu ([HL99]). So far no classification result, neither for homogeneous nor for inhomogeneous isoparametric submanifolds of Hilbert space, was known, though the analogy to the finite dimensional theory suggests that at least those of codimension greater than one should be orbits of s-representations of affine Kac-Moody symmetric spaces.

In this thesis we obtain rigidity results for a certain class of homogeneous isoparametric submanifolds in Hilbert space by proving that they are isometric to principal orbits of $P(G, H)$-actions. Essentially the additional assumption is that the eigenspaces of the shape operator are irreducible modules of the isotropy representation. This class includes any isoparametric submanifold whose affine Dynkin diagram is of type $\tilde{A}_n$ ($n \geq 2$), $\tilde{D}_n$, $\tilde{E}_k$ ($k = 6, 7, 8$), $\tilde{F}_4$ or $\tilde{G}_2$.

Moreover we obtain information about the geometry of $P(G, H)$-orbits, in particular their affine marked Dynkin diagrams and slice representations.

In CHAPTER 1 we provide the preliminaries for proving rigidity of isoparametric submanifolds. The normal homogeneous structure S (introduced by Olmos and Sánchez ([OS91]) in a single point x in M together with the second fundamental form α_x determines an isoparametric submanifold uniquely. Moreover an isoparametric submanifold M of higher codimension is determined by certain hypersurfaces, called rank-one leaves, contained in M. This leads to a strategy for a classification: First classify homogeneous isoparametric hypersurfaces (Chapters 2 and 3), then investigate how the affine Dynkin diagram of an isoparametric submanifold of higher codimension determines the type of rank-one leaves (Chapter 5).

The irreducible modules of the isotropy representation, which we treat in CHAPTER 2, are essential to understand the normal homogeneous structure just as in the finite dimensional setting cf. [LES97]. A main difference between finite and infinite dimensional isoparametric submanifolds is the different role of the space $E(0)$, which is the eigenspace of the shape operator corresponding to the eigenvalue 0. Any isoparametric submanifold of $\mathbb{R}^n$ splits as a product of $E(0)$ with a compact isoparametric submanifold. This is no longer true in the infinite dimensional setting; actually we prove that $E(0)$ is always infinite dimensional. We assume any other eigenspace of the shape operator to be irreducible under the isotropy representation. Thus the main task in Chapter 2 is to determine the splitting of $E(0)$ into irreducible modules of the isotropy representation. To do this we associate an isotropy module with a pair of eigenspaces using the covariant derivative of the shape operator.

In CHAPTER 3 we refine the results about the isoparametric hypersurfaces treated in Chapter 2 to obtain their normal homogeneous structure.

We determine in CHAPTER 4 affine marked Dynkin diagrams and slice representations of the known examples of polar representations on Hilbert space, that is the $P(G, H)$-actions. Such arise from hyperpolar actions on compact Lie groups and were classified on simple groups by Kollross in [KOL02].

We proof rigidity of isoparametric submanifolds of codimension greater than one with irreducible eigenspaces in CHAPTER 5. It turns out that they are principal orbits of Hermann actions on Hilbert space. As a by-product of this classification we determine which Hermann actions are orbit-equivalent.

Though we have not classified homogeneous isoparametric submanifold with reducible eigenspaces or whose slice representations are not s-representations, the results nourish the hope that this problem can be solved in general.

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CHAPTER 1

A Rigidity Theorem for homogeneous isoparametric submanifolds

1.1. Preliminary Definitions and Results

We will summarize the results on isoparametric submanifold, that will be used throughout the thesis, starting with the definition of isoparametric submanifolds in Hilbert space taken from [Ter89].

Definition 1.1. A submanifold M of a Hilbert space V is called *proper Fredholm* or a *PF-manifold*, if the end point map

$$Y\colon \nu M \to V$$
$$v \mapsto x + v \quad \text{if } v \in \nu_x M$$

is Fredholm and the restriction of Y to the unit disk normal bundle is proper.

A Hilbert manifold M is proper Fredholm if and only if the shape operator A_v for any normal vector v is compact. The codimension of PF-manifolds is finite.

Definition 1.2. An immersed PF submanifold M of a Hilbert space V is called *isoparametric* if

(1) the normal bundle νM is globally flat.
(2) the shape operators $A_{\xi(x)}$ and $A_{\xi(y)}$ are orthogonally equivalent for any parallel normal field ξ and any point x and y in M.

Remark. In [HLO00] it was proven, that any isoparametric submanifold is embedded, this was already stated by Terng. Moreover it is sufficient to require flatness of the normal bundle, cf. [HLO00, Theorem B].

Definition 1.3. Let V be a Hilbert space and G a Hilbert Lie group. An affine representation $\varrho\colon G \to \mathrm{Iso}(V) = \mathrm{O}(V) \rtimes V$ is called *polar* if

(1) the G-action on V is proper,
(2) the orbit maps $\omega_x\colon G \to V$ with $g \to \varrho(g)(x)$ are Fredholm for any $x \in V$ and
(3) for any regular point x the normal plane $\nu_x M$ meets every orbit and always perpendicularly.

Theorem 1.4 ([Ter89]). *A homogeneous submanifold M of a Hilbert space is isoparametric if and only if it is a principal orbit of a polar representation.*

Examples of homogeneous isoparametric submanifolds of Hilbert space were found by Terng ([Ter89]), Pinkall and Thorbergsson ([PT90]) and Terng gave in [Ter95] a fairly general construction by lifting hyperpolar actions on compact Lie groups to Hilbert space, cf. Chapter 4. *Hyperpolar* means that the action is polar with a flat section.

As for any proper action, for a polar action on Hilbert space any isotropy group G_x is compact. Since the orbits are PF-manifolds, the shape operators at a point x

are compact and since the normal bundle is flat ; therefore there is a simultaneous eigenspace decomposition of the tangential space T_xM. Moreover since the shape operators are orthogonally equivalent along parallel vector fields, this yields a splitting of the tangential bundle as

$$TM = \overline{\bigoplus_{i \in I} E(\lambda_i)}, \qquad \text{with } \dim(E(\lambda_i)) = m(\lambda_i)$$

where I is a countable set and $\lambda_i \colon \nu_x M \to \mathbb{R}$ are the eigenvalues. The eigen distributions $E(\lambda_i)$ are called *curvature distributions*. Note that 0 is always an eigenvalue and $m(0) = \infty$ is possible, whereas $m(\lambda_i) < \infty$ for any other eigenvalue. For any normal field v

$$A_v|_{E(\lambda_i)} = \langle v, v_{\lambda_i} \rangle \operatorname{id}|_{E(\lambda_i)}$$

for a well-defined parallel normal field v_{λ_i}, the so-called *curvature normal*. Throughout this thesis we will assume that the curvature normals $v_{\lambda_i}(x)$ span $\nu_x M$, therefore M is full, that is, not contained in a proper closed affine subspace of V.

The curvature distributions $E(\lambda_i)$ are autoparallel and their integral manifolds are spheres with center $c_{\lambda_i}(x) = x + (v_{\lambda_i}(x)/||v_{\lambda_i}||^2)$ and radius $1/||v_{\lambda_i}||$. These are called *curvature spheres* $S_{\lambda_i}(x)$. Note that the integral manifold of $E(0)$ is an affine plane $x + E(0)(x) \subset M$.

Remark. In finite dimensions, if 0 is an eigenvalue of the shape operator the isoparametric manifold $M \subset \mathbb{R}^n$ splits as $M = \tilde{M} \times E(0)$, where $\tilde{M}$ is a submanifold of a sphere $S^{n-\dim(E(0))}$. This is not true for infinite dimensions.

Let $l_{\lambda_i}(x) \subset x + \nu_x M$ be the normal hyperplane to v_{λ_i}, that is,

$$l_{\lambda_i}(x) = \{x + v \mid \langle v, v_{\lambda_i} \rangle = 1,\, v \in \nu_x M\}.$$

Denote by $R^x_{\lambda_i} \colon (x + \nu_x M) \to (x + \nu_x M)$ the reflection at $l_{\lambda_i}(x)$. Then the group generated by the $R^x_{\lambda_i}$ is an affine Weyl group $W(x)$ and its Coxeter graph is an affine Dyknin diagram. By the *marked affine Dynkin diagram* of an isoparametric submanifold we understand the affine Dynkin diagram of the reflection hyperplanes $l_{\lambda_i}(x)$, where a vertex associated with $l_{\lambda_i}(x)$ is marked with m_{λ_i}. Note that $m_{\lambda_i} = m_{\lambda_j}$, if there is an element in $W(x)$ mapping $l_{\lambda_i}(x)$ to $l_{\lambda_j}(x)$.

For any eigendistribution $E(\lambda_i)$, with $\lambda_i \neq 0$, there is a diffeomorphism φ_{λ_i} which maps a point x to the antipodal point of x on the curvature sphere $S_{\lambda_i}(x)$. If the hyperplane $R^x_{\lambda_j}(l_{\lambda_i}(x)) = l_{\lambda_{\sigma_j(i)}}(x)$, then

$$E(\lambda_j)(\varphi_{\lambda_i}(x)) = E(\lambda_{\sigma_j(i)})(x).$$

Since the curvature normals induce an affine Weyl group, there are only finitely many non proportional curvature normals and for any curvature normal there is an infinite family of proportional curvature normals v_n, which are of the form $v_n = \frac{v}{d+n}$, where v is some normal field and d a number which encodes the distance of the associated reflection hyperplanes. The eigenvalue associated with this family is then of the form $\lambda_n = \frac{c}{d+n}$ for $c \in \mathbb{R}$ and $d \in \mathbb{R}$ depending on the point $x \in M$.

Finally we give the definition of an s-representation.

Definition 1.5. Let $M = G/K$ be a semi-simple simply connected symmetric space, that is, the connected component $G = I^0(M)$ of the isometry group is a semi-simple Lie group. Then the isotropy representation of M is called an *s-representation*.

Let M be of compact type. If $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition, then the isotropy representation of G/K at any point is equivalent to the adjoint representation of K on $\mathfrak{p}$:

$$\begin{aligned} K \times \mathfrak{p} &\to \mathfrak{p} \\ (K, v) &\mapsto KvK^{-1} \end{aligned}$$

DEFINITION 1.6. Two representations $\rho_i : G_i \to \mathrm{SO}(n)$, $i = 1, 2$ are called *orbit-equivalent* or *ω-equivalent*, if there exists an isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(G_1(x)) = G_2(F(x))$$

for any $x \in \mathbb{R}^n$, that is, the representations ρ_i have the same orbits. Replacing $\mathbb{R}^n$ by a Hilbert space V and $\mathrm{SO}(n)$ by $\mathrm{Iso}(V)$ generalizes the definition to affine representations of Hilbert space.

1.2. Reduction of the codimension

Let M be a homogeneous isoparametric submanifold of a Hilbert space V. In [HL97] a construction is given which associates with each affine subspace of the normal space a homogeneous isoparametric submanifold of lower rank. This is done in the following manner: One chooses a point $a \in M$ and an affine subspace $P \subset \nu_a M$ which defines an distribution on M by

$$D_P = \overline{\bigoplus \{E(\lambda_i) \mid v_{\lambda_i}(a) \in P\}}.$$

This distribution is autoparallel, and we denote the leaves through $x \in M$ by $L_P(x)$ and let $W_P(x) = x + D_P(x) \oplus \mathrm{span}\,\{v_{\lambda_i}(x) \mid v_{\lambda_i}(a) \in P\}$. Then the following theorem ([HL97], Lemma 3.3.) is valid:

THEOREM 1.7. *If M is a full, irreducible isoparametric submanifold of an infinite dimensional Hilbert space with codimension at least 2, then $L_P(x)$ is an extrinsically homogeneous isoparametric submanifold of $W_P(x)$ for any affine subspace of $\nu_a M$ and any $a \in M$.*

REMARK. Theorem C in [HL99] says that these submanifold are homogeneous, if the codimension of M is greater or equal to two, even if M is not assumed to be homogeneous. This is the infinite dimensional version of the Homogeneous Slice Theorem of [HOT91] and a crucial step in proving the homogeneity of M.

If the subspace P is not linear, then $L_P(x)$ is finite dimensional since there are only finitely many non-proportional curvature normals. On the other hand, if it is linear (and contains at least one curvature normal), the leaves are infinite dimensional. Note that the distribution D_P contains $E(0)$ in this case, therefore as we will see generically $L_P(x)$ is reducible, one can split off a subdistribution of $E(0)$.

We start with the following proposition, describing generally the part of $E(0)$ by means of $\nabla\alpha$ which splits off from a given isoparametric submanifold M by Moore's Lemma. Compare with Lemma 3.1. in [HL97] where a similar construction is described using the orthogonal complement of the span of all normal spaces.

PROPOSITION 1.8. *Let M be a homogeneous isoparametric submanifold of Hilbert Space V and define*

$$\mathcal{H}(x) = \left\{ Z \in E(0)(x) \mid \big(\nabla_X \alpha\big)(Y, Z) = 0 \textit{ for all } X, Y \in T_x M \right\}.$$

Then $M = \mathcal{H} \times M_2$, where M_2 is the integral manifold of $\mathcal{H}^\perp$.

PROOF. We observe first, since $\big(\nabla_X\alpha\big)(Y,Z)=0$ for all Z if X and Y are contained in $E(0)(x)$ that we may restrict ourselves to the case $v_Y\neq 0$, where v_Y is the curvature normal associated with $E(\lambda)$, when $Y\in E(\lambda)$. We want to apply Lemma 3.1. of [HL97]. For any X and $Y\in E(0)^{\perp}(x)$ by [HL97, Lemma 2.1]

$$(\nabla_X\alpha)(Y,\mathcal{H})=\langle\nabla_XY,\mathcal{H}\rangle\, v_Y=-\langle Y,\nabla_X\mathcal{H}\rangle\, v_Y=0$$

hence $\nabla_X\mathcal{H}\subset E(0)(x)$ for any $X\in T_xM$. Denote by $\bar\nabla$ the Levi-Civita connection of V. Then, by the Gauß formula

$$\bar\nabla_X\mathcal{H}=\nabla_X\mathcal{H}+\alpha(x,\mathcal{H})=\nabla_X\mathcal{H}\subset E(0)$$

and hence $\nu_yM\subset E_i(x)\oplus\nu_xM$ for any y in any curvature sphere containing x. Therefore $\mathcal{H}(x)\perp\nu_y(M)$. The same holds trivially for $y\in x+E(0)(x)$.

In [HL99] the following equivalence relation $\sim_0$ is defined: If for two point $x=x_0$ and $y=x_n$ exists a finite number of points x_k such that x_k is contained in a curvature sphere containing x_{k-1} or $x_k\in x_{k-1}+E(0)(x_{k-1})$, then $x\sim_0 y$. The equivalence classes are denoted by $Q_0(x)$, and $\overline{Q_0(x)}=M$ ([HL99, page 163 and Theorem D]).

Therefore $\mathcal{H}(x)\perp\nu_yM$ for any $y\in M$, since we have proven orthogonality for any $y\in Q_0(x)$. Let

$$V'=\overline{\operatorname{span}\{v(y)\mid y\in M\text{ and }v(y)\in\nu_yM\}}$$

and hence $\mathcal{H}(x)\perp V'$. By Lemma 3.1. of ([HL97]) $M=M'\times(V')^{\perp}$ so it remains to prove $\mathcal{H}(x)=(V')^{\perp}$.

Since $(V')^{\perp}$ is a subdistribution of $E(0)$ and parallel (cf. proof of Lemma 3.1. in [HL97]) and moreover for any $v\in(V')^{\perp}$

$$(\nabla_X\alpha)(Y,v))=\langle\nabla_XY,v\rangle\, v_Y=-\langle Y,\nabla_Xv\rangle\, v_Y\subset\big\langle Y,(V')^{\perp}\big\rangle\, v_Y=0$$

we conclude $(V')^{\perp}\subset\mathcal{H}$, which finishes the proof. □

If we consider a leaf $L_P(x)$ for some subspace P of ν_xM, we can describe the part of $E(0)$ that splits of by the last proposition, namely

$$\mathcal{H}_P(x)=\{Z\in E(0)\mid\nabla_X\alpha(Y,Z)=0\text{ for all }X,Y\in D_P(x)\}\,.$$

By $\tilde L_P(x)$ we will denote the reduced leaf.

DEFINITION 1.9. Let P be an n-dimensional linear subspace of ν_xM, then we call $\tilde L_P(x)$ a *rank-n leaf* of M, if $\operatorname{span}\{v_i(a)\in P\}$ is n-dimensional.

Let $D_P=E_P\oplus E(0)$, then

$$E(0)=\mathcal{H}_P\oplus\left(\nabla_{E_P}E_P\right)_0,$$

where $(\cdot)_0$ denotes projection onto $E(0)$. Moreover if $P_1\perp P_2$ then $\nabla_{E_{P_1}}E_{P_2}\perp E(0)$, since

$$(\nabla_{E_{P_1}}\alpha)(E_{P_2},E(0))=\big\langle\nabla_{E_{P_1}}E_{P_2},E(0)\big\rangle\, n_2=\big\langle\nabla_{E_{P_2}}E_{P_1},E(0)\big\rangle\, n_1=0$$

by Codazzi equation and the fact that non zero curvature normals $v_1\in P_1$ and $v_2\in P_2$ are not proportional.

Later (cf. Subsection 2.5 on page 29) we will see, that this construction may be refined by considering distributions of eigenspaces, where associated curvature normals do not consist of whole proportional families.

Heintze and Liu proved in [HL99], that an isoparametric submanifold is uniquely determined by the $L_P(x)$ when P is one-dimensional. If we assume that the second fundamental form α_x, that is the affine marked Dynkin diagram is fixed, the rank-2 leaves for P, that is not linear, contain no additional information, because finite dimensional rank-2 isoparametric submanifolds are determined by their marked Dynkin diagram. This proves the following slight modification of Proposition 3.1 in [HL99]:

COROLLARY 1.10. *Let M_1 and M_2 be two irreducible isoparametric submanifolds of V with rank bigger than or equal to 2. Assume that there exist $x \in M_1 \cap M_2$ such that $T_xM_1 = T_xM_2$ and $\tilde{L}_{1l}(x) = \tilde{L}_{2l}(x)$ for any one-dimensional linear subspace $l \subset \nu_x M_1 = \nu_x M_2$ and $\alpha_1(x) = \alpha_2(x)$. Then $M_1 = M_2$.*

In other words: Two different isoparametric submanifold with same second fundamental form at one common point have to contain at least one rank-1 leaf that is different. Hence: Understanding the homogeneous isoparametric hypersurfaces is a crucial point in understanding isoparametric submanifolds of higher codimension. Therefore we concentrate on hypersurfaces in the next chapters.

1.3. Normal homogeneous structures

Our aim in this section is to show that an isoparametric homogeneous submanifold is uniquely determined by the second fundamental form α and the normal homogeneous structure S in a point. We will use the ideas described in [BCO03] chapter 7.1.b.

The investigation of (extrinsic) homogeneous structures has been started in the paper [OS91] by Olmos and Sánchez. They proved that a compact full submanifold of Euclidean space admits a normal homogeneous structure if and only if it is an orbit of an s-representation, that is, a submanifold with extrinsic homogeneous normal bundle, in particular these are homogeneous submanifold with constant principal curvature.

DEFINITION 1.11. A *normal homogeneous structure* S on a submanifold M on V is of the form $S = \nabla + \nabla^\perp - \tilde{\nabla}$, where $\tilde{\nabla} = \nabla^c + \nabla^\perp$ is a so-called *canonical connection*, i.e.

- $\tilde{\nabla}$ is a metric connection.
- α is $\tilde{\nabla}$-parallel.
- S is $\tilde{\nabla}$-parallel.

We use the operator

$$\Gamma_v X = S_v X + \alpha(v, X^T) - A_{X^\perp} v$$

for $v \in T_P M$ and $X \in V$ which encodes the information of the second fundamental form and the homogeneous structure and is $\tilde{\nabla}$-parallel.

REMARK. The more general notion *homogeneous structure* is defined likewise, where $\tilde{\nabla} = \nabla \oplus \nabla^\perp - S$ and TM is a $\tilde{\nabla}$-parallel bundle, without requiring that the connections coincide on the normal bundle.

A central point in the following discussion is, that the differential equation for $\tilde{\nabla}$-geodesic has constant coefficients, namely

$$\frac{D}{dt}B = BC,$$

where $B(t) = (B_1(t), \ldots, B_k(t), B_{k+1}(t), \ldots)$ is a $\tilde{\nabla}$-parallel Darboux frame along γ and $C_{ij} = \left\langle \Gamma_{\dot\gamma(0)} B_i(0), B_j(0) \right\rangle$. Thereby let $(B_1, \ldots, B_k)$ be a normal frame, $B_{k+1} = \dot\gamma$ and $(B_{k+1}(t), \ldots)$ be a orthonormal Schauder basis of $T_{\gamma(t)}M$.

Therefore the $\tilde{\nabla}$-geodesics starting at p and the $\tilde{\nabla}$-parallel transport along any curve through p are determined by Γ_p. The following lemma is valid, cf. [BCO03, Lemma 7.1.10], formulated for infinite dimensions.

LEMMA 1.12. *Let M be a submanifold of Hilbert space V admitting a homogeneous structure and let p and q be arbitrary points in M. Then there exists an isometry $F\colon V \to V$ mapping p to q and $F(M) \subseteq M$.*

PROOF. The same proof as in [BCO03] also applies on the infinite dimensional setting. □

The arguments in the proof show that Γ is F_*-invariant along curves in M. We modify the proof to show

THEOREM 1.13. *Let M_1 and M_2 be two connected, complete, homogeneous isoparametric submanifolds of V with normal homogeneous structures $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ respectively. Assume that there exist $x \in M_1 \cap M_2$ such that $T_xM_1 = T_xM_2$, $\Gamma_1(x) = \Gamma_2(x)$ and $\alpha_1(x) = \alpha_2(x)$. Then $M_1 = M_2$.*

PROOF. Since the second fundamental forms in x coincide so do the curvature normals, the curvature spheres and the affine subspace $E(0)(x)$. Let c be a $\tilde{\nabla}$-geodesic either in a curvature sphere or in $E(0)(x)$, which is determined by the given data $\Gamma_i(x)$.

Denote by τ the $\tilde{\nabla}_1$–parallel transport along c and by $F\colon V \longrightarrow V$ the unique isometry such that $F(x) = y$ and $F_{*p} = \tau$. Observe that we could also use $\tilde{\nabla}_2$ since $\Gamma_1(x) = \Gamma_2(x)$ and the curve c is contained in $M_1 \cap M_2$. Therefore the second fundamental form and the homogeneous structures of M_1 and M_2 coincide on the curvature spheres containing x and in $x + E(0)$, since Γ_i is F_*-invariant.

Hence the two geometric data coincide on the common dense subset of M_1 and M_2, namely on $Q_0(M_1) = Q_0(M_2)$ (cf. proof of Proposition 1.8 on page 5) and therefore $M_1 = M_2$ since the manifolds are complete. □

Therefore, to obtain a rigidity result, we have to determine the canonical connection. The ideas arise from is description of ∇^c in the finite dimensional case, i.e. for the orbits of s-representations, which is closely connected to the so-called projection connection ∇^π. The latter is defined by

$$\nabla^\pi_X Y = \sum_{n=1}^{k} (\nabla_X Y_n)_n$$

where $TM = \oplus_{n=1}^k E_n$ and $(\cdot)_i$ denotes projection onto E_i. This is the canonical connection if the restricted root system of the corresponding symmetric space is reduced. Leschke gave in [LES97] the canonical connection for any finite dimensional homogeneous isoparametric submanifold, which is almost a projection connection as well, projecting onto modules of the isotropy representation instead of onto the eigenspaces. We will describe this more closely, cf. further details [BCO03], example 3.2 on page 49ff. and example 3.4 on page 63.

Let G/K a semi-simple symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, which is a section of the polar representation K acting on $\mathfrak{p}$. Then the spaces $\mathfrak{p}_\lambda$ are the eigenspaces of $\mathrm{ad}(a)^2$ for $a \in \mathfrak{a}$, where λ is a positive restricted root and $\mathfrak{p}_{2\lambda} = 0$ if 2λ is not a root. If $M = K \cdot a$ is a principal orbit of the s-representation, then the eigenspaces of the shape operator are given by $E_\lambda = \mathfrak{p}_\lambda + \mathfrak{p}_{2\lambda}$ and this decomposition is respected by the isotopy representation.

Then the canonical connection ∇^c is the projection connection of the $\mathfrak{p}_\lambda$ with the only exception if $X \in \mathfrak{p}_{2\lambda}$ and $Y \in \mathfrak{p}_\lambda$ then

$$\nabla^c_X Y = (\nabla_X Y)_\lambda + \frac{1}{2}(\nabla_Y X)_\lambda. \tag{1.1}$$

Note that $(\nabla^c_X Y)_{\mathfrak{p}_\mu} = 0$ when $Y \in \mathfrak{p}_\lambda$ with $\lambda \neq \mu$, cf. [Les97, p. 58].

The reason for this exception is the following: Any K–invariant vector field has to be parallel with respect to the canonical connection and for such vector fields $\nabla_{\mathfrak{p}_\lambda}\mathfrak{p}_\mu = [\mathfrak{k}_\lambda, \mathfrak{p}_\mu]^T \subseteq (\mathfrak{p}_{\lambda+\mu} \oplus \mathfrak{p}_{\lambda-\mu})^T$. So after projection to $\mathfrak{p}_\mu$ this is not zero only in the case $\lambda = 2\mu$, which is the exception from above.

Our aim is to study the situation for the infinite dimensional setting, i.e. to determine the canonical connection for certain isoparametric hypersurfaces. In general the eigenspace $E(0)$ is infinite dimensional and in order to define a projection connection ∇^π with respect to irreducible modules of the isotropy representation, the main task is to describe these modules within $E(0)$. We will do this in the next chapter by means of ∇A. In fact this will prove that the normal homogeneous structure is determined by A and ∇A, at least if the eigenspaces of A are isotropy irreducible.

A projection connection in that sense is a good candidate for the canonical connection, for the latter has to respect modules of the isotropy representation:

Corollary 1.14. *Let G be a Hilbert Lie group acting polarly on a Hilbert space V and let a be a regular point. With respect to the canonical connection the modules of the isotropy representation are parallel distributions.*

Proof. Since the holonomy of the canonical connection is part of the isotropy representation, there is for any curve c a unique curve $g\colon I \to G$ such that $c(t) = g(t) \cdot c(0)$ and the parallel translation along c is given by $g(t)_* X$.

Let W be a tangential distribution, which is invariant under the isotropy representation, and $a \in M$ a regular point. Then $W(g \cdot a) = g_* W(a)$ for any $g \in G$. Let $c(t) = g(t) \cdot a$ be a curve and $X(t) = \sum_{i=1}^n \lambda_i(t) g(t)_* X_i$ an arbitrary vector field along c, where $X_1, \ldots, X_n$ is a basis of $W(a)$. Then

$$\nabla_{\dot c(t)} X(t) = \sum_{i=1}^n \lambda_i(t) \nabla_{\dot c(t)} g(t)_* X_i + \sum_{i=1}^n \dot c(t)(\lambda_i(t) g(t)_* X_i) \in W(c(t))$$

since the first summand vanishes by the choice of $g(t)$ as above. □

CHAPTER 2

The isotropy representation of isoparametric hypersurfaces

Throughout this chapter let $G \times V \to V$ be an irreducible, effective polar representation of a Hilbert Lie group G on a Hilbert space V with cohomogeneity one. Let $M = G \cdot a$ be a principal orbit hence an isoparametric hypersurface and assume that it does not split in the sense of Proposition 1.8 on page 5. Since the isotropy group G_a is compact and finite dimensional the tangent space $T_a M$ splits into finite dimensional irreducible modules of the isotropy representation. Our aim is to describe these modules, to determine the canonical connection of M.

Let

$$T_a M = \overline{\bigoplus_{n \in \mathbb{Z}} E_n \oplus E(0)},$$

where $E_n = E(\lambda_n)$ is the eigenspace associated with the curvature normal $v_n = v_{\lambda_n} = \frac{v}{d+n}$.

Remark. Note that there is an (finite dimensional) eigenspace E_0 associated with the greatest positive eigenvalue λ_0, which must not be mistaken for $E(0)$, the eigenspace associated with the eigenvalue 0. Nevertheless this notation will turn out to be very useful in this and the next chapter.

Since

$$g_*(A_\xi v) = A_{g_*\xi}(g_* v) = A_\xi g_* v$$

the eigenspaces are invariant subspaces under the isotropy representation.

The submanifold M is a hypersurface, its affine marked Dynkin diagramm is of type $\tilde{A}_1$, that is $\underset{m_1}{\circ}\overset{\infty}{-}\underset{m_2}{\circ}$. The eigenspaces E_n are of dimension m_1, if n is even and of dimension m_2 if n is odd. Note that $m_1 = m_2$ is possible.

To understand the isotropy representation, it is necessary to investigate the isotropy group closer.

2.1. Structure of the principal isotropy group

Proposition 2.1. *Let c_n be the midpoint of the curvature sphere $S_n(a)$. Then*

$$(G_{c_n})_{a-c_n} = \{g \in G \mid g \cdot c_n = c_n,\, g_*(a - c_n) = a - c_n\} = G_a$$

i.e. the principal isotropy group of the singular slice representation is the principal isotropy group of the action.

Proof. We observe that

$$g \in (G_{c_n})_{a-c_n} \iff g \cdot c_n = c_n \text{ and } g_*(a - c_n) = a - c_n \iff g \cdot a = a$$

since the action is affine which yields one inclusion, the other being clear by the same argument because $G_a \subset G_{c_n}$. □

Since G_{c_n} is compact we equip its Lie algebra $\mathfrak{g}_{c_n}$ with a biinvariant metric and decompose

$$\mathfrak{g}_{c_n} = \mathfrak{g}_{c_n}^{\mathrm{tr}} \oplus \mathfrak{g}_{c_n}^{\mathrm{eff}},$$

where $\mathfrak{g}_{c_n}^{\mathrm{tr}}$ is the Lie algebra of the subgroup of G_{c_n}, which acts trivially on $\nu_{c_n}(G \cdot c_n)$ and $\mathfrak{g}_{c_n}^{\mathrm{eff}}$ the orthogonal complement.

Then $G_n = (G_{c_n}^{\mathrm{eff}})_{a-c_n}$ is the part of G_a which acts effectively on $E_n(a)$, by the above lemma this is the principal isotropy group of the effectivized slice representation, i.e. the principal isotropy group of an action which is transitive on the curvature sphere. By the classification of actions transitive on spheres (cf. Section 2.2 on page 14), G_n consists either of one or two simple factors or of one simple factor and a one-dimensional abelian factor.

Since G_a is compact, it is clear that only finitely many of these factors G_n may be different. If $m_1 \neq m_2$, then G_{2n} is not isomorphic to G_{2n+1}, but for some low dimensional exceptions. Our aim is to show that $G_n = G_{n+2}$ for all n or all G_n are equal. First we prove

Proposition 2.2. *Let $k, n \in \mathbb{N}$ arbitrary. Then $G_n = G_{4k+n}$.*

Proof. Consider the antipodal map φ_k on the curvature sphere S_k, i.e.

$$\varphi_k(x) = x + 2\frac{v_k(x)}{||v_k(x)||^2} = x + \xi_k(x)$$
$$\varphi_{k*_x}(v) = v - A_{\xi_k(x)}v$$

Restricted to an eigenspace E_n the map φ_{k*} is equivariant, that is,

$$g_*(\varphi_{k*}(v)) = \left(1 - 2\frac{\langle v_k, v_n\rangle}{\langle v_k, v_k\rangle}\right) g_*(v) = \frac{n-2k-d}{d+n} g_*(v).$$

Since $\varphi_*(E_n) = E_{2k-n}$ and $\varphi_{k*}(E_{2k-n}) = E_{2n}$ this implies

$$G_n(a) = G_{2n-k}(\varphi_k(a))$$

since φ_k is a diffeomorphism and $G_a = G_{\varphi_k(a)}$ by the last proposition.

Let $h_k \in G$ be an element such that $h_k(a) = \varphi_k(a)$. Then

$$h_k G_n(a) h_k^{-1} = G_n(\varphi_k(a)) = G_{2k-n}(a).$$

An easy calculation shows $h_l(h_k(a)) = \varphi_k(\varphi_l(a))$ and this yields by the above equation

$$\begin{aligned} G_n(\varphi_k(\varphi_l(a))) &= G_{2k-n}(\varphi_l(a)) &&= G_{2l-2k+n}(a) = \\ G_n(h_l(h_k(a))) &= h_l G_n(h_k(a)) h_l^{-1} = G_{2l-n}(h_k(a)) &&= G_{2k-2l+n}(a) \end{aligned}$$

□

With help of the last proposition we prove

Theorem 2.3. *Let $M = G \cdot a$ be a homogeneous isoparametric submanifold, with $\dim E_{2n} = m_1$ and $\dim E_{2n+1} = m_2$. We assume that G_a acts effectively on T_aM.*

The isotropy group is a product

$$G_a = \tilde{G} \times G_0 \times G_1,$$

where $G_{2n} = \tilde{G} \times G_0$ ($G_{2n+1} = \tilde{G} \times G_1$ resp.) is the principal isotropy group of a transitive action on S^{m_1} (S^{m_2} resp.), acting effectively on the corresponding eigenspace E_{2n} (E_{2n+1} resp.).

Any of the factors of G_a may be trivial.

PROOF. We divide the proof into two steps. First we show that there is no factor acting effectively on $E(0)$ but not effectively on any E_n.

STEP 1. Assume there is an element $g \in G_a$ such that $g_*|_{E_n} = \mathrm{id}\,|_{E_n}$ but $g_* X = Y$ for $X \neq Y$, $X, Y \in E(0)$. Then $(\nabla_{E_i}\alpha)(E_j, X) = (\nabla_{E_i}\alpha)(E_j, Y)$, hence $X - Y \in \ker(\nabla_{E_i}\alpha)(E_j, \cdot)$ for all $i, j \in \mathbb{Z}$ which implies that M splits, cf. Proposition 1.8 on page 5. This contradiction proves that G_a is the product of the G_n for $n \in \mathbb{N}$.

STEP 2. We prove $G_n = G_{2k-n}$ in this part. We have already seen that $G_n = G_{4k+n}$, which yields, together with the first step, a splitting on Lie algebra level

$$\mathfrak{g}_a = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 + \mathfrak{g}_4$$

into ideals $\mathfrak{g}_i$. Moreover we have proven in the last proposition, that $h_k G_n h_k^{-1} = G_{2k-n}$, when $h_k(a) = \varphi_k(a)$ and therefore $\mathfrak{g}_1 \cong \mathfrak{g}_3$ and $\mathfrak{g}_2 \cong \mathfrak{g}_4$.

Observe that h_k commutes with G_n whenever $\mathfrak{g}_k$ and $\mathfrak{g}_n$ are disjoint, since the maximal subgroup of $G^{\mathrm{eff}}_{c_k}$ with Lie algebra $\mathfrak{g}_k$ is $G_k \cup \{g \cdot h_k \cdot g^{-1} \mid g \in G_k\}$ and contains in particular any element h_k.

So far we have proven, when $\mathfrak{g}_1 \cap \mathfrak{g}_2 = \{0\}$ then $G_n = G_{2k-n}$ for any $k, n \in \mathbb{Z}$. This corresponds to the case when $\tilde{G}$ vanishes in the statement of the theorem. We remark that the converse is also true, that is if $\mathfrak{g}_1 = \mathfrak{g}_3 = h_2 \mathfrak{g}_1 h_2^{-1}$, then $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$. This yields $\mathfrak{g}_2 = \mathfrak{g}_4$, too, since $h_1 \cdot G_2 \cdot h_1^{-1} = G_2 = G_4$ holds.

Let now $\mathfrak{g}_1 \cap \mathfrak{g}_2 \neq \{0\}$. We start with the case, when $\mathfrak{g}_1 \subset \mathfrak{g}_2$. By conjugation with h_1 the following holds: $\mathfrak{g}_1 \subset \mathfrak{g}_1 \cap \mathfrak{g}_2 \cong \mathfrak{g}_1 \cap \mathfrak{g}_4$ and therefore $\mathfrak{g}_1 \subset \mathfrak{g}_2 \cap \mathfrak{g}_4$. The same holds for $\mathfrak{g}_3$ since $h_2 \mathfrak{g}_1 h_2^{-1} = \mathfrak{g}_3 \subset h_2 \mathfrak{g}_2 h_2^{-1} = \mathfrak{g}_2$. So either $\mathfrak{g}_1 = \mathfrak{g}_3$ or $\mathfrak{g}_1 \oplus \mathfrak{g}_3 = \mathfrak{g}_2 \cap \mathfrak{g}_4$ consists of two isomorphic summands.

In the first case, by the remark above $\mathfrak{g}_2 = \mathfrak{g}_4$ and therefore $\mathfrak{g}_a = \tilde{\mathfrak{g}}$ or $\mathfrak{g}_a = \mathfrak{g}_2 \supsetneq \mathfrak{g}_1$.

In the second case $\mathfrak{g}_2 = \mathfrak{g}_4$ because any $\mathfrak{g}_i$ consists of at most two ideals or one-dimensional abelian summands, being the principal isotropy algebra of an action transitive on a sphere. But then again $\mathfrak{g}_1 = \mathfrak{g}_3$, which yields a contradiction.

It remains to analyze the case , when $\{0\} \neq \mathfrak{g}_1 \cap \mathfrak{g}_2 \neq \mathfrak{g}_i$ for $i = 1, 2$, in particular $\mathfrak{g}_i$ consists of two summands for any i. Since conjugation with appropriate h_i yields

$$\mathfrak{g}_1 \cap \mathfrak{g}_2 \cong \mathfrak{g}_1 \cap \mathfrak{g}_4 \cong \mathfrak{g}_2 \cap \mathfrak{g}_3 \cong \mathfrak{g}_3 \cap \mathfrak{g}_4,$$

either $\mathfrak{g}_a = \tilde{\mathfrak{g}} \oplus \sum_{i=1}^{4} \mathfrak{h}_i$ (where $\mathfrak{g}_{\mathrm{i}} = \tilde{\mathfrak{g}} \oplus \mathfrak{h}_i$) or $\mathfrak{g}_1 = \mathfrak{g}_1 \cap \mathfrak{g}_2 \oplus \mathfrak{g}_1 \cap \mathfrak{g}_4$. The latter case implies that $G_1 = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, for this is the only possibility with two isomorphic summands, and h_1 interchanges the two factor, by explicitly examining the s-representation $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ acting on $\mathbb{R}^8$, one sees immediately that this is not the case.

Let therefore $\mathfrak{g}_a = \tilde{\mathfrak{g}} \oplus \sum_{i=1}^{4} \mathfrak{h}_i$ and $H_i \subset G_a$ the connected, closed subgroup with Lie algebra $\mathfrak{h}_i$. Then $G_3 = \tilde{G} \times H_3 = h_2 G_1 h_2^{-1} = h_2 \tilde{G} h_2^{-1} \times h_2 H_1 h_2^{-1} = h_2 \tilde{G} h_2^{-1} \times H_1$ since $[\mathfrak{g}_2, \mathfrak{h}_1] = 0$. Therefore $\mathfrak{h}_1 = \mathfrak{h}_3$, which finishes this step. □

REMARK. Any of the four cases for the principal isotropy group, namela $G_1 = G_2$, $G_1 \subset G_2$, $\{\mathrm{id}\} \subsetneq G_1 \cap G_2 \subsetneq G_i$ and $G_1 \cap G_2 = \{\mathrm{id}\}$ does occur. We give examples (for the calculation of the diagrams and the description of the $P(G, H)$-actions, see Chapter 4) and characterize the corresponding Dynkin diagram.

We remark, that for the known examples the factor $\tilde{G}$ is $\mathrm{U}(1)$ or $\mathrm{Sp}(1)$, if either G_0 or G_1 does not vanish. Moreover in the case $\tilde{G} = \{\mathrm{id}\}$ the isotropy group is $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$.

