



Dynamic flows with adaptive route choice

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Abstract

We study dynamic network flows and introduce a notion of *instantaneous dynamic equilibrium* (IDE) requiring that for any positive inflow into an edge, this edge must lie on a currently shortest path towards the respective sink. We measure current shortest path length by current waiting times in queues plus physical travel times. As our main results, we show:

1. existence and constructive computation of IDE flows for multi-source single-sink networks assuming constant network inflow rates,
2. finite termination of IDE flows for multi-source single-sink networks assuming bounded and finitely lasting inflow rates,
3. the existence of IDE flows for multi-source multi-sink instances assuming general measurable network inflow rates,
4. the existence of a complex single-source multi-sink instance in which any IDE flow is caught in cycles and flow remains forever in the network.

Keywords Dynamic flows · Flows over time · Equilibria · Networks

Mathematics Subject Classification 05C21 · 90C35 · 65K10

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1 Introduction

Dynamic network flows have been studied for decades in the optimization and transportation literature, see the classical book of Ford and Fulkerson [7] or the more recent surveys of Skutella [22] and Peeta [17]. A fundamental model describing the dynamic flow propagation process is the so-called *deterministic queue model*, see Vickrey [24]. Here, a directed graph $G = (V, E)$ is given, where edges $e \in E$ are associated with a queue with positive rate capacity $\nu_e \in \mathbb{R}_{>0}$ and a physical transit time $\tau_e \in \mathbb{R}_{>0}$. If the total inflow into an edge $e = vw \in E$ exceeds the rate capacity ν_e , a queue builds up and agents need to wait in the queue before they are forwarded along the edge. The total travel time along e is thus composed of the waiting time spent in the queue plus the physical transit time τ_e . A schematic illustration of the inflow and outflow mechanics of an edge e is given in Fig. 1.

The fluid queue model has been mostly studied from a game-theoretic perspective, where it is assumed that agents act selfishly and travel along shortest routes under prevailing conditions. This behavioral model is known as *dynamic equilibrium* and has been analyzed in the transportation science literature for decades, see Friesz et al. [8], Meunier and Wagner [16] and Zhu and Marcotte [25]. In the past years, however, several new exciting developments have emerged: Koch and Skutella [14] elegantly characterized dynamic equilibria by their derivatives, which gives a template for their computation. Subsequently, Cominetti et al. [5] derived alternative characterizations and proved existence and uniqueness in terms of experienced travel times of equilibria even for multi-commodity networks. Very recently, Cominetti et al. [6] shed light on the behavior of steady state queues assuming single commodity networks and constant inflow rates. Sering and Vargas-Koch [21] analyzed the impact of spillbacks in the fluid queueing model and Bhaskar et al. [1] devised Stackelberg strategies in order to improve the efficiency of dynamic equilibria.

The concept ‘dynamic equilibrium’ assumes *complete knowledge and simultaneous route choice* by all travelers. Complete knowledge requires that a traveler is able to exactly forecast future travel times along the chosen path effectively anticipating the whole evolution of the flow propagation process across the network. This assumption has been justified by letting travelers learn good routes over several trips and a dynamic equilibrium then corresponds to an attractor of the underlying learning dynamic. While certainly relevant, this concept may not accurately reflect the behavioral changes caused by the wide-spread use of navigation devices. As also discussed in Marcotte et al. [15], Hamdouch et al. [11] and Unnikrishnan and Waller [23], drivers may not always learn good routes over several trips but are now informed in real-time about the current traffic situations and, if beneficial, reroute instantaneously no matter

