Escape driven by α-stable white noises

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We explore the archetype problem of an escape dynamics occurring in a symmetric double well potential when the Brownian particle is driven by white Lévy noise in a dynamical regime where inertial effects can safely be neglected. The behavior of escaping trajectories from one well to another is investigated by pointing to the special character that underpins the noise-induced discontinuity which is caused by the generalized Brownian paths that jump beyond the barrier location without actually hitting it. This fact implies that the boundary conditions for the mean first passage time (MFPT) are no longer determined by the well-known local boundary conditions that characterize the case with normal diffusion. By numerically implementing properly the set up boundary conditions, we investigate the survival probability and the average escape time as a function of the corresponding Lévy white noise parameters. Depending on the value of the skewness β of the Lévy noise, the escape can either become enhanced or suppressed: a negative asymmetry parameter β typically yields a decrease for the escape rate while the rate itself depicts a non-monotonic behavior as a function of the stability index α that characterizes the jump length distribution of Lévy noise, exhibiting a marked discontinuity at $\alpha=1$. We find that the typical factor of 2 that characterizes for normal diffusion the ratio between the MFPT for well-bottom-to-well-bottom and well-bottom-to-barrier-top no longer holds true. For sufficiently high barriers the survival probabilities assume an exponential behavior versus time. Distinct non-exponential deviations occur, however, for low barrier heights.

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I. INTRODUCTION

The noise driven escape from a deterministically metastable state is a theme that impacts many phenomena in diverse fields of natural sciences [1–4]. In particular, as a notable model for a chemical reaction Kramers [1] pioneered the problem of an escape of a Brownian particle of mass m moving in a potential $V(x)$ with local minima corresponding to an initial reactant and a final product state. In this scenario, both states are assumed to be separated by a barrier located at a position x_b. In the spatial-diffusion-limited regime, the Kramers rate theory is based on a stochastic dynamics that does not involve inertial effects and thus is described by an overdamped Langevin dynamics, reading

$$\frac{dx}{dt} = v = -\frac{1}{\eta}V'(x) + \sqrt{\frac{k_B T}{\eta}}\tilde{\zeta}(t). \qquad (1)$$

Here, $\tilde{\zeta}(t)$ constitutes a white Gaussian noise process with correlations $\langle\tilde{\zeta}(t)\tilde{\zeta}(s)\rangle=2\delta(t-s)$, representing thermal fluctuations whose intensity is scaled by the friction η. The escape problem then concerns the surmounting of an energetic barrier for stochastic trajectories that predominantly dwell the neighborhood of separating attractors which in this case are made of two neighboring potential wells. The imposed quasistationarity condition is realized by assuming special boundary conditions with respect to the time evolution equation for the probability density. In the overdamped regime, the evolution equation for the probability density $p(x,t)$ follows the Smoluchowski dynamics

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}J(x,t), \qquad (2)$$

where

$$J(x,t) = -\frac{1}{\eta}V'(x)p(x,t) - \frac{k_B T}{\eta}\frac{\partial p(x,t)}{\partial x}, \qquad (3)$$

and the stationarity approximation describes escape events which correspond to a constant, nonvanishing flux of probability $J_s=J(x)$. Those stochastic realizations of the process that have surpassed the barrier top are immediately absorbed and reinserted into the original attractor region. In this way a steady probability flow across the activated barrier state located between the locally stable states of "reactants" and "products" is established. Escaped trajectories, absorbed at a position larger than the barrier location $x_a > x_b$ require that $p_s(x_a)=0$ on the whole half-line $x_a > x_b$. The rate formulation is then based on the "flux over the population" method [2,5], yielding in this case of a Smoluchowski dynamics the celebrated result [1–4]

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$$k = \frac{\omega_b}{\eta} \frac{\omega_w}{2\pi} \exp\left[-\frac{\Delta V}{k_B T}\right], \tag{4}$$

with ΔV denoting the activation energy $\Delta V = V(x_b) - V(x_w)$ and ω_w, ω_b representing the frequencies of the particle's motion near the metastable potential minimum x_w and at the top of the crossed barrier x_b, respectively. This result represents a feasible estimate for the actual reaction rate, if all trajectories ejected by the source properly thermalize before eventual thermal fluctuations drive them out of the initial well and, most importantly, a distinct time-scale separation between escape dynamics and intrawell relaxation holds true. This latter requirement of a clear-cut time-scale separation is at the basis for the description of the escape dynamics in terms of a (time-independent) rate coefficient [2,5].

Yet another alternative to the approaches discussed above is rooted in the concept of the mean first passage time (MFPT), i.e., the average time that a random walker starting out from a point x_0 inside the initial domain of attraction, assumes in order to leave the attracting domain for the first time [2–4]. Put differently, the MFPT is the average time needed to cross the deterministic separatrix manifold for the first time [2,6,7]. At weak noise the MFPT becomes essentially independent of the starting point, i.e., $t(x_0) \approx T_{\text{MFPT}}$ for all starting configurations away from the immediate neighborhood of the separatrix. Given the fact that the crossing of the separatrix in either direction equals for *normal diffusion* one half, the total escape time equals $2T_{\text{MFPT}}$ and thus the rate of escape k itself becomes in this case

$$k = \frac{1}{2T_{\text{MFPT}}}. \tag{5}$$

Characterization of the escape rate by use of the MFPT is a rather complex notion for a general class of stochastic processes. In particular, the MFPT analysis requires the choice of a correct boundary condition [6,7]. These are well known for one-dimensional stochastic diffusion Markov processes $x(t)$ which are of the Fokker-Planck form, Eqs. (2) and (3) or for one dimensional master equations with birth and death kinetics. With generally non-Gaussian white noise the knowledge of the boundary location alone typically cannot specify in full the corresponding boundary conditions for, say, absorption or reflection, respectively [6,8]. In particular, the trajectories driven by non-Gaussian white noise depict discontinuous jumps. As a consequence, the location of the boundary itself is not hit by the majority of discontinuous sample trajectories. This implies that regimes beyond the location of the boundaries must be properly accounted for when setting up the boundary conditions. Most importantly, returns (i.e., so termed recrossings of the boundary location) from excursions beyond the specified state space back into this very finite interval where the process proceeds must be excluded.