$G_1 = G_2$: In most cases the Dynkin diagram is of type $\underset{m}{\circ}\overset{\infty}{-}\underset{m}{\circ}$ for $m \in \mathbb{N}$. An example is given by the $P(G, H)$ action with $G = \mathrm{SO}(m+1)$ and $H = (\mathrm{SO}(1) \times \mathrm{SO}(m))^2$, the principal isotropy group is then $\mathrm{SO}(m)$.

$\mathfrak{g}_a = \tilde{\mathfrak{g}}$	Diagram	Action
$\mathfrak{so}(2) = \mathfrak{u}(1)$	$\underset{2}{\circ}\overset{\infty}{—}\underset{3}{\circ}$	$G_2/(SU(3) \times SO(4))$
$\mathfrak{so}(3) = \mathfrak{su}(2) = \mathfrak{sp}(1)$	$\underset{3}{\circ}\overset{\infty}{—}\underset{5}{\circ}$	A II–III$(3,3)$
	$\underset{3}{\circ}\overset{\infty}{—}\underset{7}{\circ}$	C II$(1,2)$–II$(1,2)$
	$\underset{5}{\circ}\overset{\infty}{—}\underset{7}{\circ}$	A II–III$(3,5)$
$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$\underset{4}{\circ}\overset{\infty}{—}\underset{7}{\circ}$	rank-2 C II$(2,3)$
$\mathfrak{so}(5) = \mathfrak{sp}(2)$	$\underset{5}{\circ}\overset{\infty}{—}\underset{11}{\circ}$	E II–IV
$\mathfrak{so}(6) = \mathfrak{su}(4)$	$\underset{6}{\circ}\overset{\infty}{—}\underset{9}{\circ}$	rank-2 E II
$\mathfrak{su}(3)$	$\underset{6}{\circ}\overset{\infty}{—}\underset{7}{\circ}$	$SO(16)/Spin(9) \times (SO(2) \times SO(14))$
$\mathfrak{so}(7)$	$\underset{15}{\circ}\overset{\infty}{—}\underset{7}{\circ}$	F II–II
$\mathfrak{g}_a = \tilde{\mathfrak{g}} \oplus \mathfrak{g}_1$		
$\mathfrak{sp}(m) \oplus \mathfrak{u}(1)$	$\underset{2}{\circ}\overset{\infty}{—}\underset{4m+3}{\circ}$	$SO(4m)/Sp(m) \cdot Sp(1) \times SO(4m-2)$
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\underset{5}{\circ}\overset{\infty}{—}\underset{4m+3}{\circ}$	A II–III$(3, 2m+3)$
$\mathfrak{su}(m) \oplus \mathfrak{u}(1)$	$\underset{2}{\circ}\overset{\infty}{—}\underset{2m+1}{\circ}$	D I$(3, 2m+3)$–III
$\mathfrak{g}_a = \tilde{\mathfrak{g}} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$		
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$\underset{4}{\circ}\overset{\infty}{—}\underset{4m+3}{\circ}$	rank-2 C II(2,m)

TABLE 2.1. $P(G,H)$-action with "exotic" principal isotropy group

The fact that some low dimensional Lie algebras are isomorphic yields a second kind of examples: One is the rank-2 example of E II (cf. subsection 4.6.1 on page 57), which has isotropy group U(4) and Dynkin diagram $\underset{9}{\circ}\overset{\infty}{—}\underset{6}{\circ}$, that is U(4) acts as Spin(6) on some eigen spaces. We give the complete list of these examples in the table on this page. The last two example arise from the fact that $\mathfrak{so}(7)$ and $\mathfrak{su}(3)$ are the principal isotropy algebra of two different actions transitive on spheres respectively.

$\boldsymbol{G_1 \subset G_2}$: In most cases the Dynkin diagram is of type $\underset{1}{\circ}\overset{\infty}{—}\underset{2m+1}{\circ}$ or $\underset{3}{\circ}\overset{\infty}{—}\underset{4m+3}{\circ}$. An example is given by $G = Sp(m)$ and $H = \big(Sp(1) \times Sp(m-1)\big)^2$, the principal isotropy group is $Sp(m-2) \times Sp(1)$ and the diagram is $\underset{3}{\circ}\overset{\infty}{—}\underset{4(m-2)+3}{\circ}$.

Again the isomorphisms of low dimensional Lie algebras give another kind of example: The exceptional cohomogeneity one action with $G = SO(4m)$, $H = Sp(m) \cdot Sp(1)$ and $K = SO(4m-2)$ has isotropy group $Sp(m) \times U(1)$. The diagram is $\underset{2}{\circ}\overset{\infty}{—}\underset{4(m-2)+3}{\circ}$, that is the factor U(1) acts as SO(2) on some eigenspaces. The examples of that type are collected in the table on the current page.

$\boldsymbol{\{\mathrm{id}\} \subsetneq G_1 \cap G_2 \subsetneq G_i}$: The Dynkin diagram is of type $\underset{2m_1+1}{\circ}\overset{\infty}{—}\underset{2m_2+1}{\circ}$ or $\underset{4m_1+3}{\circ}\overset{\infty}{—}\underset{4m_2+3}{\circ}$ for any $m_i \in \mathbb{N}$. Examples of those type are given by complex Grassmannians (i.e. $G = SU(n)$, $H = S(U(k) \times U(n-k)) \times S(U(l) \times U(n-l))$ for $k \neq l$) with isotropy group $SU(m_1) \times U(1) \times SU(m_2)$ or quaternionic Grassmannians.

Again there is an exceptional example, namely the rank-2 example arising from C II$(2, m)$, whose diagram is $\underset{4}{\circ}\overset{\infty}{—}\underset{4m+2}{\circ}$ with principal isotropy group $\mathrm{Sp}(m) \times \mathrm{Sp}(1)^2$.

$G_1 \cap G_2 = \{\mathrm{id}\}$: No restrictions on the Dynkin diagrams. An example of this type is given by real Grassmannians (i.e. $G = \mathrm{SO}(n)$, $H = \mathrm{SO}(k) \times \mathrm{SO}(n-k) \times \mathrm{SO}(l) \times \mathrm{SO}(n-l)$ for $k \neq l$) with isotropy group $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$.

Remark. For a finite dimensional homogeneous isoparametric submanifold $M = G \cdot a$ the effectivized slice representation always coincides with the normal holonomy representation, hence is an s-representation ([HO92]), if one considers the maximal group G, i.e. the connected component of the full group of isometries on M. This is not true in the infinite dimensional setting although the normal holonomy representation is an s-representation (cf. [HL99, Lemma 2.1]): Let G be $\mathrm{SO}(16)$ and $H \subset G \times G$ be $\mathrm{Spin}(9) \times (\mathrm{SO}(2) \times \mathrm{SO}(14))$, the action of H on G has cohomogeneity one and is therefore hyperpolar and lifts to a $P(G, H)$ action. One of its singular slice representation is the polar action of G_2 on S^6, which is not an s-representation, the other is $\mathrm{U}(4)$ acting on S^8, that is, the diagram is $\underset{6}{\circ}\overset{\infty}{—}\underset{7}{\circ}$.

Definition 2.4. We call a homogeneous isoparametric hypersurface of a Hilbert space *elementary* if the diagram is

$$\underset{m}{\circ}\overset{\infty}{—}\underset{m}{\circ} \qquad \underset{1}{\circ}\overset{\infty}{—}\underset{2m+1}{\circ} \qquad \underset{3}{\circ}\overset{\infty}{—}\underset{4m+3}{\circ} \qquad \underset{8}{\circ}\overset{\infty}{—}\underset{15}{\circ}$$

and the isotropy group is $\mathrm{SO}(m)$, $\mathrm{U}(m)$, $\mathrm{Sp}(m) \times \mathrm{Sp}(1)$ and $\mathrm{Spin}(7)$ respectively.

Remark. Among the Hermann actions the $P(G, K \times K)$-actions are elementary, if G/K is sphere or a projective space.

We will see later for the case $G_a = \mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$, that each non elementary isoparametric hypersurface contains two elementary parts, each associated with one of the vertices of the Dynkin diagram (cf. section 2.5 on page 29). This is done by showing that the distribution $D_1 = \{X \in TM \mid G_1 \cdot X = X\} = \mathrm{Fix}_{G_1}(TM)$ is autoparallel.

2.2. Decomposition of eigenspaces E_n

To describe the decomposition of E_n in modules of the isotropy representation we only have to determine the groups acting effectively on E_n by means of Theorem 2.3. Let $\dim E_n = m$.

m arbitrary: Let the effectivized slice representation be the standard representation of the group $\mathrm{SO}(m+1)$ acting on $\mathbb{R}^{m+1}$ with principal isotropy group $\mathrm{SO}(m)$.

The effectivized isotropy representation on E_n is the standard representation of $\mathrm{SO}(m)$ on $\mathbb{R}^m$, which acts transitively on the sphere, hence E_n is an irreducible module of the isotropy representation.

$\mathbf{m = 2\tilde{m} + 1}$: Additionally the effectivized slice representation could be the s-representation of a complex projective space, i.e. $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(\tilde{m}+1)) = \mathrm{U}(m+1)$ acting on $\mathbb{C}^{\tilde{m}+1} = \mathbb{R}^{2\tilde{m}+2}$ with principal isotropy group $\mathrm{S}(\mathrm{U}(\tilde{m}) \times \mathrm{U}(1)) = \mathrm{U}(\tilde{m})$.

The effectivized isotropy representation on E_n is the representation of $\mathrm{U}(\tilde{m})$ on $\mathbb{R}^{2\tilde{m}+1}$, therefore E_n decomposes into an $2\tilde{m}$-dimensional module with the standard representation of $\mathrm{U}(\tilde{m})$ and a one-dimensional trivial one.

$\mathbf{m = 4\tilde{m} + 3}$: Additionally the effectivized slice representation could be the s-representation of a quaterionic projective space, i.e. $\mathrm{Sp}(\tilde{m}+1) \times \mathrm{Sp}(1)$ acting on $\mathbb{H}^{\tilde{m}+1} = \mathbb{R}^{4\tilde{m}+4}$ with principal isotropy group $\mathrm{Sp}(\tilde{m}) \times \mathrm{Sp}(1)$.

The effectivized isotropy representation on E_n is the representation of $\mathrm{Sp}(\tilde{m})\times \mathrm{Sp}(1)$ on $\mathbb{R}^{4\tilde{m}+3}$, therefore E_n decomposes into an $4\tilde{m}$-dimensional module with the standard representation of $\mathrm{Sp}(\tilde{m})$ and a three-dimensional module with the standard representation of $\mathrm{Sp}(1)$.

$\mathbf{m = 15 = 8 + 7}$: Additionally the effectivized slice representation could be the s-representation of the projective Cayley plane, i.e. $\mathrm{Spin}(9)$ acting on $\mathbb{R}^{16}$ with principal isotropy group $\mathrm{Spin}(7)$.

The effectivized isotropy representation on E_n is the representation of $\mathrm{Spin}(7)$ on $\mathbb{R}^{15}$, therefore E_n decomposes into an 8-dimensional module with the representation of $\mathrm{Spin}(7)$ and a 7-dimensional module with the standard representation of $\mathrm{SO}(7)$.

Finally we consider the case of a transitive action on S^{m+1}, which is not an s-representation.

$\mathbf{m = 2\tilde{m} + 1}$: Additionally the effectivized slice representation could be $\mathrm{SU}(m+1)$ acting on $\mathbb{R}^{2\tilde{m}+2}$ with principal isotropy group $\mathrm{SU}(\tilde{m})$, therefore E_n decomposes into an $2\tilde{m}$-dimensional module with the standard representation of $\mathrm{SU}(\tilde{m})$ and a one-dimensional trivial one.

$\mathbf{m = 4\tilde{m} + 3}$: Additionally the effectivized slice representation could be $\mathrm{Sp}(\tilde{m}+1)\times \mathrm{U}(1)$ acting on $\mathbb{R}^{4\tilde{m}+4}$ with principal isotropy group $\mathrm{Sp}(\tilde{m})\times \mathrm{U}(1)$. Therefore E_n decomposes into an $4\tilde{m}$-dimensional module with the standard representation of $\mathrm{Sp}(\tilde{m})$, a two-dimensional with the standard representation of $\mathrm{SO}(2)$ and a one-dimensional module.

$\mathbf{m = 4\tilde{m} + 3}$: Additionally the effectivized slice representation could be $\mathrm{Sp}(\tilde{m}+1)$ acting on $\mathbb{R}^{4\tilde{m}+4}$ with principal isotropy group $\mathrm{Sp}(\tilde{m})$. Therefore E_n decomposes into an $4\tilde{m}$-dimensional module with the standard representation of $\mathrm{Sp}(\tilde{m})$ and three one-dimensional modules.

$\mathbf{m = 6}$: Additionally the effectivized slice representation could be G_2 acting on $\mathbb{R}^7$ with principal isotropy group $\mathrm{SU}(3)$, therefore E_n is irreducible.

$\mathbf{m = 7}$: Additionally the effectivized slice representation could be $\mathrm{Spin}(7)$ acting on $\mathbb{R}^8$ with principal isotropy group G_2, therefore E_n is irreducible.

2.3. Decomposition of $E(0)$ — associated modules

We associate a module of the isotropy representation with each pair of modules by means of $\nabla\alpha$ in the following manner.

Definition 2.5. Let V_1 and V_2 be not necessarily irreducible modules of the isotropy representation and let ξ be a parallel normal vector field. Then we define

$$\begin{aligned} V_{V_1,V_2} &= \left(\bigcap_{X\in V_1, Y\in V_2} \ker(\nabla_X\alpha)(Y,\cdot) \right)^{\perp} = \operatorname{span}\{(\ker(\nabla_X\alpha)(Y,\cdot))^{\perp} \mid X\in V_1, Y\in V_2\} \\ &= \{(\nabla_X A)_\xi Y \mid X\in V_1, Y\in V_2\} \end{aligned}$$

the *module associated with V_1 and V_2*.

Remark. (1) If the V_i are modules, then V_{V_1,V_2} is a module as well.
(2) We have $\dim V_{V_1,V_2} \le \dim V_1 \cdot \dim V_2$.
(3) The Codazzi equation implies $V_{V_1,V_2} = V_{V_2,V_1}$.
(4) For now on we use the abbreviation

$$V_{n,m} = V_{E_n,E_m}.$$

Note, that $V_{n,n} = \{0\}$, since eigenspaces are autoparallel.

PROPOSITION 2.6. *Any irreducible module of the isotropy representation is contained in some associated module. Moreover the modules $V_{n,m}$ span $E(0)$.*

PROOF. The first claim means that the tangent space is spanned by the associated modules. Assume there is a vector perpendicular to all associated modules, that means it is contained in $\ker(\nabla_X \alpha)(Y, \cdot)$ for every X and Y. This is a contradiction since M does not split, cf. Proposition 1.8 on page 5.

For the second part we observe that if both modules V_i are subsets of $E(0)$ the associated module is 0 since $(\nabla_{V_1} A)_\xi(V_2) = 0$ by the autoparallelity of $E(0)$. If $V_1 \subset E(0)$ and $V_2 \subset E_n$ by the Codazzi equation the associated module is not contained in $E(0)$ which proves the second assertion. □

Thus, to describe the splitting of $E(0)$ into irreducible modules of the isotropy representation, it is sufficient to understand the representation on the modules $V_{n,m}$. As we will later see the converse of the last proposition is not true: there could be modules $V_{n,m}$ which are not subsets of $E(0)$.

The effectivized isotropy representations on E_n and E_m induces a natural action on $E_n \otimes E_m$, either by G_a or one of its factors, i.e. the map

$$\begin{aligned} \psi\colon E_n \otimes E_m &\to V_{n,m} \\ X \otimes Y &\mapsto (\nabla_X A)_\xi Y. \end{aligned}$$

is equivariant. The same group acts effectively on $V_{n,m}$: let $g_*|_{E_n \otimes E_m} = \mathrm{id}|_{E_n \otimes E_m}$ then $(\nabla_X \alpha)(Y, Z) = (\nabla_X \alpha)(Y, g_* Z)$ for all $X \in E_n, Y \in E_m$ and $Z \in T_a M$ hence $g_*|_{V_{n,m}} = \mathrm{id}|_{V_{n,m}}$, if we assume, that M does not split.

The representations on $E_n \otimes E_m$ which are tensor products of standard representations are well known, and by Schur's Lemma ψ restricted to an irreducible module is a multiple of the identity. Hence, to determine the irreducible modules within $V_{n,m}$ we have to figure out which of the modules of $E_n \otimes E_m$ vanish under ψ and whether they are subsets of $E(0)$.

Our first observation shows the close relation between the spaces $V_{n,m}$ and the involutions associated with curvature spheres. Again we denote by φ_k the antipodal map of the curvature sphere S_k. Since restricted on an eigenspace this is an equivariant map and so is ψ, the following diagram is commutative.

$$\begin{array}{ccc} E_n(a) \otimes E_m(a) & \xrightarrow{(\varphi_k)_*} & E_{2k-n}(\varphi_k(a)) \otimes E_{2k-m}(\varphi_k(a)) \\ \downarrow \psi & & \downarrow \psi \\ V_{n,m}(a) & \xrightarrow{(\varphi_k)_*} & V_{2k-n,2k-m}(\varphi_k(a)) \end{array} \tag{2.1}$$

We will use this diagram to determine in which eigenspaces the $V_{n,m}$ are contained.

PROPOSITION 2.7. *Let $n \neq m$, then the associated module $V_{n,m}$ is contained in*

$$\begin{cases} E(0) & \text{if } n - m = 0 \mod 4 \\ E(0) \oplus E_{\frac{n+m}{2}} & \text{if } n - m = 2 \mod 4 \\ E(0) \oplus E_{2m-n} \oplus E_{2n-m} & \text{if } n - m = 1 \mod 2. \end{cases}$$

PROOF. Let E_{2m} and E_{2n} be two eigenspaces and as in the proof of Theorem 2.3 denote by $\varphi = \varphi_{m+n}$ the involution interchanging the two eigenspaces. The diagram

(2.1) in this case yields:

$$\begin{array}{ccc} E_{2n}(a)\otimes E_{2m}(a) & \xrightarrow{\varphi_*} & E_{2m}(\varphi(a))\otimes E_{2n}(\varphi(a)) \\ \psi\downarrow & & \downarrow\psi \\ V_{2n,2m}(a) & \xrightarrow{\varphi_*} & V_{2m,2n}(\varphi(a)) = V_{2n,2m}(\varphi(a)) \end{array}$$

By the explicit description of φ_* namely

$$\varphi_*|_{E_{2n}} = -\frac{d+2m}{d+2n}\cdot \mathrm{id}\,|_{E_{2n}} \quad \text{and} \quad \varphi_*|_{E_{2m}} = -\frac{d+2n}{d+2m}\cdot \mathrm{id}\,|_{E_{2m}}$$

we obtain $\varphi_*|_{E_{2n}\otimes E_{2m}} = \mathrm{id}\,|_{E_{2n}\otimes E_{2m}}$.

This proves

$$V_{2n,2m} \subset E(0) \oplus E_{m+n},$$

because these are the only invariant subspaces under φ_* for which φ_* restricted to is id or $-$ id. Moreover $V_{2n,2m}(a) = V_{2n,2m}(\varphi(a))$ as linear subspaces. Similarly $V_{2n+1,2m+1} \subset E(0) \oplus E_{m+n+1}$.

We denote the eigenvalues by λ_k. The fact that $\nabla_{E_n} E_m \subset V_{n,m} \oplus E_m$ yields

$$\begin{aligned} \langle (\nabla_{E_n} A)_\xi E_{n+4m}, E_{n+2m}\rangle &= -\langle E_{n+4m}, (\nabla_{E_n} A)_\xi E_{n+2m}\rangle = \\ &= -\langle E_{n+4m}, \lambda_{n+2m}(\nabla_{E_n} E_{n+2m}) + A_\xi(\nabla_{E_n} E_{n+2m}\rangle \\ &\subset \langle E_{n+4m}, E(0) \oplus E_{n+2m} \oplus E_{n+m}\rangle = 0. \end{aligned}$$

Hence $V_{(n,m)} \subset E(0)$ for $n - m = 0 \mod 4$. The same for $4m+2$ instead of $4m$ shows that

$$\langle V_{n,n+2m+1}, E_{n+4m+2}\rangle \neq 0 \text{ if and only if } \langle V_{n,n+4m+2}, E_{n+2m+1}\rangle \neq 0.$$

Since $V_{n,n+2m+1} = V_{n+2m+1,n}$,

$$\langle V_{n,n+2m+1}, E_{n-2m-1}\rangle \neq 0 \text{ if and only if } \langle V_{n+2m+1,n-2m-1}, E_n\rangle \neq 0.$$

These are the only eigenspaces which are not orthogonal to $V_{n,n+2m+1}$. □

2.4. Modules of the isotropy representation for irreducible eigenspaces E_n

Let us consider an isoparametric hypersurface with multiplicities m_1 and m_2 and let G_a act on each eigenspace as $\mathrm{SO}(m_i)$, i.e. the eigenspaces E_n are irreducible modules of dimension m_1 or m_2 of the isotropy representation. In this and the next chapter we will study hypersurfaces of this type, in Chapter 5 infinite dimensional isoparametric submanifolds of higher codimension with isotropy irreducible eigenspaces.

Remark. Throughout the chapter we identify for convenience reasons the isotropy group G_a with $\mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$ via a Lie homomorphism Φ, such that Φ_* is a Lie algebra isomorphism.

There are three types of associated modules:

$$\begin{aligned} \mathrm{SO}(m_1) &\text{ acting on } V_{2n,2m} \\ \mathrm{SO}(m_2) &\text{ acting on } V_{2n+1,2m+1} \\ \mathrm{SO}(m_1)\times\mathrm{SO}(m_2) &\text{ acting on } V_{2n+1,2m} \end{aligned}$$

If $m_1 = m_2$ and $G_a = \mathrm{SO}(m_1)$ only one of the module types exists so some of the following arguments are redundant.

PROPOSITION 2.8. *The associated modules $V_{2n,2m}$ and $V_{2n+1,2m+1}$ decompose into at most three submodules:* $\mathrm{tr}_{2n(+1),2m(+1)}$ *(which is one-dimensional) and the* $\frac{m_i(m_i-1)}{2}$ *dimensional module* $\Lambda_{2n(+1),2m(+1)}$ *where* $\mathrm{SO}(m_i)$ *acts as the adjoint representation and* $S^2{}_{2n(+1),2m(+1)}$*, where* $\mathrm{SO}(m_i)$ *acts as the s-representation of the symmetric space A I.*

All modules are contained in $E(0)$, if G_a consist of two factors or $m_1 = m_2 > 3$. We list the multiplicities and modules which may project non trivially to $E_{2m+2n+1}$:

m_1	m_2	*module*
3	*3*	$\Lambda_{4n+2,4m}$
2	*2*	$S^2{}_{4n+2,4m}$
m	*1*	$\mathrm{tr}_{4n+2,4m}$
2	*1*	$\mathrm{tr}_{4n+2,4m}$ *or* $\Lambda_{4n+2,4m}$

If $V_{2n(+1),2m(+1)} \subset E(0)$ then

$$\psi(X \otimes Y) = \lambda_{2m(+1)}(\nabla_X Y)_{E(0)}$$

for any $X \in E_{2n(+1)}, Y \in E_{2m(+1)}$, where $(\cdot)_{E(0)}$ denotes the projection to $E(0)$.

If $m_1 = m_2$ the statement is also valid for modules $V_{2n,2m+1}$.

PROOF. Let us study $V_{2n,2m}$ the other being treated similarly. The action of $\mathrm{SO}(m_1)$ on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_1}$ by conjugation decomposes into three modules: the antisymmetric, denoted by Λ (adjoint action), the symmetric traceless denoted by S^2 (s-representation AI) and the trace (trivial). We have seen in Proposition 2.7 any of these modules has to be contained in $E(0)$ if $m - n = 0 \mod 2$ and in $E(0) \oplus E_{m+n}$ otherwise.

Assume the image under ψ of one of the three modules projected to E_{m+n} is not zero, in particular this implies that $m+n$ is odd and that both dimension and representation coincide. Since on eigenspaces E_{2k+1} the effective group acting is $\mathrm{SO}(m_2)$ while on $V_{2n,2m}$ acts $\mathrm{SO}(m_1)$ this only can happen in two cases: If $m_1 = m_2$ and $G_a = \mathrm{SO}(m_1)$ or if $m_2 = 1$ for a one-dimensional module of $V_{n,m}$.

We start with the case $m_2 = 1$. The trace module $\mathrm{tr}_{n,m}$ is always one-dimensional and if $m_1 = 2$ the antisymmetric module as well and one of those could be not orthogonal to E_{m+n}, if $m_2 = 1$.

Let now $m_1 = m_2$ and $G_a = \mathrm{SO}(m_1)$. If $m_2 > 3$ none of the modules of $\mathrm{SO}(m_2)$ acting on $\mathbb{R}^{m_2} \otimes \mathbb{R}^{m_2}$ coincides with the standard representation of $\mathrm{SO}(m_2)$.

- If $m_1 = m_2 = 3$ the antisymmetric module is three dimensional. This is an equivalent representation to the standard representation, so $\langle \Lambda_{2n,2m}, E_{n+m}\rangle \neq 0$ is possible.
- If $m_1 = 2$ The symmetric traceless module is two dimensional, therefore $\langle S^2{}_{2n,2m}, E_{n+m}\rangle \neq 0$ is possible.
- If $m_1 = 1$ the only module is the trace and could be not orthogonal to E_{m+n}.

Let us consider the generic case, i.e. $V_{n,m} \subset E(0)$ and $X \in E_n$, $Y \in E_m$, since $\nabla_X Y \subset V_{n,m} \oplus E_m$ the following holds

$$\psi(X \otimes Y) = (\nabla_X A_a)Y = \nabla_X(A_a Y) - A_a(\nabla_X Y) = (\lambda_m \,\mathrm{id} - A_a)\nabla_X Y = \lambda_m(\nabla_X Y)_{E(0)}.$$

□

PROPOSITION 2.9. *Let $m_1 \neq m_2$. The associated module $V_{2n,2m+1}$ is irreducible and* $\dim V_{2n,2m+1} = m_1 m_2$ *or* 0. *If $V_{2n,2m+1} \subset E(0)$, then*

$$V_{2n,2m+1}(a) = V_{2(k-n),2(k-m)-1}(\varphi_k(a)).$$

If $m_1 > 1$ and $m_1 > 1$, then $V_{2n,2m+1} \subset E(0)$.

If $m_1 = 1$ and $m_1 > 3$, then either $V_{2n,2m+1} = E_{4n-2m-1}$ or $V_{2n,2m+1} \subset E(0)$.

If $m_1 = 1$ and $1 \leq m_1 \leq 3$, then $V_{2n,2m+1} \subset E_{4n-2m-1} \oplus E(0)$.

PROOF. Since the action of $\mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$ on $\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$ is irreducible, ψ is just a multiple of the identity and the first part of the proposition is proven. Considering the diagram (2.1) in this situation proves the second assertion, since $\varphi_{k*}|_{E(0)} = \mathrm{id}\,|_{E(0)}$.

If both multiplicities are different from 1, because of dimensional reasons $V_{2n,2m+1}$ can not be an eigenspace E_k, hence $V_{2n,m+1}$ is a subset of $E(0)$.

If $m_1 = 1$ then $\dim(V_{2n,2m+1}) = m_2$. If additionally $m_2 > 3$ then the $V_{2n+1,2m+1}$ does not contain an m_2-dimensional module, so either $V_{2n,2m+1}$ is contained completely in $E(0)$ or $E_{\mathbb{Z}}$. In the latter case, by Proposition 2.7, the candidates are $E_{4m+2-2n}$ or $E_{4n-2m-1}$, but the first is one-dimensional.

If $m_2 \in \{1,2,3\}$ there are m_2-dimensional modules in $V_{2n+1,2m+1}$, so $V_{2n,2m+1}$ is contained in $E(0)\oplus E_{4n-2m-1}$, but maybe diagonally. □

REMARK. We give examples among the known $P(G,H)$-actions where the exceptional cases of the last propositions arise.

(1) The action $G = \mathrm{SO}(7)$ and $H = \mathrm{G}_2 \times \mathrm{U}(3)$ has isotropy group $\mathrm{SO}(3)$ and $m_1 = m_2 = 3$. Here the antisymmetric modules $\Lambda_{4m+2,4n} = E_{2m+2n+1}$.
(2) Consider the σ-action of $\mathrm{SU}(3)$, that is $G = \mathrm{SU}(3)$ and $H = \{(g,\sigma(g)\}$, where σ is complex conjugation. Then ${S^2}_{4m+2,4n} = E_{2m+2n+1}$ and $V_{2n,2m+1} = E_{4m-2n+2}$.
(3) The action of type A I–III (i.e. $G = \mathrm{SU}(n)$, $H = \mathrm{SO}(n)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n-1))$ has multiplicities $n-2$ and 1. Here $\mathrm{tr}_{4m+2,4n} = E_{2m+2n+1}$ and $V_{2n,2m+1} = E_{4m-2n+2}$.
(4) The exceptional cases $m_1 = 1$ and $m_2 \leq 3$ do not occur, we will exclude them in Proposition 2.19 on page 28.

2.4.1. The singular isotropy representations for isotropy group $\mathrm{SO}(n)$. The simplest case among isoparametric submanifolds with irreducible eigenspaces, are the elementary submanifolds with diagram $\underset{n}{\circ}\overset{\infty}{-}\underset{n}{\circ}$ and principal isotropy group $\mathrm{SO}(n)$, which we will investigate in this subsection.

It will help to understand first the modules the singular isotropy representation to determine those of principal isotropy representations. As before let a be a regular point and denote by $c_k(a)$ the midpoint of the curvature sphere $S_k(a)$. First assume $G_a = \mathrm{SO}(n)$, that is any eigenspace is n-dimensional.

Let $G_{c_k} = \mathrm{SO}(n+1)$ and $G_a = \mathrm{SO}(n)\times\mathrm{SO}(1)$, let us assume embedded in G_{c_k} in the standard way, i.e. $(A,1) \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Modules of the singular isotropy consist of the span of some modules of the principal isotropy, which are trivial($\mathbb{R}$), standard ($\mathbb{R}^n$), adjoint (Λ^2) or A I (S^2), i.e. the representation of $\mathrm{SO}(n)$ on symmetric traceless $n\times n$-matrices. Hence we have to check which representations of $\mathrm{SO}(n+1)$ if restricted to $\mathrm{SO}(n)$ decomposes into those modules.

PROPOSITION 2.10. *Any isotropy group G_p acts on the tangent space T_pM as trivial, standard, Λ^2 or S^2-representations. Moreover the modules of the singular isotropy representation if restricted to a principal isotropy group decompose in the following way:*

$$\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$$
$$\Lambda^2(n+1) = \Lambda^2(n) \oplus \mathbb{R}^n$$
$$S^2(n+1) = S^2(n) \oplus \mathbb{R}^n \oplus \mathbb{R}$$

The only exception for $n = 3$ is

$$\Gamma_{(2,\pm 2)} = S^2(3)$$

PROOF. We use the classical Branching Theorem for the restriction of representations of SO$(n+1)$ to SO(n) (cf. [KNA01] p. 424 f). Let the root space be spanned by a orthonormal basis $e_1, \dots, e_k$, where $k = \lfloor \frac{n}{2} \rfloor$ is the rank of SO$(n+1)$. Then a representation Γ_λ is uniquely determined by the highest weight $\lambda = (a_1, \dots, a_k)$ (that is, $a_1 e_1 + \cdots + a_k e_k$),

$$\text{where} \begin{cases} a_1 \geq a_2 \geq \cdots \geq a_k \geq 0 & \text{if } n+1 = 2k+1 \\ a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq |a_k| & \text{if } n+1 = 2k. \end{cases}$$

When restricted to SO(n), the representation Γ_λ decomposes into $\oplus_{\bar\lambda} \Gamma_{\bar\lambda}$, where the sum is over all $\bar\lambda = (c_1 \dots, c_k)$ or $(c_1 \dots, c_{k-1})$resp. which fulfill the following condition:

$$\begin{cases} a_1 \geq c_1 \geq a_2 \geq c_2 \cdots \geq a_{k-1} \geq c_{k-1} \geq a_k \geq |c_k| & \text{if } n+1 = 2k+1 \\ a_1 \geq c_1 \geq a_2 \geq c_2 \cdots \geq a_{k-1} \geq c_{k-1} \geq |a_k| & \text{if } n+1 = 2k \end{cases} \tag{2.2}$$

That is: For a module Γ_λ of the singular isotropy representation each $\bar\lambda$ has to be either $(0, 0, \dots, 0)$, $(1, 0, \dots, 0)$, $(1, 1, \dots, 0)$ or $(2, 0, \dots, 0)$ (if $n > 4$).

Since $\bar\lambda = \lambda$ for $n = 2k+1$ and $\bar\lambda = (a_1, \dots a_{k-1})$ for $n = 2k$ is always possible, the only representations Γ_λ are the once mentioned in the statement, if $k > 2$.

We list the exceptions for low dimensions:

$n = 5$: For the representation $\lambda = (1, 1, 1)$ as well as for $(1, 1, -1)$ of SO(6) is $\bar\lambda = (1, 1)$ the only possibility. But these representations are excluded since they are not of real type (cf. [BTD95] p. 276).

$n = 4$: Here the adjoint action of SO(4) decomposes into highest roots $(1, 1)$ and $(1, -1)$, but if one is a valid $\bar\lambda$ so is the other. The situation stays the same as in the general case.

$n = 3$: For SO(3) the standard and the adjoint representation are equivalent with highest root (1), the A I-representation has highest root (2). Possible representations of SO(4) are therefore (a_1, a_2) with $a_1 \leq 2$. Among these only $(2, \pm 2)$ is real (cf. [FH91] p.26)

$n = 1, 2$: The branching rule (2.2) applies without problems. Note that for $n = 2$ it is not clear what the weight of the representation in this case is, but we will prove later (cf. Proposition 2.15 on page 24 an preceding remark), that in fact only the representation $\Gamma_{(c_1)}$ for $c_1 = 0, 1, 2$ occur.

□

REMARK. For the rest of this and the following subsection we will exclude the case of one-dimensional eigenspaces, they will be treated in Subsection 2.4.3 on page 28.

We denote by $G_k \subset G_{c_k}$ the set mapping a to its antipodal point $\varphi_k(a)$ on $S_k(a)$. If $G_{c_k} = \mathrm{SO}(n+1)$ and $G_a = \mathrm{SO}(n) \times \mathrm{SO}(1)$, then $G_k = \left(\begin{smallmatrix} A & 0 \\ 0 & -1 \end{smallmatrix}\right)$, where $A \in \mathrm{O}^-(n)$, i.e. $A \in \mathrm{O}(n)$ and $\det A = -1$.

In Table 2.2 on the next page we collect the behavior under G_k for the modules of the principal isotropy representation V in dependence of the extension to a module $\tilde V$ of the singular isotropy representation. That means e.g. : Let V be a standard module of G_a contained in a module $\tilde V$ of G_{c_k}, x a vector in V and $g = \left(\begin{smallmatrix} A & 0 \\ 0 & -1 \end{smallmatrix}\right) \in G_k$. Then, if $\tilde V$ is a standard module $g_* x = Ax$ while if $\tilde V$ is a Λ^2- or S^2-module $g_* x = -Ax$.

By these properties we will be able to study the behavior of the modules of the isotropy representations closer. First of all we need to know which modules V of the

$x \in$	$V = \mathrm{tr}$	$V = \mathbb{R}^n$	$V = \Lambda^2(n)$	$V = S^2(n)$
$\tilde{V} = \mathrm{tr}$				
$\tilde{V} = \mathbb{R}^{n+1}$	$-x$	Ax		
$\tilde{V} = \Lambda^2(n+1)$		$-Ax$	AxA^T	
$\tilde{V} = S^2(n+1)$	x	$-Ax$		AxA^T

TABLE 2.2. Extension of modules from $\mathrm{SO}(n)$ to $\mathrm{SO}(n+1)$

principal isotropy representation admit an extension to a module $\tilde{V}$ of the singular isotropy representation. We call such a V an *extendable module* and a necessary condition for extendability is invariance under G_k.

Let $g \in G_k$, then $g_*(E_m(a)) = E_m(\varphi_k(a)) = E_{2k-m}(a)$. Hence the module of the singular isotropy representation, which contains $E_m(a)$ has to contain $E_{2k-m}(a)$ as well and is therefore not irreducible. This means eigenspaces are not extendable, but the $2n$-dimensional space $E_m(a) \oplus E_{2k-m}(a)$ has to contain two n-dimensional extendable modules.

To describe these, we have to choose first an appropriate basis for $E_m(a) \oplus E_{2k-m}(a)$. Let Φ be the Lie homomorphisms from G_a to $\mathrm{SO}(m)$ and choose f_m such that the following diagram is commutative.

$$\begin{array}{ccc} G_a & \curvearrowright & E_m(a) \\ \Phi \downarrow & & \downarrow f_m \\ \mathrm{SO}(n) & \curvearrowright & \mathbb{R}^n \end{array} \tag{2.3}$$

By choosing a fixed basis $e_1, \dots, e_n$ of $\mathbb{R}^m$ this gives a bases $X_i^m = f_m^{-1}(e_i)$ of $E_m(a)$, we will call such a basis *natural*. Note that f_m is only determined uniquely up to sign. Let g be an element of $\mathrm{SO}(1) \times \mathrm{O}^-(n-1) \times \mathrm{O}^-(1)$, then $X_1^{2k-m} = g_* X_1^m$, the rest of the basis of $E_{2k-m}(a)$ is defined likewise. These bases are equivariant, that is the linear map $X_i^m \mapsto X_i^{2k-m}$ is equivariant.

PROPOSITION 2.11. *Let $\tilde{V}$ be a module of the singular isotropy representation in the point $c_k(a)$ and $V \subset \tilde{V}$ a standard module of the principal isotropy representation at the point a. Denote by $\{X_i^m \mid i = 1, \dots, n\}$ and $\{X_i^m \mid i = 1, \dots, n\}$ equivariant bases of the eigenspaces $E_m(a)$. Then*

$$V = \mathrm{span}\left\{X_i^m + X_i^{2k-m} \mid i = 1, \dots, n\right\} = \mathrm{diag}^+(E_m, E_{2k-m}) \quad \textit{or}$$
$$V = \mathrm{span}\left\{X_i^m - X_i^{2k-m} \mid i = 1, \dots, n\right\} = \mathrm{diag}^-(E_m, E_{2k-m})$$

and $\tilde{V}$ is a standard module in the first case and a Λ^2- or S^2-module in the second case.

PROOF. Note that the spaces $\mathrm{diag}^+(E_m, E_{2k-m})$ and $\mathrm{diag}^-(E_m, E_{2k-m})$ are both invariant under G_a and G_k and they are the only n-dimensional subspaces of T_aM with that property, hence the only candidates for V in $E_m \oplus E_{2k-m}$.