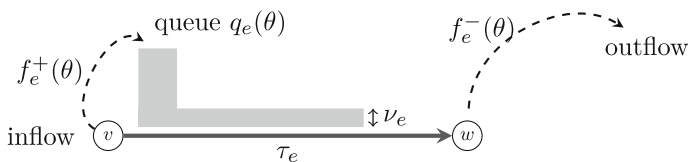


Fig. 1 An edge $e = vw$ with a nonempty queue at time θ

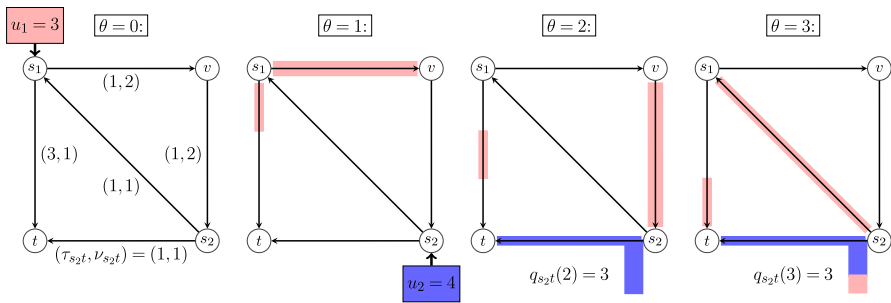


Fig. 2 The evolution of an IDE flow over the time horizon $[0, 3]$

how good or bad that route was in hindsight. Also, the information available to a navigation device is usually not complete, that is, congestion information is available only as an aggregate (estimated waiting times for road traversal) but the individual routes and/or source and destinations of travelers are unknown—for good reason.¹

In this paper, we consider an adaptive route choice model, where at every node (intersection), travelers may alter their route depending on the current network conditions, that is, based on current travel times and queuing delays. The needed information is anonymous and indeed available by navigation devices. We assume that, if a traveler arrives at the end of an edge, she may change the current route and opt for a currently shorter one. This type of reasoning does neither rely on personalized information nor on the capability of unraveling the future flow propagation process. We term a dynamic flow an *instantaneous dynamic equilibrium (IDE)*, if for every point in time and every edge with positive inflow (of some commodity), this edge lies on a currently shortest path towards the respective sink. In the following, we illustrate IDE in comparison to classical dynamic equilibrium with an example.

1.1 An example

Consider the network in Fig. 2 (left). There are two source nodes s_1 and s_2 with constant inflow rates $u_1(\theta) \equiv 3$ for times $\theta \in [0, 1)$ and $u_2(\theta) \equiv 4$ for $\theta \in [1, 2)$. Commodity 1 (red) has two simple paths connecting s_1 with the sink t . Since both have equal length ($\sum_e \tau_e = 3$), in an IDE both can be used by commodity 1. In Fig. 2, the flow takes the direct edge to t with a rate of one, while edge $s_1 v$ is used at a rate of two. This is actually the only split possible in an IDE, since any other split (different in more than just a subset of measure zero of $[0, 1)$) would result in a queue forming on one of the two edges, which would make the respective path longer than the other one. At time $\theta = 1$, the inflow at s_1 stops and a new inflow of commodity 2 (blue) at s_2 starts. This new flow again has two possible paths to t , however, here the direct path ($\sum_e \tau_e = 1$) is shorter than the alternative ($\sum_e \tau_e = 4$). So all flow enters edge $s_2 t$ and starts to form a queue. At time $\theta = 2$, the first flow particles of commodity 1 arrive at s_2 with a rate of 2. Since the flow of commodity 2 has built up a queue of length 3

¹ Some navigation systems have a large population of customers from which they infer even personalized information, but complete knowledge over all travelers seems unrealistic.

on edge s_2t by this time, the estimated travel times $\sum_e(\tau_e + q_e(\theta))$ are the same on both simple s_2-t paths. Thus, the red flow is split evenly between both possible paths. This results in the queue-length on edge s_2t remaining constant and therefore this split gives us an IDE flow for the interval $[2, 3)$. At time $\theta = 3$, red particles will arrive at s_1 again, thus, completing a full cycle (namely s_1, v, s_2, s_1). This example shows that IDE flows may involve a flow decomposition along cycles. In contrast, the (classical) dynamic equilibrium flow will just send more of the red flow along the direct path s_1, t since the future queue growth at edge s_2t of the alternative path is already anticipated.