Following our reasoning in discussing Lévy-Brownian motion on finite intervals [8], we present in this paper the analysis of escape events of a noninertial, generalized diffusion process which is driven by Lévy noise dwelling a symmetric double-well potential. Our investigation thus complements and extends earlier studies of the escape problem driven by symmetric Lévy white noises [9–11].

II. WHITE LÉVY NOISE

The standard definition of the Gaussian white noise specifies the latter as a time derivative of the Wiener process [i.e., a derivative of a stationary process with independent and Gaussian-distributed increments whose covariance is given by $\langle W(t)W(s)\rangle = \min(t,s)$]. A classical Brownian motion (the Wiener process) can be therefore represented as a limit in distribution of independent Gaussian jumps taken at infinitesimally short time intervals of nonrandom length $1/n$. Alternatively (following, e.g., the definition of Feller [12–14]), the Wiener process can be thought of as a limiting process of random Gaussian jumps at random Poissonian jump times

$$W(t) = \lim_{n\to\infty} \sum_{k=1}^{[tn]} W_k\left(\frac{1}{n}\right) = \lim_{n\to\infty} W\left(\frac{N(nt)}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{N(tn)} W_k. \tag{6}$$

The symbol $[tn]$ stands for an integer number of jumps which for a Poisson counting process $N(nt), t \geqslant 0$ with the mean $\langle N(nt)\rangle = nt$ are assumed to be independent (decoupled) of *i.i.d* random variables W_k sampled from the Gaussian distribution. The equality sign in Eq. (6) denotes a limit in distribution sense. For $n\to\infty$, the Poisson distribution becomes peaked around $k = nt$ and the limiting process $W(t), t \geqslant 0$ tends to a Brownian motion diffusion for which the cumulative distribution function reads [15]

$$\begin{aligned}
\text{Prob}\{W(t) \leqslant w\} &= \lim_{n\to\infty} \sum_{k=0}^{\infty} \text{Prob}\left\{ W\left(\frac{k}{n} \leqslant w\right)\right\} \\
&\quad \times \text{Prob}\{N(nt) = k\} \\
&= \lim_{n\to\infty} \int_{-\infty}^{w} \sum_{k=0}^{\infty} (2\pi k/n)^{-1/2} \\
&\quad \times \exp(-nx^2/2k) \frac{(nt)^k}{k!} \exp(-nt) dx \\
&= \int_{-\infty}^{w} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} dx.
\end{aligned} \tag{7}$$

Here, we introduce a non-Gaussian white noise as a derivative of the generalized Wiener process $W_{\alpha,\beta}(t)$, i.e., a non-Gaussian random process with stationary and independent increments. The increments of such generalized Wiener process have the α-stable distribution with the stability index α and the time increment $\Delta t^{1/\alpha}$ as a scale parameter

$$W_{\alpha,\beta}(t) = \int_0^t \zeta(s) ds = \int_0^t dL_{\alpha,\beta}(s) \approx \sum_{i=0}^{N-1} (\Delta s)^{1/\alpha} \zeta_i, \tag{8}$$

where ζ_i are independent random variables distributed with the stable, Lévy probability density function (PDF) $L_{\alpha,\beta}(\zeta;\sigma,\mu=0)$ and $N\Delta s = t - t_0$. The parameter α denotes the stability index, yielding the asymptotic power law for the jump length distribution for $\alpha<2$ being proportional to $|\zeta|^{-(1+\alpha)}$. The parameter σ characterizes a scale, β defines an asymmetry (skewness) of the distribution, whereas μ denotes the location parameter. Throughout the paper, we deal only with strictly stable distributions not exhibiting a drift regime; this implies a vanishing location parameter $\mu=0$ in the remaining part of this work. For $\alpha\neq 1$, the characteristic function $\phi(k)=\int_{-\infty}^{\infty}d\zeta e^{-ik\zeta}L_{\alpha,\beta}(\zeta;\sigma,\mu)$ of an α-stable random variable ζ can be represented by

$$\phi(k)=\exp\left[-\sigma^{\alpha}|k|^{\alpha}\left(1-i\beta\,\mathrm{sgn}(k)\tan\frac{\pi\alpha}{2}\right)\right], \quad (9)$$

while for $\alpha=1$ this expression reads

$$\phi(k)=\exp\left[-\sigma|k|\left(1+i\beta\frac{2}{\pi}\,\mathrm{sgn}(k)\ln|k|\right)\right]. \quad (10)$$

The three remaining parameters vary within the allowed regimes $\alpha\in(0,2]$, $\beta\in[-1,1]$, $\sigma\in(0,\infty)$. Random variables ζ corresponding to the characteristic functions (9) and (10) can be generated using the Janicki-Weron algorithm [16,17].

For $\alpha=2$ [and an arbitrary skewness parameter β, cf. Eq. (9)], the generalized Lévy-Brownian motion $W_{\alpha,\beta}(t)$ becomes a standard Gaussian Wiener process whose time derivative leads to the well known Gaussian white noise limit. Following this interpretation, the stochastic source term in Eq. (1) can be represented as a sum of independent pulses (having stable distribution) acting on equally spaced times. As a consequence, such Lévy noise is white, i.e., its autocorrelation function is formally a Dirac-delta function of time.