Let $V = \mathrm{diag}^+(E_m, E_{2k-m})$ and $v_i = X_i^m + X_i^{2k-m} \in V$ and choose $g \in G_k$ such that $g_* X_i^m = \varepsilon_i X_i^{2k-m}$ with $\varepsilon_i = \pm 1$ for all i, i.e. a diagonal matrix within G_k. Since we have chosen equivariant bases $g_* X_i^{2k-m} = \varepsilon_i X_i^m$ and therefore $g_* v_i = \varepsilon_i v_i$, that means $\tilde{V}$ is a standard module (cf. Table 2.2). On the other hand if $w_i = X_i^m - X_i^{2k-m} \in V = \mathrm{diag}^-(E_m, E_{2k-m})$ then $g_* w_i = -w_i$ and $\tilde{V}$ is a Λ^2- or S^2-module. □

Next we study the extendability of modules in $V_{k,m} \oplus V_{k,2k-m}$, where we choose natural bases for E_m and E_{2k-m} as above. Let $X_1^k, \dots, X_n^k$ be a natural basis of E_k, where the choice of sign does not matter. Then

$$\begin{aligned}
\mathrm{tr}_{k,m} &= \mathrm{span} \sum_{i=1}^{n} \psi(X_i^k \otimes X_i^m) \\
\Lambda_{k,m} &= \mathrm{span} \left\{ \psi(X_i^k \otimes X_j^m - X_j^k \otimes X_i^m) \mid 1 \le i < j \le n \right\} \\
{S^2}_{k,m} &= \mathrm{span} \left\{ \psi(X_i^k \otimes X_j^m + X_j^k \otimes X_i^m) \mid 1 \le i < j \le n \right\} \\
&\quad \oplus \mathrm{span} \left\{ \psi(X_i^k \otimes X_i^m - X_j^k \otimes X_j^m) \mid 1 \le i < j \le n \right\}
\end{aligned}$$

and $\mathrm{diag}^{\pm}$ is used as in the last proposition. The calculations are similar to those in the case of eigenspaces, using additionally $g_* X_i^k = -\varepsilon_i X_i^k$. This is because E_k is contained in a standard module and $X_i^k(a) = -X_i^k(\varphi_k(a)) = (\varphi_k)_* X_i^k(a)$. Comparing with Table 2.2 on the page before yields the following table:

Module V	possible extension $\tilde{V}$
$\mathrm{diag}^+(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$	tr or $\mathbb{R}(n+1)$
$\mathrm{diag}^-(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$	tr or $S^2(n+1)$
$\mathrm{diag}^+(\Lambda_{k,m}, \Lambda_{k,2k-m})$	none
$\mathrm{diag}^-(\Lambda_{k,m}, \Lambda_{k,2k-m})$	$\Lambda(n+1)$
$\mathrm{diag}^+({S^2}_{k,m}, {S^2}_{k,2k-m})$	none
$\mathrm{diag}^-({S^2}_{k,m}, {S^2}_{k,2k-m})$	$S^2(n+1)$

We treat the generic case, i.e. associated modules are contained in $E(0)$ first.

PROPOSITION 2.12. *Assume that $V_{n,m}$ is a subset of $E(0)$. Any isotropy group G_p acts on the tangent space T_pM as trivial, standard or Λ^2-representation for both singular and regular points p. Choose natural bases for $E_m(a)$, $E_{2k-m}(a)$ and $E_k(a)$ as above, then the irreducible modules of the singular isotropy representation in $c_k(a)$ are*

$$\begin{aligned}
V_+ &:= \mathrm{diag}^+(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m}) \oplus \mathrm{diag}^+(E_m, E_{2k-m}) \qquad \textit{and} \\
V_- &:= \mathrm{diag}^-(\Lambda_{k,m}, \Lambda_{k,2k-m}) \oplus \mathrm{diag}^-(E_m, E_{2k-m}).
\end{aligned}$$

PROOF. Any module of the principal isotropy representation, whose possible extension is of unique type is extendable or vanishes, that holds for the $\mathrm{diag}^+(E_m, E_{2k-m})$, $\mathrm{diag}^-(\Lambda_{k,m}, \Lambda_{k,2k-m})$ and $\mathrm{diag}^-({S^2}_{k,m}, {S^2}_{k,2k-m})$.

Since $\nabla_{E_k} E_m \subset V_{k,m} \oplus E_m$ and $\nabla_{E_k} E_{2k-m} \subset V_{k,2k-m} \oplus E_{2k-m}$ it is clear that a module containing $\mathrm{diag}^+(E_m, E_{2k-m})$ is a subset of $E_m \oplus E_{2k-m} \oplus (V_{k,m} + V_{k,2k-m})$. Since its extension is $(n+1)$-dimensional and $\mathrm{diag}^+(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$ is the only one-dimensional modules admitting such an extension, the first part of the statement is proven, when $n \ge 3$.

As we have seen in the last table $\mathrm{diag}^+(\Lambda_{k,m}, \Lambda_{k,2k-m})$ and $\mathrm{diag}^+({S^2}_{k,m}, {S^2}_{k,2k-m})$ are not extendable and therefore have to vanish. Using the basis vectors from above we deduce for any $i \neq j$

$$\begin{aligned}
\psi(X_i^k \otimes X_j^m) - \psi(X_j^k \otimes X_i^m) &= -\psi(X_i^k \otimes X_j^{2k-m}) + \psi(X_j^k \otimes X_i^{2k-m}), \\
\psi(X_i^k \otimes X_j^m) + \psi(X_j^k \otimes X_i^m) &= -\psi(X_i^k \otimes X_j^{2k-m}) - \psi(X_j^k \otimes X_i^{2k-m}), \\
\psi(X_i^k \otimes X_i^m) - \psi(X_j^k \otimes X_j^m) &= -\psi(X_i^k \otimes X_i^{2k-m}) + \psi(X_j^k \otimes X_j^{2k-m})
\end{aligned}$$

and therefore

$$\psi(X_i^k \otimes X_j^m) = -\psi(X_i^k \otimes X_j^{2k-m}) \tag{2.4}$$

$$\psi(X_i^k \otimes X_i^m) + \psi(X_i^k \otimes X_i^{2k-m}) = \psi(X_j^k \otimes X_j^m) + \psi(X_j^k \otimes X_j^{2k-m}) \tag{2.5}$$

for any $i \neq j$.

Both the possible extension of $\mathrm{diag}^-(\Lambda_{k,m}, \Lambda_{k,2k-m})$ and $\mathrm{diag}^-(S^2_{k,m}, S^2_{k,2k-m})$ (for $n \neq 3$, see below for $n = 3$) have to contain $\mathrm{diag}^-(E_m, E_{2k-m})$, therefore one of them has to vanish. This yields by the equation (2.4):

$$\psi(X_i^k \otimes X_j^m) = \varepsilon \psi(X_j^k \otimes X_i^m)$$

for any $i \neq j$ and $\varepsilon \in \{1, -1\}$.

Using Proposition 2.8 on page 18 and the Gauß equation, shows:

$$\begin{aligned}\left\langle \psi(X_i^m \otimes X_j^k), \psi(X_j^m \otimes X_i^k) \right\rangle &= \lambda_k \lambda_m \left\langle \nabla_{X_i^m} X_j^k, \nabla_{X_i^k} X_j^m \right\rangle = \\ = -\lambda_k \lambda_m \left\langle X_j^k, \nabla_{X_i^m} \nabla_{X_i^k} X_j^m \right\rangle &= -\lambda_k \lambda_m \left\langle X_j^k, \nabla_{X_i^k} \nabla_{X_i^m} X_j^m + \nabla_{[X_i^m, X_i^k]} X_j^m \right\rangle = \\ &= \lambda_k \lambda_m \left\langle \nabla_{X_i^k} X_j^k, \nabla_{X_i^m} X_j^m \right\rangle - \lambda_k \lambda_m \left\langle X_j^k, \nabla_{[X_i^m, X_i^k]} X_j^m \right\rangle\end{aligned}$$

The first summand vanishes since the eigenspaces are autoparallel. For the second summand only the projection onto $E(0)$ of $[X_j^m, X_i^m]$ does matter and an easy calculation using Lemma 5.2. from [HL99] proves that $\nabla_{[X_i^m, X_i^k]} X_j^m = -\nabla_{X_j^m} \nabla_{X_i^k} X_i^m$ (for the projection), which yields

$$\left\langle \psi(X_i^m \otimes X_j^k), \psi(X_j^m \otimes X_i^k) \right\rangle = -\left\langle \psi(X_i^k \otimes X_i^m), \psi(X_j^m \otimes X_j^k) \right\rangle .$$

Both sides are always positive or always negative independent of $i \neq j$.

Since all $\psi(X_i^m \otimes X_j^k)$ have the same length by the equivariance of ψ, this proves $\psi(X_i^k \otimes X_i^m) = -\varepsilon \psi(X_j^m \otimes X_j^k)$. This yields $\varepsilon = -1$ if $n \geq 3$, and therefore the module $\mathrm{diag}^-(S^2_{k,m}, S^2_{k,2k-m})$ as well as $\mathrm{diag}^-(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$ vanishes.

In the second part of the proof, we treat the exceptions in low dimensions. Assume first $n = 3$, the representations $\Gamma_{(2,\pm 2)}$ were not excluded by the branching rule in Proposition 2.10 on page 19. If it occurs as a singular isotropy representation module $\tilde{V}$, it is contained completely in $E(0)$, since $V = \tilde{V} = S^2(3)$ in that case. Generally modules $\tilde{V}$ are invariant under ∇_{E_k} and for $\tilde{V} \subset E(0)$ this yields $\psi(E_k \otimes \tilde{V}) = 0$. But then $\psi(E_k \otimes T_a M) \perp \tilde{V}$, in particular $\tilde{V} \perp V_{k,m}$ for any m. Therefore $\mathrm{diag}^{\pm}(S^2_{k,m}, S^2_{k,2k-m})$ are not $\Gamma_{(2,\pm 2)}$-modules of the singular isotropy representation in the case $n = 3$ and have to vanish by the arguments on general n above.

Now we study the case $n = 2$. Since G_k acts on the one-dimensional module $\mathrm{diag}^+(\Lambda_{k,m}, \Lambda_{k,2k-m})$ as id, not as $-\mathrm{id}$, the module $\mathrm{diag}^+(E_m, E_{2k-m})$ has to extend to V_+ as in the general case. Moreover $\psi(X_i^k \otimes X_i^m) = \psi(X_j^m \otimes X_j^k)$ holds, since otherwise $\mathrm{diag}^+(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$ vanishes. Again this excludes any possibility except V_- as an extension of the module $\mathrm{diag}^-(E_m, E_{2k-m})$. □

Eventually we collect the results on $V_{n,m}$.

COROLLARY 2.13. *For any $k, m \in \mathbb{Z}$ the modules $V_{k,m} = V_{k,2k-m}$, while $V_{k,m}$ is orthogonal $V_{k,\tilde{m}}$ for any other $\tilde{m} \in \mathbb{Z}$. Moreover $V_{k,m} = \mathrm{tr}_{k,m} \oplus \Lambda^2_{k,m}$ is of dimension $1 + \frac{n(n-1)}{2}$.*

In particular the space $E(0)$ is infinite dimensional.

PROOF. The proof of the last proposition shows that $S^2_{k,m}$ vanishes, as well as $\mathrm{diag}^-(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$ and $\mathrm{diag}^+(\Lambda_{k,m}, \Lambda_{k,2k-m})$. Therefore $V_{k,m} = V_{k,2k-m}$. Any other space $V_{k,\tilde{m}}$ is orthogonal, since it is contained in a different modules of the singular isotropy representation in $c_k(a)$. □

Observe that the identity $V_{k,m} = V_{k,2k-m}$ may be iterated, yielding for example $V_{0,1} = V_{1,2} = V_{2,3} = \cdots$. More general the following theorem holds

THEOREM 2.14. *Under the condition of proposition 2.12 the spaces $V_{n,m}$ depend only on $|n-m|$, i.e. for any $k, n, m \in \mathbb{Z}$*

$$V_{n,m} = V_{k-n,k-m}$$

and

$$E(0) = \sum_{n,m\in\mathbb{Z}} V_{n,m} = \sum_{n\in\mathbb{Z}} V_{k,n} = \bigoplus_{n>k} V_{k,n}$$

for any fixed $k \in \mathbb{Z}$.

PROOF. We proof the statement by induction over $l = |n-m|$. The case $l = 1$ is already proven, so we assume the statement to be true for any $l \le l_0$ for a fixed $l_0 \in \mathbb{N}$.

Assume V_{0,l_0+1} is perpendicular to V_{1,l_0+2}. Using the Gauss equation and Proposition 2.8 this yields:

$$\begin{aligned} 0 &= \langle \nabla_{E_0} E_{l_0+1}, \nabla_{E_1} E_{l_0+2} \rangle = -\langle E_{l_0+1}, \nabla_{E_0}\nabla_{E_1} E_{l_0+2} \rangle = \\ &= \langle \nabla_{E_1} E_{l_0+1}, \nabla_{E_0} E_{l_0+2} \rangle + \left\langle E_{l_0+1}, \nabla_{[E_0,E_1]} E_{l_0+2} \right\rangle \end{aligned}$$

Since by the induction hypothesis $V_{1,l_0+1} = V_{0,l_0} \perp V_{0,l_0+2}$ the first summand has to vanish and

$$\left\langle E_{l_0+1}, \nabla_{[E_0,E_1]} E_{l_0+2} \right\rangle = \left\langle E_{l_0+1}, \nabla_{V_{0,1}} E_{l_0+2} \right\rangle = 0,$$

which means $\psi(V_{0,1} \otimes E_{l_0+2}) \perp E_{l_0+1}$. This is a contradiction, since $V_{0,1} = V_{l_0+1,l_0+2}$.

The rest of the statement follows by the last corollary. □

To finish this section we study the case when $V_{n,m}$ is not a subset of $E(0)$.

PROPOSITION 2.15. *Let $G_a = \mathrm{SO}(2)$ and let $E_{\frac{k_0+m_0}{2}} \subset \psi(E_{k_0}, E_{m_0})$ for at least one pair (k_0, m_0) with $k_0 - m_0 = 2 \mod 4$. With out loss of generality let k_0 be even.*

Then $E_{\frac{k+m}{2}} \subset \psi(E_k, E_m)$ holds precisely for any pair (k, m) of even numbers with $k - m = 2 \mod 4$. For k even and m odd $\psi(E_k, E_m) = E_{2m-k}$.

Choose natural bases for $E_m(a)$, $E_{2k-m}(a)$ and $E_k(a)$ as above, then modules of the singular isotropy representation are the same as in Proposition 2.12, if k is odd or $k - m = 0 \mod 4$ and otherwise

$$V_+ := \mathrm{diag}^+(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m}) \oplus \mathrm{diag}^+(E_m, E_{2k-m}) \oplus \mathrm{diag}^+(E_{\frac{k+m}{2}}, E_{\frac{3k-m}{2}}) \quad \text{and}$$

$$V_- := \mathrm{diag}^-(\Lambda_{k,m}, \Lambda_{k,2k-m}) \oplus \mathrm{diag}^-(E_m, E_{2k-m}) \oplus \mathrm{diag}^-(E_{\frac{k+m}{2}}, E_{\frac{3k-m}{2}}).$$

Denote by $V^0_{n,m}$ the projection onto $E(0)$ of $V_{n,m}$, then:

$$\begin{aligned} V^0_{n,m} &= 0 && \text{for } n-m = 1 \mod 2 \\ V^0_{n,m} &= V^0_{2k-n,2k-m} && \text{for } n-m = 0 \mod 2 \\ V^0_{n,m} &\perp V^0_{n+1,m+1} && \text{and} \\ \mathrm{tr}_{4n+2,4m+2} &= \mathrm{tr}_{2n+1,2m+1}\,. \end{aligned}$$

PROOF. In this case it is a priori not clear with which weight SO(2) acts on the eigenspaces (see remark below for the case when associated modules are subsets of $E(0)$), but it acts with same weight on eigenspaces E_{2n} and E_{2n+1} respectively.

If $k - m = 0 \mod 4$, by Proposition 2.7 on page 16 the associated module is contained in $E(0)$ and the situation stays the same as in the statement of Proposition 2.12.

Since $E_{\frac{k_0+m_0}{2}} = {S^2}_{k_0,m_0}$ the isotropy group SO(2) acts on E_{2m+1} as $\Gamma_{(2c)}$ while on E_{2m} as $\Gamma_{(c)}$, that is with double rate on eigenspaces of odd index. But $c = 1$ for otherwise G_a does not act effectively. Therefore $\psi(E_{2n+1}, E_{2m+1}) \subset E(0)$, since $\Gamma_{(2)} \otimes \Gamma_{(2)} = \Gamma_{(0)} \oplus \Gamma_{(0)} \oplus \Gamma_{(4)}$ contains no module of type $\Gamma_{(1)}$.

Let $E_{\frac{k+m}{2}} \subset \psi(E_k, E_m)$, then $\mathrm{diag}^{\pm}({S^2}_{k,m}, {S^2}_{k,2k-m}) \perp E(0)$ and the equations (2.4), (2.5) and their analogues for diag^- prove that

$$\psi(X_i^k \otimes X_i^m)_{E(0)} = \psi(X_j^k \otimes X_j^m)_{E(0)} = \psi(X_i^k \otimes X_i^{2k-m})_{E(0)}$$
$$\psi(X_i^k \otimes X_j^m)_{E(0)} = -\psi(X_j^k \otimes X_i^m)_{E(0)}$$

Since $\mathrm{diag}^+(\Lambda_{k,m}, \Lambda_{k,2k-m})$ as well as $\mathrm{diag}^-(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m})$ are subset of $E(0)$, this proves they both have to vanish.

Since $\Gamma_{(1)} \otimes \Gamma_{(2)} = \Gamma_{(1)} \oplus \Gamma_{(3)}$ the module $\psi(E_k \otimes E_{\frac{k+m}{2}})$ contains at most one 2-dimensional module in $E(0)$besides E_m (cf. proposition 2.7). We observe that any module of the singular isotropy representations contains precisely one 1-dimensional module and up to two 2-dimensional modules. The module in the statement provides the only possibility for $\mathrm{diag}^+(E_m, E_{2k-m}) \oplus \mathrm{diag}^+(E_{\frac{k+m}{2}}, E_{\frac{3k-m}{2}})$ and $\psi(E_k, \cdot)$ of this space only contains one 1-dimensional module. The same holds for diag^-. Moreover $\psi(E_k, E_{\frac{k+m}{2}}) = E_m$, since there are no one-dimensional modules left.

Next we prove that the modules behave that way for any pair of even number (k, m) with $k - m = 2 \mod 4$. Observe first that it this is true for any pair with $|k - m| = |k_0 - m_0|$ by using antipodal maps φ_{l*}. Therefore for any odd number l there is a pair (k_l, m_l) such that $E_l \subset \psi(E_{k_l}, E_{m_l})$. Hence for any even k the space $E_l \oplus E_{2k-l} \oplus V_{k,l} \oplus V_{k,2k-l}$ consists of two-dimensional modules and is therefore neither invariant under $\mathrm{SO}(3) = G_{c_k}$ nor under $\psi(E_k \otimes \cdot)$. Since $\psi(E_k \otimes E(0)) \perp E(0)$ this means that $V_{k,l}$ can not be contained entirely in $E(0)$, so $E_{2l-k} \subset V_{k,l}$ (the eigenspace E_{2k-l} is not possible, for $2k-l$ is odd). This is equivalent to $E_l \subset V_{2l-k,k}$ by Proposition 2.7.

Finally we investigate which spaces $V_{n,m}^0$ coincide. For $n - m = 0 \mod 4$ the proof of Theorem 2.14 holds (induction step $l \to l + 4$), proving $V_{n,m} = V_{4k-n,4k-m}$. The same holds for odd n, m with $n - m = 2 \mod 4$ and induction step $l \to l + 2$, i.e. $V_{n,m} = V_{2k-n,2k-m}$ in that case.

We use the same calculation as in Theorem 2.14, that is

$$\langle V_{n_1,m_1}, V_{n_2,m_2} \rangle = \langle V_{n_1,m_2}, V_{m_1,n_2} \rangle + \langle V_{n_1,m_2}, V_{m_1,n_2} \rangle . \tag{2.6}$$

We have omitted factors in this equation, but as long as one of the summands on the right hand sight vanishes, this is sufficient for our arguments.

Setting $n_2 = n_1 + 1$ and $m_2 = m_1 + 1$ in equation (2.6) proves $V_{n,m}^0 \perp V_{n+1,m+1}^0$ for n and m of the same parity:

$$\langle V_{2n,2m}, V_{2n+1,2m+1} \rangle = \langle E_{4m-2n+2}, E_{4n-2m+2} \rangle + \langle E_{2n+2}, E_{2m+2} \rangle = 0.$$

The last statement of the theorem is proven by

$$\langle V_{4n+2,4m+2}, V_{2n+1,2m+1} \rangle = \langle E_{4m-4n}, E_{4n-4m} \rangle + \langle E_0, E_0 \rangle \neq 0$$

and finally

$$\langle V_{4n,4m}, V_{4n+2,4m+2} \rangle = \langle V_{4n,4m}, V_{2n+1,2m+1} \rangle = \langle E_{4m-4n+2}, E_{4n-4m+2} \rangle + \langle E_2, E_2 \rangle \neq 0$$

□

REMARK. In Propositions 2.10 and 2.12 we have assumed in the case $n = 2$, that the eigenspaces are $\Gamma_{(1)}$-modules. This is justified by the same argument as we used in the last proof: since any modules of the singular isotropy representation has to contain one-dimensional modules, therefore the representation on each eigenspace has the same weight. If the weight is not 1, the principal isotropy group has a non trivial effectivity kernel, but we have assumed, that it acts effectively.

REMARK. We will not treat the exceptional case $G_a = \mathrm{SO}(3)$ with associated modules not contained in $E(0)$ since it is of no relevance for rigidity results of higher codimension.

Now we are prepared for the more general case of non simple isotropy groups.

2.4.2. The singular isotropy representations for isotropy group $\mathbf{SO}(m_1)\times$ $\mathbf{SO}(m_2)$. Let k be even and $G_{c_k} = \mathrm{SO}(m_1+1)\times\mathrm{SO}(m_2)$, $G_a = \mathrm{SO}(m_1)\times\mathrm{SO}(1)\times\mathrm{SO}(m_2)$ and $G_k = \mathrm{O}^-(m_1)\times\mathrm{O}^-(1)\times\mathrm{O}^-(m_2)$, let us assume both groups embedded in G_{c_k} in the standard way. Furthermore let $\dim E_{2n} = m_1$ and $\dim E_{2n+1} = m_2$. We will see that for the modules E_{2n+1} and $V_{2n+1,2m+1}$ the situation is the same as in the case of simple isotropy group, for $\mathrm{SO}(m_1)$ acts trivially on these spaces. So our main focus in this paragraph lies on the eigenspaces E_{2m} and on $V_{2n,2m}$ and $V_{2n,2m+1}$. We remark that Corollary 2.13 on page 23 is also valid for $V_{2n,2m}$.

Let $\rho : G_{c_k}\times\tilde V\to\tilde V$ be an irreducible representation, hence $\rho = \rho_1\otimes\rho_2$, where $\rho_1 : \mathrm{SO}(m_1+1)\times\tilde V_1\to\tilde V_1$ and $\rho_2 : \mathrm{SO}(m_2)\times\tilde V_2\to\tilde V_2$ are irreducible representations with $\tilde V = \tilde V_1\otimes\tilde V_2$. Moreover $\tilde V$ is the span of irreducible modules of G_a, denoted by W_i.

$$\tilde V = \tilde V_1\otimes\tilde V_2 =$$
$$W_1\oplus W_2\oplus\cdots\oplus W_k = (W_1^1\otimes W_1^2)\oplus\cdots\oplus(W_k^1\otimes W_k^2)$$

The spaces W_i^j are irreducible modules of $\mathrm{SO}(m_i)$ and $\tilde V_i = \sum_{j=1}^k W_j^i$. Since $\tilde V_2$ is an irreducible modules of $\mathrm{SO}(m_2)$, it consist only of one summand, while $\tilde V_1$ may consist of at most two summands by the previous discussion in Proposition 2.12 on page 22, which generalizes in the following way.

COROLLARY 2.16. *Any irreducible module of G_a, on which $\mathrm{SO}(m_1)$ acts trivially and $\mathrm{SO}(m_2)$ does not, is also an irreducible module of the singular isotropy representation or its extension contains a subspace of $\sum_{i,j\in\mathbb{Z}} V_{2i,2j+1}$.*

Any irreducible module of G_a, on which $\mathrm{SO}(m_2)$ acts trivially and $\mathrm{SO}(m_1)$ does not, extends to a module of the singular isotropy representation as is described in Proposition 2.12.

PROOF. Without loss of generality let $\tilde V_2 = W_1^2$. The space $W_1 = W_1^1\oplus W_1^2$ is by definition an irreducible G_a-module, and either one of the $\mathrm{SO}(m_i)$-factors acts trivially or $W_1 = V_{2n,2m+1}$ for some $n, m\in\mathbb{Z}$. □

Hence we mainly need to determine the extension of modules in $\sum_{i,j\in\mathbb{Z}} V_{2i,2j+1}$. We collect the knowledge on the modules of the singular isotropy representations in the following theorem.

THEOREM 2.17. *Let M be an homogeneous isoparametric hypersurface with principal isotropy group $G_a = \mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$ and $\dim E_{2n} = m_1$ and $\dim E_{2n+1} = m_2$. Let k be even and $c_k(a)$ the mid point of the curvature sphere $S_k(a)$ with $G_{c_k} =$*

$SO(m_1+1)\times SO(m_2)$. *Then the irreducible modules of the singular isotropy representation are*

$$\mathrm{diag}^+(\mathrm{tr}_{k,2m},\mathrm{tr}_{k,2k-2m})\oplus\mathrm{diag}^+(E_{2m},E_{2k-2m})$$
$$\mathrm{diag}^-(\Lambda_{k,2m},\Lambda_{k,2k-2m})\oplus\mathrm{diag}^-(E_{2m},E_{2k-2m})$$
$$\mathrm{diag}^-(V_{2m+1,k},V_{2k-2m-1,k})\oplus\mathrm{diag}^-(E_{2m+1},E_{2k-2m-1})$$
$$\mathrm{diag}^+(E_{2m+1},E_{2k-2m-1})$$
$$\mathrm{diag}^-(\Lambda_{k+1,2m+1},\Lambda_{k+1,2k-2m+1})$$

Moreover $V_{n,m}=V_{2k-n,2k-m}$ *and* $\mathrm{tr}_{n,m}=\mathrm{tr}_{n+1,m+1}$.

REMARK. An isoparametric submanifold with isotropy group $SO(m)$ and Dynkin diagram $\underset{m}{\circ}\overset{\infty}{-}\underset{m}{\circ}$ is a special case of the theorem if one allows the multiplicities to be 0, that is $m_1=m$ and $m_2=0$, compare to proposition 2.12 on page 22.

PROOF. Invariant under G_k are the spaces

$$\mathrm{diag}^+(V_{2m+1,k},V_{2k-2m-1,k})=\mathrm{span}\left\{\psi(X_i^{2m+1}\otimes Y_j+X_i^{2k-2m-1}\otimes Y_j)\mid i,j=1,\dots,n\right\},$$
$$\mathrm{diag}^-(V_{2m+1,k},V_{2k-2m-1,k})=\mathrm{span}\left\{\psi(X_i^{2m+1}\otimes Y_j-X_i^{2k-2m-1}\otimes Y_j)\mid i,j=1,\dots,n\right\}.$$

We remark that we choose the basis of $E_{2k-2m-1}$ with respect to that of E_{2m+1} by requiring that any element g of $G_k=O^-(m_1)\times O^-(1)\times SO(m_2)$ of type $A\times(-1)\times E$ fulfills $g_*X_i^{2m+1}=X_i^{2k-1m-1}$.

Let $\tilde V$ be an extension of one of those spaces. By

$$\mathrm{diag}^\pm(V_{2m+1,k},V_{2k-2m-1,k})=\psi(E_k\otimes\mathrm{diag}^\pm(E_{2m+1},E_{2k-2m-1}))$$

follows that $W_1^1=E_k$ and $W_1^2=\mathrm{diag}^\pm(E_{2m+1},E_{2k-2m-1})$. Therefore $\tilde V_1=\nu_aM\oplus E_k$ for this is the module of $SO(m_1+1)$ containing E_k, and

$$\tilde V_2=\psi(\mathrm{diag}^\pm(E_{2m+1},E_{2k-2m-1})\otimes\nu_aM)=\mathrm{diag}^\pm(E_{2m+1},E_{2k-2m-1}).$$

Consider an element g as described above, then

$$g_*(X_i^{2m+1}\pm X_i^{2k-2m-1})=\pm(X_i^{2m+1}\pm X_i^{2k-2m-1}).$$

Comparing this with the standard representation of $SO(m_1+1)\times SO(m_2)$ yields that only $\mathrm{diag}^-(E_{2m+1},E_{2k-2m-1})$ is extendable and $\mathrm{diag}^+(E_{2m+1},E_{2k-2m-1})$ is an irreducible module of the singular isotropy representation. Moreover the modules of type $\mathrm{diag}^+(V_{2m+1,k},V_{2k-2m-1,k})$ vanishes for there is no $m_1\cdot m_2$ dimensional module of $SO(m_1)\times SO(m_2+1)$.

Eventually we have to discuss the spaces $\mathrm{diag}^+(\mathrm{tr}_{2n+1,2m+1},\mathrm{tr}_{2n+1,2k-2m+1})$. The reason, why they do not arise in the list in the theorem is, that they coincide with $\mathrm{diag}^+(\mathrm{tr}_{2n,2m},\mathrm{tr}_{2n,2k-2m})$.

To prove this, we show that $\mathrm{tr}_{2m,2n}=\mathrm{tr}_{2m+1,2n+1}$.

$$\left\langle\nabla_{X_i^{2m}}X_i^{2n},\nabla_{X_i^{2m+1}}X_i^{2n+1}\right\rangle=-\left\langle X_i^{2n},\nabla_{X_i^{2m}}\nabla_{X_i^{2m+1}}X_i^{2n+1}\right\rangle=$$
$$=-\left\langle X_i^{2n},\nabla_{X_i^{2m+1}}\nabla_{X_i^{2m}}X_i^{2n+1}\right\rangle+\left\langle X_i^{2n},\nabla_{[X_i^{2m},X_i^{2m+1}]}X_i^{2n+1}\right\rangle=$$
$$=-\left\langle\nabla_{X_i^{2m+1}}X_i^{2n},\nabla_{X_i^{2m}}X_i^{2n+1}\right\rangle-\left\langle X_i^{2n},\nabla_{\mathrm{tr}_{2m,2m+1}}X_i^{2n+1}\right\rangle$$

Since $\mathrm{tr}_{2m,2m+1}=\mathrm{tr}_{2n,2n-1}$ by the first part, the second summand is not zero. Therefore either the left hand side does not vanish ($\mathrm{tr}_{2m,2n}=\mathrm{tr}_{2m+1,2n+1}$) or the first summand

$(\mathrm{tr}_{2m+1,2n} = \mathrm{tr}_{2m,2n+1})$. Using Theorem 2.14, which holds by the same proof for even k and assuming $\mathrm{tr}_{2m+1,2n} = \mathrm{tr}_{2m,2n+1}$ we deduce

$$\mathrm{tr}_{1,2(m-n)+2} \overset{k=2m+2}{=} \mathrm{tr}_{2m+1,2n} = \mathrm{tr}_{2m,2n+1} \overset{k=2m}{=} \mathrm{tr}_{0,2(m-n)-1} = \mathrm{tr}_{1,2(m-n)-2},$$

which contradicts Proposition 2.12 on page 22. □

We summarize the results on the irreducible modules in $E(0)$.

THEOREM 2.18. *If the isotropy group $G_a = \mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$ then $E(0)$ decomposes into irreducible modules of the isotropy representation in the following way:*

$$E(0) = \bigoplus_{n\in 2\mathbb{N}} \mathrm{tr}_{0,n} \oplus \bigoplus_{n\in 2\mathbb{N}} \Lambda^2_{0,n} \oplus \bigoplus_{n\in 2\mathbb{N}\oplus 1} \Lambda^2_{1,n} \oplus \bigoplus_{n\in 2\mathbb{N}} V_{1,2n} = \mathrm{Tr}\oplus\Lambda(m_1)\oplus\Lambda(m_2)\oplus V_{1,2\mathbb{Z}}.$$

The eigenspace $E(0)$ is infinite dimensional.

2.4.3. The singular isotropy representations for one-dimensional eigenspaces. One-dimensional eigenspaces do not fit into the context of the treatment in the preceding subsections for some reasons: first the distinction between isotropy groups with one or two factors does not make sense, second the choice of equivariant bases for E_m and E_{2k-m} is not possible as we have done it. This point may be solved easily: Assume the spaces E_{2n} are one-dimensional and k is even. Choose a unit vector X^{2m} of E_{2m} and define $X^{2k-2m} = -g_* X^{2m}$, where $g \in G_k = \{g\}$, then the spaces $\mathrm{diag}^{\pm}$ are defined and behave just as in the last subsections. For $m_1 = m_2 = 1$ proposition 2.12 holds, the modules in a singular point $c_k(a)$ are

$$\begin{aligned} V_+ &:= \mathrm{diag}^+(\mathrm{tr}_{k,m}, \mathrm{tr}_{k,2k-m}) \oplus \mathrm{diag}^+(E_m, E_{2k-m}) \quad \text{and} \\ V_- &:= \mathrm{diag}^-(E_m, E_{2k-m}). \end{aligned}$$

Moreover $V_{n,m} = V_{k-n,k-m}$ by the same proof.

REMARK. There is no analogue for Theorem 2.17 in the case $m_1 = m_2 = 1$, more precisely its statement is the same as proposition 2.12, if one writes $\mathrm{tr}_{2n,2m+1}$ instead of $V_{2n,2m+1}$ and observes that the choice of signs in Theorem 2.17 is different for odd numbers, interchanging $\mathrm{diag}^{\pm}$.

If $m_1 = 1 < m_2$ Theorem 2.17 holds by the same proof.

Finally we consider the case $G_a = \mathrm{SO}(n)$ with diagram $\underset{1}{\circ}\overset{\infty}{-}\underset{n}{\circ}$, when the associated modules are not always contained in $E(0)$.

PROPOSITION 2.19. *Let $G_a = \mathrm{SO}(n)$, $\dim(E_{2m}) = n$ and $\dim(E_{2m+1}) = 1$ and let $E_{\frac{k_0+m_0}{2}} \subset \psi(E_{k_0}, E_{m_0})$ for at least one pair (k_0, m_0) with $k_0 - m_0 = 2 \mod 4$. Then k_0 is even, and $E_{\frac{k+m}{2}} \subset \psi(E_k, E_m)$ holds precisely for any pair (k,m) of even numbers with $k - m = 2 \mod 4$. For k even and m odd $\psi(E_k, E_m) = E_{2m-k}$.*

Choose natural bases for $E_m(a)$, $E_{2k-m}(a)$ and $E_k(a)$ as above, then modules of the singular isotropy representation are the same as in Theorem 2.17 if $k - k_0 = 1 \mod 2$ or k is odd and otherwise

$$\begin{aligned} V_+ &: = \mathrm{diag}^+(E_m, E_{2k-m}) \oplus \mathrm{diag}^+(E_{\frac{k+m}{2}}, E_{\frac{3k-m}{2}}) \quad \textit{and} \\ V_- &: = \mathrm{diag}^-({S^2}_{k,m}, {S^2}_{k,2k-m}) \oplus \mathrm{diag}^-(E_m, E_{2k-m}) \oplus \mathrm{diag}^-(E_{\frac{k+m}{2}}, E_{\frac{3k-m}{2}}). \end{aligned}$$

Denote by $V^0_{n,m}$ the projection onto $E(0)$ of $V_{n,m}$, then:

$$\begin{aligned} V^0_{n,m} &= 0 && \textit{for } n-m=1 \mod 2\\ V^0_{n,m} &= V^0_{2k-n,2k-m} && \textit{for } n-m=0 \mod 2\\ V^0_{n,m} &\perp V^0_{n+1,m+1} && \textit{and}\\ V^0_{4n+2,4m+2} &= V^0_{2n+1,2m+1}. \end{aligned}$$

Proof. Assume first $n > 3$. The pair (k_0, m_0) consists of even numbers, since $V_{2n+1,2m+1}$ is one-dimensional and E_{2m} is n-dimensional. Therefore $\mathrm{tr}_{k_0,m_0} = E_{\frac{k_0+m_0}{2}}$ and $V_{k_0,\frac{k_0+m_0}{2}} = E_{m_0}$ for dimensional reasons, cf. Propositions 2.8 and 2.9. Modules $V_{2n+1,2m+1}$ are always subsets of $E(0)$.

Both modules $\mathrm{diag}^+(\mathrm{tr}_{k_0,m_0}, \mathrm{tr}_{k_0,2k_0-m_0})$ and $\mathrm{diag}^-(\mathrm{tr}_{k_0,m_0}, \mathrm{tr}_{k_0,2k_0-m_0})$ do not vanish, therefore the given modules provide the only possibility.

Let k and m be even numbers, such that $|k-m| = |k_0 - m_0|$. The equation (2.6) yields

$$0 = \langle V_{k,m}, V_{k+1,m+1}\rangle = \langle V_{k,m+1}, V_{k+1,m}\rangle + \langle V_{k,k+1}, V_{m,m+1}\rangle .$$

Since the first summand on the right hand side vanishes (cf. the end of the proof of Theorem 2.17) $\langle V_{k,k+1}, V_{m,m+1}\rangle = 0$. If they both are contained in $E(0)$ they have to coincide, therefore $V_{k,k+1} = E_{k+2}$ and by Proposition 2.7 on page 16 $V_{k,k+2} \supset E_{k+1}$. Using again the equation (2.6)

$$\langle \mathrm{tr}_{0,2}, \mathrm{tr}_{-2,4}\rangle = \langle \mathrm{tr}_{0,-2}, \mathrm{tr}_{2,4}\rangle + \langle \mathrm{tr}_{0,4}, \mathrm{tr}_{2,-2}\rangle = \langle E_1, E_3\rangle + \langle \mathrm{tr}_{0,4}, \mathrm{tr}_{2,-2}\rangle \neq 0.$$

proves $\mathrm{tr}_{-2,4} = E_1$. Inductively we derive $\mathrm{tr}_{4k,4m+2} = E_{2k+2m+1}$ for any k and m.

The proof, which spaces $V^0_{n,m}$ coincide works as in the proof of Proposition 2.15 on page 24.

Finally we treat the cases $n \leq 3$. For $n = 3$, we remark that again $\mathrm{tr}_{k_0,m_0} = E_{\frac{k_0+m_0}{2}}$ by Proposition 2.8 and the modules $V_\pm$ are the same. This proves $V_{k_0,\frac{k_0+m_0}{2}} = E_{m_0}$ for $n = 3$, since the 3-dimensional modules $\Lambda^2(\mathbb{R}^3)$ vanish.

Let $n = 2$, then the same arguments as in Proposition 2.12 on page 22 prove that $\mathrm{tr}_{k_0,m_0} = E_{\frac{k_0+m_0}{2}}$, since $\mathrm{diag}^+(\Lambda)$-modules are not extendable to standard modules of $\mathrm{SO}(3)$. The space $E(0)$ contains the two-dimensional modules ${S^2}_{4k,4l+2}$, therefore it is a priori not clear that $V_{2n+2m+1,4m} = E_{4n+2}$. But the representation of $\mathrm{SO}(2)$ on eigenspaces is $\Gamma_{(1)}$, while it is $\Gamma_{(2)}$ on S^2-modules, which proves the assertion in that case, too.

Let $n = 1$, then $\mathrm{tr}_{k_0,m_0} = E_{\frac{k_0+m_0}{2}}$ holds, assume k_0 to be even. Then $V_{2n+1,4m+3} \subset E(0)$ since otherwise $V_{2n,2m+1}$ is not orthogonal to $E_{4m-2n+2}$ and to $E_{4n-2m-1}$. Since modules of the singular isotropy representation are at most two dimensional and invariant under ∇_{E_k}, this is not possible. Moreover by the by the same argument, it follows that $V_{2n,2m+1} = E_{4m-2n+2}$. □

2.5. Reduction to elementary isoparametric hypersurfaces

Let $G_a = \mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$. We define two tangential distributions of TM, associated with the families of eigenspaces of even respectively odd index.

$$D_1 = \{X \in T_aM \mid g_*X = X \text{ for all } g \in G_2\}$$
$$D_2 = \{X \in T_aM \mid g_*X = X \text{ for all } g \in G_1\}$$

Observe that neither $D_1 \cap D_2 = \{0\}$ (one-dimensional modules belong to both distributions) nor $D_1 + D_2 = T_a M$, since the modules of type $V_{2n,2m+1}$ are missing.