Note, that cycles can appear even in the case of only a single commodity (and therefore a single source and sink) and a constant inflow rate over a single interval—see Example 2 in the online appendix for such an instance. This shows that the differences between the two equilibrium concepts are quite fundamental and occur even in very simple examples.

1.2 Related work

In the transportation science literature, the idea of an instantaneous user or dynamic equilibrium has already been proposed since the late 80's, see Ran and Boyce [18, § VII-IX], Boyce et al. [2,19], Friesz et al. [9]. These works develop an optimal control-theoretic formulation and characterize instantaneous user equilibria by Pontryagin's optimality conditions. However, not much is known regarding IDE existence and their structural properties. In fact, the underlying equilibrium concept of Boyce et al. [2,19] and Friesz et al. [9] is different from ours. While the verbally written concept of an IDE is similar to the one we use here, the mathematical definition of an IDE in [2,9,19] requires that instantaneous travel times are minimal only for *used paths* towards the sink. A path is used, if every arc of the path has positive flow. As, for instance, the authors in Boyce et al. [2, p.130] admit: "Specifically with our definition of a used route, it is possible that no route is ever 'used' because vehicles stop entering the route before vehicles arrive at the last link on the route. Thus, for some networks every flow can be in equilibrium." Ran and Boyce [18, § VII, pp.148] present a link-based definition of IDE. They define node labels at nodes $v \in V$ indicating the current shortest travel time from the source node to some intermediate node v and require that whenever edge vw has positive flow, edge vw must be contained in a shortest $s-w$ path, where s is the flow's source. This is different from our definition of an IDE, because we require that whenever there is positive inflow into an edge vw , it must be contained in a currently shortest $v-t$ path, where t is the sink of the considered inflow.

Another important difference to our model is the assumed time horizon. The previous works [2,9,18,19] all assume a *finite time horizon* on which the control problems are defined, thus, only describing the flow trajectories over the given time horizon. All numerical studies and simulation results appearing in these works further implicitly assume that for given finitely lasting bounded inflow rates, there exists a finite time horizon $[0, T]$ with T large enough so that eventually all travelers reach their destination. Our results reveal that this is in fact not true: there are multi-commodity instances with finitely lasting bounded inflows that admit IDE flows cycling forever. For the discrete version of this model using the natural discrete version of our equi-

librium concept, such a behavior was already discovered in Ismaili [12, Theorem 8], though his instance makes critical use of edges with zero transit time (which we do not allow) and the instantaneous travel time always increases when there is positive inflow into an edge, even when the edge inflow rate is smaller than the rate capacity. In particular, players may observe increased instantaneous travel time, although no player is in fact delayed. This is not possible in our model. Based on this flexibility, the generated instance is considerably simpler than ours but one can show that IDE flows in our sense do terminate in that instance.

1.3 Our results

In this paper we introduce the concept of an *instantaneous dynamic equilibrium* (IDE for short). We call a feasible flow over time an IDE, if at any point in time, every edge with positive inflow (of some commodity) lies on a currently shortest path towards the respective sink.

Our first main result (Theorem 3.4) shows that IDE exist for multi-source single-sink networks with piecewise constant inflow rates (generating the volume of agents originating at the sources). The existence proof relies on a constructive method extending any IDE flow up to time θ to an IDE flow on a strictly larger interval $\theta + \epsilon$ for some $\epsilon > 0$. The key insight for the extension procedure relies on solving a sequence of nonlinear programs, each associated with finding the right outflow split for given node inflows. We also show that such solutions can be found by a simple water filling procedure. With the extension property we can apply a limit argument on the real numbers implying the existence of IDE on the whole $\mathbb{R}_{\geq 0}$.

Given that, unlike the classical dynamic equilibrium, IDE flows may involve cycling behavior (see the example in Fig. 2), we turn to the issue of whether it is possible that positive flow volume remains forever in the network (assuming finitely lasting bounded inflows). Our second main result (Theorem 4.6) shows that for multi-source single-sink networks, this is impossible: Even for arbitrary bounded and finitely lasting inflow rate functions, there always exists a finite time $T > 0$ at which the network is cleared, that is, all flow particles have reached their destination.