For the sake of clarification, we mention here that there exist different representations of other Lévy noises that are non-Markovian in nature. As an example, so called fractional Gaussian (or Lévy) noise is sometimes defined in literature as the time derivative of a fractional Brownian motion process [18–22]. In contrast to the Gaussian (or our Lévy) white noise, the fractional Gaussian noise (fractional Lévy stable motion) may exhibit slowly decaying time correlations. This is however not the case with the type of Lévy noise addressed in this work, where the noise source at the level of the Langevin equation is a white noise process.

III. ESCAPE IN A DOUBLE WELL: SURVIVAL PROBABILITY AND MEAN FIRST PASSAGE TIME

Let us consider a Brownian, overdamped particle in an external potential $V(x)$, i.e.,

$$V(x)=-\frac{a}{2}x^2+\frac{b}{4}x^4, \quad (11)$$

which is driven by Lévy stable white noise. The sample trajectories are then obtained by a direct integration of Eq. (1):

$$x(t)=x_0-\int_{t_0}^{t}V'(x(s))ds+\int_{t_0}^{t}dL_{\alpha,\beta}(s), \quad (12)$$

using the standard techniques of integration of stochastic differential equation with respect to the Lévy stable PDFs [8,9,16,17,23–25]. The first passage time problem has been

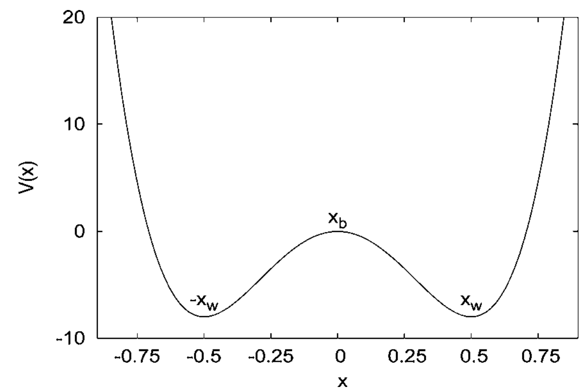

FIG. 1. The generic double well potential $V(x)=-\frac{a}{2}x^2+\frac{b}{4}x^4$ for inspection of the Kramers problem discussed in the paper. To assure sufficiently high barrier separating the stationary states $-x_w, x_w$, the following set of parameters has been chosen: $a=128, b=512, \Delta V=V(0)-V(-\sqrt{a/b})=8$.

analyzed as $\tau=\inf\{t\geq 0\,|\,x(t)\geq x_b\}$ with trajectories $x(t)$ starting at $x=x_0$ and subject to corresponding boundary conditions. In particular, an absorbing boundary condition is realized by stopping the trajectory whenever it reaches the boundary, or, more typically, whenever it has jumped beyond that very boundary location. The role of reflection, which in the case of a free diffusion [8] has been assured by wrapping the hitting (or crossing) trajectory around the boundary location, while preserving its assigned length, see in Refs. [8,25], is taken over naturally here by the confining potential walls of the symmetric double well. The details of our employed numerical scheme for stochastic differential equations driven by Lévy white noise has been detailed elsewhere [8]. In the following we shall omit cases when $\alpha=1$ with $\beta\neq 0$. In fact, this parameter set is known to induce instabilities in the numerical evaluation of corresponding trajectories [9,16,17,23,25].

In the following we consider several differing situations. For a particle that starts out at a well bottom $x_0=-x_w$, see Fig. 1 and makes excursions toward the neighboring well bottom, taken as an absorbing boundary we use the subscript notation w-w. Likewise, for a particle staring out at well bottom and being absorbed at barrier top location we use the notation w-b. Using the statistics of the first passage time events we shall next study the two corresponding survival probabilities:

$$S^{w\text{-}b}(t)=1-\mathcal{F}^{w\text{-}b}(t), \quad (13)$$

$$S^{w\text{-}w}(t)=1-\mathcal{F}^{w\text{-}w}(t), \quad (14)$$

where $\mathcal{F}^{\cdot\cdot}(t)$ is the corresponding cumulative first passage time distribution, i.e.,

$$\mathcal{F}^{w\text{-}b,w\text{-}w}(t)=\int_{-\infty}^{x_b,x_w}p^{w\text{-}b,w\text{-}w}(x,t)dx, \quad (15)$$

with $p^{w\text{-}b,w\text{-}w}(t)$ being the corresponding first passage time density.

In order for the escape to become dominated by a clear-cut, single time scale we use a sufficiently high potential

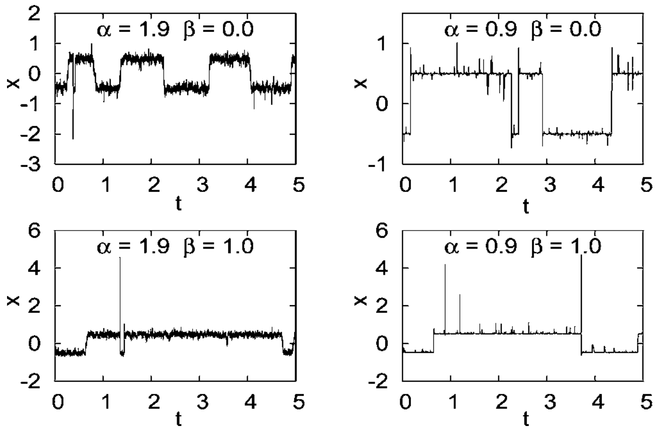

FIG. 2. Sample trajectories of random walks in the generic double well potential subjected to Lévy stable noise with $\alpha=1.9$ (left panel) and $\alpha=0.9$ (right panel) with various β ($\beta=0$, top panel, $\beta=1.0$, bottom panel). The scale parameter is $\sigma=\sqrt{2}$, the chosen time step of integration is $dt=10^{-4}$.

barrier, thereby enforcing rare escape events. A too low barrier would involve too many recrossing events with the escape then being ruled by many time scales. Our used parameters for the potential are the parameter set a,b with $a=128, b=512$, yielding a barrier height of $\Delta V=V(0)-V(-\sqrt{a/b})=\frac{a^2}{4b}=8$. This symmetric potential well is schematically depicted with Fig. 1.