THEOREM 2.20. *The distributions D_1 and D_2 are autoparallel and therefore integrable with totally geodesic leaves. In other words: A homogeneous isoparametric submanifold contains two totally geodesic submanifolds which are elementary isoparametric. Moreover if G_1 is the group acting effectively on E_{2n} and if we assume additionally that associated modules are subspaces of $E(0)$:*

$$D_1 = \bigoplus_{n \in \mathbb{Z}} E_{2n} + \bigoplus_{n,m \in \mathbb{Z}} V_{2n,2m}$$

$$D_2 = \bigoplus_{n \in \mathbb{Z}} E_{2n+1} + \bigoplus_{n,m \in \mathbb{Z}} V_{2n+1,2m+1}$$

PROOF. The autoparallelity follows easily since for all X and $Y \in D_1$ and $g \in G_2$:

$$g_*(\nabla_X Y) = \nabla_{g_* X} g_* Y = \nabla_X Y$$

Therefore $\nabla_X Y \in D_1$. For the alternative description of the distributions we observe that G_2 acts trivially on $\bigoplus_{n \in \mathbb{Z}} E_{2n} + \bigoplus_{n,m \in \mathbb{Z}} V_{2n,2m}$ and on non of the other modules, except the trace modules in $V_{2n+1,2m+1}$. But we have proven in Theorem 2.17 that those coincide with the trace factors of $V_{2n,2m}$ which finishes the proof. $\square$

CHAPTER 3

Canonical connections of isoparametric hypersurfaces

In this chapter we describe the canonical connections of certain homogeneous isoparametric hypersurfaces. Together with Theorem 1.13 on page 8 this yields a rigidity result for those hypersurfaces. We have already seen the close relation between canonical connections and projection connections for s-representations in Section 1.3, similar constructions work in the infinite dimensional setting. We consider the case when the isotropy representation acts irreducibly as the standard representation of $\mathrm{SO}(n)$ on any eigenspace except $E(0)$, more precisely the principal isotropy group is of the form $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$ or $\mathrm{SO}(m)$, by Theorem 2.3 on page 11.

For a finite dimensional homogeneous isoparametric submanifold the following is true, cf. [Les97] and [BCO03, Exercise 7.4.4] :

Proposition 3.1. *Let G/K be a semi-simple symmetric space and let $M = K \cdot a$ be a principal orbit of its isotropy representation. Then the projection connection $\nabla^{\pi}_X Y = \sum_{i=1}^{g} (\nabla_X Y_i)_i$, where g is the number of the curvature normals and $(\cdot)_i$ denotes projection onto the eigendistribution E_i, is the canonical connection if and only if the restricted root system of G/K is reduced.*

Since the eigenspaces of the shape operator of M are of the form $E_\lambda = \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$ and the isotropy representation respects this splitting (cf. page 9), having a reduced root system is equivalent to the fact that the eigenspaces are irreducible modules of the isotropy representation. Since in infinite dimensions $E(0)$ is never irreducible, one has to examine its behavior more closely.

Definition 3.2. Let G be a Hilbert Lie group acting polarly on a Hilbert space V and let a be a regular point. Moreover let $T_a M = \oplus_{i \in \mathbb{Z}} V_i$ where the V_i are irreducible modules of the isotropy representation and the V_i are subsets of an eigenspace of the shape operator. Moreover the $V_i \subset E(0)$ are contained in associated modules of two eigenspaces in the sense of definition 2.5 on page 15.

The *projection connection* ∇^{π} is defined by $\nabla^{\pi}_X Y = \sum_{i \in \mathbb{Z}} (\nabla_X Y_i)_i$ where $(\cdot)_i$ denotes projection onto V_i.

We denote by $S^{\pi} = \nabla - \nabla^{\pi}$ the corresponding normal homogeneous structure, then the tensor S^{π} is G-invariant, since we project onto modules of the isotropy representation. Moreover since the eigenspaces are ∇^{π}–parallel so is α. Therefore it would be sufficient to show that the holonomy representation of ∇^{π} is contained in the isotropy representation (i.e. G-invariant vector fields are ∇^{π}–parallel) because this yields that any G–invariant tensor field (especially S^{π}) is ∇^{π}–parallel and thus ∇^{π} would be the canonical connection. In fact we will show that G-invariant vector field in any E_n are ∇^{π}–parallel, but this is not true for G-invariant vector fields in $E(0)$. We give the description of the canonical connection ∇^c after some preliminary propositions.

First we give an alternative description of the associated modules.

PROPOSITION 3.3. *Let E_i and E_j be eigenspaces and V_{ij} the associated module. Let $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$ be vector fields, then $\nabla_X Y \in V_{ij}$ if and only if X and Y are G-invariant vector fields.*

PROOF. By the definition of V_{ij} it is obvious that $\nabla_{E_i} E_j \subset E_j \oplus V_{ij}$. Moreover $\{\nabla_X Y \mid X \in \Gamma(E_i), Y \in \Gamma(E_j)$ are G-invariant vector fields$\}$ is a module of the isotropy representation, therefore equals V_{ij}. □

REMARK. For the rest of this chapter by $\nabla_V W$, where V and W are modules of the isotropy representation, we mean

$$\{\nabla_X Y \mid X \in \Gamma(V), Y \in \Gamma(W) \text{ are } G\text{-invariant vector fields}\}$$

By Proposition 2.7 on page 16 $\psi(V, W) = \nabla_V W$ for eigenspaces except for an constant factor.

We now have to check whether $\nabla_{V_i} V_j$ (for G-invariant vector fields) is orthogonal to V_j, because this yields ∇^π–parallelity. This is evident if V_i and V_j are eigenspaces, because either their associated module are contained in $E(0)$ or it is an eigenspace not equal to V_i or V_j, cf. Proposition 2.7 on page 16.

REMARK. In the next paragraphs we will only consider the case of associated modules lying in $E(0)$, the other cases we will be solved in proposition 3.9 on page 36. Moreover we will not mention explicitly the case $G_a = \mathrm{SO}(n)$ with diagram $\underset{n}{\circ}\overset{\infty}{-}\underset{n}{\circ}$, but the conclusions hold for this case as well, cf. also the remark after Theorem 2.17 on page 26. Only modules of type $\mathrm{tr}_{0,n}$ and $\Lambda_{0,n}$ do exist (n even or odd), the other equations being of no relevance in that case. Therefore we consider an isoparametric hypersurface $G \cdot a$ with isotropy group $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$.

DEFINITION 3.4. We choose bases $\left\{X_1^0, \ldots, X_{m_1}^0\right\}$ for E_0 and $\left\{X_1^1, \ldots, X_{m_2}^1\right\}$ for E_1 as on page 21 in (2.3), where the choice of sign does not matter. Then there are bases $\left\{X_1^{2n}, \ldots, X_{m_2}^{2n}\right\}$ for E_{2n} and $\left\{X_1^{2n+1}, \ldots, X_{m_1}^{2n+1}\right\}$ for E_{2n+1}, defined as described on page 21. We will call these bases *natural*. Moreover they give rise to a choice of natural bases on the irreducible modules on $E(0)$, that is e.g. $\psi(X_i^0 \otimes X_j^{2n})$ is the natural bases for $\Lambda^2_{0,2n}$.

The next proposition solves the case of one eigenspace E_k and one module V in $E(0)$. Remember that by Theorem 2.18 it is sufficient to consider only modules V either associated with E_0 or E_1 for these span $E(0)$.

PROPOSITION 3.5. *Let k and n be even and E_k be an eigenspace of dimension m_1 and V a module in $E(0)$. Then the associated module of E_k and V is orthogonal to E_k and to V. The following table contains the precise information:*

$V =$	$\mathrm{tr}_{0,n} = \mathrm{tr}_{1,n+1}$	$\Lambda^2_{0,n}$	$\Lambda^2_{1,n+1}$	$V_{0,n+1}$
$\nabla_{E_k} V =$	$\mathrm{diag}^+(E_{k+n}, E_{k-n})$	$\mathrm{diag}^-(E_{k+n}, E_{k-n})$	0	$\mathrm{diag}^-(E_{k+n+1}, E_{k-n-1})$
$\nabla_V E_k =$	$\mathrm{diag}^-(E_{k+n}, E_{k-n})$	$\mathrm{diag}^+(E_{k+n}, E_{k-n})$	0	$\mathrm{diag}^+(E_{k+n+1}, E_{k-n-1})$

More precisely $\nabla_{X_r^k} \psi(X_s^0 \otimes X_s^n) = ||v_k|| \, (X_r^{k+n} + X_r^{k-n})$ and for $s \neq t$:

$$\nabla_{X_r^k} \psi(X_s^0 \otimes X_t^{n(+1)}) = \begin{cases} ||v_k|| \, (X_t^{k+n(+1)} - X_t^{k-n(-1)}) & \text{if } r = s \\ ||v_k|| \, (-X_s^{k+n(+1)} + X_s^{k-n(-1)}) & \text{if } r = t \\ 0 & \text{if } s \neq r \neq t \end{cases}$$

Similar statements hold when k is odd.

Proof. We start with the modules of type $\nabla_{E_k} V$. Generally, if $\nabla_{E_k} V$ is orthogonal to E_l for some l, then $V_{k,l}$ is orthogonal to V. This is because the connection is metric:

$$0 = \langle \nabla_{E_k} V, E_l \rangle = - \langle V, \nabla_{E_k} E_l \rangle$$

This proves immediately that $\nabla_{E_k} V$ is orthogonal to E_k since $V_{k,k} = \{0\}$. Moreover by the same argument and the fact that $E(0)$ is autoparallel, the associated module of E_k and V is orthogonal to $E(0)$, in particular to V. Now we study the situation more closely:

The module $V_{0,n} = \nabla_{E_0} E_n$ is the associated module of E_k and E_{k+n} as well as of E_k and E_{k-n}. These are the only possibilities involving E_k and therefore

$$\nabla_{E_k} V_{0,n} \subset E_{k+n} \oplus E_{k-n}.$$

The statements (1) and (2) follow since the modules $\operatorname{diag}^-(\operatorname{tr}_{k,k+n}, \operatorname{tr}_{k,k-n})$ and $\operatorname{diag}^+(\Lambda^2_{k,k+n}, \Lambda^2_{k,k-n})$ vanish (cf. Proposition 2.12 on page 22 and especially Equation (2.4)).

Now we consider the case $V = \Lambda^2_{1,n+1}$, the associated module of V and E_k is zero by the discussion above, since $\Lambda_{1,n+1} \perp V_{k,l}$ for any l.

The precise statement on $\nabla_X Y$ follows, since the following diagram is commutative (up to a constant factor) if we choose natural bases

$$\begin{array}{ccc} \mathbb{R}^{m_1} \otimes \Lambda^2(m_1) & \xrightarrow{\;\Phi\;} & \mathbb{R}^{m_1} \\ \downarrow & & \downarrow \\ E_k \otimes \Lambda^2_{(0,n)} & \xrightarrow{\;\nabla\;} & \operatorname{diag}^-(E_{k+n}, E_{k-n}) \end{array} \tag{3.1}$$

Thereby is Φ an equivariant map — the projection onto the irreducible module within the tensor representation $\mathbb{R}^{m_1} \otimes \Lambda^2(m_1)$, that is

$$\Phi\big(e_r \otimes (e_s \otimes e_t - e_t \otimes e_s)\big) = \begin{cases} e_t & \text{if } r = s \\ -e_s & \text{if } r = t \\ 0 & \text{if } s \neq r \neq t. \end{cases}$$

The behavior of ∇ for natural bases is the same as for Φ up to a constant factor, which is $||v_k||$. This is since $\Lambda^2_{(0,n)} \oplus \operatorname{diag}^-(E_{k+n}, E_{k-n})$ is an irreducible modules of the singular isotropy representation at the midpoint of the curvature sphere $S_k(a)$, the radius of which is $\frac{1}{||v_k||}$. For the other cases similar arguments hold.

Finally we consider the case $V = \nabla_{E_0} E_{n+1}$, where the conclusion is proven as in the first case using the fact that $\operatorname{diag}^+(V_{k,k+n+1}, V_{k,k-n-1})$ vanishes, cf. Theorem 2.17 on page 26.

For modules of type $\nabla_V E_k$, we use the fact that $\psi(X \otimes Y) = \psi(Y \otimes X)$, which holds by the Codazzi-equation and therefore if $X \in E_x$ and $Y \in E_y$

$$(\lambda(y) \cdot \operatorname{id} - A_\xi) \nabla_X Y = (\lambda(x) \cdot \operatorname{id} - A_\xi) \nabla_Y X$$

Now let $X \in E_k$ and $Y \in \mathrm{tr}_{0,n}$ then $\nabla_X Y = Z_{k+n} + Z_{k-n}$ by the first part of the proof ($Z_{k\pm n}$ isotropy equivalent vectors in $E_{k\pm n}$). Hence

$$\begin{aligned} -\nabla_Y X &= (A_\xi - \lambda(k))^{-1} A_\xi (Z_{k+n} + Z_{k-n}) = \\ &= (A_\xi - \lambda(k))^{-1} (\lambda(k+n) Z_{k+n} + \lambda(k-n) Z_{k-n}) = \\ &= \frac{\lambda(k+n)}{\lambda(k+n) - \lambda(k)} Z_{k+n} + \frac{\lambda(k-n)}{\lambda(k-n) - \lambda(k)} Z_{k-n} = \\ &= -\frac{d+k}{n} Z_{k+n} + \frac{d+k}{n} Z_{k-n}, \end{aligned}$$

which proves case (1). We have used the description of the eigenvalue given in Section 1.1 on page 3, namely $\lambda_n = \frac{c}{d+n}$.

The other cases are treated likewise. □

Eventually we study the case of two modules in $E(0)$, where the situation is slightly more complicated.

PROPOSITION 3.6. *Denote* $\mathrm{Tr} = \bigoplus_{n\in\mathbb{N}} \mathrm{tr}_{(0,2n)} \subset E(0)$*, then for G-invariant vector fields holds:*

$$\nabla_{E(0)} E(0) \perp \mathrm{Tr} \quad \textit{and} \quad \nabla_{\mathrm{Tr}} E(0) = 0$$

PROOF. First note that the two statements are equivalent by the autoparallelity of $E(0)$.

Let $v \in E(0)(a)$ and choose an $h \in G$ such that $h \cdot a = a + v$, then $\mathrm{Tr}(h\cdot a) = h_* \mathrm{Tr}(a)$ since $G_{a+v} = h G_a h^{-1}$. For any vector in $\mathrm{Tr}(a)$ is of type $\nabla_{X_n} X_m$ for appropriate vector fields in some eigenspaces E_n and E_m, which are isotropy equivalent. Then $h_* X_m$ and $h_* X_n$ are isotropy equivalent vector field as well lying in the orthogonal complement of $E(0)(a) = E(0)(h \cdot a)$, that is $h_*(\nabla_{X_n} X_m)$ is a subset of $\mathrm{Tr}(h \cdot a)$ as well as of $\mathrm{Tr}(a)$. Therefore $\mathrm{Tr}(a + v) = \mathrm{Tr}(a)$ for any v. Hence Tr is a parallel distribution within $E(0)$ and it remains to prove $\nabla_{\mathrm{Tr}} \mathrm{Tr} = 0$ for G-invariant vector fields. This is since the above argument holds as well for the distributions $\mathrm{tr}_{n,m}$. □

PROPOSITION 3.7. *Let V and W be irreducible modules in $E(0)$ associated to eigenspaces of same parity and let w.l.o.g. n, m be even numbers.*

(0) If V or W is a trace module, then $\nabla_V W$ and $\nabla_W V$ vanish.

(1) If $V = \Lambda^2_{0,n}$ and $W = \Lambda^2_{0,m}$, then
$$\nabla_V W = \mathrm{diag}^-(\Lambda^2_{0,m+n}, \Lambda^2_{0,m-n}) = \mathrm{diag}^+(\Lambda^2_{0,m+n}, \Lambda^2_{0,n-m}) \textit{ if } m_1 \neq 2$$
while $\nabla_V W$ and $\nabla_W V$ vanish if $m_1 = 2$.

(2) If $V = \Lambda^2_{1,n+1}$ and $W = \Lambda^2_{1,m+1}$, then
$$\nabla_V W = \mathrm{diag}^-(\Lambda^2_{1,n+m-1}, \Lambda^2_{1,m-n+1}) = \mathrm{diag}^+(\Lambda^2_{1,n+m-1}, \Lambda^2_{1,n-m-1}).$$

(3) If $V = \Lambda^2_{0,n}$ and $W = \Lambda^2_{1,m+1}$, then $\nabla_V W$ and $\nabla_W V$ vanish.

Let $V = \nabla_{E_0} E_{n+1}$ be a module in $E(0)$ associated to eigenspaces of different parity.

(4) If $W = \Lambda^2_{0,m}$ or $\Lambda^2_{1,m+1}$, then
$$\nabla_V W = \mathrm{diag}^+(\nabla_{E_0} E_{m+n+1} \oplus \nabla_{E_0} E_{m-n+1}),$$
$$\nabla_W V = \mathrm{diag}^-(\nabla_{E_0} E_{m+n+1} \oplus \nabla_{E_0} E_{m-n+1}).$$

(5) If $W = \nabla_{E_0} E_{m+1}$ then
$$\nabla_V W = \mathrm{diag}^-(\Lambda^2_{0,m+n+2}, \Lambda^2_{0,m-n}) \oplus \mathrm{diag}^-(\Lambda^2_{1,m+n+3}, \Lambda^2_{1,m-n+1}).$$

Only in case (1) and (2) $\nabla_V W$ does intersect V if $m = 2n$, W if $n = 2m$.

Precise formulas for the covariant derivative of G-invariant vector fields (natural bases) may be exhibit explicitly as in Proposition 3.5 on page 32.

PROOF. All modules $\nabla_V W$ are contained in $E(0)$ since this is an autoparallel distribution.

The statement (0) is obvious by the last proposition, (3) by dimension reasons, except the case, when at least one of the multiplicities is 2. We will treat this case later.

The dimension of the modules $\nabla_V W$ and $\nabla_W V$ are determined by the decompositions of tensor representations of $\mathrm{SO}(m)$ and $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$ respectively , which hold for not too small dimensions:

$$\Lambda^2 \otimes \Lambda^2 = \mathrm{tr} \oplus \Lambda^2 \oplus S^2 \oplus \Lambda^4 \oplus \Gamma_{(1,0,1,0\ldots)} \oplus \Gamma_{(0,2,0,\ldots)}$$
$$\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2} \otimes \Lambda^2(m_2) = (\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}) \oplus (\mathbb{R}^{m_1} \otimes \Lambda^3(m_2)) \oplus (\mathbb{R}^{m_1} \otimes \Gamma_{(1,1,0,\ldots)})$$
$$(\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}) \otimes (\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}) = (\mathrm{tr} \oplus \Lambda^2(m_1) \oplus S^2(m_1)) \otimes (\mathrm{tr} \oplus \Lambda^2(m_2) \oplus S^2(m_2))$$

It is not difficult to check that in the low-dimensional cases the modules Λ^2 , $\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}$ and $\Lambda^2(m_1) \oplus \Lambda^2(m_2)$, respectively, are contained in the decomposition of the tensor-representations as well. Remember that tr-modules do not arise by the last proposition.

Let V_k be an arbitrary module in $E(0)$ (associated to E_k). For the precise statement we use the Gauß equation and the last proposition:

$$\begin{aligned}
\left\langle \nabla_{\Lambda^2_{0,n}} \nabla_{E_0} E_m, V_k \right\rangle &= \left\langle \nabla_{E_0} \nabla_{\Lambda^2_{0,n}} E_m, V_k \right\rangle + \left\langle \nabla_{[E_0, \Lambda^2_{0,n}]} E_m, V_k \right\rangle = \\
&= \left\langle \nabla_{E_0} \nabla_{\Lambda^2_{0,n}} E_m, V_k \right\rangle + \left\langle \nabla_{\nabla_{E_0} \Lambda^2_{0,n}} E_m, V_k \right\rangle - \left\langle \nabla_{\nabla_{\Lambda^2_{0,n}} E_0} E_m, V_k \right\rangle = \\
&\overset{\text{Prop.3.5}}{=} \left\langle \nabla_{\nabla_{E_0} \Lambda^2_{0,n}} E_m, V_k \right\rangle \subset \left\langle \mathrm{diag}^-(\nabla_{E_0} E_{m+n}, \nabla_{E_0} E_{m-n}), V_k \right\rangle .
\end{aligned}$$

Statement (1) is hence proven by the dimension argument above, (2) and (4) are proven in a similar manner. Non of these modules vanishes, which is proven by using the same calculation for the natural bases together with proposition 3.5 on page 32, which gives the precise description of the projection connection. Moreover this proves the statement also for the case when at least one of the multiplicities is 2.

To prove statement (5) we use statement (4) and the fact that the connection is metric:

$$\begin{aligned}
\left\langle \nabla_V W, \mathrm{diag}^\pm(\Lambda^2_{0,m+n+2}, \Lambda^2_{0,m-n}) \right\rangle &= - \left\langle W, \nabla_V \mathrm{diag}^\pm(\Lambda^2_{0,m+n+2}, \Lambda^2_{0,m-n}) \right\rangle = \\
= - \left\langle W, \mathrm{diag}^\pm \left(\mathrm{diag}^-(V_{2n+m+3}, V_{m+1}), \mathrm{diag}^-(V_{m+1}, V_{m-2n-1}) \right) \right\rangle &
\end{aligned}$$

This is only nonzero for $\mathrm{diag}^-(\Lambda^2_{0,m+n+2}, \Lambda^2_{0,m-n})$. Since the same holds for the module $\mathrm{diag}^-(\Lambda^2_{1,m+n+3}, \Lambda^2_{1,m-n+1})$ and only for these two spaces, it is clear that $\nabla_V W$ is contained in the direct sum. Since both of them are irreducible modules of the isotropy representation on which different subgroups of G_a act effectively, the only possibility is the one stated. □

THEOREM 3.8. *Let M be a homogeneous isoparametric submanifold of Hilbert space with isotropy group $\mathrm{SO}(m)$ or $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$. Then the canonical connection is*

$$\nabla^c_X Y = \begin{cases} \nabla^\pi_X Y - \frac{1}{2}(\nabla^\pi_Y X)_n & \text{if } X \in \Lambda_{k,k+2n},\ Y \in \Lambda_{k,k+n} \\ \nabla^\pi_X Y & \text{otherwise} \end{cases}$$

PROOF. As we have seen at the beginning of the chapter $\nabla^c_X Y = \nabla^\pi_X Y$ for vectors X and Y in modules V_X and V_Y respectively, if $\psi(V_X, V_Y)$ is orthogonal to V_Y. The propositions 3.5 and 3.7 prove, that this is true except the case when $X \in \Lambda_{0,2n}$ and $Y \in \Lambda_{0,n}$ ($X \in \Lambda_{1,2n+1}$ and $Y \in \Lambda_{1,n+1}$ respectively).

Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$ and $\{v_{ij} = e_i \otimes e_j - e_j \otimes e_i \mid 1 \leq i < j \leq n\}$ be a basis of $\Lambda^2(n)$, then it is easy to check that the map $(v_{ij}, v_{ik}) \mapsto v_{jk}$ is equivariant and describes the projection from $\Lambda^2(n) \otimes \Lambda^2(n) \to \Lambda^2(n)$. Therefore we only have to deal with $\nabla_{(\nabla_{X_i^0} X_j^{2n})}(\nabla_{X_i^0} X_k^n)$.

The idea is the same as in finite dimensions, cf. equation (1.1) on page 9: we subtract the interfering part $(\nabla_X Y)_{V_Y}$ but to ensure that the result is a tensor in Y we interchange the roles of X and Y, i.e. subtract $\mu(\nabla_Y X)_{V_Y}$, where μ is a constant factor, such that $\mu(\nabla_Y X)_{V_Y} = (\nabla_X Y)_{V_Y}$. The factor μ is easily calculated by Codazzi-equation, when either one of the vectors is orthogonal to $E(0)$. In our case we do not exchange the vectors themselves but only X_j^{2n} and X_k^n, again using Gauß-equation:

$$\left\langle \nabla_{(\nabla_{X_i^0} X_j^{2n})}(\nabla_{X_i^0} X_k^n), \Lambda_{0,n} \right\rangle =$$

$$= \left\langle \nabla_{X_i^0}(\nabla_{(\nabla_{X_i^0} X_j^{2n})} X_k^n), \Lambda_{0,n} \right\rangle + \left\langle \nabla_{[\nabla_{X_i^0} X_j^{2n}, X_i^0]} X_k^n, \Lambda_{0,n} \right\rangle = (1) + (2)$$

The map $(e_i, v_{ij}) \mapsto e_j$ is the projection of $\mathbb{R}^n \otimes \Lambda^2(n) \to \mathbb{R}^n$ (bases chosen as above). Therefore the term (1) vanishes. For the vector $[\nabla_{X_i^0} X_j^{2n}, X_i^0]$ only the projection onto E_{2n} plays a role, and using Lemma 5.1. of [HL99] shows:

$$[\nabla_{X_i^0} X_j^{2n}, X_i^0] := -\frac{d+2n}{2n} \nabla_{X_i^0} \nabla_{X_i^0} X_j^{2n} = -\frac{d+2n}{2n} \cdot \mu_0 X_j^{2n}$$

where $\mu_i = ||v_i||$ only depends on E_i, cf. proof of Proposition 3.5 on page 32. Therefore

$$(2) = -\frac{d+2n}{2n} \cdot \mu_0 \left\langle \nabla_{X_j^{2n}} X_k^n, \Lambda_{0,n} \right\rangle \underset{[\mathrm{HL99}]}{\overset{5.1}{=}} -\frac{d+2n}{2n} \cdot \mu_0 \cdot \frac{d+n}{d+2n} \left\langle \nabla_{X_k^n} X_j^{2n}, \Lambda_{0,n} \right\rangle =$$

$$= -\frac{d+n}{2n} \cdot \mu_0 \left\langle \nabla_{X_k^n} X_j^{2n}, \Lambda_{0,n} \right\rangle = -\frac{d+n}{2n} \cdot \mu_0 \cdot \mu_0^{-1} \left\langle \nabla_{\nabla_{X_i^0} \nabla_{X_i^0} X_k^n} X_j^{2n}, \Lambda_{0,n} \right\rangle =$$

$$\underset{[\mathrm{HL99}]}{\overset{5.1}{=}} -\frac{d+n}{2n} \cdot \frac{-n}{d+n} \left\langle \nabla_{[\nabla_{X_i^0} X_k^n, X_i^0]} X_j^{2n}, \Lambda_{0,n} \right\rangle = \frac{1}{2} \left\langle \nabla_{[\nabla_{X_i^0} X_k^n, X_i^0]} X_j^{2n}, \Lambda_{0,n} \right\rangle$$

Finally the Gauß-equation yields

$$\nabla_{(\nabla_{X_i^0} X_j^{2n})}(\nabla_{X_i^0} X_k^n) = \frac{1}{2} \nabla_{(\nabla_{X_i^0} X_k^n)}(\nabla_{X_i^0} X_j^{2n})$$

□

Eventually we treat the exceptional cases described in propositions 2.15 on page 24 and 2.19 on page 28.

Proposition 3.9. *Let M be a homogeneous isoparametric submanifold of Hilbert space with isotropy group* SO(2) *and affine Dynkin diagram* $\underset{2}{\circ}\overset{\infty}{-}\underset{2}{\circ}$ *or* SO(m) *with* $\underset{1}{\circ}\overset{\infty}{-}\underset{m}{\circ}$*, where associated modules* $V_{4n,4m+2} \supset E_{2n+2m+1}$*. Then the canonical connection is*

$$\nabla_X^c Y = \begin{cases} \nabla_X^\pi Y - \frac{1}{2}(\nabla_Y^\pi X)_n & \text{if } X \in \Lambda_{k,k+2n},\ Y \in \Lambda_{k,k+n} \quad \text{for } \underset{2}{\circ}\overset{\infty}{-}\underset{2}{\circ} \\ & \text{if } X \in S^2{}_{k,k+2n},\ Y \in S^2{}_{k,k+n} \quad \text{for } \underset{m}{\circ}\overset{\infty}{-}\underset{1}{\circ} \\ \nabla_X^\pi Y & \text{otherwise} \end{cases}$$

Proof. If the principal isotropy group is SO(2) and the modules are as in Proposition 2.15, the statements of the Propositions 3.5 and 3.7 hold by essentially the same proof: the trace modules do behave differently (e.g. $\mathrm{tr}_{4m+2,4n+2} = \mathrm{tr}_{2m+1,2n+1}$), which changes the statement of Proposition 3.5 slightly, but this does not matter for Proposition 3.7, where trace modules are of no importance. The precise description of $\nabla^\pi_X Y$ may be exhibited by diagrams similar to (3.1).

The same holds also for the exceptional hypersurfaces with diagram $\underset{1}{\circ}\overset{\infty}{-\!-}\underset{m}{\circ}$, whose modules are described in Proposition 2.19: The Λ-modules have to be replaced by $S^2(m)$-modules, whose behavior is similar, since

$$S^2(m) \otimes S^2(m) = \mathrm{tr} \oplus \Lambda^2(m) \oplus S^2(m) \oplus S^4 \oplus \Gamma_{(2,1,0,0\dots)} \oplus \Gamma_{(0,2,0,\dots)}$$
$$\mathbb{R}^m \otimes S^2(m) = \mathbb{R}^m \oplus S^3(m) \oplus \Gamma_{(1,1,0,\dots)}$$

□

Uniqueness of the canonical connection yields a rigidity result for infinite dimensional homogeneous isoparametric hypersurfaces, which is all about the same as the result of Exercise 7.4.5. in [BCO03], where the assumption on the isotropy representation is formulated in terms of the restricted root systems.

Theorem 3.10. *Let $M = G{\cdot}a$ be a complete, connected, homogeneous isoparametric submanifold of a Hilbert space. Assume that the isotropy representation acts as standard representations of* $\mathrm{SO}(m)$ *on each eigenspace of the shape operator except $E(0)$. Then M is uniquely determined by the second fundamental form and its covariant derivative in the point a.*

Remark. Instead of the condition on the isotropy representation, we may assume likewise that the singular slice representation are standard representations of $\mathrm{SO}(m{+}1)$.

Proof. The second fundamental form determines the curvature normals, spheres and the eigenspaces of the shape operator. Moreover we have seen in section 2.1 on page 10, that the isotropy group is either $\mathrm{SO}(m)$ or $\mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$, if it acts as standard representation. The covariant derivative of α determines the irreducible modules of the isotropy representation as described in the last chapter, especially distinguishes hypersurfaces with affine Dynkin diagram $\underset{m}{\circ}\overset{\infty}{-\!-}\underset{m}{\circ}$ with isotropy group $\mathrm{SO}(m)$ from those with isotropy group $\mathrm{SO}(m) \times \mathrm{SO}(m)$, as well as from those cases where the associated modules $(\nabla_{E_n} A.)_\xi E_m$ are not always contained in $E(0)$. To choose natural bases for the irreducible modules (within associated modules) of the isotropy representation, we only have to choose an Lie isomorphism between G_a and $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$, then as we have seen in proposition 3.5 on page 32 especially in equation (3.5), the projection connection of G-invariant vector fields are uniquely determined by the projections onto irreducible modules of certain tensor representations. So far we have proven that the projection connection is uniquely determined by the given geometric data. But Theorem 3.8 on page 35 gives the canonical connection, i.e. the normal homogeneous structure in terms of the projection connection, which is therefore uniquely determined as well. Theorem 1.13 on page 8 finishes the proof. □

CHAPTER 4

Slice Representations and Dynkin Diagrams of $P(G,H)$-Actions

In this chapter we will determine the affine marked Dynkin diagrams and singular slice representations of the known homogeneous isoparametric submanifolds in Hilbert space, i.e. the principal orbits of the $P(G,H)$–actions described by Terng in [Ter95]. We give a brief description of these actions and refer for further details to [Ter95].

Let G be a compact, connected, semi-simple Lie group, equipped with a biinvariant metric and $H \subset G \times G$ a closed connected subgroup acting hyperpolarly on G by

$$(h,k)\cdot g = hgk^{-1}.$$

For simple G such actions where classified by Kollross in [Kol02].

The most important class of a hyperpolar action is the following: If the subgroup is of type $H = K_1 \times K_2$, where both K_1 and K_2 are symmetric subgroups of G the action is called a Hermann action ([Her60]). We refer to such actions by terms like A I–II, where the letter stands for the group G and the roman numbers for the two involved symmetric subgroup, cf. [Hel01] for the list of compact symmetric spaces and Table A.1 on page 77 and A.3 on page 79 for a list of the Hermann examples.

A σ–action is given by a subgroup $G(\sigma) = \{(g,\sigma(g)) \mid g \in G\}$ where σ is an outer automorphism of G or $\sigma = \mathrm{id}$. These actions also may be seen as Hermann actions on $G \times G$ with $K_1 = G(\sigma)$ and $K_2 = \Delta(G) = G(\mathrm{id})$, since $G(\sigma)$ is the fixed point set of the map $(x,y) \mapsto (\sigma^{-1}x, \sigma y)$ and therefore a symmetric subgroup of $G \times G$.

If the cohomogeneity is greater than one, then the only examples are Hermann actions or σ–actions, whereas in the cohomogeneity one case one has a short list of exceptions and examples arising from isotropy representations of symmetric spaces of rank 2, cf. [Kol02, Theorem A].

Remark. There exist hyperpolar actions of cohomogeneity one on non-simple groups, though they are not classified. Let for example

$$\begin{aligned} G &= \mathrm{Spin}(8) \times \mathrm{Spin}(8) \times \mathrm{Spin}(8),\\ K_1 &= \mathrm{Spin}(7) \times \mathrm{Spin}(7) \times \mathrm{Spin}(7) \text{ and}\\ K_2 &= \left\{(g, \alpha(g), \alpha^2(g)) \mid g \in \mathrm{Spin}(8)\right\}, \end{aligned}$$

where α is the diagram automorphism of order 3 of Spin(8). Then $G/K_1 = S^7 \times S^7 \times S^7$ and Spin(8) acts transitively on S^7 with principal isotropy group Spin(7). The group Spin(7) acts transitively on S^7 with principal isotropy group G_2 and G_2 acts with cohomogeneity one on S^7; hence the action of K_2 on G/K_1 is a cohomogeneity one action.

This kind of actions may be lifted to Hilbert space in the following way. Let $\hat{G} = H^1([0,1],G)$ and $V = H^0([0,1],\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G. Then the action of the group

$$P(G,H) = \left\{ g \in \hat{G} \mid (g(0), g(1)) \in H \right\}$$

on V by gauge transformations ($g.v = gvg^{-1} - g'g^{-1}$) is proper Fredholm with the same cohomogeneity as the H–action on G. The $P(G,H)$-action is polar if and only if H acts hyperpolarly on G, cf. [Ter95] Theorem 1.2. and preceding remarks.

Remark. Some of these examples are reducible in the sense of Proposition 1.8 on page 5, i.e. there is a subspace of $E(0)$ which splits off.

To determine singular slice representations for Hermann actions we use frequently the following proposition.

Proposition 4.1. *Let σ and τ be involutions such that $K_1 = G^\sigma$ and $K_2 = G^\tau$ are the fixed point groups. Then $(G^{\sigma\circ\tau}, K_1 \cap K_2)$ is a symmetric pair and the associated s-representation is equivalent to the slice representation at* 0 *of the $P(G, K_1 \times K_2)$–action. The cohomogeneity is equal the rank of $(G^{\sigma\circ\tau}/(K_1 \cap K_2)$.*

Proof. This is a simple consequence of Proposition 3.1 in [Kol02] and Theorem 1.8 in [Ter95]. □

As we have already remarked on page 14 (cf. also [HO92, Theorem 2]) for a finite dimensional polar representation, slice representations and normal holonomy representations are equivalent. Although this does not hold in general in infinite dimensions, it is true at least for actions of Hermann type.

Proposition 4.2. *Let K_1 and K_2 be symmetric subgroups of G and consider the action of $P(G, K_1 \times K_2)$ on the Hilbert space. Then the (effectively made) slice representation at some point a is equivalent to the normal holonomy representation.*

Proof. It is sufficient to consider the singular point 0, both slice representation and normal holonomy representation being trivial in regular points. Let $M = P(G,H)\cdot 0$. The isotropy group is then $G_0 = K_1 \cap K_2$, the normal space is $\nu_0 M = \mathfrak{p}_1 \cap \mathfrak{p}_2$ where $\mathfrak{g} = \mathfrak{k}_i \oplus \mathfrak{p}_i$ are Cartan decompositions. Therefore the slice representation is an s-representation ($\tilde{\mathfrak{g}} = \mathfrak{k}_1 \cap \mathfrak{k}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{p}_2$ is a Cartan decomposition with respect to $\tau_1 \circ \tau_2$, when K_i is the fixed group under τ_i), cf. also [Kol05, Lemma 11.1]. The normal holonomy representation (cf. [BCO03, section 4.2.]). By the Homogeneous Slice Theorem these two representations are orbit-equivalent, therefore equivalent or transitive on an odd dimensional sphere.

In the latter case, let $\mathfrak{a} \subset \nu_0 M$ be a maximal abelian subalgebra and $a \in \mathfrak{a}$ be a regular point, that is $\nu_a P(G,H)(a) = \mathfrak{a}$. Both s-representations give rise to root space decompositions of $\nu_0 M = \mathfrak{p}_1 \cap \mathfrak{p}_2$ with respect to a maximal abelian subalgebra $\mathfrak{a} \subset \nu_0 M$. Since the eigenspaces of $\mathrm{ad}(a)^2$ do not depend on the representation, they are equivalent. □

Definition 4.3. Consider a polar representation of cohomogeneity k on a Hilbert space. Then we call a slice representation at a point p *most singular*, if it is of the same cohomogeneity k. The point p is called a *most singular point.*

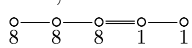
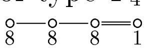
4.1. Possible marked affine Dynkin diagrams

Before dwelling on the calculations of the Dynkin diagrams of the known examples, we give a list of all marked affine Dynkin diagrams which may arise. The results of this section are similar to Theorem 8.7.6. in [PT88], but restricted to homogeneous submanifolds. By [HL99, Theorem A] this determines any marked affine Dynkin diagram of isoparametric submanifolds, for the inhomogeneous ones are of codimension one.

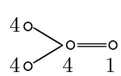
The main a priori restriction is the following: If we consider a subdiagram of the given marked affine Dynkin diagram, i.e. if we omit one or more vertices from the affine Dynkin diagram together with the lines originating from them, this determines by the Slice Theorem [PT88, Theorem 6.5.9.] an finite dimensional isoparametric submanifold of lower rank, more precisely the principal orbit of a singular slice representation. Therefore any subdiagram has to be the Dynkin diagram of some s-representation. In [HPT88] one finds a complete list of the Dynkin diagrams of s-representations, we have summarized the results in Table A.5 in the Appendix.

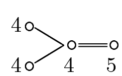
This argument was used by Terng as well, but we exclude some diagrams of type $\tilde{C}_2$ by means of the isotropy representation. The result are summarized in Table 4.1 on the next page. Multiplicities given there for higher rank are always possible for lower rank as well.

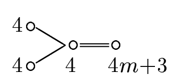
4.1.1. Diagrams with uniform multiplicity. Vertices in a Dynkin diagram joined by a single or triple line have the same multiplicity. Therefore isoparametric submanifolds with diagram $\tilde{A}_k$ $(k > 1)$, $\tilde{D}_k$, $\tilde{E}_k$ $(k = 6, 7, 8)$ and $\tilde{G}_2$ have uniform multiplicity. By omitting a certain vertex we obtain a singular slice representation with diagram A_k, D_k, E_k or G_2, respectively. Therefore the restrictions on the multiplicity are just the same as in finite dimensions, i.e. the multiplicity is 1 or 2, except for $\tilde{A}_k$ where it also might be 4 for any k and 8 for $\tilde{A}_2$. Similarly $\tilde{B}_k$-, $\tilde{C}_k$- and $\tilde{F}_2$-diagrams with uniform multiplicity permit only multiplicity 1 or 2.

4.1.2. Diagrams with at most two different multiplicities — $\tilde{F}_4$, $\tilde{B}_k$ and $\tilde{A}_1$. We start with $\tilde{F}_4$ and assume $m_1 \neq m_2$, the diagram contains a subdiagram of type F_4 one of whose multiplicities is 1, the other 2, 4 or 8. This yields six different diagrams of type $\tilde{F}_4$, which are all valid except  which contains a subdiagram  of type B_4 that is not the Dynkin diagram of an s-representation.

A diagram of type $\tilde{B}_k$ contains one subdiagram of type B_k and one of type D_k, this yields that the only possibilities are $(m, 1)$ and $(2m + 1, 2)$, except $k = 3$, because of $D_3 = A_3$ also multiplicity 4 is allowed. Hence the diagrams



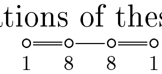
,


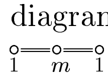
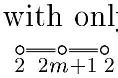
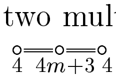
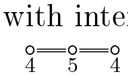
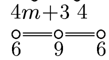
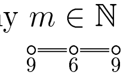
and


may also occur.

For a homogeneous isoparametric hypersurface (i.e. diagram $\tilde{A}_1$) in Hilbert space there are no restrictions on the multiplicities.

4.1.3. Diagrams with three different multiplicities — $\tilde{C}_k$. First assume $k > 3$, then a diagram of type $\tilde{C}_k$ contains two subdiagrams of type B_k. Hence for the vertices in the middle, the only possible multiplicities are 1, 2 and 4. If it is 1 there are no restrictions on the multiplicities at the boundary vertices, if it is 2 they are either 2 or odd, if it is 4 they are 1,5, or $4m + 3$. All combinations of these are possible.

For $k = 3$ there is an additional diagram, namely  , arising from the s-representation of E VI.

Now let $k = 2$, of course all examples for general k occur here as well. Hence we restrict ourselves to the case, when the middle vertex has a multiplicity which is not 1, 2 or 4. All general diagrams with only two multiplicities arise here with interchanged multiplicities, i.e.  ,  ,  for any $m \in \mathbb{N}$ and  . Of the same type but possible only for $k = 2$ is  and  .

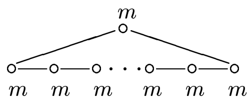
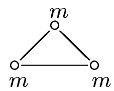
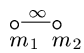
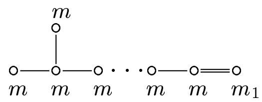
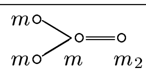
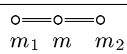
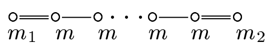
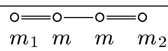
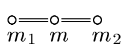
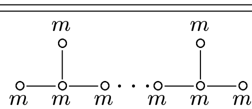
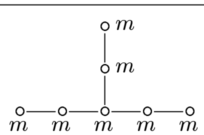
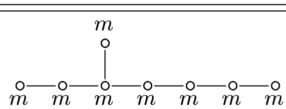
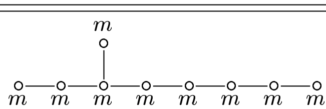
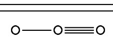
Dynkin Diagram		rank k	m_1	m	m_2
$\tilde{A}_k$		> 2		1,2 or 4	
		2		8	
		1	arb.		arb.
$\tilde{B}_k$		> 3		1	arb.
				2	2 or $2m+1$
		3		4	1,5 or $4m+3$
		2	cf. $\tilde{C}_2$		
$\tilde{C}_k$		> 3	arb.	1	arb.
			2 or $2m+1$	2	2 or $2m+1$
			1,5 or $4m+3$	4	1, 5 or $4m+3$
		3	1	8	1
		2	1	arb.	1
			2	$2m+1$	2
			4	$4m+3$	4
			6	9	2 or 6
			9	6	1 or 9
			2 or 4	5	1 or 4
			1	3	4
$\tilde{D}_k$		> 4		1 or 2	
$\tilde{E}_6$		6		1 or 2	
$\tilde{E}_7$		7		1 or 2	
$\tilde{E}_8$		8		1 or 2	
$\tilde{F}_4$		4	1		1,2,4 or 8
			2		1 or 2
			4		1
$\tilde{G}_2$		2		1 or 2	

TABLE 4.1. Possible marked Dynkin diagrams for homogeneous isoparametric submanifolds of Hilbert space

PROPOSITION 4.4. *Let $M = G \cdot a$ be an infinite dimensional homogeneous isoparametric submanifold with marked Dynkin diagram* $\underset{m_1}{\circ}=\underset{m}{\circ}=\underset{m_2}{\circ}$. *The vertex marked m belongs to two irreducible subdiagrams* $\underset{m_1}{\circ}=\underset{m}{\circ}$ *and* $\underset{m_2}{\circ}=\underset{m}{\circ}$, *that is, two different s-representation. The m-dimensional eigenspaces of these s-representations are of the form $\mathfrak{p}^i_\lambda \oplus \mathfrak{p}^i_{2\lambda}$ for $i = 1, 2$, cf. page 9. Then* $\dim(\mathfrak{p}^1_{2\lambda}) = \dim(\mathfrak{p}^2_{2\lambda})$.

PROOF. We fix an m-dimensional eigenspace, say E_k, of the infinite dimensional manifold, together with its curvature normal v_k. The isotropy group G_a is the principal isotropy group of any singular slice representation (Proposition 2.1 on page 10 holds for any singular point), hence the dimensions of the irreducible modules within E_k of the isotropy representation are determined by the root system of any singular slice representation.

Therefore to prove that the reducibility for both types of slice representations is the same, we have to find two singular points q_i with $E_k \subset \nu_{q_i}(G \cdot q_i)$, such that the effectivized slice representation at q_i is the s-representation with diagram $\underset{m_i}{\circ}=\underset{m}{\circ}$. This is possible since any two eigendistributions associated with non proportional curvature normals may be focalized simultaneously without focalizing any other eigendistribution. Applying this to E_k and an m_i-dimensional eigenspace leads to the point q_i as the focal point of a. $\square$

The proposition excludes such possibilities as $\underset{1}{\circ}=\underset{2m+1}{\circ}=\underset{2}{\circ}$ or $\underset{2}{\circ}=\underset{4m+3}{\circ}=\underset{4}{\circ}$, but we remark that the list in [EH99] of polar representations, that are not s-representations gives rise to two additional examples $\underset{2}{\circ}=\underset{5}{\circ}=\underset{1}{\circ}$ and $\underset{4}{\circ}=\underset{5}{\circ}=\underset{1}{\circ}$, since among those examples is one with diagram $\underset{1}{\circ}=\underset{5}{\circ}$ where the 5-dimensional eigenspace is reducible, cf. Table 5.4 on page 73.

The possible Dynkin diagrams are stated in table 4.1 on the preceding page.

4.2. Actions of type $K_1 = K_2$

We determine the affine marked Dynkin diagrams in the case of a subgroup of type $K \times K$, where K is a symmetric subgroup of G. These actions were studied first by Pinkall and Thorbergsson in [PT90]. To determine the singular slice representations of this class of $P(G,H)$-actions is fairly easy since an explicit description of the eigenspaces is computable without much effort. Together with Proposition 4.1, which yields that one most singular slice representation is the isotropy representation of G/K or the adjoint representation of $G^\sigma = \{g \in G \mid g = \sigma(g)\}$ for the σ–actions, this determines the marked Dynkin diagram.

The eigenspaces of σ–actions where described by Terng in [TER89] for $\sigma = \mathrm{id}$ and in [TER95] for general σ. Here we give the eigenspaces explicitly for the other $K_1 = K_2$ cases, i.e. with simple G.

4.2.1. Actions on a simple Lie group G. Let K be a symmetric subgroup of a compact Lie group G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. Morover let $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$ which is a section of the $P(G, K \times K)$-action. We denote by $\Lambda \subset \mathfrak{a}^*$ the restricted root system with respect to $\mathfrak{a}$, which may be non-reduced. Moreover let

$$\mathfrak{k}_\lambda = \left\{X \in \mathfrak{k} \mid \mathrm{ad}(a)^2 X = \lambda(a)^2 X \text{ for all } a \in \mathfrak{a}\right\}$$
$$\mathfrak{p}_\lambda = \left\{X \in \mathfrak{p} \mid \mathrm{ad}(a)^2 X = \lambda(a)^2 X \text{ for all } a \in \mathfrak{a}\right\},$$

where $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$ and $\mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}$. We choose a regular $a_0 \in \mathfrak{a}$ and define $\Lambda_0 = \{\lambda \in \Lambda \mid \lambda(a_0) = 0\}$ and $\Lambda_+ = \{\lambda \in \Lambda \mid \lambda(a_0) > 0\}$. Then the eigenspaces of $K \cdot a_0$

(that is for a principal orbit of the s-representation of G/K) are given by $E_\lambda = \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$ for any $\lambda \in \Lambda_+$, where $\mathfrak{p}_{2\lambda} = 0$ if $2\lambda \notin \Lambda_+$. See for example [BCO03, Examples 3.2 and 3.4].

To describe the eigenspaces of the $P(G, K \times K)$–action we choose bases $X_1^\lambda, \dots, X_{m_\lambda}^\lambda$ of $\mathfrak{k}_\lambda$ and $Y_1^\lambda, \dots, Y_{m_\lambda}$ of $\mathfrak{p}_\lambda$ such that

$$[a, X_i^\lambda] = -\lambda(a) Y_i^\lambda$$
$$[a, Y_i^\lambda] = \lambda(a) X_i^\lambda.$$

By m_λ we denote the dimension of $\mathfrak{p}_\lambda$. It is then easy to verify that the curvature normals are given by (cf. [PT90])

$$v_{\lambda,n}(a) = -\frac{\lambda}{\lambda(a) + n} \qquad \text{for } a \in \mathfrak{a}, n \in \mathbb{N}, \lambda \in \Lambda_+.$$

Note that $v_{\lambda,n} = v_{2\lambda,2n}$ for non reduced roots. Let

$$\tilde{E}_{\lambda,n} = \operatorname{span}\left\{\vartheta \mapsto \cos n\vartheta Y_i^\lambda - \sin n\vartheta X_i^\lambda \mid i = 1, \dots, m_\lambda\right\},$$

then the eigenspaces are given by $E_{\lambda,2n} = \tilde{E}_{\lambda,n} \oplus \tilde{E}_{2\lambda,2n}$ and $E_{\lambda,2n+1} = \tilde{E}_{2\lambda,2n+1}$ if λ is not reduced and $E_{\lambda,n} = \tilde{E}_{\lambda,n}$ if λ is reduced.

The eigenspace associated with the eigenvalue 0 is given by

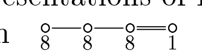
$$E(0) = \operatorname{span}\left\{\vartheta \mapsto \cos n\vartheta K_i, \ \vartheta \mapsto \sin n\vartheta H_i \mid n \in \mathbb{N}_0\right\},$$

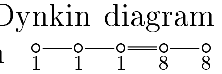
where $\{K_i\}$ is a basis of $\mathfrak{k}_0$ and $\{H_i\}$ is a basis of $\mathfrak{p}_0$ and therefore $E(0)$ is always infinite dimensional.

The Dynkin diagrams of the s-representation may be found in Table A.5 on page 81, the affine Dynkin diagram of the associated $P(G, K \times K)$–action has to contain that diagram as a subdiagram. Remember that the cohomogeneity of both actions is equal. It is therefore true that isoparametric submanifolds arising from an s-representation with Dynkin diagram A_k, E_k, F_4 or G_2 have a diagram of type $\tilde{A}_k, \tilde{E}_k, \tilde{F}_4$ or $\tilde{G}_2$ respectively. The multiplicities stay the same, except $\tilde{F}_4$ with two different multiplicities, where it is a priori not clear which multiplicity belongs to the additional vertex. We will solve this case later. All results (affine Dynkin diagrams and slice representations) may be found in the Tables A.1 to A.4 in the Appendix.

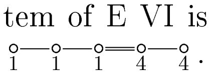
By the description of the eigenspaces of the $P(G, H)$–action we know that no new families arise, which excludes the possibility of the finite dimensional action having a D_k-diagram and the corresponding infinite dimensional isoparametric submanifold having a diagram of type $\tilde{B}_k \supset D_k$, therefore it has a $\tilde{D}_k$-diagram with the same multiplicity. Note that this is not true for the σ–actions.

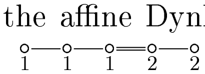
What remains are the cases of $\tilde{F}_4$-diagrams with two multiplicities and of B_k-diagrams, where we have to determine whether the $P(G, H)$–action has $\tilde{B}_k$- or $\tilde{C}_k$-diagram and in the latter case what the multiplicity of the new vertex is.

We start with $\tilde{F}_4$ which only contains three examples (corresponding multiplicities in brackets): the s-representations of E II(1,2), E VI(1,4) and E IX(1,8). First observe that a Dynkin diagram 

does not exists and therefore E IX has to have diagram 

.

Since $(\mathrm{E}_8, \mathrm{E}_7)$ and $(\mathrm{E}_7, \mathrm{SO}'(12))$ are both symmetric pairs, the reduced root system of E VI is contained in that of E IX and the affine Dynkin diagram of E VI is 

. The same argument shows that 

is the diagram of E II.

Next we consider the B_k cases, which are the Grassmannians A III–III, BD I–I and C II–II (which will be solved in the subsection 4.4.1 on page 47), D III(1,4) or

(5,4), E III(9,6) with $k=2$ and E VII(8,1) with $k=3$. For E VII and D III we see immediately that only a $\tilde{C}_k$-diagram is possible since neither A_3 with multiplicity 8 nor D_k with multiplicity 4 are valid Dynkin diagrams for s-representations. Since no new multiplicity occurs for the new vertex, the affine diagrams are $\underset{1}{\circ}=\underset{8}{\circ}-\underset{8}{\circ}=\underset{1}{\circ}$ for E VII and $\underset{1}{\circ}=\underset{4}{\circ}-\underset{4}{\circ}\cdots\underset{4}{\circ}-\underset{4}{\circ}=\underset{1}{\circ}$ for D III (k even). If k is odd, the multiplicity 5 belongs to a non reduced root λ with $\dim(\mathfrak{p}_\lambda)=4$ and $\dim(\mathfrak{p}_{2\lambda})=1$. The description of the eigenspaces yields that within the family $E_{n,\lambda}$ of the $P(G,H)$–action the multiplicities 5 and 1 alternate and therefore the affine Dynkin diagram is $\underset{5}{\circ}=\underset{4}{\circ}-\underset{4}{\circ}\cdots\underset{4}{\circ}-\underset{4}{\circ}=\underset{1}{\circ}$ for D III with odd k. For the same reason $\underset{9}{\circ}=\underset{6}{\circ}=\underset{1}{\circ}$ is the affine Dynkin diagram of E III.

4.2.2. σ–actions. Denote by G^σ the fixed point group $\{g\in G \mid g=\sigma(g)\}$. The cases where the adjoint representation of G^σ has diagram A_k, E_k, G_2 are solved by the same arguments as in the last section, also F_4 since these diagrams have uniform multiplicity 2.

The $P(G,\Delta(G))$-action for $G=\mathrm{SO}(2n)$ has affine diagram $\tilde{D}_k$, which may be easily seen by the description of the eigenspaces given in [Ter89] — there are no families of focal hyperplanes with a 45° angle between them.

We consider the $P(G,\Delta(G))$-actions of $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(n)$ both having finite Dynkin diagram of type B_k. Let $l_{n,\lambda}(a)$ be the focal hyperplanes, then the distance d_λ between adjacent focal hyperplanes $l_{n,\lambda}$ and $l_{n+1,\lambda}$ is $\frac{1}{||\lambda||}$. The new vertex arising in the affine diagram represents a family of focal hyperplanes with the smallest distance d, that is, in this case the familiy associated with the longest root. Therefore the affine Dynkin diagrams of the $P(G,\Delta(G))$-actions of $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(n)$ are $\tilde{B}_n$ and $\tilde{C}_n$. We remark that the finite dimensional actions are not distinguishable by their Dynkin diagram, while this is possible for their infinite dimensional lifts.

Remark. We have proven now that the lifts of the adjoint action of G to a $P(G,\Delta(G))$-action has an affine Dynkin diagram of the same type as the Dynkin diagram of the Lie algebra $\mathfrak{g}$.

By explicit calculations it is possible to find a second most singular slice representation for the σ–actions of $\mathrm{SU}(n)$ and $\mathrm{SO}(2n)$. We conjugate the group $\sigma(G)$ by an appropriate involution J, then the adjoint representation of $G\cap J\sigma(G)J$ is a slice representation of the $P(G,G(\sigma))$-action at some point.

First consider the σ–action on $\mathrm{SO}(2n)$ with $G^\sigma=\mathrm{SO}(2n-1)$. The outer involution σ is given by conjugation with the matrix $\Sigma=\left(\begin{smallmatrix}E & 0\\ 0 & -1\end{smallmatrix}\right)$. Let $J_p=\begin{pmatrix}-E_{2p} & 0\\ 0 & E_{2n-2p}\end{pmatrix}$ for $p=0,\dots,n-1$, then the involution $J\Sigma J$ has fixed point group $G^p=\mathrm{SO}(2p+1)\times\mathrm{SO}(2n-2p-1)$ and the Dynkin diagram of the adjoint action of G^p has two connected components — both having B_k-diagrams. Therefore the affine Dynkin diagram of that action is of type $\tilde{C}_{n-1}$ with uniform multiplicity 2.

The outer involution of the σ–action on $\mathrm{SU}(n)$ is the complex conjugation and $G^\sigma=\mathrm{SO}(n)$. For J we define $\left(\begin{smallmatrix}0 & E_n\\ -E_n & 0\end{smallmatrix}\right)$ on $\mathrm{SU}(2n)$ and $\left(\begin{smallmatrix}0 & E_{n+1}\\ -E_n & 0\end{smallmatrix}\right)$ on $\mathrm{SU}(2n+1)$, then the new fixed point group is $\mathrm{Sp}(n)$ or $\mathrm{Sp}(n)\times\mathrm{U}(1)$ respectively. Both adjoint actions have diagram B_n and therefore the affine Dynkin diagrams are $\tilde{B}_n$ for $\mathrm{SU}(2n)$ and $\tilde{C}_n$ for $\mathrm{SU}(2n+1)$ with uniform multiplicity 2. The diagrams for the σ-actions are listed in Table 4.2 on the next page.

G	SO($2n$)	SU($2n$)	SU($2n+1$)	E_6	Spin(8)
G^σ	SO($2n-1$)	SO($2n$)	SO($2n+1$)	F_4	G_2
Dynkin diagram	$\tilde{C}_{n-1}$	$\tilde{B}_n$	$\tilde{C}_n$	$\tilde{F}_4$	$\tilde{G}_2$

TABLE 4.2. Dynkin diagrams of σ-actions

4.3. Geometry of $K_1 \neq K_2$-Actions

The explicit description of the eigenspaces for these actions is not necessary to determine most of the slice representations, as we will see in the next sections. Nevertheless we will give this description at least for actions with commuting involutions and remark that the only cases where the involutions do not commute are A II–III and D I–III with k odd, D III–III' with n odd and D_4 I–I' with k, l even (k, l, n refer to the dimensions as listed in Table A.1 on page 77), cf. [CON69].

Let G be a simple Lie group and σ and τ commuting involutions with fixed point sets K_1 and K_2 and Cartan decomposition

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{k}_2 \oplus \mathfrak{p}_2 = (\mathfrak{k}_1 \cap \mathfrak{k}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{p}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{k}_2) =: \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Here we used the fact, that the involutions commute. If they do not commute there are additional summands. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}_1 \cap \mathfrak{p}_2$, which is a section of the $P(G, K_1 \times K_2)$ action.

The subspace $\mathfrak{g}_1$ is a Lie algebra, let Λ_1 be its restricted root system with respect to $\mathfrak{a}$, $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ its root spaces just as in the case $K_1 = K_2$, cf. Section 4.2.1 on page 42. This yields eigenspaces $E_{\lambda,n}$ in the same way.

The subspace $\mathfrak{g}_2$ is invariant under $\mathfrak{a}$ in the sense that $[\mathfrak{g}_2, \mathfrak{a}] \subset \mathfrak{g}_2$. Let Λ_2 be the restricted root system of $\mathfrak{g}_2$ with respect to $\mathfrak{a}$ and let $\mathfrak{m}_\lambda \subset \mathfrak{k}_1 \cap \mathfrak{p}_2$ and $\mathfrak{n}_\lambda \subset \mathfrak{p}_1 \cap \mathfrak{k}_2$ the corresponding root spaces. We remark that $\Lambda_2 \subset \Lambda_1 \cup 2 \cdot \Lambda_1$, this is because in any case e is a most singular point with slice representation $G_1/K_1 \cap K_2$, where G_1 is a Lie group with Lie algebra $\mathfrak{g}_1$. To determine the eigenspaces one has to assure that the boundary values are contained in $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively . Therefore we restrict the parameter ϑ to $[0, \frac{\pi}{2}]$.

Let

$$\begin{aligned} X_1^\lambda, \dots, X_{m_\lambda}^\lambda \quad &\text{be a basis of } \mathfrak{k}_\lambda \\ Y_1^\lambda, \dots, Y_{m_\lambda}^\lambda \quad &\text{be a basis of } \mathfrak{p}_\lambda \\ U_1^\lambda, \dots, U_{m_\lambda}^\lambda \quad &\text{be a basis of } \mathfrak{m}_\lambda \\ V_1^\lambda, \dots, V_{m_\lambda}^\lambda \quad &\text{be a basis of } \mathfrak{n}_\lambda \end{aligned}$$

then $\cos(2n)\vartheta Y - \sin(2n)\vartheta X$ and $\cos(2n+1)\vartheta V - \sin(2n+1)\vartheta U$ are tangential vectors. There are four possible types of eigenspaces:

$$\begin{aligned} E_{\lambda,4n} &= \operatorname{span}\left\{\cos(2n)\vartheta Y_i^\lambda - \sin(2n)\vartheta X_i^\lambda, \cos(4n)\vartheta Y_i^{2\lambda} - \sin(4n)\vartheta X_i^{2\lambda}\right\} \\ E_{\lambda,4n+1} &= \operatorname{span}\left\{\cos(4n{+}1)\vartheta V_i^{2\lambda} - \sin(4n{+}1)\vartheta U_i^{2\lambda}\right\} \\ E_{\lambda,4n+2} &= \operatorname{span}\left\{\cos(2n{+}1)\vartheta V_i^{\lambda} - \sin(2n{+}1)\vartheta U_i^{\lambda}, \cos(4n{+}2)\vartheta Y_i^{2\lambda} - \sin(4n{+}2)\vartheta X_i^{2\lambda}\right\} \\ E_{\lambda,4n+3} &= \operatorname{span}\left\{\cos(4n{+}3)\vartheta V_i^{2\lambda} - \sin(4n{+}3)\vartheta U_i^{2\lambda}\right\} \end{aligned}$$

The dimension of the eigenspaces are alternating

$$m_\lambda^1 + m_{2\lambda}^1 \qquad m_{2\lambda}^2 \qquad m_\lambda^2 + m_{2\lambda}^1 \qquad m_{2\lambda}^2,$$

where the upper index denotes the root system. Except m^1_λ any of these numbers may be zero, if $m^2_\lambda = 0$ for any λ we have the special case $K_1 = K_2$.

Since within a family of proportional curvature normals there are at most two different (alternating) multiplicities, if $m^2_{2\lambda} \neq 0$ then $m^1_\lambda = m^2_\lambda$.

The eigenspace of the eigenvalue 0 is given by

$$E(0) = \operatorname{span}\left\{\cos 2n\vartheta K_i,\ \sin 2n\vartheta H_i,\ \cos(2n+1)\vartheta M_i,\ \sin(2n+1)\vartheta N_i \mid n \in \mathbb{N}_0\right\},$$

where $\{K_i\}$ is a basis of $\mathfrak{k}_0$ and $\{H_i\}$ of $\mathfrak{p}_0$, $\{M_i\}$ of $\mathfrak{m}_0$ and $\{N_i\}$ of $\mathfrak{n}_0$ and therefore $E(0)$ is always infinite dimensional.

Next we answer the following question: If we have a given marked Dynkin diagram arising from a Hermann action with commuting involutions (including the $K_1 = K_2$-actions), how many possibilities for $m^1_\lambda, m^1_{2\lambda}, m^2_\lambda, m^2_{2\lambda}$ are there?

For a vertex associated with a family of eigenspaces of the same dimension, especially for a vertex which is joined by only single lines to all neighboring vertices, there are two possibilities. The root λ is always reduced and either $m^2_\lambda = 0$ or $m^2_\lambda = m^1_\lambda$.

A pair of examples for this type are the $P(G, \Delta(G))$-action for $\mathrm{SU}(n)$ ($m^2_\lambda = 0$ since it is of type $K_1 = K_2$) and the action of type A I–II, which has the same diagram (cf. Section 4.4).

Now consider a vertex associated with a family of eigenspaces with alternating dimensions, i.e. a boundary vertex joined by a doule line to its neighboring vertex of a $\tilde{C}_k$-diagram or a vertex of a diagram $\tilde{A}_1$.

- $\underset{m}{\circ}\overset{\infty}{-\!\!-}\underset{\tilde m}{\circ}$: Since the roots are restricted, the only possibility for the multiplicities are $m^1_\lambda = m$ and $m^2_\lambda = \tilde m$. If $m = \tilde m$ then also $m^2_\lambda = 0$ is possible, this is precisely the difference between principal isotropy group $\mathrm{SO}(m)$ and $\mathrm{SO}(m) \times \mathrm{SO}(m)$.
- $\underset{2m+1}{\circ}\overset{\infty}{-\!\!-}\underset{1}{\circ}$ or $\underset{4m+3}{\circ}\overset{\infty}{-\!\!-}\underset{3}{\circ}$ or $\underset{15}{\circ}\overset{\infty}{-\!\!-}\underset{7}{\circ}$: Either $\lambda \notin \Lambda_2$, i.e. $m^1_\lambda = 2m$ and $m^1_{2\lambda} = 1$ (analogous for the other dimensions) or $m^1_\lambda = m^2_\lambda$ and $m^1_{2\lambda} = m^2_{2\lambda}$.
- $\underset{2m+1}{\circ}\overset{\infty}{-\!\!-}\underset{2\tilde m+1}{\circ}$ or $\underset{4m+3}{\circ}\overset{\infty}{-\!\!-}\underset{4\tilde m+3}{\circ}$: Here $m^1_\lambda = 2m$, $m^2_\lambda = 2\tilde m$ and $m^1_{2\lambda} = m^2_{2\lambda} = 1$ (analogous for the other dimensions). Among the $P(G, H)$-action the latter case does not arise.
- $\underset{2m+1}{\circ}\overset{\infty}{-\!\!-}\underset{\tilde m}{\circ}$ or $\underset{4m+3}{\circ}\overset{\infty}{-\!\!-}\underset{\tilde m}{\circ}$ or $\underset{15}{\circ}\overset{\infty}{-\!\!-}\underset{\tilde m}{\circ}$: Here $m^1_\lambda = m^2_\lambda = 2m$, $m^1_{2\lambda} = 1$ and $m^2_{2\lambda} = \tilde m$ (analogous for the other dimensions).

Remark. The case of a family of eigenspaces with alternating dimensions and both types are reducible with different dimensions of the smaller space (e.g. $\underset{4m+3}{\circ}\overset{\infty}{-\!\!-}\underset{5}{\circ}$, this is the only example of such a hypersurface which belongs to a whole family of isoparametric submanifolds of growing codimension) is only possible if the involutions do not commute, or the action is not of Hermann type.

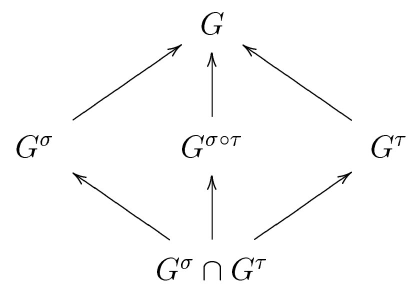
In the next sections we continue the calculation of the singular slice representations and affine marked Dnykin diagrams, starting with the $P(G, H)$-actions arising from Hermann actions. In Section 4.6 on page 57 we study the exceptional actions of cohomogeneity one.

4.4. Actions on the Classical Lie Groups

In this section we start to determine the slice representations of Hermann actions of type $K_1 \neq K_2$. Note that one most singular slice representation of any such action is listed in [Kol05, Table 5].

As for the σ–actions it is here possible to calculate most singular slice representation explicitly. Sometimes the following proposition is useful, cf. [Kol02, Proposition 3.3]:

Proposition 4.5. *Let G be a compact Lie group, σ and τ different, commuting involutions of G. Then we have the following diagram, where all arrows denote inclusions of symmetric subgroup:*

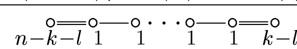
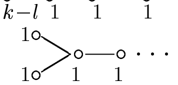


From such a diagram one can read off by Proposition 4.1 a slice representation of all three Hermann actions arising, e.g.: the s-representation of $G^{\sigma\circ\tau}/G^\sigma \cap G^\tau$ is a slice representation of the $P(G, H)$-action with $H = G^\sigma \times G^\tau$.

We will use this proposition by considering a known slice representation, say the s-representation of K'/H', of a Hermann action $(G, K_1 \times K_2)$, draw the associated diagram and read off the slice representations for $(G, K_1 \times K')$ or $(G, K' \times K_2)$. Thereby one has to assure, that K' is a symmetric subgroup of G, i.e. we have to choose an appropriate (most singular) slice representation. In many cases this will be a reducible but most singular slice representation of a $K_1 = K_2$–type action which then yields the slice representation of a $K_1 \neq K_2$–action.

4.4.1. Slice Representations of the actions A III–III, BD I–I, C II–II. We focus on the real case BD I–I, the complex and quaterionic case may be treated in an analogous way. Therefore consider an $P(G, H)$-action with $G = \mathrm{SO}(n)$ and $H = \big(\mathrm{SO}(k) \times \mathrm{SO}(n)\big) \times \big(\mathrm{SO}(l) \times \mathrm{SO}(n-l)\big)$ where we assume $k \leq l \leq \frac{n}{2}$. Let $(A, B) \in \mathrm{SO}(k) \times \mathrm{SO}(n-k)$ be embedded in $\mathrm{SO}(n)$ in the usual way $(A, B) \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. For the first slice representation we embed $\mathrm{SO}(l) \times \mathrm{SO}(n-l)$ in the same manner, while for the second one we use $(A, B) \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & A \end{smallmatrix}\right)$. In both cases the point e turns out to be most singular, and the (irreducible) slice representation at this point is easily calculated to be:

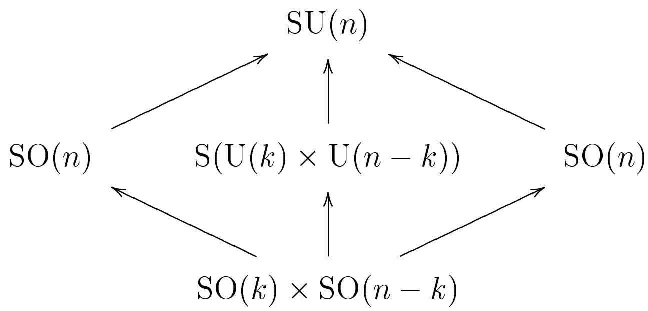
Action	first slice represention	second slice represention
A III–III	$\mathrm{SU}(n+k-l)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-l))$	$\mathrm{SU}(k+l)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(l))$
BD I–I	$\mathrm{SO}(n+k-l)/\mathrm{SO}(k) \times \mathrm{SO}(n-l)$	$\mathrm{SO}(k+l)/\mathrm{SO}(k) \times \mathrm{SO}(l)$
C II–II	$\mathrm{Sp}(n+k-l)/\mathrm{Sp}(k) \times \mathrm{Sp}(n-l)$	$\mathrm{Sp}(k+l)/\mathrm{Sp}(k) \times \mathrm{Sp}(l)$

Therefore the affine Dynkin diagram (for BD I–I) is  , where multiplicity 0 at one or both ends of the diagram denotes  . We will use this convention throughout the rest of the chapter.

4.4.2. Slice Representations of Hermann actions on Grassmannians. In this section we deal with the remaining hyperpolar actions on real, complex and quaterionic Grassmannian manifolds of k-dimensional linear subspaces of $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. To determine slice representations we use the known slice representations of Hermann $K_1 = K_2$-actions and the actions of the last subsection.

First consider **A I–III**, we remark that for **C I–II** the same arguments are valid. Consider the action A I–I and its reducible most singular slice representation of Typ

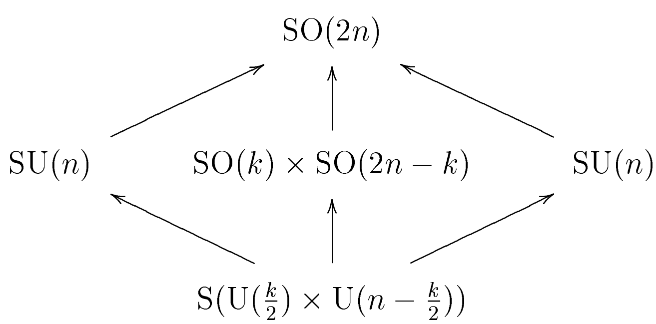
$A_k + A_{n-k} \subset \tilde{A}_n$, which gives the following diagram.



From this we can read off the s-representation of the symmetric space $\mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ as a slice representation of A I–III and $\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k))$ as a slice representation of C I–II, respectively. In terms of proposition 4.1 on page 39, this slice representation occurs, when we embed $\mathrm{SO}(n)$ and $\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k))$ in the standard way in $\mathrm{SU}(n)$, their intersection then being $\mathrm{SO}(k)\times\mathrm{SO}(n-k)$.

We claim that the second slice representation is the s-representation of $\mathrm{Sp}(k)/\mathrm{U}(k)$ for A I–III and the adjoint action of $\mathrm{Sp}(k)$ for C I–II. This can be proven by an appropriate embedding of $\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k))$ or $\mathrm{Sp}(k)\times\mathrm{Sp}(n-k)$, respectively. To be precise, we embed $\mathrm{U}(k)\subset V$ and $\mathrm{U}(n-k)\subset V^{\perp}$, where V is the k–dimensional linear subspace of $\mathbb{C}^n$ given by $\mathrm{span}\{e_1 - ie_{k+1}, \ldots, e_k - ie_{2k}\}$ in the complex case. Then the intersection of $\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k))$ embed that way with a standardly embedded $\mathrm{SO}(n)$ is $\mathrm{U}(k)$ and one can calculate explicitly, that the slice representation is the one we have stated above. This proves, that the affine marked Dynkin diagram is $\underset{1}{\circ}=\underset{1}{\circ}-\underset{1}{\circ}\cdots\underset{1}{\circ}-\underset{1}{\circ}=\underset{n-k}{\circ}$ or $\underset{2}{\circ}=\underset{2}{\circ}-\underset{2}{\circ}\cdots\underset{2}{\circ}-\underset{2}{\circ}=\underset{2(n-k)+1}{\circ}$ respectively.

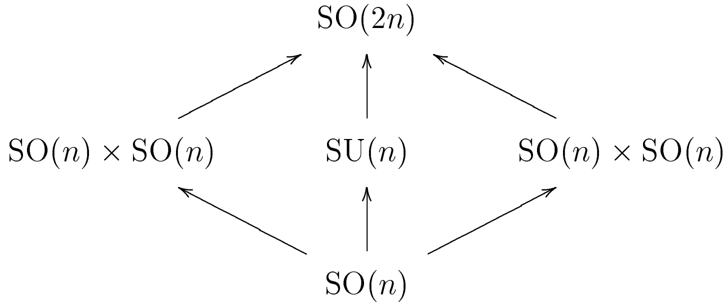
The other pair of actions on Grassmannians is **D I–III** and **A II–III**, which can be treated simultaneously. Therefore we restrict our attention to the action of type D I–III and start with the case $k > 2$ even and a reducible slice representation of D III–III of type $C_{\frac{k}{2}} + C_{\lfloor \frac{n+1}{2} \rfloor - \frac{k}{2}}$.



The above diagram shows that $\underset{1}{\circ}-\underset{1}{\circ}-\underset{1}{\circ}\cdots\underset{1}{\circ}-\underset{1}{\circ}=\underset{2(n-k)+1}{\circ}$ is the Dynkin diagram of a most singular slice representation of DI–III, $\underset{4}{\circ}-\underset{4}{\circ}-\underset{4}{\circ}\cdots\underset{4}{\circ}-\underset{4}{\circ}=\underset{4(n-k)+3}{\circ}$ of A II–III.

Another slice representation may be found for the special case $k = n$, if we consider the most singular slice representation of a certain action of type BD I–I with diagram

$A_{n-1} \subset \tilde{C}_n$.



Together this leads to the conjecture, that D I–III has affine Dynkin diagram $\tilde{B}_{\frac{k}{2}}$ with multiplicities $(2(n-k)+1, 1)$ for k even and $\tilde{C}_{\frac{k-1}{2}}$ with multiplicities $(2(n-k)+1, 1, 1)$ for k odd, where the most singular slice representation, which arises by omitting the vertex marked $2(n-k)+1$, is the adjoint action of $\mathrm{SO}(\lfloor \frac{k}{2} \rfloor)$. This can be proven by an explicit calculation of the slice representation at the (most singular) point e with standard embedding of the symmetric subgroups on one hand (which yields the slice representation found above) and on the other hand with embedding $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ such that $\mathrm{SO}(k) \subset V \subset \mathbb{R}^{2n}$, where $V = \mathrm{span}\{e_1, \ldots, e_{k-\lfloor \frac{k}{2} \rfloor}, e_{n+1}, \ldots, e_{n+\lfloor \frac{k}{2} \rfloor}\}$.

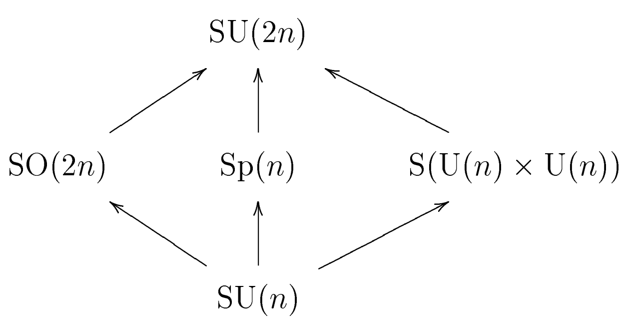
In the case of the hypersurface D I–III with $k = 2$ it is not difficult to compute the eigenspaces with help of the description in Section 4.3 on page 45 and see that there is only one type of most singular slice representation, i.e. the affine Dynkin diagramm is $\underset{2n-1}{\circ} \overset{\infty}{—} \underset{2n-1}{\circ}$.

We finish this section by a summary of the results in the following table.

Action	first slice represention	second slice represention
A I–III	$\mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k)$	$\mathrm{Sp}(k)/\mathrm{U}(k)$
A II–III	$\mathrm{Sp}(n+k)/\mathrm{Sp}(n) \times \mathrm{Sp}(k)$	$\mathrm{SO}(2k)/\mathrm{SU}(k)$
D I–III	$\mathrm{SU}(n-\frac{k}{2}+\lfloor \frac{k}{2} \rfloor)/\mathrm{S}(\mathrm{U}(\lfloor \frac{k}{2} \rfloor) \times \mathrm{U}(n-\lceil \frac{k}{2} \rceil))$	$\mathrm{SO}(\lfloor \frac{k}{2} \rfloor) \times \mathrm{SO}(\lfloor \frac{k}{2} \rfloor)/\mathrm{SO}(\lfloor \frac{k}{2} \rfloor)$
C I–II	$\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k))$	$\mathrm{SO}(2k+1) \times \mathrm{SO}(2k+1)/\mathrm{SO}(2k+1)$

4.4.3. Slice Representations of A I–II, D III–III', $\mathbf{D_4}$ I–I'. In this section we determine the affine Dynkin diagrams of the remaining actions on the classical Lie groups.