A. Case of normal diffusion

For $\alpha=2$, the process driven by white Lévy stable noise approaches the Brownian limit. The α-stable white Lévy noise, as used in this study, then leads to the standard Gaussian white noise with intensity $2\sigma^2$, i.e., $\langle\zeta(t)\zeta(s)\rangle_{\alpha=2}=2\sigma^2\delta(t-s)$. In this case, the corresponding values of the MFPT from the potential minimum $-x_w$ to the top of the absorbing potential barrier, i.e., x_b or to the neighboring, absorbing minimum, $x=x_w$, can be calculated from the following quadrature formulas:

$$T_{\mathrm{MFPT}}(-x_w \to x) = \frac{1}{\sigma^2}\int_{-x_w}^{x} \exp[V(z)/\sigma^2] \times \int_{-\infty}^{z} \exp[-V(y)/\sigma^2] dy dz. \quad (16)$$

Note that for a system driven by white Gaussian noise the energy difference ΔV measured in units of $k_B T$ may be directly related to the intensity of the noise σ^2. In contrast, for Lévy stable noises with $\alpha<2$, the scale parameter σ is no longer "thermodynamically" related to the system temperature and consequently becomes a free parameter of the model. We have set throughout this study this value to $\sigma=\sqrt{2}$.

B. Survival probabilities for α-stable noise driven escape

Typical sample trajectories of the stochastic process defined by Eq. (1) are depicted in Fig. 2. We observe that a nonzero β parameter induces a dynamical asymmetry for the escape dynamics occurring in a *symmetric* double well potential. It is reflected in our numerical simulations by the emergence of stochastic trajectories that spend more time in the vicinity of one of the potential minima. As a consequence, one of the attracting states of the process ($\pm x_w$) becomes favored over the other. This kind of behavior can be also detected in our other figures. Notably, for a decreasing value of the stability index α we observe larger fluctuations of the particle positions. These occasional long jumps of trajectories may be of the order of, or even larger than the distance $2x_w$ separating the two minima of the symmetric potential $V(x)$.

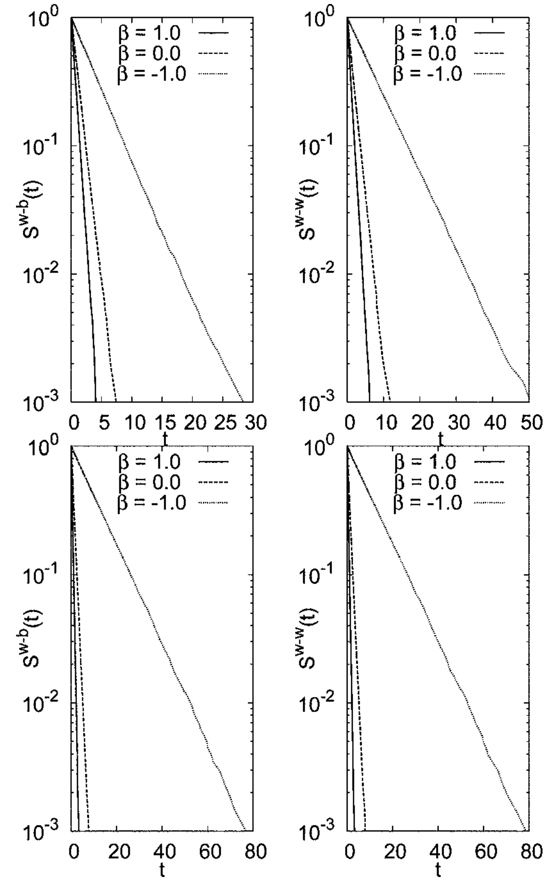

FIG. 3. Survival probability densities $S^{w-b}(t)$ (left panel) and $S^{w-w}(t)$ (right panel) for a particle dwelling the generic double well potential subjected to Lévy stable noise with $\alpha=1.9$ (top panel) and $\alpha=0.9$ (bottom panel). The simulation parameters are $dt=10^{-5}$, $N=2\times10^4$, $\sigma=\sqrt{2}$.

From the ensemble of single trajectories, which are subject to the boundary conditions discussed above, and presented in Fig. 2, we estimated the survival probability densities $S^{w-b}(t)$ and $S^{w-w}(t)$, see Fig. 3. The behavior of the survival probability is consistent with the results as predicted by inspection of the corresponding stochastic trajectories: We clearly detect from Fig. 3 that at a chosen value of α a decrease of the skewness β parameter of a driving white noise causes a distinct decrease the rate of escape of the particle from the left potential minimum; thus stabilizing the starting position at $x=-x_w$. Most importantly, at sufficient high barrier heights a visible exponential decay of the survival probability occurs,—according to $S(t)=\exp(-t/T_{\mathrm{MFPT}})$. This is the expected exponential behavior for regular Brownian motion and has been detected already previously for symmetric Lévy noises [10,11] and for totally skewed (with $\beta=1$) one sided Lévy motions [26]. Here we observe it as well for skewed stable noises. Notably, the characteristic exponent T_{MFPT} clearly depends on both noise parameters, i.e., on the value of the stability index α and also on the skewness parameter β.