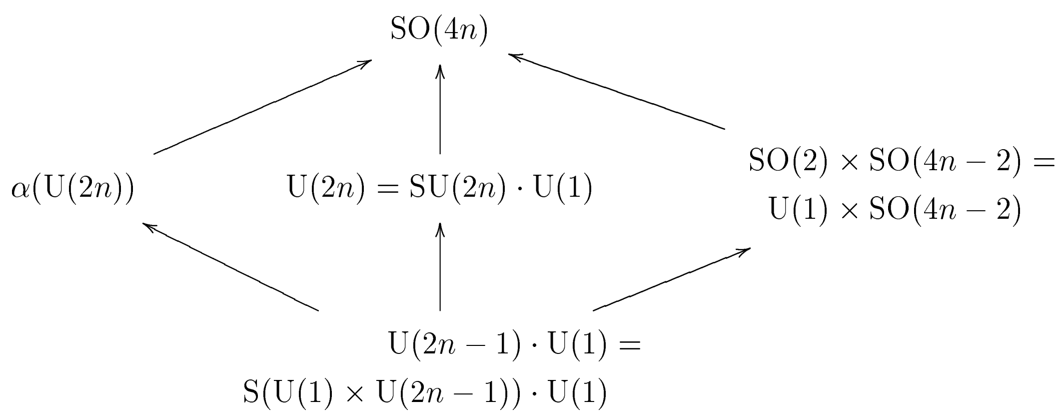
We continue with the action **A I–II**, obtaining a slice representation by a most singular slice representation of A I–III (for even dimension $2n$ and $k = n$).



Therefore A I–II has a most singular slice representation of type $\underset{2}{\circ}—\underset{2}{\circ}\cdots\underset{2}{\circ}—\underset{2}{\circ}$ and its affine marked Dynkin diagram is $\tilde{A}_{n-1}$ with multiplicity 2. The action has only one type of most singular orbits.

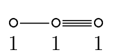
Next we consider the action **D III–III'** arising from the diagram automorphisms α of $\mathrm{SO}(2n)$, that is D III' denotes $\mathrm{SO}(2n)/\alpha(\mathrm{SU}(n))$. First we note that for the action D I–III there is no difference in using $\alpha(\mathrm{U}(n))$ instead of $\mathrm{U}(n)$, the same holds for D III-III' if n is odd, cf. [Kol02, 3.1.1.]. Hence let n be even, the involution α is then given by $\mathrm{diag}(-1, 1, \ldots, 1)$.

The following diagram is given by the special case $k = 2$ of the action D I–III



and yields a reducible slice representation of D III–III' with Dynkin diagram C_{n-1} and multiplicities $(5, 4)$. The affine Dynkin diagram is $\tilde{C}_{n-1}$ with multiplicities $(5, 4, 5)$, which may be seen by embedding $\mathrm{U}(2n)$ standardly and using $\alpha = \begin{pmatrix} -E_{2p+1} & 0 \\ 0 & E_{4n-2p-1} \end{pmatrix}$, then the intersection of $\mathrm{U}(2n)$ and $\alpha(\mathrm{U}(2n))$ is the group $\mathrm{U}(2p+1) \times \mathrm{U}(2n-2p-1)$, where the rank of both groups is odd and the slice representation of D III–III' is of type D $\mathrm{III}(2p+1) \oplus$D $\mathrm{III}(2n-2p-1)$.

The last Hermann action **$\mathbf{D}_4$ I–I'** on the classical groups arises from the order 3 automorphisms τ on $\mathrm{Spin}(8)$ with fixed point group G_2. The only case when this is not equivalent to some Hermann action is $G = \mathrm{Spin}(8)$ and $H = (\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)) \times \tau(\mathrm{Spin}(5) \cdot \mathrm{Spin}(3))$ which is an action of cohomogeneity 2 with one slice representation equivalent to the s-representation $G_2/\mathrm{SO}(4)$, therefore the affine Dynkin diagram is



.

If the column "second slice representation" is left empty, there is only one most singular orbit type.

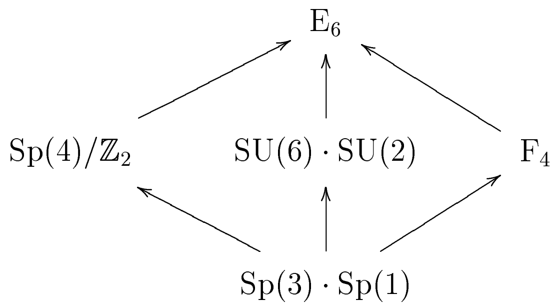
Action	first slice represention	second slice represention
A I–II	$\mathrm{SU}(n-1) \times \mathrm{SU}(n-1)/\mathrm{SU}(n-1)$	
D III–III'	$\mathrm{SO}(2n-2)/\mathrm{U}(n-1)$	
D_4 I–I'	$\mathrm{G}_2/\mathrm{SO}(4)$	$\mathrm{SU}(3)/\mathrm{SO}(3)$

4.5. Actions on the Exceptional Lie Groups

Since explicit calculations are more difficult here (but can be done by using a computer algebra system, e.g. MAPLE, our main tool in this section is Proposition 4.5.

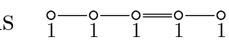
4.5.1. Slice Representations of Hermann Actions on $\mathbf{E_6}$. Let σ and τ denote the commuting outer involutions on E_6 with fixed point groups $\mathrm{Sp}(4)/\mathbb{Z}_2$ and F_4 respectively. Hence $\sigma \circ \tau$ is an inner involution with fixed point group either $\mathrm{Spin}(10) \cdot \mathrm{SO}(2)$ or $\mathrm{SU}(6) \cdot \mathrm{SU}(2)$. Since the only common symmetric subgroup of $\mathrm{Sp}(4)$ and F_4 is

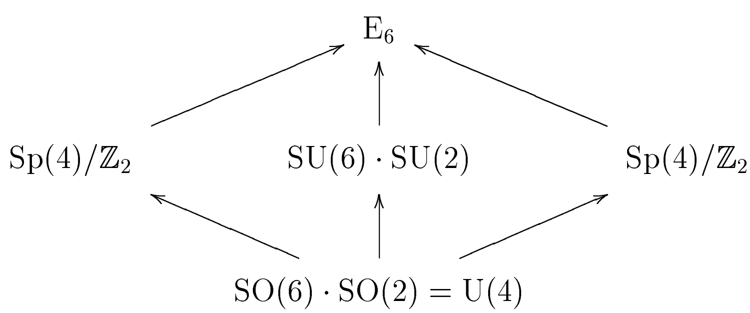
Sp(3) · Sp(1) and this group has to be a symmetric subgroup of the fixed point group of $\sigma \circ \tau$, the only possibility is SU(6) · SU(2). This leads to the following diagram (cf. [Kol02], page 607), where we can read off one slice representation of E I–II, E I–IV and E II–IV:



(4.1)

For the Hermann action **E I–IV** we obtain as slice representation the s-representation of SU(6)/Sp(3) which has Dynkin diagram A_2 with multiplicity 4. Since the only affine Dynkin diagram of rank 2 containing A_2 as a subdiagram is $\tilde{A}_2$, we conclude that the Dynkin diagram of E I–IV is $\tilde{A}_2$ with multiplicity 4. The action has only one type of most singular orbits, i.e only one type of most singular slice representations.

The action **E I–II** has cohomogeneity 4 and one of its slice representation is the s-representation of F_4/Sp(3) · Sp(1), which has diagram F_4 with uniform multiplicity 1. Hence the affine Dynkin diagram of E I–II is  . There are two types of most singular orbits, the second slice representation is the s-representation of Sp(4)/U(4) not that of SO(9)/SO(4) × SO(5) (having the same diagram C_4), which can be seen from the following diagram.



We know that E I–I has diagram $\tilde{E}_6$ with multiplicity 1, hence admits a (most singular reducible) slice representation with diagram $(A_5 + A_1)$, which leads to the above diagram.

The last slice representation which can be read off diagram (4.1) is the s-representation of Sp(4)/Sp(1) × Sp(3) which is a slice representation of the cohomogeneity one action **E II–IV**. Hence one of the multiplicities of the related $\tilde{A}_1$–diagram is $11 = 8+3$. The principal isotropy group of all slice representations has to be Sp(2) · Sp(1) as for Sp(4)/Sp(3) · Sp(1). Reducing SU(6) · SU(2) to SU(6) leads to an orbit equivalent action (cf. [Kol05, Table 1]), whose principal isotropy is Sp(2) $\simeq$ Spin(5). Therefore the second multiplicity may be either 5 or 11. Using Borel-De Siebenthal theory as in [Kol05, Section 10.1.] shows that there exists a singular slice representation of type SO(7)/SO(6).

Now we want to determine the affine Dynkin diagram of the action **E II–III**. Both involutions are inner, so their composition has to be inner, too. We use the known slice represention of E II–II and E III–III to obtain two most singular slice representations of this action, as we did for E I–II. Let us start with E II–II, which has diagram $\tilde{F}_4$ with multiplicities 1 and 2. We need the slice representation with B_4-diagram, i.e. the s-representation of $\mathrm{SO}(10)/\mathrm{SO}(4)\times\mathrm{SO}(6)$ and thus obtain:

$$\begin{array}{ccccc}
 & & \mathrm{E}_6 & & \\
 & \nearrow & \uparrow & \nwarrow & \\
\mathrm{SU}(6)\cdot\mathrm{SU}(2) & & \mathrm{Spin}(10)\cdot\mathrm{SO}(2) & & \mathrm{SU}(6)\cdot\mathrm{SU}(2) \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \mathrm{S}(\mathrm{U}(4)\times\mathrm{U}(2))\cdot\mathrm{SU}(2) = & & \\
 & & \mathrm{Spin}(6)\cdot\mathrm{Spin}(4)\cdot\mathrm{SO}(2) & &
\end{array}$$

Hence we have proven that the s-representation of $\mathrm{SU}(6)/\mathrm{S}(\mathrm{U}(4)\times\mathrm{U}(2))$ with diagram C_2 and multiplicities $(5,2)$ is a most singular slice representation of E II–III.

The action E III–III has diagram $\tilde{C}_2$ with multiplicities $(9,6,1)$, we use its reducible slice representation with diagram (A_1+A_1) for the following diagram (remember that $9=8+1$ belongs to a non-reduced root, hence the related slice representation is the one stated below).

$$\begin{array}{ccccc}
 & & \mathrm{E}_6 & & \\
 & \nearrow & \uparrow & \nwarrow & \\
\mathrm{Spin}(10)\cdot\mathrm{SO}(2) & & \mathrm{SU}(6)\cdot\mathrm{SU}(2) & & \mathrm{Spin}(10)\cdot\mathrm{SO}(2) \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(5))\cdot\mathrm{SO}(2) = & & \\
 & & \mathrm{U}(5)\cdot\mathrm{SO}(2) & &
\end{array}$$

Therefore the affine Dynkin diagram of E II–III is $\underset{4}{\circ}\!=\!\underset{5}{\circ}\!=\!\underset{2}{\circ}$.

Next we determine the diagram of **E I–III**, the first slice representation can be found again with help of a slice representation of E I–I namely that with D_5-diagram.

$$\begin{array}{ccccc}
 & & \mathrm{E}_6 & & \\
 & \nearrow & \uparrow & \nwarrow & \\
\mathrm{Sp}(4)/\mathbb{Z}_2 & & \mathrm{Spin}(10)\cdot\mathrm{SO}(2) & & \mathrm{Sp}(4)/\mathbb{Z}_2 \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \mathrm{Sp}(2)\cdot\mathrm{Sp}(2) = & & \\
 & & \mathrm{Spin}(5)\times\mathrm{Spin}(5) & &
\end{array}$$

The s-representation $\mathrm{Sp}(4)/\mathrm{Sp}(2)\times\mathrm{Sp}(2)$, whose diagram is $\underset{3}{\circ}\!=\!\underset{4}{\circ}$, we found that way, is a most singular slice representation of E I–III, its principal isotropy group is $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$. The other candidates for the second slice representation are therefore $\underset{1}{\circ}\!=\!\underset{3}{\circ}$, $\underset{1}{\circ}\!=\!\underset{4}{\circ}$, $\underset{5}{\circ}\!=\!\underset{4}{\circ}$ or $\underset{4}{\circ}\!=\!\underset{4m+3}{\circ}$. The principal isotropy group of the s-representation with diagram $\underset{5}{\circ}\!=\!\underset{4}{\circ}$ and $\underset{4}{\circ}\!=\!\underset{4m+3}{\circ}$ for $m > 0$ are larger than $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, which excludes these possibilities. For the others we check whether they fulfills the necessary condition for the dimensions

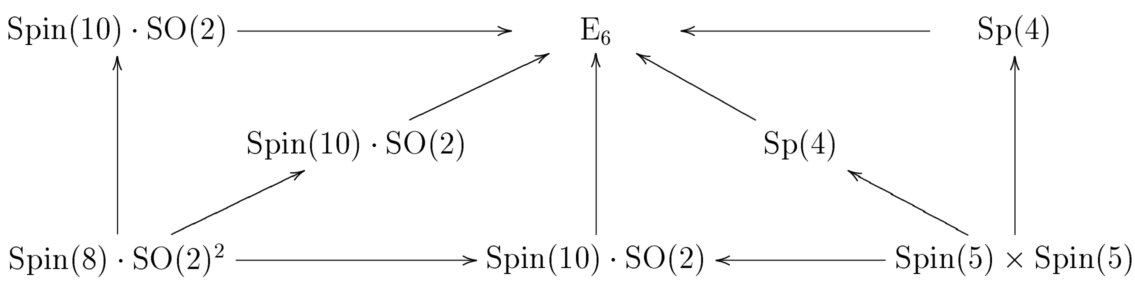
$$\dim G^{\sigma\tau} - 2\dim(K_1 \cap K_2) = \dim G - \dim(K_1 \times K_2),$$

which is a consequence of Proposition 4.1 (cf. [Kol02, page 606]). The right hand side of the equation, which is independent of the embedding of the K_i, in this case is $78 - (36+46) = -4$. This excludes $\underset{1}{\circ}\!=\!\underset{4}{\circ}$, that is the s-representation of $\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{SO}(6)$, since the left hand side is then $28 - 2(1+15) = -2$. The rank of $\mathrm{SO}(2) \times \mathrm{SO}(6)$ is 4, hence it can not be enlarged by trivially acting $\mathrm{SO}(2)$-factors in order to achieve -4 on the left hand side. By similar arguments we can exclude the slice representation $\underset{4}{\circ} + \underset{4}{\circ}$, and for this reason $\underset{4}{\circ}\!=\!\underset{3}{\circ}\!=\!\underset{4}{\circ}$ is not the affine marked Dynkin diagram of E I-III.

Note that we will prove in the next chapter, that in fact there exists no isoparametric submanifold whose diagram is $\underset{4}{\circ}\!=\!\underset{3}{\circ}\!=\!\underset{4}{\circ}$, cf. Section 5.4 on page 72.

Therefore the marked affine Dynkin diagram of the action E I–III is either $\underset{3}{\circ}\!=\!\underset{4}{\circ}\!=\!\underset{3}{\circ}$ or $\underset{1}{\circ}\!=\!\underset{3}{\circ}\!=\!\underset{4}{\circ}$, the equation above is fulfilled for any of the most singular slice representations. Note that the slice representation associated with $\underset{1}{\circ}\!=\!\underset{3}{\circ}$ is $\mathrm{SO}(7) \times \mathrm{SO}(3)/\mathrm{SO}(2) \times \mathrm{SO}(3) \times \mathrm{SO}(5)$ whose principal isotropy group is $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$.

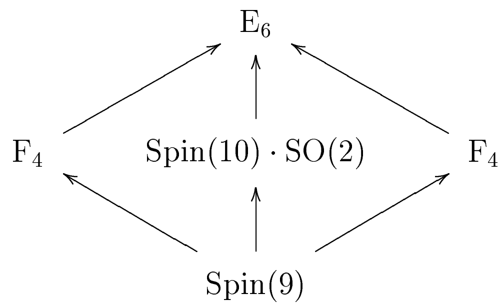
The following diagram is a combination of the above diagram, together with the diagram arising from the slice representation of E III-III with diagram $\underset{1}{\circ}\!=\!\underset{6}{\circ}$:



Choose a fixed root system Λ of $\mathfrak{e}_6$ with positive roots Λ_+. Consider the outer involution τ with fixed point algebra $\mathfrak{sp}(4)$, that is, τ maps any root α to $-\alpha$. On the other hand the root system of $\mathfrak{spin}(10) \oplus \mathfrak{so}(2)$ is a subset Λ_i of the root system of $\mathfrak{e}_6$. The associated inner involution σ_i is identity on the roots in Λ_i and on the maximal torus, and $-\mathrm{id}$ on the roots in $\Lambda \setminus \Lambda_i$. Therefore any such σ_i commutes with τ. To derive the above diagram, where all occurring involutions commute, we choose the involutions σ_i, with $G^{\sigma_i} = \mathrm{Spin}(10) \cdot \mathrm{SO}(2)$, such that their intersection is $\mathrm{Spin}(8) \cdot \mathrm{SO}(2)^2$ (denote by thick lines in the following picture)

This proves that the $P(G,H)$-action in the second row (whose diagram is $\underset{3}{\circ}\!=\!\underset{1}{\circ}\!=\!\underset{3}{\circ}$), is contained totally geodesic in E I–III, by the explicit description of the eigenspaces in Section 4.3 on page 45, E I-III has to contain one-dimensional eigenspaces as well. Therefore its affine marked Dynkin diagram is $\underset{1}{\circ}\!=\!\underset{3}{\circ}\!=\!\underset{4}{\circ}$.

The only remaining Hermann action on E_6 is cohomogeneity one action **E III–IV**, we use the rank-1 slice representation of E IV–IV with multiplicity 8 to obtain:



Since the principal isotropy group of $\mathrm{F}_4/\mathrm{Spin}(9)$ is Spin(7) the only other possible slice representation of E III–IV is the s-representation of SO(9)/SO(8) (with principal isotropy group SO(7)), hence the affine Dynkin diagram is $\tilde{A}_1$ with multiplicities $(15, 15)$ or $(15, 7)$. With help of the description of the eigenspaces in section 4.3 on page 45 and calculation of the dimension the second possibility can be excluded in the following way:

Let $K_1 = \mathrm{Spin}(10) \cdot \mathrm{SO}(2)$ and $K_2 = \mathrm{F}_4$, embedded as for the above diagram. Then the dimension of the spaces $\mathfrak{k}_1 \cap \mathfrak{p}_1 = \mathfrak{spin}(9)$, $\mathfrak{k}_1 \cap \mathfrak{p}_2$, $\mathfrak{k}_2 \cap \mathfrak{p}_1$ and $\mathfrak{k}_2 \cap \mathfrak{p}_2$ are

$\cap$	$\mathfrak{k}_2$	$\mathfrak{p}_2$	Σ
$\mathfrak{k}_1$	36	10	46
$\mathfrak{p}_1$	16	16	32
Σ	52	26	78

The root system $\{\lambda, 2\lambda\} \supset \Lambda_2$ fulfills

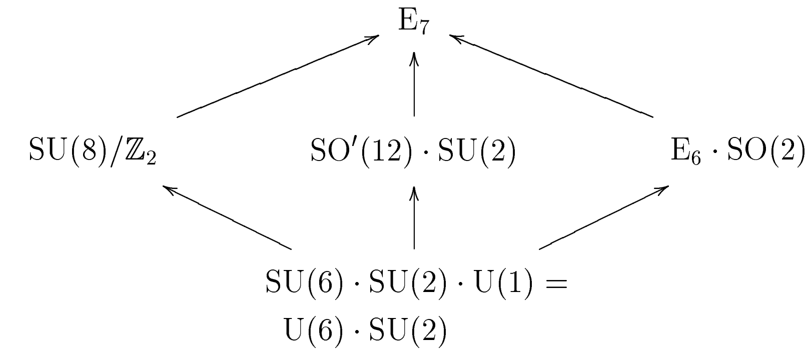
$$m_\lambda^2 + m_{2\lambda}^2 \leq \min\{\dim(\mathfrak{k}_1 \cap \mathfrak{p}_2), \dim(\mathfrak{k}_2 \cap \mathfrak{p}_1)\} = 10$$

and we already know that $m_\lambda^1 = 8$ and $m_{2\lambda}^1 = 7$. Assume that the second singular slice representation is SO(9)/SO(8), then $m_\lambda^2 = 8$ and $m_{2\lambda}^2 = 7$, which contradicts the above inequality. Hence the diagram is $\underset{15}{\circ}\overset{\infty}{-}\underset{15}{\circ}$, i.e. $m_\lambda^2 = 8$ and $m_{2\lambda}^2 = 0$.

Finally we summarize the obtained slice representations of Hermann actions of type $K_1 \neq K_2$ on E_6 in the following table.

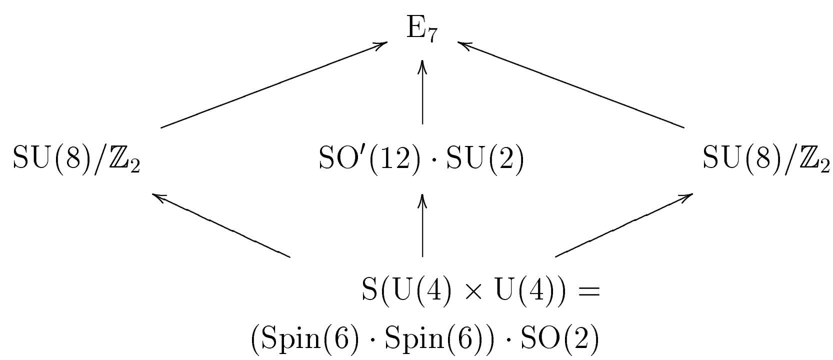
Action	first slice representation	second slice representation
E I–II	$\mathrm{F}_4/\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$	$\mathrm{Sp}(4)/\mathrm{U}(4)$
E I–III	$\mathrm{Sp}(4)/\mathrm{Sp}(2) \times \mathrm{Sp}(2)$	$\mathrm{SO}(7)/\mathrm{SO}(2) \times \mathrm{SO}(5)$
E I–IV	$\mathrm{SU}(6)/\mathrm{Sp}(3)$	
E II–III	$\mathrm{SO}(10)/\mathrm{U}(5)$	$\mathrm{SU}(6)/\mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(2))$
E II–IV	$\mathrm{Sp}(4)/\mathrm{Sp}(1) \times \mathrm{Sp}(3)$	$\mathrm{SO}(7)/\mathrm{SO}(6)$
E III–IV	$\mathrm{F}_4/\mathrm{Spin}(9)$	

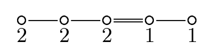
4.5.2. Slice Representations of Hermann Actions on E_7. The three involutions on E_7 are all inner. From the diagram (4.2) on the next page one most singular slice representation of any of the Hermann actions E V–VI, E V–VII and E VI–VII can be read off. (The existence of this diagram can be proven by the same methods as were used to determine the diagram of E I–III: The action E V–VII is contained totally geodesic in E VIII–IX, whose affine marked Dynkin diagram we will determine in the next subsection.)



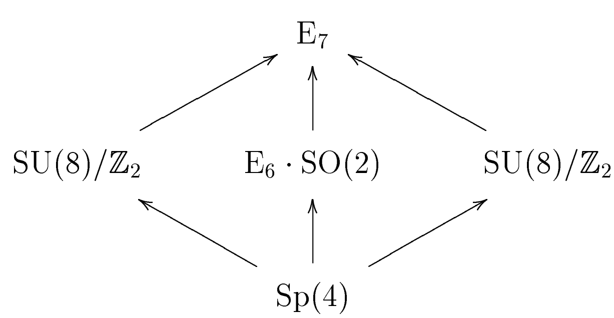
(4.2)

For a second most singular slice representation of the cohomogeneity-4 action **E V–VI** we consider a rank-6 slice representation of the action E V–V, namely that with diagram $D_6 \subset \tilde{E}_7$.



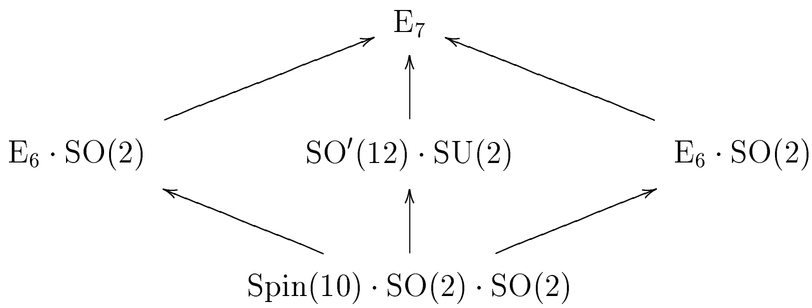
Hence the affine marked Dynkin diagram we are looking for is  .

Again with help of a slice representation of E V–V (with diagram E_6), we obtain the Dynkin diagram A_3 with multiplicity 4 to be a subdiagram of the affine marked Dynkin diagram of **E V–VII**:



Together with the subdiagram from (4.2), we conclude that the $P(G, H)$-action E V–VII has a affine marked Dynkin diagram of type $\tilde{B}_3$ with multiplicities $(1, 4)$.

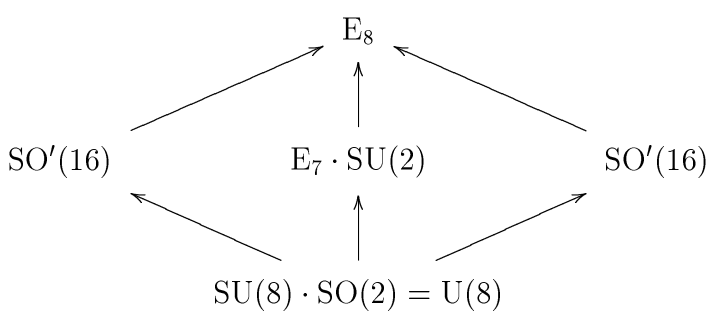
It remains to find a second slice representation of **E VI–VII**, which is done by means of E VII–VII and its slice representation of type $\underset{1}{\circ}\!=\!\underset{8}{\circ} \oplus \underset{1}{\circ} \subset \underset{1}{\circ}\!=\!\underset{8}{\circ}\!-\!\underset{8}{\circ}\!=\!\underset{1}{\circ}$:



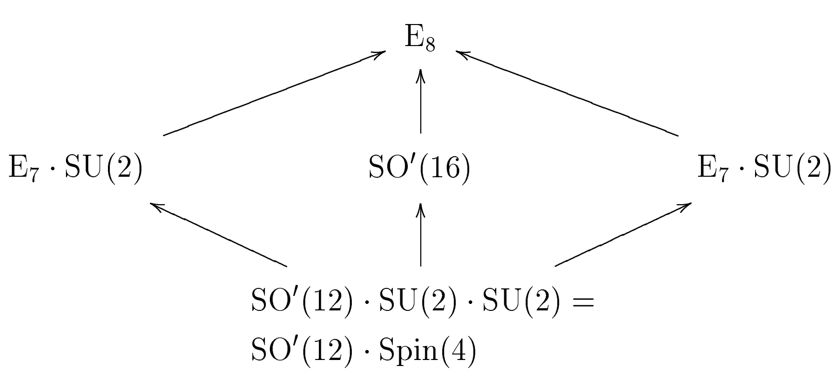
Hence we have proven, that the marked affine Dynkin diagramm of E VI–VII is $\underset{6}{\circ}\!=\!\underset{9}{\circ}\!=\!\underset{2}{\circ}$. The following table contains the most singular slice representations found in this section.

Action	first slice represention	second slice represention
E V–VI	$E_6/SU(6)\cdot SU(2)$	$SU(8)/S(U(4)\times U(4))$
E V–VII	$SO(12)/U(6)$	$SU(8)/Sp(4)$
E VI–VII	$E_6/Spin(10)\cdot SO(2)$	$SU(8)/S(U(6)\times U(2))$

4.5.3. Slice Representations of Hermann Actions on E_8. The group E_8 has only two symmetric subgroups, hence we only have to consider the action **E VIII–IX**. First we use the slice representation of E VIII–VIII belonging to the subdiagram $E_7 + A_1$ of $\tilde{E}_8$, namely:



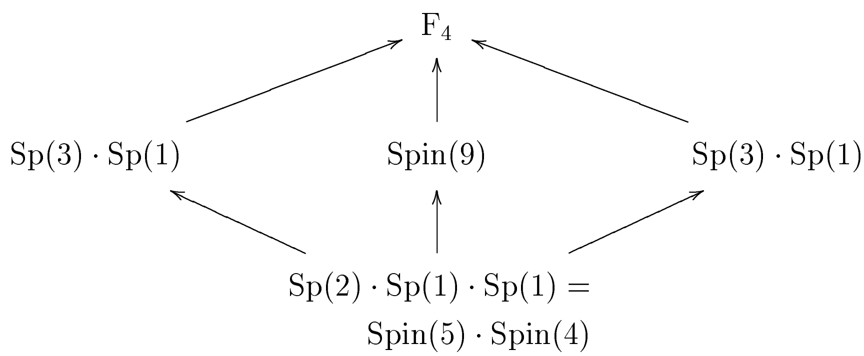
The other slice representation may be obtained from the following diagram, which arises from the slice representation with diagram $\underset{1}{\circ}\!-\!\underset{1}{\circ}\!-\!\underset{1}{\circ}\!=\!\underset{8}{\circ} \subset \underset{1}{\circ}\!-\!\underset{1}{\circ}\!-\!\underset{1}{\circ}\!=\!\underset{8}{\circ}\!-\!\underset{8}{\circ}$ of E IX–IX:



Therefore the two most singular orbits of E VIII–IX are described by:

Action	first slice represention	second slice represention
E VIII–IX	$E_7/SO'(12) \cdot SU(2)$	$Spin(16)/SU(8)$

4.5.4. Slice Representations of Hermann Actions on $\mathbf{F_4}$. Here we have to obtain the two multiplicities of the cohomgeneity one action **F I–II**. One, namely $7 = 4 + 3$, might be easily read off the following diagram which is determined by the slice representation of F I–I with Dynkin diagram $C_4 \subset \tilde{F}_4$ and uniform multiplicity 1:



It is not possible to determine the second slice representation of F I–II with help of the action F II–II. But we can use the principal isotropy group $Sp(1)^3$ of the known slice representation to obtain some restrictions: Since one of those Sp(1)-factors acts trivially, it has to act effectively on the other eigenspaces. It might act as SO(3), then the second multiplicity is 3, as SO(4) with multiplicity 4 or as $Sp(1)(\times Sp(1))$ then the second multiplicity is 7. Observe that it is not possible that it acts as SU(2) by Remark(3) on page 46, since the involutions commute.

Similar arguments as for E III–IV exclude multiplicity 3 and 4, since $\dim(\mathfrak{k}_1 \cap \mathfrak{p}_2) = 8$. By the description of the eigenspaces in Section 4.3 on page 45 $m_\lambda^2 = 4$, and the rank of $K_1 = 4$, therefore $\dim(\mathfrak{k}_1 \cap \mathfrak{p}_2)_0 \geq 3$, that is $m_{2\lambda}^2 = 0$ or 1.

Action	first slice represention	second slice represention
F I–II	$Sp(3)/Sp(2) \cdot Sp(1)$	

4.6. Cohomogeneity one actions

In this section we describe slice representations and Dynkin diagrams of the cohomogeneity one actions which are not Hermann actions.

4.6.1. Actions arising from rank-2 symmetric spaces. Let G/K be a semi-simple symmetric space of rank two, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition and n the dimension of $\mathfrak{p}$. Moreover let $\rho : K \to SO(n)$ be equivalent to the isotropy representation of G/K, that is, $\rho(K)$ acts with cohomogeneity two on $\mathbb{R}^n$ and therefore with cohomogeneity one on $S^n = SO(n)/SO(n-1)$, cf. [Kol02, Theorem A]. Lifting this action to Hilbert space, i.e. considering $P(SO(n), \rho(K) \times SO(n-1))$ acting on $H^0([0,1], \mathfrak{so}(n))$, yields examples of polar actions.

The principal isotropy group is the same as for the s-representation and may be found in Table A.5 on page 81. Let m_1 and m_2 be the (not necessarily distinct) multiplicities of the s-representation, then the action on Hilbert space has Dynkin diagram $\underset{m_1}{\circ}\overset{\infty}{-}\underset{m_2}{\circ}$. Table 4.3 on the following page lists all examples together with their multiplicities and principal isotropy groups.

action	G/K	m_1	m_2	isotropy group	hermitian
A I	$\mathrm{SU}(3)/\mathrm{SO}(3)$	1	1	$\mathbb{Z}_2^2$	
A II	$\mathrm{SU}(6)/\mathrm{Sp}(3)$	4	4	$\mathrm{Spin}(4)$	
A III	$\mathrm{SU}(m+4)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(m+2))$	$2m+1$	2	$\mathrm{SU}(m)\times\mathrm{U}(1)^2$	✓
BD I	$\mathrm{SO}(m+4)/\mathrm{SO}(2)\times\mathrm{SO}(m+2)$	m	1	$\mathrm{SO}(m)$	✓
D III	$\mathrm{SO}(10)/\mathrm{U}(5)$	5	4	$\mathrm{SU}(2)^2\times\mathrm{U}(1)$	✓
C II	$\mathrm{Sp}(m+4)/\mathrm{Sp}(2)\times\mathrm{Sp}(m+2)$	$4m+3$	4	$\mathrm{Sp}(m)\times\mathrm{Spin}(4)$	
E III	$\mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{SO}(2)$	9	6	$\mathrm{U}(4)$	✓
E IV	$\mathrm{E}_6/\mathrm{F}_4$	8	8	$\mathrm{Spin}(8)$	
G	$\mathrm{G}_2/\mathrm{SO}(4)$	1	1	$\mathbb{Z}_2^2$	

TABLE 4.3. Multiplicities of actions arising from rank-2 symmetric spaces

REMARK. The abelian factors U(1) in the principal isotropy groups of A III and D III may be eliminated by replacing K with $K' = \mathrm{SU}(2)\times\mathrm{SU}(m+2)$ or $\mathrm{SU}(5)$, respectively. These subactions are orbit equivalent.

In case C II it is not possible to reduce the singular slice representation of dimension $4m+3$, since the Sp(1)-factor acts non-trivially on the other eigenspace as a part of Spin(4). For the same reason it is not possible to get rid of more than one of the two U(1)-factors in A III.

The four actions arising from hermitian symmetric space give rise to a second type of cohomogeneity one action, namely after removing the abelian factor of $\rho(K)$ the group acts on $S^n = \mathrm{SU}(\frac{n}{2})/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(\frac{n}{2}-1))$. The multiplicities stay the same, the isotropy group is the same except for the abelian factor. We remark that if we apply this procedure to the action given by the s-representation of BD I, viewed as the s-representation of an hermitian symmetric space, this is precisely the Hermann action of type AI–III.

4.6.2. Exceptional actions on simple groups. We give the complete list of examples of these type, cf. [KOL02, p. 46], together with the multiplicities in Table 4.4. The multiplicities may be obtained in the following way: we regard the action of K_1 on

No	G	K_1	K_2	m_1	m_2	isotropy group
1	G_2	$\mathrm{SU}(3)$	$\mathrm{SU}(3)$	5	5	$\mathrm{SU}(2)$
2	G_2	$\mathrm{SU}(3)$	$\mathrm{SO}(4)$	2	3	$\mathrm{SO}(2)$
3	$\mathrm{SO}(7)$	G_2	G_2	6	6	$\mathrm{SU}(3)$
4	$\mathrm{SO}(7)$	G_2	$\mathrm{SO}(4)\times\mathrm{SO}(3)$	3	3	$\mathrm{SU}(2)$
5	$\mathrm{SO}(7)$	G_2	$\mathrm{U}(3)$	5	1	$\mathrm{U}(2)$
6	$\mathrm{SO}(16)$	$\mathrm{Spin}(9)$	$\mathrm{SO}(14)\times\mathrm{SO}(2)$	7	6	$\mathrm{U}(3)$
7	$\mathrm{SO}(4n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathrm{SO}(4n-2)\times\mathrm{SO}(2)$	$4(n-2)+3$	2	$\mathrm{Sp}(n-2)\cdot\mathrm{SO}(2)^2$

TABLE 4.4. Multiplicities of exceptional actions on simple groups

G/K_2, hence the dimension of a principal orbit is $\dim(G/K_2)-1$ and the dimension of the principal isotropy group is $\dim(K_1)-\dim(G/K_2)+1$. Now we describe the action in detail.

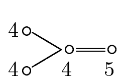
(1) This is an action of type $K_1 = K_2$, hence one of its singular slice representations is $K_1 = \mathrm{SU}(3)$ acting transitively on the sphere S^5, with principal isotropy group $\mathrm{SU}(2)$. Hence one of the multiplicities is 5, but since the dimension of a principal orbit is 5, too, this is the only multiplicity, i.e. there is only one singular orbit type.
(2) The principal isotropy group is $\mathrm{SO}(2)$. As an action of $\mathrm{SO}(4)$ on S^6 the action (2) is orbit-equivalent to the action of $\mathrm{SO}(4) \times \mathrm{SO}(3)$ on S^6 therefore the multiplicities are 2 and 3.
(3) Analogous to (1).
(4) The principal isotropy group is three-dimensional and its rank is at most 2, therefore its Lie algebra is $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. By using an explicit description of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, it is easy to see that $K_1 \cap K_2 = \mathrm{SO}(4)$, hence one multiplicity is 3 and so is the other. Moreover one can explicitly calculate the eigenspaces and the associated module of two eigenspaces of "different" type and finds that the (three-dimensional) antisymmetric module is not contained in $E(0)$.

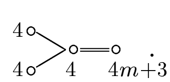
   This shows that the action is not orbit-equivalent to $\mathrm{SO}(5)/\mathrm{SO}(4) \times \mathrm{SO}(4)$, which has the same diagram and singular slice representations.
(5) The principal isotropy group is four-dimensional and its rank is at most 2, therefore its Lie algebra is $\mathfrak{u}(2) = \mathfrak{so}(2) \oplus \mathfrak{so}(3)$. As an action of $\mathrm{U}(3)$ on S^7 it is orbit-equivalent to the action of $\mathrm{SO}(6) \times \mathrm{SO}(2)$ on S^7 therefore the multiplicities are 1 and 5.
(6) We consider the action of $\mathrm{Spin}(9)$ on the Stiefel manifold $V_2(\mathbb{R}^{16})$ as in [Kol02, p. 38]. Choose a vector e_1, then $(\mathrm{Spin}(9)_{e_1}) = \mathrm{Spin}(7)$ for this is the p.i.g. of $\mathrm{Spin}(9)$ acting on $\mathbb{R}^{16}$. The orthogonal complement of e_1 is an $\mathbb{R}^{15} = \mathbb{R}^7 \oplus \mathbb{R}^8$ where $\mathrm{Spin}(7)$ acts as standard or spin representation respectively. Choosing a vector e_2 from the $\mathbb{R}^7$ gives $(\mathrm{Spin}(9))_{(e_1,e_2)} = (\mathrm{Spin}(7))_{e_2} = \mathrm{Spin}(6) = \mathrm{SU}(4)$ as an singular isotropy group. This proves that $\mathrm{SU}(3)$ is the principal isotropy of the action on the Stiefel manifold and $\mathrm{U}(3)$ on the Grassmannian manifold and therefore one multiplicity is 7. The other is 6, which may be seen similar by choosing $e_2 \in \mathbb{R}^8$.
(7) As for the last action we study here again the corresponding action on $V_2(\mathbb{R}^{4n})$ and determine the singular isotropy groups: If e_1 and e_2 are quaternionic linear depended, the isotropy group is $\mathrm{Sp}(n-1) \cdot \mathrm{SO}(2)^2$ (i.e. multiplicity $4(n-2)+3$), if they are quaternionic linear independent, it is $\mathrm{Sp}(n-2) \cdot \mathrm{Sp}(1) \cdot \mathrm{SO}(2)$ and the multiplicity is 2.