Our numerical analysis indicates that typically the survival probability $S^{w\text{-}w}(t)$ for the "well-bottom-to-well-bottom" setup exceeds the survival probability $S^{w\text{-}b}(t)$ for the "well-bottom-to-barrier-top" setup. This observation can be explained by features of a noise-driven dynamics: A particle performing the motion under the influence of noise needs more time to dwell the neighboring potential minimum than is needed to reach the top of the barrier. This kind of behavior can be nevertheless weakened by diminishing the stability parameter α. For $\alpha<2$ the trajectory of the random particle becomes discontinuous and at sufficiently small α a particle, on average, completes its escape from the left potential minimum by a long jump which can overpass the right potential minimum. Consequently, both survival curves start to overlap converging to the same function at a given small stability index α.

For the Cauchy limit, i.e., $\alpha=1$, an approximate result for the mean crossing time as a function of the noise strength σ has been given in Ref. [10] by use of the fractional Fokker-Planck equation. In order to compare results obtained in this work with the former studies presented by Chechkin *et al.* [10], we have used a rescaled form of Eq. (1)

$$\frac{dx}{dt}=(x-x^3)+\zeta(t), \tag{17}$$

obtained by a set of transformations $x\to x/x_w$, $t\to t/\tau$ with $x_w^2=a/b$, $\tau=\eta/a$ and consequently $\sigma^\alpha=D\to\sqrt{\frac{k_BT}{\eta}}\tau/x_w^\alpha$. In these units $\sigma\to\frac{2\sigma}{128^{1/\alpha}}$, thus, $\sigma_{\alpha=2}=\frac{1}{4}$ and $\sigma_{\alpha=1}=\frac{\sqrt{2}}{64}$. From the approximate formula derived in Ref. [10] for a strictly weak noise case at $\alpha=1$ one obtains $T_{\mathrm{MFPT}}\approx 141.71$, whereas our simulations for this noise strength yield $T_{\mathrm{MFPT}}\approx 132.86$, which differs only by around 7%. Exemplary survival probabilities $S^{w\text{-}b}(t)$ and $S^{w\text{-}w}(t)$ for this Cauchy-Lévy noise are presented in Fig. 4. The data indicate that for the Cauchy noise the MFPT for well-bottom-to-barrier-top scenario is somewhat smaller than the MFPT for the case of well-bottom-to-well-bottom, see Fig. 4.

Our prior studies of a *free* Lévy-Brownian motion driven by α stable noise has shown that for some parameterizations of the Lévy noise (totally skewed stable distributions with $\alpha<1$) a non-exponential survival probability density emerges. Therefore, in this study we tested whether for a particle driven by a Lévy-Smirnoff noise, i.e., with setting $\alpha=0.5$, $\beta=1$, a decrease of the potential barrier may induce visible deviations from the exponential distribution. The results of the test are displayed in Fig. 5 where the survival probability densities $S^{w\text{-}b}(t)$ and $S^{w\text{-}w}(t)$ for various heights of the potential barrier ΔV are presented. For decreasing ΔV the motion of a particle indeed approaches the behavior of a free diffusion; consequently also distinct deviations from the exponential character of survival distributions do show up.

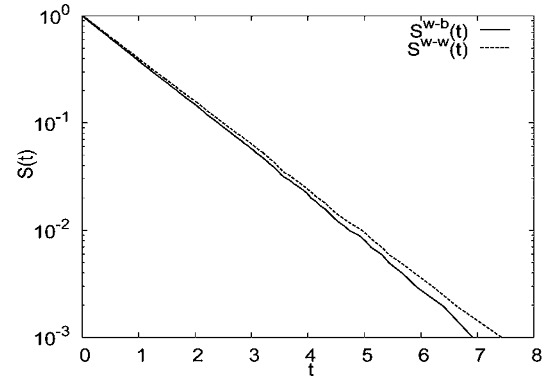

FIG. 4. Survival probability densities $S^{w\text{-}b}(t)$ and $S^{w\text{-}w}(t)$ for a particle wandering in the generic double well potential subjected to Cauchy noise, i.e., Lévy stable noise with $\alpha=1.0$. $T^{w\text{-}b}_{\mathrm{MFPT}}=1.038\pm0.008$ and $T^{w\text{-}w}_{\mathrm{MFPT}}=1.086\pm0.008$. The simulation parameters are $dt=10^{-5}$, $N=2\times10^4$, $\sigma=\sqrt{2}$.

C. Behavior for the MFPT

From the ensembles of collected first passage times we have also evaluated directly the mean values of the distributions. Our findings for MFPTs are presented in Figs. 6 and 7. The depicted results corroborate with the decrease of the escape rate (i.e., the inverse of the MFPT) upon increasing the skewness β. Moreover, for the stability index $\alpha=2$ (normal Brownian white noise) and for any skewness parameter β, the Gaussian case is properly reconfirmed, see Figs. 6 and 7.

The statistics of first passage times for well-bottom-to-barrier-top cases was collected solely on those trajectories which reached or overcame the barrier top (the half-line beyond the location of the barrier top was then treated as an absorbing interval), therefore no recrossings were registered

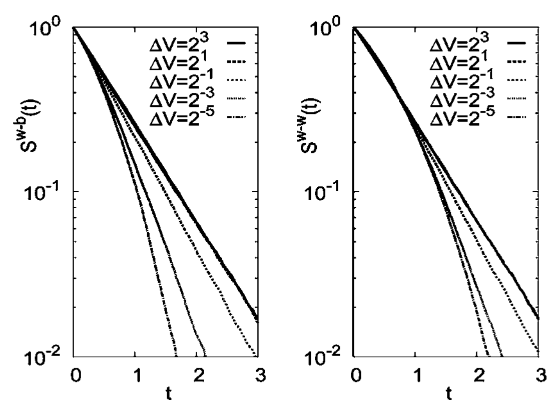

FIG. 5. Survival probabilities $S^{w\text{-}b}(t)$ (left panel) and $S^{w\text{-}w}(t)$ (right panel) for various barrier heights separating two wells of the potential: $\Delta V=\{2^3,2^1,2^{-1},2^{-3},2^{-5}\}$ (from the top to the bottom). The system is driven by Lévy-Smirnoff noise, i.e., Lévy stable noise with $\alpha=0.5$ and $\beta=1$. The simulation parameters are $dt=10^{-5}$, $N=2\times10^4$, $\sigma=\sqrt{2}$.

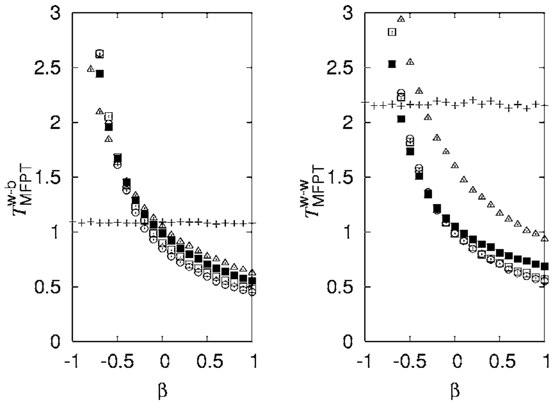

FIG. 6. Mean first passage times for well-bottom-to-barrier-top $T^{w\text{-}b}_{\text{MFPT}}$ (left panel) and mean first passage times for well-bottom-to-well-bottom $T^{w\text{-}w}_{\text{MFPT}}$ (right panel) as a function of the skewness parameter β. Simulation parameters $dt=10^{-5}$, $N=2\times 10^4$, $\sigma=\sqrt{2}$. The error bars are estimated using bootstrap method with $N_b=2\times 10^3$. The white Lévy stable noise with the stability index $\alpha=2$ and any allowed value of β is equivalent to the white Gaussian noise, what is manifested by the independence of $T_{\text{MFPT}}(\beta)$ for $\alpha=2$. The various symbols represent the various values of the stability index α: (+) $\alpha=2.0$, (△) $\alpha=1.9$, (○) $\alpha=1.5$, (□) $\alpha=1.3$, and (■) $\alpha=1.1$.

in these situations. In turn, trajectories traversals contributing to the well-bottom-to-well-bottom statistics included also the recrossings events (i.e., possible multiple traversals over the barrier before reaching the left/right potential minimum).

Note, that our method of simulating skewed stable white noises assumes a specific representation of an α stable variable whose characteristic function is described by four parameters α,β,σ,μ. Two of these have been preset in the simulations to $\sigma=\sqrt{2}$ and $\mu=0$ resulting in the noise intensity (as interpreted at the level of a discrete form of the Langevin equation) changing with various α [see Eqs. (8)–(10)] and fulfilling the PDF scaling $L_{\alpha,\beta}(\zeta;\sigma,0)=\frac{1}{\sigma}L_{\alpha,\beta}(\zeta/\sigma;1,0)$. Alternatively, one can use a constant noise intensity which, with varying α, effectively yields a corresponding change for the scale parameter σ. For symmetric Lévy stable white noises both procedures comfort with a diffusion analogy [28], when the intensity of the noise (σ parameter of the Lévy distribution) corresponds to the diffusion coefficient of a fractional Fokker Planck equation $D=\sigma^{\alpha}$. This analogy breaks down, however, for asymmetric noises when the scaling noise parameter at the level of the Langevin (or random walk) description does no longer relate in any obvious way to the "noise intensity," understood here as the strength of the diffusion parameter on the level of the FFPE. More precisely, the asymmetry of Lévy-noise impacts on the trajectory $x(t)$ implies, on one hand side, a contribution to the drift of the probability flow (which intuitively can be understood as a consequence of the biased directionality of the Lévy flight) and, on the other, adds to the diffusion term with a fractional derivative [27,28]. This nontrivial noise contribution composed of a mixed drift and diffusive term calls for a more detailed analysis of a relationship between the noise intensity entering the integral Eq. (12) and fractional diffusion for asymmetric Lévy sources; a task that is the subject for future studies.

In line with the above comment, the MFPTs, as observed in Figs. 6 and 7, have been estimated based on a constant $\sigma=\sqrt{2}$ which results in a changing "noise intensity," depending on a value of parameter α. For $\beta=0$ our results suggest that T_{MFPT} for a well-bottom-to-barrier-top is essentially independent of α, see left panel of Fig. 6

Moreover, for $\alpha=1$, or adequately $\alpha\approx 1$, the procedure of simulating skewed random variables becomes unstable, as reported elsewhere [16,17,23]. It can be readily explained by examining the form of the noise-term characteristic function $\phi(t)$—see Eqs. (9) and (10). The exponentiated functions are no longer continuous functions of the parameters and exhibit discontinuities when $\alpha=1$, $\beta\neq 0$. This feature in turn induces the discontinuities for $T_{\text{MFPT}}(\alpha)$, as it can be observed with Fig. 7.