   One of the $\mathrm{SO}(2)$-factors of the isotropy group acts trivially (to be more precise: there is an orbit-equivalent action with $K_2' = \mathrm{SO}(4n-2)$, where this factor vanishes), leaving $\mathrm{Sp}(n-2) \times \mathrm{U}(1)$. In terms of Theorem 2.3 on page 11: the whole group acts nontrivially on the $4(n-2)+3$ dimensional eigenspaces and the $\mathrm{U}(1)$-factor acts as $\mathrm{SO}(2)$ on the other.

4.7. Dynkin diagrams not arising from $P(G,H)$-actions

We compare the affine marked Dynkin diagrams of the $P(G,H)$-actions with the possible Dynkin diagrams of section 4.1. All the missing diagrams are of "exotic type" in the sense that they are only possible for cohomogeneity two or three. The following affine diagrams of type $\tilde{B}_3$ do not arise



and


.

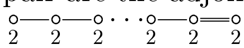
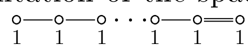
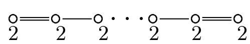
Moreover there are six diagrams of type $\tilde{C}_2$, who do not occur as the Dynkin diagram of a $P(G,H)$-action, namely

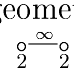
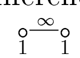
Vertices marked with 5 or 9 always belong to reduced roots, i.e. the eigenspaces are reducible modules of the isotropy representation.

In Section 5.4 we will show that most of these marked Dynkin diagrams do not occur as the diagram of any homogeneous isoparametric submanifold of Hilbert space.

4.8. Actions with equal marked affine Dynkin diagram

In this section we are interested in $P(G,H)$-actions whose marked Dynkin diagram coincides. We also want to briefly explain their geometric differences, for details see the next chapter.

4.8.1. Different slice representations. Remember that in finite dimensions there are only two pairs of s-representation with equal Dynkin diagram. The first pair are the adjoint representations of $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$, whose diagrams are both  . The second pair arises from the first by the involution which maps every root to its negative, namely the s-representation of the spaces $\mathrm{Sp}(n)/\mathrm{U}(n)$ and $\mathrm{SO}(2n+1)/\mathrm{SO}(n+1)\times\mathrm{SO}(n)$ with diagram  . Hence any affine Dynkin diagram with one of these as a subdiagram allows different slice representation, any of these combinations occurs among the $P(G,H)$-actions. For example, an action with diagram  contains two irreducible most singular slice representations of type B and multiplicity 2 and any of three possible combinations arises: Both singular slice representations of the $P(G,\Delta(G))$-action for $G=\mathrm{Sp}(n)$, are the adjoint representation of $\mathrm{Sp}(n)$. Similarly both singular slice representations of the σ-action of $\mathrm{SO}(2n+2)$ are of type $\mathrm{SO}(2n+1)$. Finally the σ-action of $\mathrm{SU}(n+1)$ has both the adjoint of $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$ as a singular slice representation.

See Table 4.5 on the next page for all examples with the same Dynkin diagram, but different slice representations. To explain the geometric difference of these actions, we have to consider only the hypersurfaces of type  and  , cf. Corollary 1.10 on page 7.

In the next chapter especially in Lemma 5.10 on page 69 we will see, that it depends on the length of the roots occurring — if there are only two different lengths (i.e. the two most singular slice representation are equal) the rank-1 leaves are isometric to the $P(G,\Delta(G))$-action of $\mathrm{SU}(2)$, while if they are different there are some rank-1 leaves isometric to the σ-action of $\mathrm{SU}(3)$.

4.8.2. Equal slice representations. In this section we give the four examples of pairs of $P(G,H)$-actions whose singular slice representations agree in Table 4.6 on the facing page. We will see in the next chapter that some of these examples are orbit-equivalent despite the first and the last. Two different examples only occur when the diagram admits different possibilities for the principal isotropy group — this determines whether the rank-1 leaves have isotropy group $\mathrm{SO}(m)$ or $\mathrm{SO}(m)\times\mathrm{SO}(m)$.

Note that in any pair there is one action which is of type $K_1=K_2$ and the other is of type $K_1\neq K_2$.

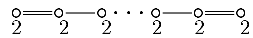
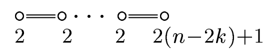
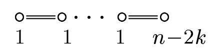
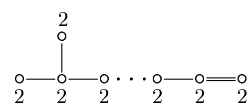
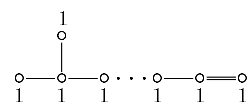
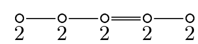
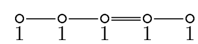
Action	slice repr.	Diagram	slice repr.
$P\big(\mathrm{Sp}(n), \Delta(\mathrm{Sp}(n))\big)$	$\mathrm{Ad}(\mathrm{Sp}(n))$		$\mathrm{Ad}(\mathrm{Sp}(n))$
$P\big(\mathrm{SU}(2n{+}1), \Delta^\sigma(\mathrm{SU}(2n{+}1))\big)$			$\mathrm{Ad}(\mathrm{SO}(2n{+}1))$
$P\big(\mathrm{SO}(2n{+}2), \Delta^\sigma(\mathrm{SO}(2n{+}2))\big)$	$\mathrm{Ad}(\mathrm{SO}(2n{+}1))$		
C I–I(n)	C I (n)		C I(n)
A I– III$(n, n+1)$			BD I$(n, n+1)$
BD I$(n, n+2)$– I$(n+1, n+1)$	BD I$(n, n+1)$		
C I–II$(k, n-k)$	$\mathrm{Ad}(\mathrm{Sp}(k))$		A III$(k, n-k)$
D I–III(k ungerade)	$\mathrm{Ad}(\mathrm{SO}(2n{+}1))$		
A I– III$(k, n-k)$	C I(k)		BD I$(k, n-k)$
BD I$(k, n{+}1{-}k)$– I$(k{+}1, n{-}k)$	BD I$(k, k+1)$		
$P\big(\mathrm{SO}(2n{+}1), \Delta(\mathrm{SO}(2n{+}1))\big)$	$\mathrm{Ad}(\mathrm{SO}(2n))$		$\mathrm{Ad}(\mathrm{SO}(2n{+}1))$
$P\big(\mathrm{SU}(2n), \Delta^\sigma(\mathrm{SU}(2n))\big)$			$\mathrm{Ad}(\mathrm{Sp}(n))$
A I–III(n, n)	BD I(n, n)		C I(n)
BD I$(n, n+1)$–I$(n, n+1)$			BD I$(n, n+1)$
$P\big(\mathrm{F}_4, \Delta(\mathrm{F}_4)\big)$	$\mathrm{Ad}(\mathrm{SO}(9))$		$\mathrm{Ad}(\mathrm{F}_4)$
$P\big(\mathrm{E}_6, \Delta^\sigma(\mathrm{E}_6)\big)$	$\mathrm{Ad}(\mathrm{Sp}(4))$		
F I–I	BD I $(4, 5)$		F I
E I–II	C I (4)		

TABLE 4.5. $P(G, H)$-actions with the same affine marked Dynkin diagram but different singular slice representations

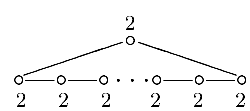
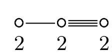
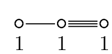
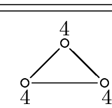
Action	Diagram
$P\big(\mathrm{SU}(n), \Delta(\mathrm{SU}(n))\big)$	
A I–II$(2n)$	
$P\big(\mathrm{G}_2, \Delta(\mathrm{G}_2)\big)$	
$\sigma(\mathrm{Spin}(8))$	
$\mathrm{G}_2/\mathrm{SO}(4) \times \mathrm{SO}(4)$	
D_4 I–I'$(k = l = 3)$	
A II–II	
E I–IV	

TABLE 4.6. $P(G, H)$-actions with the same affine marked Dynkin diagram and the same singular slice representations

CHAPTER 5

Rigidity of isoparametric submanifolds

In this chapter we give a classification of homogeneous isoparametric submanifold with isotropy irreducible eigenspaces, by proving that they are isometric to a principal orbit of a $P(G, H)$-action. In particular we investigate for a given affine marked Dynkin diagram how many different infinite dimensional homogeneous isoparametric submanifolds with that diagram exist. Moreover we determine which among the Hermann actions with the same Dynkin diagram are in fact orbit-equivalent, cf. Section 4.8 on page 60 for a complete list of these.

The strategy for solving this question is developed in Corollary 1.10 on page 7 and Theorem 1.13 on page 8: Different isoparametric submanifolds have to contain at least one rank-1 leaf that is different. Therefore we have to determine which kinds of rank-1 leaves for a given diagram are possible. Hypersurfaces in turn are determined by their normal homogeneous structure, for a special class of them we have proven rigidity in Chapter 3, namely for those with principal isotropy group $\mathrm{SO}(m)$ or $\mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$. This class is almost the same as hypersurfaces whose eigenspaces are irreducible modules of the isotropy representation.

Therefore we restrict ourselves to isoparametric submanifolds of higher codimension whose principal isotropy group are of type $\mathrm{SO}(m_1)^{k_1} \times \mathrm{SO}(m_2)^{k_2} \times \mathrm{SO}(m_3)^{k_3}$, where the m_i are the multiplicities. Note, that this implies in particular, that the slice representation are equivalent to s-representation.

The assumption, we have possed on the class of isoparametric manifolds, we are studying is equivalent to requiring that any slice representation has principal isotropy group of that type, therefore one can check Table A.5 on page 81, which affine Dynkin diagrams from the list 4.1 on page 41 belong to this class.

Throughout this chapter we denote the rank-1 leaves which may occur in the following way:

	$P(G,H)$-action	isotropy group	modules
$S(m)$	BD I$(1, m+1)$–I$(1, m+1)$	$\mathrm{SO}(m)$	Prop. 2.12
$\tilde{S}(1)$	A I–III$(1, 2)$		Prop. 2.19
$\tilde{S}(2)$	$\sigma(\mathrm{SU}(3))$	$\mathrm{SO}(2)$	Prop. 2.15
$S(m_1, m_2)$	BD I$(1, m_1+m_2+1)$–I(m_1+1, m_2+1)	$\mathrm{SO}(m_1)\times\mathrm{SO}(m_2)$	Thm. 2.17.
$\tilde{S}(1, m)$	A I–III$(1, m+1)$	$\mathrm{SO}(m)$	Prop. 2.19

The term $\tilde{S}$ means that there are associated modules that are not subspaces of $E(0)$. With help of the Dynkin diagrams for the known examples and explicit calculations of the associated modules, developed in the last chapter, we can establish which $P(G, H)$-action of cohomogeneity one belongs to which kind of isoparametric hypersurface in the above table.

Each affine marked Dynkin diagram describes an infinite reflection group, more precisely an affine Weyl group. Any of the reflection hyperplanes

$$l_i = \{a + v \mid v \in \nu_a M,\ \langle v, v_i(a)\rangle = 1\}$$

is associated with a curvature normal v_i and therefore with an eigenspace $E_i(a)$ of the shape operator.

Let $P_i = \operatorname{span}\{v_i\}$ and $\tilde{L}_{P_i}$ the reduced rank-1 leaf (cf. Theorem 1.7 on page 5 and Definition 1.9), this associates with each vertex of the Dynkin diagram an isoparametric hypersurface. We remark that $\tilde{L}_{P_i}$ and $\tilde{L}_{P_j}$ are isometric if there exists an element within the affine Weyl group mapping l_i to l_j. This is always the case if the vertices are joined by a single or a triple line, therefore there are at most two different kinds of hypersurfaces within an isoparametric submanifold of higher codimension with diagram $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{F}_4$ and only one for the others. More precisely a submanifold with Dynkin diagram $\tilde{B}_n$ or $\tilde{F}_4$ and multiplicities m_1 and m_2 contains two rank-1 leaves with diagram $\underset{m_i}{\circ}\overset{\infty}{—}\underset{m_i}{\circ}$, while for $\tilde{C}_n$ ($\underset{m_1}{\circ}=\underset{m_2}{\circ}—\underset{m_2}{\circ}\cdots\underset{m_2}{\circ}—\underset{m_2}{\circ}=\underset{m_3}{\circ}$) it contains $\underset{m_1}{\circ}\overset{\infty}{—}\underset{m_3}{\circ}$ and $\underset{m_2}{\circ}\overset{\infty}{—}\underset{m_2}{\circ}$, since if one considers a reflection hyperplane marked by m_1 the multiplicities m_1 and m_3 alternate within the family of parallel hyperplanes.

We start with rigidity of isoparametric submanifolds with uniform multiplicity 2, among the $P(G,H)$-action only σ-actions are of that type. This class is especially interesting for we have seen at the end of the last chapter, that many examples admitting the same Dynkin diagram are of this class.

5.1. Uniform multiplicity 2

In Chapters 2 and 3 we have proven that there exist three different infinite dimensional isoparametric hypersurfaces with affine Dynkin diagram $\underset{2}{\circ}\overset{\infty}{—}\underset{2}{\circ}$, in the last chapter we have seen that those are the principal orbits of the following $P(G,H)$-actions

	G	H	isotropy group	modules described by
$S(2)$	SU(2)	$\Delta(\mathrm{SU}(2))$	SO(2)	Proposition 2.12 on page 22
$\tilde{S}(2)$	SU(3)	$\Delta^\sigma(\mathrm{SU}(3))$	SO(2)	Proposition 2.15 on page 24
$S(2,2)$	SU(4)	$\mathrm{SO}(4)\times\mathrm{Sp}(2)$	$\mathrm{SO}(2)\times\mathrm{SO}(2)$	Theorem 2.17 on page 26.

We remark that there are other descriptions of the first and third action, namely the first is orbit-equivalent to the lift of the adjoint action of SO(3), to the σ-action of SO(4) and to the action $G = \mathrm{SO}(4)$, $H = \mathrm{SO}(3)\times\mathrm{SO}(3)$, while the third to $G = \mathrm{SO}(6)$, $H = (\mathrm{SO}(3)\times\mathrm{SO}(3))\times\mathrm{SO}(5)$.

In this section we use Corollary 1.10 on page 7 to determine all isoparametric submanifold with uniform multiplicity 2. Therefore we have to determine for a given affine Dynkin diagram which isotropy groups the hypersurfaces may admit and if it admits hypersurfaces with isotropy group SO(2) whether it is possible that the corresponding hypersurface is of type $\tilde{S}(2)$. This point is solved by the following criterion.

Proposition 5.1. *Let $S = \tilde{L}_P$ be a hypersurface within an isoparametric submanifold M of higher codimension with uniform multiplicity 2, where P is the span of some curvature normal. Assume that the effective part of the isotropy group acting on TS is* SO(2)*, and let $\{v_i \mid i \in \mathbb{Z}\}$ be the curvature normals in P.*

Then S is isometric to $S(2)$ if there is an element α in the affine Weyl group of M such that $\alpha|_P$ is the translation $l_i \mapsto l_{i+1}$, where l_i is the reflection hyperplane associated with v_i.

Proof. We only have to exclude that S is isometric to $\tilde{S}(2)$. In Proposition 2.15 we have seen, that $V_{4n+2,4m} \supset E_{2n+2m+1}$ while $V_{2n+1,2m+1} \subset E(0)$. If $\alpha(E_i) = E_{i+1}$ this is a contradiction for α does not commute with $\psi = \nabla A$. □

Remark. Such an element α as in the Proposition exists for any family of eigenspace except the ones belonging to the vertices marked in black in $\tilde{C}$-diagrams

, (5.1)

that is the black vertices represent the only hypersurfaces contained in an isoparametric submanifold of higher codimension which might be of type $\tilde{S}(2)$.

We note for later use, that this argument holds for the other $\tilde{S}$-hypersurfaces as well.

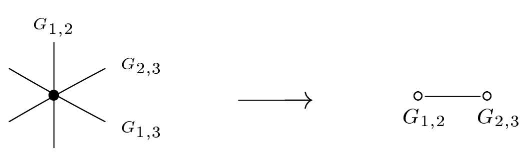
Throughout the rest of the section let $M = G \cdot a$ be an isoparametric submanifold of Hilbert space, with cohomogeneity greater than one and uniform multiplicity two.

Proposition 5.2. *Let $M = G \cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{A}_n$. Then the principal isotropy group is either* $\mathrm{SO}(2)^n$ *or* $\mathrm{SO}(2)^{n+1}$*. In the first case any rank-1 leaf $\tilde{L}_P$ contained in M is of type $S(2)$, while in the second case it is of type $S(2,2)$.*

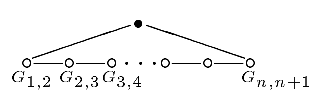
The manifold M is isometric to a principal orbit the $P(G,H)$-action with $G = \mathrm{SU}(n+1)$, $H = \Delta(\mathrm{SU}(n+1))$ in the first case and to A I–II in the second.

Proof. First we determine the possible principal isotropy groups G_a. Associated with each vertex in the affine Dynkin diagram is curvature normal together with an eigenspace and therefore a factor of G_a which acts effectively on that eigenspace.

Any most singular slice representation is the adjoint representation of $\mathrm{SU}(n+1)$ whose principal isotropy group is the maximal torus $\mathrm{SO}(2)^n$. Let G_{ij} be the group of diagonal matrices in $\mathrm{SU}(n+1)$ where the i-th entry is $\theta \in S^1$ and the j-th $\bar{\theta}$. One sees easily that on each of the $\frac{n(n+1)}{2}$ eigenspaces of the adjoint representation of $\mathrm{SU}(n+1)$ one of these groups acts effectively. We mark each reflection hyperplane in the affine Weyl group with the factor of the isotropy group acting effectively on the corresponding eigenspace, see the figure for $\mathrm{SU}(3)$, where the same is also done for the Dynkin diagram.



For arbitrary rank in the affine Dynkin diagram this looks like



We have to determine the group acting effectively on the family of eigenspaces corresponding to the black vertex. Since for two orthogonal curvature normals the corresponding groups are orthogonal as well, it has to be orthogonal to $G_{2,3}, \dots, G_{n-1,n}$, since not joined by a line to any of these.

There are two possibilities: either it is the group $G_{1,n+1}$ or it is a new $\mathrm{SO}(2)$-factor isomorphic to $\tilde{G}_{1,n+1}$ (first and last entry θ) both is compatible with the slice representations. For this purpose we look at the slice representation corresponding to the black vertex and the one marked with $G_{1,2}$, which is the adjoint representation of $\mathrm{SU}(3)$. On the third family of eigenspaces, that is on the one not represented by a vertex in the diagram, the effective acting part of the isotropy group is either $G_{2,n+1}$ or $\tilde{G}_{2,n+1}$. This proves, that the principal isotropy group of M is $\mathrm{SO}(2)^n$ in the first case and $\mathrm{SO}(2)^{n+1}$ in the second.

At a most singular point in the affine Weyl group meet $\frac{n(n+1)}{2}$ reflection hyperplanes whose effective isotropy group, that is acting effectively on the corresponding eigenspace, are different. For the case $n = 2$ see the figure above.

Assume the principal isotropy group is $G_a = \mathrm{SO}(2)^n$, containing $\frac{n(n+1)}{2}$ subgroups of type G_{ij}. Since a reflection hyperplane in the affine Weyl group meets any non parallel hyperplane at some point, that means only parallel hyperplanes correspond to the same SO(2)-factor within G_a. Hence any rank-1 leaf has effective isotropy group of type SO(2) and by the last proposition is isometric to $S(2)$.

Finally assume $G_a = \mathrm{SO}(2)^{(n+1)}$. Since there are $\frac{n(n+1)}{2}$ different rank-1 leaves, but more groups of type G_{ij} or $\tilde{G}_{ij}$, which are all effective isotropy groups of some eigenspace, there has to be at least one rank-1 leaf with effective isotropy group $\mathrm{SO}(2) \times \mathrm{SO}(2)$. For the affine Weyl group maps any rank-1 leaf to any other, all have to be of the same type, that is isometric to $S(2,2)$.

There are two examples among the $P(G,H)$-actions fulfilling the conditions of the proposition and it is not difficult to determine their isotropy groups: It is $\mathrm{SO}(2)^r$ where r is the rank of $K_1 \cap K_2$, which is $\mathrm{SU}(n+1)$ for $P(\mathrm{SU}(n+1), \Delta(\mathrm{SU}(n+1))$ and $\mathrm{Sp}(n+1) \cap \mathrm{SO}(2n+1) = \mathrm{U}(n+1)$ for A I–II. This proves the last statement. □

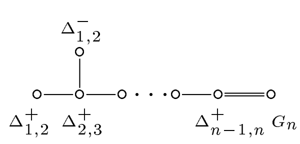
Remark. The part of isotropy group acting effectively on a family of eigenspaces corresponds to the root system of the Lie algebra associated with the (non affine) Dynkin diagram. Let $G_a = \mathrm{SO}(2)^n = G_1 \times \cdots \times G_n$ and choose the factors G_i such that for a basis of the roots system $\{e_1, \dots e_n\}$ the factor G_i acts trivially on e_j for $i \neq j$. Then the groups G_{ij} from the last proof correspond to the roots $e_j - e_i$. That way it is not difficult to determine the factor acting effectively on a certain eigenspace.

The new vertex corresponds to the highest root, hence the effectively acting group may always correspond to that root, e.g. $G_{1,n+1} \simeq e_{n+1} - e_1$ in the $\tilde{A}_n$-case. We have to investigate whether there are other possibilities, e.g. $\tilde{G}_{1,n+1} \simeq e_{n+1} + e_1$ in the above example.

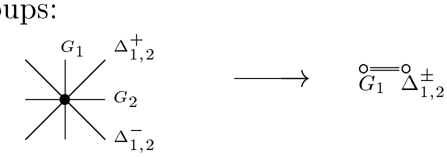
Proposition 5.3. *Let $M = G \cdot a$ be an infinite dimensional isoparametric submanifold of rank $n \geq 3$ with affine Dynkin diagram $\tilde{B}_n$. Then the principal isotropy group is $\mathrm{SO}(2)^n$ and the rank-1 leaves are of type $S(2)$.*

The manifold M is isometric to a principal orbit of the $P(G,H)$-action with $G = \mathrm{SO}(n+1)$, $H = \Delta(\mathrm{SO}(2n+1))$ or $G = \mathrm{SU}(2n)$, $H = \Delta^\sigma(\mathrm{SU}(2n))$, these action are orbit-equivalent.

Proof. The n vertices on the left side (forming a D_n-diagram) represent a most singular slice representation which is the adjoint representation of $\mathrm{SO}(2n)$ with principal isotropy group the maximal torus $\mathrm{SO}(2)^n = G_1 \times \cdots \times G_n$ of $\mathrm{SO}(2n)$. Denote by $\Delta_{ij}^{\pm} = \Delta^{\pm}(G_i, G_j) = \{g \cdot \phi^{\pm 1}(g) \mid g \in G_i\}$ for a Lie group isomorphism ϕ between G_i and G_j, then the effectively acting parts correspond to the vertices in the following way, which may be seen by an easy calculation:



The right boundary vertex has to by marked by G_n since the adjoint action of SO(5) has effectively isotropy groups:

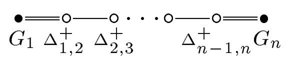


Hence we have proven that $G_a = \mathrm{SO}(2)^n$ for affine Dynkin diagram $\tilde{B}_n$. By the same argument as in the last proposition (each reflection hyperplane meets any other, which is not parallel, in some point) any rank-1 leaf has isotropy group $\mathrm{SO}(2)$. Checking the known examples finishes the proof. □

PROPOSITION 5.4. *Let $M = G \cdot a$ be an infinite dimensional isoparametric submanifold of rank $n \geq 2$ with affine Dynkin diagram $\tilde{C}_n$. Then the principal isotropy group is $\mathrm{SO}(2)^n$. The rank-1 leaves corresponding to white vertices in (5.1) are of type $S(2)$, while the black ones are either of type $S(2)$ or $\tilde{S}(2)$.*

The manifold M is isometric to a principal orbit of the $P(G,H)$-action with $G = \mathrm{Sp}(n)$, $H = \Delta(\mathrm{Sp}(n))$ or $G = \mathrm{SO}(2n+2)$, $H = \Delta^\sigma(\mathrm{SO}(2n+2))$ in the first case (those actions are orbit-equivalent) and $G = \mathrm{SU}(2n+1)$, $H = \Delta^\sigma(\mathrm{SU}(2n+1))$ in the second.

PROOF. In a similar manner as in the last propositions, by checking effectively acting parts of the isotropy group of the slice representations, one derives the following diagram as the only possibility.

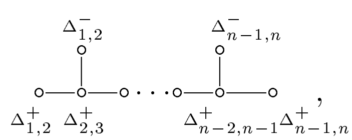


The black vertices are mapped onto each other by an appropriate element of the affine Weyl group, therefore either both are of type $S(2)$ or both are of type $\tilde{S}(2)$. □

PROPOSITION 5.5. *Let $M = G \cdot a$ be an infinite dimensional isoparametric submanifold of rank $n \geq 4$ with affine Dynkin diagram $\tilde{D}_n$ or with diagram $\tilde{E}_n$ for $n \in \{6,7,8\}$. Then the principal isotropy group is $\mathrm{SO}(2)^n$ and the rank-1 leaves are of type $S(2)$.*

The manifold M is isometric to a principal orbit of the $P(G,H)$-action with $G = \mathrm{SO}(2n)$, $H = \Delta(\mathrm{SO}(2n))$ for $\tilde{D}_n$-diagram and with $G = \mathrm{E}_n$, $H = \Delta(\mathrm{E}_n)$ for $\tilde{E}_n$ diagram.

PROOF. In the $\tilde{D}$-case the only possibility for the effectively acting part of the isotropy group is



therefore the principal isotropy group is $\mathrm{SO}(2)^n$. Again any rank-1 leaf is of type $S(2)$ as in the last propositions.

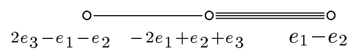
For the manifolds with $\tilde{E}_n$-diagrams, we only have to remark that they contain rank-5 leaves with diagram $\tilde{D}_5$, therefore any rank-1 leaf is of type $S(2)$ and the isotropy group is $\mathrm{SO}(2)^n$. □

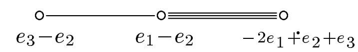
PROPOSITION 5.6. *Let $M = G \cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{F}_4$ or $\tilde{G}_2$. Then the principal isotropy group is $\mathrm{SO}(2)^4$ or $\mathrm{SO}(2)^2$ respectively and the rank-1 leaves are of type $S(2)$.*

The manifold M is isometric to a principal orbit of the $P(G,H)$-action with $G = \mathrm{F}_4$, $H = \Delta(\mathrm{F}_4)$ or $G = \mathrm{E}_6$, $H = \Delta^\sigma(\mathrm{E}_6)$ in the first case (these actions are orbit-equivalent), and $G = \mathrm{G}_2$, $H = \Delta(\mathrm{G}_2)$ or $G = \mathrm{Spin}(8)$, $H = \Delta^\sigma(\mathrm{Spin}(8))$ (these actions are orbit-equivalent) in the second.

PROOF. An isoparametric submanifold with diagram $\tilde{F}_4$ contains a rank-3 leaf with diagram $\tilde{B}_3$ and effectively isotropy group $\mathrm{SO}(2)^3$, therefore any rank-1 leaf is of type $S(2)$, the isotropy group is $\mathrm{SO}(2)^4$.

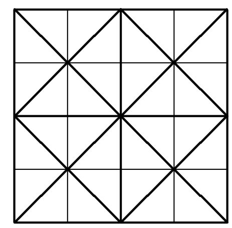
Let the diagram be $\tilde{G}_2$, then the only possibilities are





Thereby we have used that for a rank-1 leaf with effective isotropy group $\mathrm{SO}(2)^2$, the two factors have to be orthogonal, excluding possibilities as $2e_3 - e_1 - e_2 + e_4$ for the new vertex. In both cases the isotropy group is $\mathrm{SO}(2)^2$ and by Proposition 5.1 on page 63 rank-1 leaves are of type $S(2)$. □

Remark. We have seen, that in the $\tilde{B}_n$ $\tilde{C}_n$, $\tilde{F}_4$ and $\tilde{G}_2$ case there are two orbit-equivalent examples among the σ-actions. This may be seen geometrically in the following way:



The root systems of $\mathfrak{c}_2$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$ consist of roots of different lengths and the number of short roots equals the number of long roots. The length corresponds to different distances between reflection hyperplane within the affine Weyl group. Taking all families of reflection hyperplanes with the greater distance and bisecting the distance, that is put a new one in between any of the old, gives the same affine Weyl group with interchanged roles of the short and long roots. The adjoining figure shows the situation for $\tilde{C}_2$, where thin lines denote the new reflection hyperplanes.

Consider for example the lift of the adjoint action of F_4 and the σ-action of E_6. The latter has two different types of eigenspaces cf. [Ter95]: Let $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus \mathfrak{h}$ be the Cartan-decomposition, $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{e}_6$, and Δ and $\tilde{\Delta}$ resp. the set of roots with respect to $\mathfrak{a}$ of $\mathfrak{f}_4$ and $\mathfrak{h}$ respectively. Both root systems give rise to eigenspaces, those belonging to $\mathfrak{f}_4$ are also eigenspaces of the adjoint action of F_4. Those belonging to $\tilde{\Delta}$ bisect the distance of the longer roots as described above, but this does not change the geometry of the manifold. Observe that $\dim(\mathfrak{h}) = 2 + 2 \cdot 12$, therefore 12 families of new eigenspaces arise from $\tilde{\Delta}$. The two supernumerous dimensions belong to a maximal abelian subalgebra of E_6 containing $\mathfrak{a}$, therefore belong to $E(0)$ and provide the new tr- and Λ-modules associated with the eigenspaces of $\tilde{\Delta}$.

For diagrams $\tilde{B}_n$ and $\tilde{C}_n$ despite $n = 2$, this description does not hold, $\mathrm{Sp}(n)$ is not the fixed point set under the diagram automorphism of $\mathrm{SO}(2n+2)$, nevertheless it is possible to explicate the orbit-equivalence, which we will omit here.

5.2. Uniform multiplicity 1, 4 and 8

The rigidity of isoparametric submanifold with uniform multiplicity 1 works similar to the case of uniform multiplicity 2. The two hypersurfaces are

	G	H	modules
$S(1)$	$\mathrm{SU}(2)$	$\mathrm{SO}(2) \times \mathrm{SO}(2)$	Proposition 2.12 on page 22
$\tilde{S}(1)$	$\mathrm{SU}(3)$	$\mathrm{SO}(3) \times \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2))$	Proposition 2.19 on page 28

We recall that there is no analogue for $S(1,1)$ by the discussion in Subsection 2.4.3. The natural candidate for an action of this type is the $P(G,H)$-action with $G = \mathrm{SO}(4)$ and $H = \mathrm{SO}(3) \times (\mathrm{SO}(2) \times \mathrm{SO}(2))$, since the $P(G,H)$-action with $G = \mathrm{SO}(2m+2)$ and $H = \mathrm{SO}(2m+1) \times (\mathrm{SO}(m+1) \times \mathrm{SO}(m+1))$ is of type $S(m,m)$. It is not difficult to prove that for $m = 1$ it is orbit-equivalent to $S(1)$. Moreover we remark that $\tilde{S}(1) = \tilde{S}(1,1)$.

Since Proposition 5.1 stays valid for uniform multiplicity 1, we are done with the classification, which we will summarize in Table 5.1 on the following page.

Diagram	G	K_1	K_2	ω-equiv.
$\tilde{A}_n$	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$\mathrm{SO}(n)$	
$\tilde{B}_n$	$\mathrm{SU}(2n)$	$\mathrm{SO}(2n)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))$	✓
	$\mathrm{SO}(2n+1)$	$\mathrm{SO}(n)\times\mathrm{SO}(n+1)$	$\mathrm{SO}(n)\times\mathrm{SO}(n+1)$	✓
$\tilde{C}_n$	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	$\mathrm{U}(n)$	✓
	$\mathrm{SO}(2n+2)$	$\mathrm{SO}(n)\times\mathrm{SO}(n+2)$	$\mathrm{SO}(n+1)\times\mathrm{SO}(n+1)$	✓
	$\mathrm{SU}(2n+1)$	$\mathrm{SO}(2n+1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n+1))$	
$\tilde{D}_n$	$\mathrm{SO}(2n)$	$\mathrm{SO}(n)\times\mathrm{SO}(n)$	$\mathrm{SO}(n)\times\mathrm{SO}(n)$	
$\tilde{E}_6$	E_6	$\mathrm{Sp}(4)$	$\mathrm{Sp}(4)$	
$\tilde{E}_7$	E_7	$\mathrm{SU}(8)$	$\mathrm{SU}(8)$	
$\tilde{E}_8$	E_8	$\mathrm{Spin}(16)$	$\mathrm{Spin}(16)$	
$\tilde{F}_4$	F_4	$\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$	$\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$	✓
	E_6	$\mathrm{Sp}(4)$	$\mathrm{SU}(6)\cdot\mathrm{SU}(2)$	✓
$\tilde{G}_2$	G_2	$\mathrm{SO}(4)$	$\mathrm{SO}(4)$	✓
	$\mathrm{Spin}(8)$	$\mathrm{Spin}(3)\times\mathrm{Spin}(5)$	$\tau(\mathrm{Spin}(3)\times\mathrm{Spin}(5))$	✓

TABLE 5.1. Isoparametric submanifolds with uniform multiplicity one

Any of the isoparametric submanifolds with uniform multiplicity 2 has its analogue among these examples. The only exception is A I–II, whose rank-1 leaves are of type $S(2,2)$, the reason is that there is no hypersurface of type $S(1,1)$.

Finally we study uniform multiplicities 4 and 8, which occur only if the diagram is of type $\tilde{A}_n$.

PROPOSITION 5.7. *Let $M = G\cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{A}_2$ and multiplicity 8. Then the principal isotropy group is* $\mathrm{Spin}(8)$.

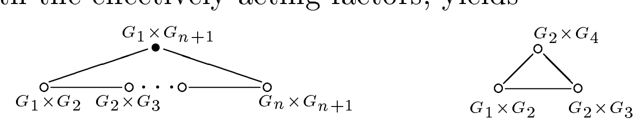
The manifold M is isometric to a principal orbit of the $P(G,H)$-action E IV–IV.

PROOF. Any singular slice representation is the s-representation of $\mathrm{E}_6/\mathrm{F}_4$, whose principal isotropy group is $\mathrm{Spin}(8)$ (cf. Table A.5 on page 81); therefore $G_a = \mathrm{Spin}(8)$ and any rank-1 leaf is isometric to a principal orbit of the $P(G,H)$-action with $G = \mathrm{SO}(10)$, $H = \mathrm{SO}(9)\times\mathrm{SO}(9)$, that is $S(9)$. □

PROPOSITION 5.8. *Let $M = G\cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{A}_n$ and multiplicity 4. Then the principal isotropy group is* $\mathrm{SO}(3)^{n+1}$ *for $n>2$. If $n=2$ then the principal isotropy group is either* $\mathrm{SO}(3)^3$ *or* $\mathrm{SO}(3)^4$.

The manifold M is isometric to a principal orbit of the $P(G,H)$-action A II–II or, if $n=2$ and $G_a = \mathrm{SO}(3)^4$, of the action E I–IV.

PROOF. The singular slice representation of rank n is the s-representation of type A II(n), whose principal isotropy group is $\mathrm{SO}(3)^{n+1} = G_1\times\cdots\times G_{n+1}$. Drawing the diagram together with the effectively acting factors, yields



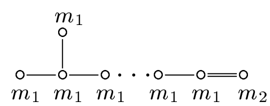
We have to determine the effective group associated with the black vertex and observe that it has to have one common factor with both adjacent vertices and none with the other, which leaves $G_1 \times G_{n+1}$ as only possibility. If $n = 2$ there is another possibility namely $G_2 \times G_4$.

In the general case (i.e. $G_a = \mathrm{SO}(3)^{n+1}$) any rank-1 leaf has effectively acting isotropy group SO(4) hence is isometric to $S(4)$, that is a principal orbit of A II(2) = BD I(1,5)–I(1,5). In the case $n = 2$ and $G_a = \mathrm{SO}(4)^2$, the rank-1 leaves are of type $S(4,4)$, that is principal orbits of BD I(1,9)–I(5,5). □

5.3. Nonuniform multiplicities

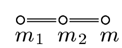
In this section we deal with isoparametric submanifolds whose eigenspaces are irreducible modules with at least two different multiplicities. Therefore the affine Dynkin diagram is of type $\tilde{B}_n$, $\tilde{C}_n$ or $\tilde{F}_4$.

PROPOSITION 5.9. *Let $M = G \cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{B}_n$ with multiplicities $m_1 \neq m_2$.*



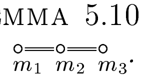
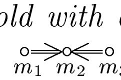
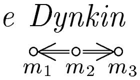
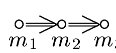
If $n > 3$ then either $m_1 = 1$, m_2 arbitrary or $m_1 = 2$, $m_2 = 1$, if $n = 3$ additionally $m_1 = 4$, $m_2 = 1$ is possible. The rank-1 leaves are of type $S(m_1)$ or $S(m_2)$, respectively.

The manifold M is isometric to a principal orbit of the $P(G,H)$-action BD I–I($k = l$) in the first case, D I—III(k even, $n = k$) in the second and E V–VII in the case with diagram $\tilde{B}_3(4,1)$.

PROOF. Since the diagram is of type $\tilde{B}$, associated modules are contained in in $E(0)$, excluding $\tilde{S}(1)$ and $\tilde{S}(2)$ as rank-1 leaves. Hence we only have to check whether the rank-1 leaves whose multiplicity is not equal to one is of type $S(m,m)$ or $S(m)$. Observe that the distance of the families of parallel reflection hyperplanes associated with m_1 is less then those of m_2. That is, there is a rank-2 leaf whose diagram is , therefore $S(m_2)$ is the only possibility for the rank-1 leaf by Proposition 5.1 on page 63. This solves the case $m_1 = 1$.

The principal isotropy group is the principle isotropy group of the most singular slice representation with diagram D_n or A_3, for adding the m_2-vertex ($m_2 = 1$) does not extend the isotropy group. Therefore it is $\mathrm{SO}(2)^n$ for $m_1 = 2$ and $\mathrm{SO}(3)^4$ for $m_1 = 4$. The same arguments as for $\tilde{A}$-diagrams (cf. proposition 5.2) prove, by considering rank-2 leaves with diagram $\tilde{A}_2$, that the remaining rank-1 leaves are of type $S(m_1)$. □

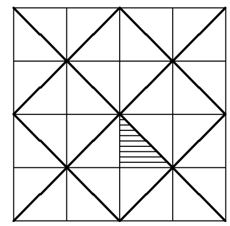
For the case of $\tilde{C}$-diagrams we start with a lemma connecting the irreducible slice representations of rank 2 with the associated modules ∇A:

LEMMA 5.10. *Let $G \cdot a$ be an isoparametric submanifold with affine Dynkin diagram* *. Then the rank 1-leaf is $S(m_1, m_3)$ when*  *or*  *and $\tilde{S}(m_1, m_3)$, when* *, where the arrows denote the length of the roots in the rank-2 slice representation.*

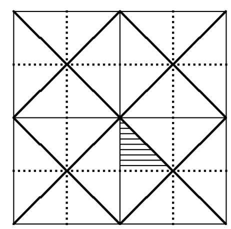
PROOF. Let q be a singular point such that slice representation at q is of type . Remember that the eigenspaces of the s-representation of G/K are given by $\mathfrak{p}_\lambda$, the eigenspaces of $\mathrm{ad}(a)^2$, when $a \in \mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$, cf. Subsection 4.2.1. In the rank-2 case the roots λ are always of the form e_1, e_2, $e_1 + e_2$

and $e_1 - e_2$, where $\mathfrak{a} = \operatorname{span}\{e_1, e_2\}$. Denote by Φ an equivariant map between the slice representation at the point q and the corresponding s-representation. Then $\nu_q(G\cdot q) = \nu_a(G\cdot a) \oplus E_{e_1} \oplus E_{e_2} \oplus E_{e_1+e_2} \oplus E_{e_1-e_2}$, where $\mathfrak{a} = \Phi(\nu_a G\cdot a)$ and $\mathfrak{p}_\lambda = \Phi(E_\lambda)$.