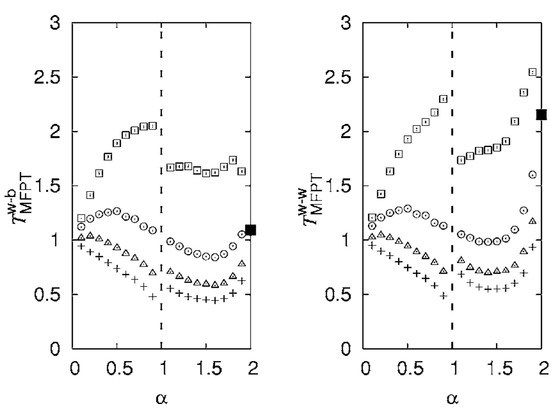

FIG. 7. Mean first passage times for well-bottom-to-barrier-top $T^{w\text{-}b}_{\text{MFPT}}$ (left panel) and mean first passage times for well-bottom-to-well-bottom $T^{w\text{-}w}_{\text{MFPT}}$ (right panel) as a function of the stability index α. Simulation parameters $dt=10^{-5}$, $N=2\times 10^4$, $\sigma=\sqrt{2}$, and $N_b=2\times 10^3$. The black squares indicate equality of MFPTs for $\alpha=2$ with any β. The various symbols represent the various values of the skewness parameter β: (+) $\beta=1.0$, (△) $\beta=0.5$, (○) $\beta=0.0$, and (□) $\beta=-0.5$.

D. MFPT ratio no longer obeying normal diffusion behavior

Next, we investigate the behavior of the value for the ratio $R=T^{w\text{-}w}_{\text{MFPT}}/T^{w\text{-}b}_{\text{MFPT}}$ of the $T^{w\text{-}w}_{\text{MFPT}}$ between the case with well-to well and well-to-bottom. Note that from the exact expression in Eq. (16), for a symmetric potential $V(x)$ with normal Gaussian white noise fluctuations (i.e., for $\alpha=2$ and an arbitrary value of the skewness parameter β) this ratio yields the commonly known factor of $R=2$. Put differently, in this case the random walker needs twice the time to reach the other potential minimum than to reach to the top of the potential barrier. In contrast, with a decreasing value of α the ratio R now distinctly deviates from 2, being always smaller than for a Gaussian diffusion case. Our numerical results are depicted in Fig. 8. This deviation can be understood by noticing that for $\alpha<2$, the stochastic escape trajectories of the random walks in the double well potential become discontinuous, meaning that the continuous movement of a particle becomes interrupted with long jumps. These occasional jumps are more probable to occur for small stability index α and ex-

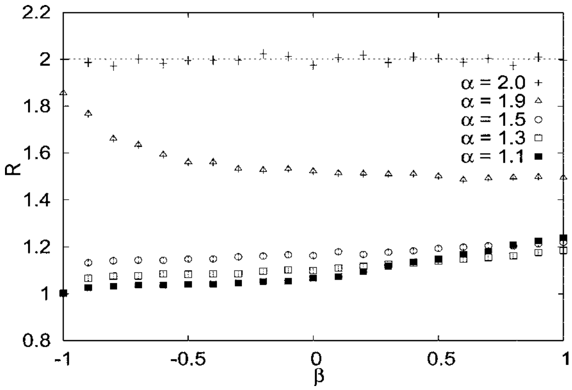

FIG. 8. Ratio $R = T^{w\text{-}w}_{\text{MFPT}} / T^{w\text{-}b}_{\text{MFPT}}$ of MFPTs to the neighboring potential well ($T^{w\text{-}w}_{\text{MFPT}}$) and to the barrier top ($T^{w\text{-}b}_{\text{MFPT}}$) as a function of the skewness parameter β. The simulation parameters are $dt = 10^{-5}$, $N = 2 \times 10^4$, $\sigma = \sqrt{2}$, and $N_b = 2 \times 10^3$.

plain the observable statistics of the first passage times, implying in turn that $R < 2$. In particular, an increasing probability of long jumps over the barrier in the overall statistics of passages tends to equalize $T^{w\text{-}w}_{\text{MFPT}}$ with $T^{w\text{-}b}_{\text{MFPT}}$, yielding R values approaching 1. The effect of the skewness parameter β which is responsible for the asymmetric stochastic dynamics can be interpreted as an additional contribution to the directionality of random impact pulses which push the Lévy-Brownian particle within the potential well. For negative (positive) $\beta \approx \pm 1$ with $\alpha < 1$, the driving Lévy white noise becomes a one-sided Lévy process with strictly negative (positive) increments which stabilize trajectories around the starting position $x = \pm x_w$.

IV. CONCLUSIONS

In this paper we have studied the survival probabilities and the mean values of the mean first passage times for escape from a symmetric double well potential when the overdamped dynamics is driven by general Lévy white noise. The statistics of escaping trajectories is investigated by a numerical analysis of a Lévy noise driven Langevin equation with properly implemented boundary conditions. Extending former studies on the escape problem driven by symmetric Lévy white noise [9–11] the first passage problem for one-sided Lévy motions [26], our work provides an analysis of the Kramers problem with an arbitrary set of α, and asymmetry β parameters that characterize the white stable noise-source entering the Langevin dynamics. Following the Markovian character of the stochastic dynamics—at sufficiently high barriers—the time dependence of the survival probabilities within the potential well assume an exponential law. As can be expected, distinct deviations from the exponential behavior become detectable, however, for low barrier heights when, similarly to a normal Gaussian diffusion case, many recrossing of the barrier are possible.