For K-invariant vector fields in $K\cdot a$ holds $\nabla_{\mathfrak{p}_\lambda}\mathfrak{p}_\mu \subseteq \mathfrak{p}_{\lambda\pm\mu}$, therefore $\nabla_{E_{e_1+e_2}}E_{e_1-e_2} = 0$, whereas $\nabla_{E_{e_1}}E_{e_2} \subseteq E_{e_1+e_2} \oplus E_{e_1-e_2}$. In that way the slice representation determines the behavior of associated modules.



In the adjoining figure we denote by thick lines reflection hyperplanes belonging to long roots, with thin lines hyperplanes belonging to short root, i.e. the Weyl group associated with $\underset{m_1}{\circ}\!\Leftarrow\!\underset{m_2}{\circ}\!\Rightarrow\!\underset{m_3}{\circ}$. The diagram $\underset{m_1}{\circ}\!\Rightarrow\!\underset{m_2}{\circ}\!\Leftarrow\!\underset{m_3}{\circ}$ is the same with interchanged thick and thin lines. Thereby the hatched triangle represents the three vertices of the affine Dynkin diagram. In any family of parallel reflection hyperplanes the length of the root is constant. We proof that this implies that associated modules of the corresponding family of eigenspaces are subspaces of $E(0)$. We denote eigenspaces by $E_{(\lambda,i)}$ where λ is a root of any singular slice representation containing $E_{(\lambda,i)}$. It is obvious that for G-invariant vector fields $\nabla_{E_{(\lambda,i)}}E_{(\mu,j)} \subseteq \oplus_{k\in\mathbb{Z}}E_{(\lambda\pm\mu,k)}$ for $\lambda \neq \mu$, since any two reflection hyperplanes which are not parallel intersect in some point. By the Gauß-equation the same holds for $\lambda = \mu$ and therefore $\nabla_{E_{(\lambda,i)}}E_{(\lambda,j)} \subset E(0)$.



The Weyl group for the Dynkin diagram $\underset{m_1}{\circ}\!\Rightarrow\!\underset{m_2}{\circ}\!\Rightarrow\!\underset{m_3}{\circ}$ is shown on the side, where roots of length 1 are denote by thin lines, of length $\sqrt{2}$ by thick lines and of length 2 by dotted lines. Within the families of hyperplanes with the smaller distance roots of length 1 and 2 alternate. In the same manner as in the last case it is proven that associated modules then do not have to be contained in $E(0)$, more precisely do have to contain a certain eigenspace as described in Propositions 2.15 and 2.19. □

Remark. The last proposition is valid for any multiplicities, e.g. for uniform multiplicity 2, where all examples $\circ\!\Rightarrow\!\circ\!\Leftarrow\!\circ$, $\circ\!\Leftarrow\!\circ\!\Rightarrow\!\circ$ and $\circ\!\Rightarrow\!\circ\!\Rightarrow\!\circ$ arise among the σ-actions. This illustrates once more that the lift of the adjoint action of $\mathrm{Sp}(n)$ and the σ-action of $\mathrm{SO}(2n+2)$ are orbit-equivalent, even though, that they have different slice representations, cf. Section 4.8 on page 60.

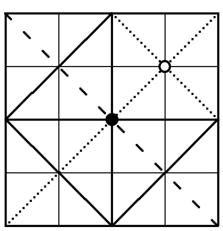
Proposition 5.11. *Let $M = G\cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{C}_n$ with multiplicities m_1, $m_2 \neq m_3$. The possible multiplicities, together with the rank-1 leaves and examples among the $P(G,H)$-actions are given in Table 5.2 on the facing page.*

Proof. By Proposition 5.1 on page 63, it is proven that the rank-1 leaf associated with the vertex in the middle is either $S(m_2)$ or $S(m_2, m_2)$, if $m_2 = 1$ it is $S(1)$. Hence case (1) is solved for $m_1 \neq m_3$. If $m = m_1 = m_3$ the isotropy group is $\mathrm{SO}(m)\times\mathrm{SO}(m)$, since there is a reducible rank-1 leaf with diagram $\underset{m}{\circ} \oplus \underset{m}{\circ}$, and therefore the infinite dimensional rank-1 leaves are of type $S(m,m)$. For $m_1 = 1$, i.e. case (2), additionally $\tilde{S}(1, m_3)$ occurs.

In case (3) the principal isotropy group is $\mathrm{SO}(2)^{n-1}$, the additional families with multiplicity one do not extend the isotropy group. Therefore the rank-1 leaf associated with a vertex in the middle is $S(2)$ as in the $\tilde{A}_n$-case (cf. Proposition 5.2).

	Diagram	rank-1	rank-1	ω-equivalent $P(G,H)$-action
(1)	$\underset{m_1}{\circ}=\underset{1}{\circ}-\underset{1}{\circ}\cdots\underset{1}{\circ}-\underset{1}{\circ}=\underset{m_3}{\circ}$	$S(1)$	$S(m_1,m_3)$	BD I$(1,m_1{+}m_2{+}1)$–I$(m_1{+}1,m_2{+}1)$
(2)	$\underset{1}{\circ}=\underset{1}{\circ}-\underset{1}{\circ}\cdots\underset{1}{\circ}-\underset{1}{\circ}=\underset{m_3}{\circ}$	$S(1)$	$\tilde{S}(1,m_3)$	A I–III$(2,m_3+2)$
(3)	$\underset{1}{\circ}=\underset{2}{\circ}-\underset{2}{\circ}\cdots\underset{2}{\circ}-\underset{2}{\circ}=\underset{1}{\circ}$	$S(2)$	$S(1)$	A III$(\frac{n}{2},\frac{n}{2})$–III$(\frac{n}{2},\frac{n}{2})$
(4)	$\underset{2}{\circ}=\underset{2}{\circ}-\underset{2}{\circ}\cdots\underset{2}{\circ}-\underset{2}{\circ}=\underset{1}{\circ}$	$S(2,2)$	$S(1,2)$	C I–II$(k=\frac{n}{2})$
		$S(2,2)$	$\tilde{S}(1,2)$	D I$(2n+1,2n+1)$–III
(5)	$\underset{1}{\circ}=\underset{4}{\circ}-\underset{4}{\circ}\cdots\underset{4}{\circ}-\underset{4}{\circ}=\underset{1}{\circ}$	$S(4)$	$S(1)$	D III –III$(2n)$ or D III–III'$(2n+1)$
(6)	$\underset{1}{\circ}=\underset{4}{\circ}-\underset{4}{\circ}\cdots\underset{4}{\circ}-\underset{4}{\circ}=\underset{3}{\circ}$	$S(4)$	$S(1,3)$	A II–III$(2n,2n)$
(7)	$\underset{3}{\circ}=\underset{4}{\circ}-\underset{4}{\circ}\cdots\underset{4}{\circ}-\underset{4}{\circ}=\underset{3}{\circ}$	$S(4)$	$S(3)$	C II$(\frac{n}{2},\frac{n}{2})$–II$(\frac{n}{2},\frac{n}{2})$
(8)	$\underset{1}{\circ}=\underset{8}{\circ}-\underset{8}{\circ}=\underset{1}{\circ}$	$S(8)$	$S(1)$	E VII–VII
(9)	$\underset{1}{\circ}=\underset{m_2}{\circ}=\underset{1}{\circ}$	$S(m_2)$	$S(1)$	BD I$(2,m_2+2)$–I$(2,m_2+2)$
(10)	$\underset{4}{\circ}=\underset{3}{\circ}=\underset{4}{\circ}$			does not exist
(11)	$\underset{1}{\circ}=\underset{3}{\circ}=\underset{4}{\circ}$	$S(3)$	$\tilde{S}(1,4)$	E I–III

TABLE 5.2. Actions with Diagram $\tilde{C}$ and nonuniform multiplicity



The principal isotropy group in case (4) for rank 2 (which proves the assertion for higher rank as well) is SO(2) × SO(2). Associated with each line in the affine Weyl group is a curvature normal and therefore an eigenspace together with the factor of the principal isotropy group acting effectively on this eigenspace. Thick lines in the figure stand for a two-dimensional eigenspace. The vertex marked black in the figure represents a singular slice representation of type $\underset{2}{\circ}=\underset{2}{\circ}$, where we indicate the different SO(2)-factors by dotted and dashed lines respectively. Then the singular slice representation of the circled vertex is of type $\underset{1}{\circ}=\underset{2}{\circ}$, and only dotted lines pass through this vertex (the principal isotropy group of the s-representation of $SO(6)/SO(2)\times SO(4)$ is SO(2)). Therefore in the family of eigenspaces associated with diagonal lines in the affine Weyl group the effectively acting factors of the isotropy group alternate, that is the hypersurface is of type $S(2,2)$.

By similar arguments it is easy to determine the hypersurfaces associated with the vertex m_2 in the other cases. It remains to analyze whether it is possible for the occurring rank-1 leaves to be of type $\tilde{S}$. We use Lemma 5.10 on page 69, therefore we need the lengths of the roots of the slice representations, which are $\underset{1}{\circ}\Rightarrow\underset{m}{\circ}$ and $\underset{4}{\circ}\Leftarrow\underset{3}{\circ}$.

For case (10) see the next section. □

REMARK. By our methods we can not exclude the affine Dynkin diagram $\underset{1}{\circ}=\underset{3}{\circ}=\underset{4}{\circ}$, but among the known examples there is no isoparametric submanifold with those diagram.

PROPOSITION 5.12. *Let $M = G\cdot a$ be an infinite dimensional isoparametric submanifold with affine Dynkin diagram $\tilde{F}_4$ with multiplicities $m_1 \neq m_2$.*

$$\underset{m_1}{\circ}-\underset{m_1}{\circ}-\underset{m_1}{\circ}=\underset{m_2}{\circ}-\underset{m_2}{\circ}$$

Diagram	isotropy	rank-1	ω-equivalent $P(G,H)$-action
1 — 1 — 1 = 2 — 2	$\mathrm{SO}(2)^4$	$S(2,2)$	E V–V
1 — 1 — 1 = 4 — 4	$\mathrm{SO}(3)^4$	$S(4)$	E VIII–IX
2 — 2 — 2 = 1 — 1	$\mathrm{SO}(2)^2$	$S(2)$	E II–II
4 — 4 — 4 = 1 — 1	$\mathrm{SO}(3)^3$	$S(4)$	E VII–VII
8 — 8 — 8 = 1 — 1	$\mathrm{Spin}(8)^4$	$S(8)$	E IX–IX

TABLE 5.3. Actions with Diagram $\tilde{F}_4$ and nonuniform multiplicity

Any rank-1 leaf is of type $S(m_i)$ except in the case $m_1 = 2$, $m_2 = 1$, where the rank-1 leaves are $S(2,2)$ and $S(1)$. The possible multiplicities, together with the rank-1 leaves and examples among the $P(G,H)$-actions are given in Table 5.3

PROOF. Since the diagram is of type $\tilde{F}_4$, associated modules are subset of $E(0)$, excluding $\tilde{S}(1)$, $\tilde{S}(2)$ and $\tilde{S}(1,m)$ as a rank-1 leaves.

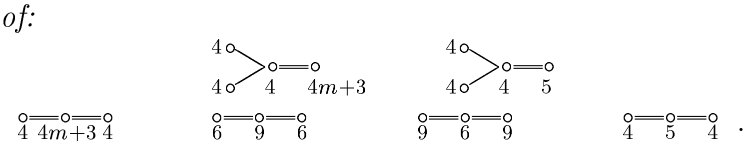
The principal isotropy group is the principle isotropy group of the most singular slice representation with diagram F_4 or C_4, that occurs by omitting the boundary vertex of multiplicity one and may be read off from Table A.5.

Any such manifold contains either a leaf with diagram $\tilde{A}_2$ or $\tilde{A}_3$ with multiplicity 2, 4 or 8. Using Propositions 5.2, 5.7 and 5.8 together with the information about the isotropy group yields the rank-1 leaves. □

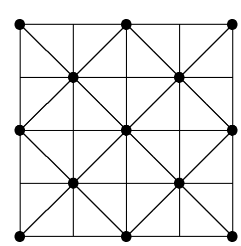
5.4. Exclusion of some Dynkin Diagrams

We exclude in this section some of the marked Dynkin diagrams who do not arise among $P(G,H)$-actions.

PROPOSITION 5.13. *There is no isoparametric submanifold whose marked Dynkin diagram is one of:*

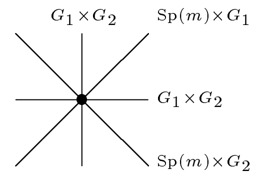


PROOF. We start with the diagrams $9 = 6 = 9$ (the diagram $6 = 9 = 6$ does not exist by the same arguments).



There are two different types of most singular slice representations: the principal isotropy group of the s–representation with diagram $6 = 9$ is $\mathrm{U}(4)$ (vertices marked black in the adjoining figure), while the one with diagram $9 \oplus 9$ is $\mathrm{U}(4) \times \mathrm{U}(4)$. Therefore the principal isotropy group of the manifold is $\mathrm{U}(4) \times \mathrm{U}(4)$. In black vertices only eigenspaces meet whose effectively acting factor is the same, while in the others two different meet. One sees immediately that this is not possible, since any two non parallel and non orthogonal hyperplanes do meet in some black vertex.

We remark, that the same holds if the slice representation is not the s-representation of E III, but its orbit-equivalent subaction with principal isotropy group $\mathrm{SU}(4)$.



Next we exclude the diagram $\underset{4}{\circ}\!=\!\underset{4m+3}{\circ}\!=\!\underset{4}{\circ}$, which also excludes the $\tilde{B}_3$-diagram with multiplicities 4 and $4m+3$. The principal isotropy group of the s-representation $\underset{4}{\circ}\!=\!\underset{4m+3}{\circ}$ is $\mathrm{Sp}(m) \times \mathrm{SO}(3) \times \mathrm{SO}(3)$, we denote the two SO(3)-factors by G_1 and G_2. Then in a black vertex the effectively acting parts are as the figure shows. Again this provides a contradiction: for the line marked $\mathrm{Sp}(m) \times G_1$ intersects any vertical line, any of those has to contain the factor G_1. This contradicts the fact that in singular point of type $\underset{4}{\circ} \oplus \underset{4}{\circ}$ the lines have non effectively acting factor in common.

The exclusion of $\underset{4}{\circ}\!=\!\underset{5}{\circ}\!=\!\underset{4}{\circ}$ works by the same arguments, replacing $\mathrm{Sp}(m)$ by U(1). Remark that the U(1)-factor is not essentially for the contradiction, that is, the argument does work if the slice representation of type $\underset{5}{\circ}\!=\!\underset{4}{\circ}$ is the isotropy representation of $G/K' = \mathrm{SO}(10)/\mathrm{SU}(5)$. □

Open Problem 5.14. *Comparing the last proposition with the possible affine Dynkin diagrams, who do not arise among the $P(G,H)$-action (cf. section 4.7 on page 59), leads to the following question:*
Is it possible to have an infinite dimensional isoparametric submanifold, whose affine Dynkin diagram is either

$$\underset{1}{\circ}\!=\!\underset{5}{\circ}\!=\!\underset{4}{\circ} \qquad \underset{1}{\circ}\!=\!\underset{5}{\circ}\!=\!\underset{2}{\circ} \; ?$$

Note that these examples have to have slice representations that are not s-representations.

5.5. Some remarks on slice representations, that are not s-representations

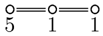
We have listed the actions, which are transitive on spheres in Section 2.2 on page 14, most exceptional cohomogeneity one examples (cf. Subsection 4.6.2) have slice representations of that type. For cohomogeneity greater than one there is a short list of polar representation, that are not s-representation, cf. [EH99]. In Table 5.4 we have

Range	G	K	isotr.	K'	isotr.
	SO(9)	$\mathrm{SO}(2) \times \mathrm{SO}(7)$	SO(5)	$\mathrm{SO}(2) \times \mathrm{G}_2$	SU(2)
	SO(10)	$\mathrm{SO}(2) \times \mathrm{SO}(8)$	SO(6)	$\mathrm{SO}(2) \times \mathrm{Spin}(7)$	SU(3)
	SO(11)	$\mathrm{SO}(3) \times \mathrm{SO}(8)$	SO(5)	$\mathrm{SO}(3) \times \mathrm{Spin}(7)$	SU(2)
$m \neq 0$	$\mathrm{SU}(m{+}2k)$	$\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(m{+}k))$	$\mathrm{U}(m)$	$\mathrm{SU}(k)\times\mathrm{SU}(m{+}k)$	$\mathrm{SU}(m)$
n odd	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$\mathrm{SU}(2)^n \cdot \mathrm{U}(1)$	$\mathrm{SU}(n)$	$\mathrm{SU}(2)^n$
	E_6	$\mathrm{SO}(2) \cdot \mathrm{Spin}(10)$	U(4)	Spin(10)	SU(4)

Table 5.4. Orbit equivalent subactions of polar representations

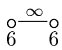
listed these examples, which arise from an s-representation by restricting the symmetric subgroup K to a group $K' \subset K$, together with their principal isotropy groups. Note that only in the second example the eigenspaces remain to be irreducible modules of the eigenspaces. Hence our assumption, that the slice representation is an s-representation for irreducible eigenspaces is not very restrictive.

In [Kol05, Table 1] Kollross gave a list of orbit-equivalent actions of Hermann actions of a group H action on a symmetric space G/K whose rank is greater than one. Assume we have a Hermann action with a most singular slice representation that is an s-representation which admits a orbit-equivalent subrepresentation. Then in most cases the list in [Kol05] shows that one can restrict one of the groups K_i to a subgroup

K_i', and thus restrict the most singular slice representation as described in Table 5.4. The only exceptions of codimension at least 2 are the actions A I-III with diagram  and D III-III for odd n.

There are no examples of codimension greater than one known with slice representations that are not s-representations, which are not orbit-equivalent to Hermann examples.

It is a priori not clear whether orbit-equivalence of the Hermann actions yields orbit-equivalence of the $P(G,H)$-action. In fact this is not true for some examples of cohomogeneity one. We will briefly explain this by an example with irreducible eigenspaces:

Consider the $P(G,H)$-action with $G = \mathrm{Spin}(7)$ and $H = \mathrm{G}_2 \times \mathrm{G}_2$, whose diagram is . Let $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{p}$ be the orthogonal decomposition, although this is not a Cartan decomposition, the eigenspaces may be derived quite similar as described in Section 4.2. The Lie algebra $\mathfrak{g}_2$ has dimension 14, 6 belong to eigenspaces E_n, which leaves an 8-dimensional subspace $\mathfrak{h}$, commuting with the section $\mathfrak{a} \subset \mathfrak{p}$. The eigenspace $E(0) = L^2(\mathfrak{h} \oplus \mathfrak{a})$ (respecting the boundary values) and the associated modules are one- and 8-dimensional: the isotropy representation on eigenspaces is the 6-dimensional representation of $\mathrm{SU}(3)$ (acting as a subgroup of $\mathrm{SO}(6)$). Remember that the modules on $E(0)$ arise as irreducibles modules of the tensor product decomposition. If $\mathrm{SO}(6)$ is restricted to $\mathrm{SU}(3)$, then the $\Lambda^2(6)$-module, which is 15-dimensional, decomposes into a 7- and a 8-dimensional irreducible module. The 7-dimensional has to vanish here, vaguely speaking since there is no space left for them in $L^2(\mathfrak{h} \oplus \mathfrak{a})$.

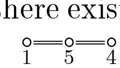
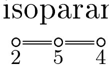
On the other hand the $P(G,H)$ action with $G = \mathrm{SO}(8)$ and $H = \mathrm{SO}(7) \times \mathrm{SO}(7)$ has the same diagram, but its irreducible modules in $E(0)$ are one- and 15-dimensional. Moreover the difference of the dimensions of $\mathrm{SO}(8)$ and $\mathrm{Spin}(7)$ is 7, these contain precisely the part of $E(0)$, that is missing in the other case.

The orbit-equivalent subactions of Hermann type of higher codimension are different, here the group G stays always the same. Consider for example $G = \mathrm{SO}(n)$, $K_1 = \mathrm{SO}(2) \times \mathrm{SO}(n-2)$ and $K_2 = \mathrm{SO}(8) \times \mathrm{SO}(n-8)$, then the action with $K_2' = \mathrm{Spin}(7) \times \mathrm{SO}(n-8)$ is orbit-equivalent. The description of the eigenspaces of the lifted action bases upon the decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{k}_2 \oplus \mathfrak{p}_2 = (\mathfrak{k}_1 \cap \mathfrak{k}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{p}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{k}_2),$$

cf. Section 4.3 on page 45. Replacing K_2 by K_2' changes the dimensions of these: Those involving $\mathfrak{k}_2$ are decreased by 7 dimension, while the other are increased by 7 dimensions. This does not change the eigenspaces, since the multiplicities stay the same, but alternates $E(0)$ in the sense that some basis vectors of the form $\sin 2n\vartheta K_i$ are replaced by $\cos 2n\vartheta K_i$. Although the 15-dimensional modules of the original action decompose into a 7- and a 8-dimensional one, $E(0)$ provides enough space for both of them. I conjecture that this does not change the geometry of the action.

Conjecture. Any polar representation on a Hilbert space with cohomogeneity at least two, whose singular slice representations are not necessarily s-representations is orbit-equivalent to a polar representation whose singular slice representations are s-representations, in fact to a $P(G,H)$-action.

In particular there exists no isoparametric submanifold whose marked affine Dynkin diagram is either  or .

The proof of this conjecture is twofold: First it is necessary to study isoparametric submanifold of Hilbert space, whose eigenspace are not irreducible modules of the isotropy representation, but whose slice representations are s-representations. This could be done along the same line as for the isotropy irreducible case, by first studying hypersurfaces (Chapters 2 and 3) and then investigate the rank-1 leaves of isoparametric submanifolds of higher codimension (where only 11 different affine marked Dynkin diagram are possible, cf. Table 4.1 on page 41). This would prove that in fact any isoparametric submanifold with cohomogeneity greater than one is isometric to a principal orbit of some $P(G, H)$-action.

Moreover it would be interesting to investigate homogeneous isoparametric submanifolds whose slice representations are not s-representations. If the above conjecture is true, it remains to classify the polar infinite dimensional cohomogeneity one actions. Most likely these will turn out to be the principal orbits of exceptional cohomogeneity one actions of $P(G, H)$-type.

APPENDIX

Tables

In this appendix we collect the geometric data of Hermann-actions developed in Chapter 4. Table A.1 contains the affine marked Dynkin diagrams of $P(G,H)$-action for classical Lie groups SO(n), SU(n) and Sp(n). D III' denotes $\mathrm{SO}(2n)/\alpha(\mathrm{SU}(n))$ where α is the non-trivial diagram automorphism of SO($2n$) and D_4 I' denotes the symmetric space $\mathrm{Spin}(8)/\tau(\mathrm{Spin}(l) \times \mathrm{Spin}(8-l))$, with τ the diagram automorphism of order three of Spin(8).

Multiplicities of the form $2n+1$, $4m+3$ or 5 do always belong to reducible eigenspaces, that is the effectively acting factor of the principal isotropy group is U(m), Sp(m) or U(2) respectively, cf. also table A.2, where the (effectivized) irreducible most singular slice representation are listed. Note that if the column "second slice representation" is left empty, there is only one most singular orbit type.

In Table A.3 and A.4 the same is done for Hermann-actions on the exceptional Lie groups.

Table A.5 contains Dynkin diagrams and principal isotropy groups of s-representations, taken from [HH70]. We remark that $\mathrm{SO}'(2n)$ denotes the image of a half-spin representation of Spin($2n$). The rank of the examples on classical groups is always n. There is only stated the isomorphism class of the connected component of the principal isotropy group, if it is not finite.

Action	G	K_1	K_2	Case	Diagram	m_1	m	m_2
A I–I	SU(n)	SO(n)	SO(n)		$\tilde{A}_{n-1}$		1	
A I–II	SU($2n$)	SO($2n$)	Sp(n)		$\tilde{A}_{n-1}$		2	
A I–III	SU(n)	SO(n)	S(U(k)×U($n-k$))	$1 \leq k < \frac{n}{2}$	$\tilde{C}_k$	$n-2k$	1	1
				$k = \frac{n}{2}$	$\tilde{B}_k$	1	1	
A II–II	SU($2n$)	Sp(n)	Sp(n)		$\tilde{A}_{n-1}$		4	
A II–III	SU($2n$)	Sp(n)	S(U(k)×U($2n-k$))	k even	$\tilde{C}_{\frac{k}{2}}$	$4(n-k)+3$	4	1
				k odd	$\tilde{C}_{\frac{k-1}{2}}$	$4(n-k)+3$	4	5
A III–III	SU(n)	S(U(k)×U($n-l$))	S(U(l)×U($n-l$))	$1 \leq k \leq l \leq \frac{n}{2}$	$\tilde{C}_k$	$2(n-l-k)+1$	2	$2(l-k)+1$
BD I–I	SO(n)	SO(k)×SO($n-k$)	SO(l)×SO($n-l$)	$1 < k < l \leq \frac{n}{2}$	$\tilde{C}_k$	$n-k-l$	1	$l-k$
				$1 = k < l \leq \frac{n}{2}$	$\tilde{A}_1$	$n-l-1$		$l-1$
				$1 = k = l$	$\tilde{A}_1$	$n-2$		$n-2$
				$k = l = 2,\ n \geq 5$	$\tilde{C}_2$	1	$n-4$	1
				$k = l = 3,\ n = 6$	$\tilde{A}_3$		1	
				$k = l \leq \frac{n}{2}$	$\tilde{B}_k$	$n-2k$	1	
				$k = l = \frac{n}{2} \geq 4$	$\tilde{D}_k$		1	
D I–III	SO($2n$)	SO(k)×SO($n-k$)	U(n)	$k = 2$	$\tilde{A}_1$	$2(n-2)+1$		$2(n-2)+1$
				$k = 4$	$\tilde{C}_2$	2	$2(n-4)+1$	2
				$k \geq 6$ even	$\tilde{B}_{\frac{k}{2}}$	$2(n-k)+1$	2	
				$k = 3$	$\tilde{A}_1$	$2(n-3)+1$		2
				$k \geq 5$ odd	$\tilde{C}_{\frac{k-1}{2}}$	$2(n-k)+1$	2	2
D III–III	SO($2n$)	U(n)	U(n)	n even	$\tilde{C}_{\frac{n}{2}}$	1	4	1
				n odd	$\tilde{C}_{\frac{n-1}{2}}$	1	4	5
D III–III'	SO($2n$)	U(n)	α(U(n))	n even	$\tilde{C}_{\frac{n}{2}-1}$	5	4	5
D_4 I–I'	Spin(8)	Spin(3)×Spin(5)	τ(Spin(3)×Spin(5))	$k = l = 3$	$\tilde{G}_2$	1		1
C I–I	Sp(n)	U(n)	U(n)		$\tilde{C}_n$	1	1	1
C I–II	Sp(n)	U(n)	Sp(k)×Sp($n-k$)	$k \leq \frac{n}{2}$	$\tilde{C}_k$	$2(n-2k)+1$	2	2
C II–II	Sp(n)	Sp(k)×Sp($n-k$)	Sp(l)×Sp($n-l$)	$k \leq l \leq \frac{n}{2}$	$\tilde{C}_k$	$4(n-k-l)+3$	4	$4(l-k)+3$

TABLE A.1. Affine marked Dynkin diagrams of Hermann-actions on the classical Lie groups

Action	first slice represention	second slice represention
A I–I	$\mathrm{SU}(n+1)/\mathrm{SO}(n+1)$	
A I–II	$\mathrm{SU}(n-1)\times\mathrm{SU}(n-1)/\mathrm{SU}(n-1)$	
A I–III	$\mathrm{SO}(n)/\mathrm{SO}(k)\times\mathrm{SO}(n-k)$	$\mathrm{Sp}(k)/\mathrm{U}(k)$
A II–II	$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	
A II–III	$\mathrm{Sp}(n+k)/\mathrm{Sp}(n)\times\mathrm{Sp}(k)$	$\mathrm{SO}(2k)/\mathrm{SU}(k)$
A III–III	$\mathrm{SU}(n+k-l)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-l))$	$\mathrm{SU}(k+l)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(l))$
BD I–I	$\mathrm{SO}(n+k-l)/\mathrm{SO}(k)\times\mathrm{SO}(n-l)$	$\mathrm{SO}(k+l)/\mathrm{SO}(k)\times\mathrm{SO}(l)$
D I–III	$\mathrm{SU}(n-\frac{k}{2}+\lfloor\frac{k}{2}\rfloor)/\mathrm{S}(\mathrm{U}(\lfloor\frac{k}{2}\rfloor)\times\mathrm{U}(n-\lceil\frac{k}{2}\rceil))$	$\mathrm{SO}(\lfloor\frac{k}{2}\rfloor)\times\mathrm{SO}(\lfloor\frac{k}{2}\rfloor)/\mathrm{SO}(\lfloor\frac{k}{2}\rfloor)$
D III–III	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathrm{SO}(2n-2)/\mathrm{U}(n-1)$ if n odd
D III–III’	$\mathrm{SO}(2n-2)/\mathrm{U}(n-1)$	
D_4I–I’	$\mathrm{G}_2/\mathrm{SO}(4)$	$\mathrm{SU}(3)/\mathrm{SO}(3)$
C I–I	$\mathrm{Sp}(n)/\mathrm{U}(n)$	
C I–II	$\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k))$	$\mathrm{SO}(2k+1)\times\mathrm{SO}(2k+1)/\mathrm{SO}(2k+1)$
C II–II	$\mathrm{Sp}(n+k-l)/\mathrm{Sp}(k)\times\mathrm{Sp}(n-l)$	$\mathrm{Sp}(k+l)/\mathrm{Sp}(k)\times\mathrm{Sp}(l)$

TABLE A.2. Most singular slice representations of Hermann-actions on the classical Lie groups

Action	G	K_1	K_2	Diagram	m_1	m	m_2
E I–I	E_6	$Sp(4)/\mathbb{Z}_2$	$Sp(4)/\mathbb{Z}_2$	$\tilde{E}_6$		1	
E I–II	E_6	$Sp(4)/\mathbb{Z}_2$	$SU(6)\cdot SU(2)$	$\tilde{F}_4$	1		1
E I–III	E_6	$Sp(4)/\mathbb{Z}_2$	$Spin(10)\cdot SO(2)$	$\tilde{C}_2$	1	3	4
E I–IV	E_6	$Sp(4)/\mathbb{Z}_2$	F_4	$\tilde{A}_2$		4	
E II–II	E_6	$SU(6)\cdot SU(2)$	$SU(6)\cdot SU(2)$	$\tilde{F}_4$	1		2
E II–III	E_6	$SU(6)\cdot SU(2)$	$Spin(10)\cdot SO(2)$	$\tilde{C}_2$	2	5	4
E II–IV	E_6	$SU(6)\cdot SU(2)$	F_4	$\tilde{A}_1$	11		5
E III–III	E_6	$Spin(10)\cdot SO(2)$	$Spin(10)\cdot SO(2)$	$\tilde{C}_2$	9	6	1
E III–IV	E_6	$Spin(10)\cdot SO(2)$	F_4	$\tilde{A}_1$	15		15
E IV–IV	E_6	F_4	F_4	$\tilde{A}_2$		8	
E V–V	E_7	$SU(8)/\mathbb{Z}_2$	$SU(8)/\mathbb{Z}_2$	$\tilde{E}_7$		1	
E V–VI	E_7	$SU(8)/\mathbb{Z}_2$	$Spin(12)\cdot SU(2)$	$\tilde{F}_4$	2		1
E V–VII	E_7	$SU(8)/\mathbb{Z}_2$	$E_6\cdot SU(2)$	$\tilde{B}_3$	1		4
E VI–VI	E_7	$SO'(12)\cdot SU(2)$	$SO'(12)\cdot SU(2)$	$\tilde{F}_4$	1		4
E VI–VII	E_7	$SO'(12)\cdot SU(2)$	$E_6\cdot SO(2)$	$\tilde{C}_2$	6	9	2
E VII–VII	E_7	$E_6\cdot SO(2)$	$E_6\cdot SO(2)$	$\tilde{C}_3$	1	8	1
E VIII–VIII	E_8	$SO'(16)$	$SO'(16)$	$\tilde{E}_8$		1	
E VIII–IX	E_8	$SO'(16))$	$E_7\cdot SU(2)$	$\tilde{F}_4$	4		1
E IX–IX	E_8	$E_7\cdot SU(2)$	$E_7\cdot SU(2)$	$\tilde{F}_4$	1		8
F I–I	F_4	$Sp(3)\cdot Sp(1)$	$Sp(3)\cdot Sp(1)$	$\tilde{F}_4$	1		1
F I–II	F_4	$Sp(3)\cdot Sp(1)$	$Spin(9)$	$\tilde{A}_1$	7		7
F II–II	F_4	$Spin(9)$	$Spin(9)$	$\tilde{A}_1$	15		7
G I–I	G_2	$SO(4)$	$SO(4)$	$\tilde{G}_2$		1	

TABLE A.3. Affine marked Dynkin diagrams of Hermann-actions on the exceptional Lie groups

Action	first slice represention	second slice represention	third slice rep.
E I–I	$E_6/(Sp(4)/\mathbb{Z}_2)$		
E I–II	$F_4/Sp(3)\cdot Sp(1)$	$Sp(4)/U(4)$	
E I–III	$Sp(4)/Sp(2)\times Sp(2)$	$SO(7)/SO(2)\times SO(5)$	
E I–IV	$SU(6)/Sp(3)$		
E II–II	$E_6/SU(6)\cdot SU(2)$	$SO(10)/SO(4)\times SO(6)$	
E II–III	$SO(10)/U(5)$	$SU(6)/S(U(4)\times U(2))$	
E II–IV	$Sp(4)/Sp(1)\times Sp(3)$	$SO(7)/SO(6)$	
E III–III	$E_6/Spin(10)\cdot SO(2)$	$SO(10)/SO(2)\times SO(8)$	
E III–IV	$F_4/Spin(9)$		
E IV–IV	E_6/F_4		
E V–V	$E_7/(SU(8)/\mathbb{Z}_2)$	$SU(8)/SO(8)$	
E V–VI	$E_6/SU(6)\cdot SU(2)$	$SU(8)/S(U(4)\times U(4))$	
E V–VII	$SO(12)/U(6)$	$SU(8)/Sp(4)$	
E VI–VI	$E_7/SO'(12)\cdot SU(2)$	$SO(12)/SO(4)\times SO(8)$	
E VI–VII	$E_6/Spin(10)\cdot SO(2)$	$SU(8)/S(U(6)\times U(2))$	
E VII–VII	$E_7/E_6\cdot SO(2)$		
E VIII–VIII	$E_8/SO'(16)$	$SO(16)/SO(8)\times SO(8)$	$SU(9)/SO(9)$
E VIII–IX	$E_7/SO'(12)\cdot SU(2)$	$SO(16)/SU(8)$	
E IX–IX	$E_8/E_7\cdot SU(2)$	$SO(16)/SO(4)\times SO(12)$	
F I–I	$F_4/Sp(3)\cdot Sp(1)$	$SO(9)/SO(4)\times SO(5)$	
F I–II	$Sp(3)/Sp(2)\cdot Sp(1)$		
F II–II	$F_4/Spin(9)$	$SO(9)/SO(1)\times SO(8)$	
G I–I	$G_2/SO(4)$	$SO(5)/SO(2)\times SO(3)$	

TABLE A.4. Most singular slice representations of Hermann-actions on the exceptional Lie groups

Type	G/K	Diagram	Isotropy
A I	$\mathrm{SU}(n+1)/\mathrm{SO}(n+1)$	1—1—1···1—1—1	$\mathbb{Z}_2^n$
A II	$\mathrm{SU}(2n+2)/\mathrm{Sp}(n+1)$	4—4—4···4—4—4	$\mathrm{Sp}(1)^{n+1}$
A III	$\mathrm{SU}(2n+m)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n+m))$	2—2—2···2—2═$2m{+}1$	$\mathrm{SU}(m)\cdot\mathrm{U}(1)^n$
BD I	$\mathrm{SO}(2n+m)/\mathrm{SO}(n)\times\mathrm{SO}(n+m)$	1—1—1···1—1═m	$\mathrm{SO}(m)$
BD I	$\mathrm{SO}(2n)/\mathrm{SO}(n)\times\mathrm{SO}(n)$	1—1—1···1—1—1 (with 1 attached to the fifth node)	$\mathbb{Z}_2^n$
D III	$\mathrm{SO}(4n)/\mathrm{U}(2n)$	4—4—4···4—4═1	$\mathrm{SU}(2)^n$
D III	$\mathrm{SO}(4n+2)/\mathrm{U}(2n+1)$	4—4—4···4—4═5	$\mathrm{SU}(2)^n\cdot\mathrm{U}(1)$
C I	$\mathrm{Sp}(n)/\mathrm{U}(n)$	1—1—1···1—1═1	$\mathbb{Z}_2^n$
C II	$\mathrm{Sp}(2n+m)/\mathrm{Sp}(n)\times\mathrm{Sp}(n+m)$	4—4—4···4—4═$4m{+}3$	$\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)^n$
E I	$\mathrm{E}_6/\left(\mathrm{Sp}(4)/\{\pm1\}\right)$	1—1—1—1—1 (with 1 attached to the middle node)	$\mathbb{Z}_2^6$
E II	$\mathrm{E}_6/\mathrm{SU}(6)\cdot\mathrm{SU}(2)$	1—1═2—2	$\mathbb{Z}_2^2\times\mathrm{SO}(2)^2$
E III	$\mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{SO}(2)$	9═6	$\mathrm{U}(4)$
E IV	$\mathrm{E}_6/\mathrm{F}_4$	8—8	$\mathrm{Spin}(8)$
E V	$\mathrm{E}_7/\left(\mathrm{SU}(8)/\{\pm1\}\right)$	1—1—1—1—1—1 (with 1 attached to the third node)	$\mathbb{Z}_2^7$
E VI	$\mathrm{E}_7/\mathrm{SO}'(12)\cdot\mathrm{SU}(2)$	1—1═4—4	$\mathbb{Z}_2^2\times\mathrm{Sp}(1)^3$
E VII	$\mathrm{E}_7/\mathrm{E}_6\cdot\mathrm{SO}(2)$	8—8═1	$\mathrm{Spin}(8)$
E VIII	$\mathrm{E}_8/\mathrm{SO}'(16)$	1—1—1—1—1—1—1 (with 1 attached to the third node)	$\mathbb{Z}_2^8$
E IX	$\mathrm{E}_8/\mathrm{E}_7\cdot\mathrm{SU}(2)$	8—8═1—1	$\mathbb{Z}_2^2\times\mathrm{Spin}(8)$
F I	$\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$	1—1═1—1	$\mathbb{Z}_2^4$
F II	$\mathrm{F}_4/\mathrm{Spin}(9)$	15	$\mathrm{Spin}(7)$
G	$\mathrm{G}_2/\mathrm{SO}(4)$	1≡1	$\mathbb{Z}_2^2$

TABLE A.5. Dynkin diagrams and principal isotropy groups of s-representations

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