An asymmetry of the Lévy noise perturbing the dynamics is shown to either enhance or also suppress the escape events and corroborates the intuitive influence of the β parameter which yields an additional, biasing contribution to the particles' motion due to the directionality of stochastic impact pulses. Moreover, the rate of escape is shown to exhibit a nonmonotonic behavior as a function of the stability index α, with a discontinuity occurring at the value $\alpha = 1$.

In clear contrast to a normal diffusion behavior as typified by systems driven by white Gaussian noise, for which the random walker requires twice the time to reach the other potential minimum as compared to the mean time to reach to the (absorbing) top of the barrier, Lévy white noise with $\alpha < 2$ now causes a decrease in this ratio of the mean first passage times: The ratio $R = T^{w\text{-}w}_{\text{MFPT}} / T^{w\text{-}b}_{\text{MFPT}}$ lies consistently below the "normal" value of $R = 2$. This remarkable result holds true for any chosen value of the skewness parameter β.

In summary, our numerical considerations demonstrate the richness of the bistable kinetics resulting from driven α-stable white noises. The observed features are rooted in the non-local jump lengths taken from a distribution which exhibits fat tails which in turn rule the random passages between the attracting states of a double well potential.

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[1] H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).
[2] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[3] P. Talkner and P. Hänggi, *New Trends in Kramers Reaction Rate Theory* (Kluwer Academic, Boston, 1995); G. R. Fleming and P. Hänggi, *Activated Barrier Crosing* (World Scientific, Singapore, 1993); P. Hänggi and J. Troe, Ber. Bunsenges. Phys. Chem. **95**, 225 (1991).
[4] P. Hänggi and P. Jung, Adv. Chem. Phys. **89**, 239 (1995).
[5] P. Reimann, G. J. Schmid and P. Hänggi, Phys. Rev. E **60**, R1 (1999).
[6] C. Van den Broeck and P. Hänggi, Phys. Rev. A **30**, 2730 (1984); P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B: Condens. Matter **56**, 333 (1984); P. Hänggi and P. Riseborough, Phys. Rev. A **27**, 3379 (1983); V. Balakrishnan, C. Van den Broeck, and P. Hänggi, *ibid.* **38**, 4213 (1988).

[7] P. Müller, P. Talkner, and P. Reimann, Physica A **247**, 338 (1997); P. Hänggi and P. Talkner, Phys. Rev. A **32**, 1934 (1985).
[8] B. Dybiec, E. Gudowska-Nowak, and P. Hänggi, Phys. Rev. E **73**, 046104 (2006).
[9] P. D. Ditlevsen, Phys. Rev. E **60**, 172 (1999); P. D. Ditlevsen, Geophys. Res. Lett. **26**, 1441 (1999); P. D. Ditlevsen, M. S. Kristensen, and K. K. Andersen, J. Clim. **18**, 2594 (2005); J. Fajans and A. Schmidt, Nucl. Instrum. Methods Phys. Res. A **521**, 318 (2004).
[10] A. V. Chechkin, V. Yu. Gonchar, J. Klafter, and R. Metzler, Europhys. Lett. **72**, 348 (2005).
[11] P. Imkeller and I. Pavlyukevich, J. Phys. A **39**, L237 (2006).
[12] W. Feller, *An Introduction to Probability Theory and its Applications* (John Wiley, New York, 1968).
[13] V. M. Zolotarev, *One Dimensional Stable Distributions* (American Mathematical Society, Providence, 1986).
[14] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Adison-Wesley, Reading, MA, 1954).
[15] The result follows a general observation for separable continuous time random walks (CTRW) which lead to normal stochastic diffusion for finite mean waiting time for a jump to occur and finite step variance [13].
[16] R. Weron, Stat. Probab. Lett. **28**, 165 (1996).
[17] A. Janicki and A. Weron, Stat. Sci. **9**, 109 (1994); *Simulation and Chaotic Behavior of α-Stable Stochastic Processes* (Marcel Dekker, New York, 1994); A. Janicki, *Numerical and Statistical Approximation of Stochastic Differential Equations with Non-Gaussian Measures* (Hugo Steinhaus Centre for Stochastic Methods, Wrocław, 1996).
[18] B. B. Mandelbrot, SIAM Rev. **10**, 422 (1968).
[19] S. Chaudhury and B. J. Cheravil, J. Chem. Phys. **125**, 024904 (2006).
[20] *Lévy Flights and Related Topics in Physics*, edited by M. Shlesinger, G. M. Zaslavsky, and U. Frisch (Springer Verlag, Berlin, 1994).
[21] I. M. Sokolov and J. Klafter, Chaos **15**, 026103 (2005); I. M. Sokolov, Phys. Rev. E **63**, 011104 (2000).
[22] S. C. Kou and X. S. Xie, Phys. Rev. Lett. **93**, 180603 (2004).
[23] A. Weron and R. Weron, Lect. Notes Phys. **457**, 379 (1995).
[24] M. E. J. Newman and G. T. Barkema, *Monte Carlo Methods in Statistical Physics* (Clarendon Press, Oxford, 1999).
[25] B. Dybiec and E. Gudowska-Nowak, Phys. Rev. E **69**, 016105 (2004).
[26] I. Eliazar and J. Klafter, Physica A **336**, 219 (2004).
[27] V. V. Yanovsky, A. V. Chechkin, D. Schertzer, and A. V. Tur, Physica A **282**, 13 (2000).
[28] D. Schertzer, M. Larchevêque, J. Duan, V. V. Yanovsky, and S. Lovejoy, J. Math. Phys. **42**, 200 (2001).