We then turn to general multi-commodity networks. For given piece-wise constant network inflow rates, we also can extend an IDE flow to a strictly larger time interval by determining the edge inflow rates and the derivatives of the current shortest path distances. This extension relies on a solution of a system of equations, which is guaranteed to exist by Kakutani's fixed point theorem. The equations are inspired by those defining a "thin flows with resetting" for Nash flows over time that were introduced by Koch and Skutella [14] and refined by Cominetti et al. [4,5]. Again, a limit argument proves the existence of IDE flows over the whole $\mathbb{R}_{\geq 0}$ (Theorem 5.5). Furthermore, we give a brief idea, how these extensions can be obtained by a mixed integer formulation. We then extend this result to arbitrary locally-integrable network inflow rate functions by using a theorem about the existence of a solution to a variational inequality in infinite-dimensional function spaces (Theorem 5.6).

Finally, we show that for bounded and finitely lasting inflow rates, termination in finite time is *not* guaranteed anymore as soon as the network contains more than one

sink (Theorem 6.1). We construct a quite complex instance where all IDE flows are caught in cycles and travel forever. This instance reveals that the assumption of a finite time horizon $[0, T]$ made previously in the transportation literature cannot be made without loss of generality. We also show that the instance can be modified in such a way that only a single-source is needed.

2 The flow model

In the following, we describe a fluid queuing model as used before by Koch and Skutella [14] and Cominetti et al. [5] and introduce the notation we will use throughout this paper.

We consider a finite directed graph² $G = (V, E)$ with positive rate capacities $v_e \in \mathbb{R}_{>0}$ and positive transit times $\tau_e \in \mathbb{R}_{>0}$ for every $e \in E$. Note, that we require all transit times to be strictly positive. We impose this restriction since—intuitively—our existence proofs all require that there is always some non-zero time between two decisions of a particle, see Remark 3.3 for a more detailed discussion. There is a finite set of commodities $I = \{1, \dots, n\}$, each with a commodity-specific source node $s_i \in V$ and a commodity-specific sink node $t_i \in V$. We will always assume that there is at least one s_i - t_i path for each $i \in I$. The (infinitesimally small) agents of every commodity $i \in I$ enter the network according to an integrable network inflow rate function $u_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$.

A *flow over time* is a tuple $f = (f^+, f^-)$, where $f^+, f^- : \mathbb{R}_{\geq 0} \times E \times I \rightarrow \mathbb{R}_{\geq 0}$ are integrable functions modeling the edge inflow rate $f_{i,e}^+(\theta)$ and edge outflow rate $f_{i,e}^-(\theta)$ of commodity i of an edge $e \in E$ at time $\theta \geq 0$.

The *queue length* of edge e at time θ is given by

$$q_e(\theta) := \sum_{i \in I} F_{i,e}^+(\theta) - \sum_{i \in I} F_{i,e}^-(\theta + \tau_e) \quad \text{for all } \theta \in \mathbb{R}_{\geq 0}, \tag{1}$$

where

$$F_{i,e}^+(\theta) := \int_0^\theta f_{i,e}^+(z) dz \quad \text{and} \quad F_{i,e}^-(\theta) := \int_0^\theta f_{i,e}^-(z) dz$$

denote the *cumulative (edge) inflow* and *cumulative (edge) outflow*. We implicitly assume $f_{i,e}^-(\theta) = 0$ for all $\theta \in [0, \tau_e)$, which will ensure together with Constraint (4) (see below) that the queue lengths are always non-negative. Furthermore, we define the cumulative network inflow rate by $U_i(\theta) := \int_0^\theta u_i(z) dz$ and, for the sake of simplicity, we denote the aggregated flow over all commodities by $f_e^+ := \sum_{i \in I} f_{i,e}^+$ and $f_e^- := \sum_{i \in I} f_{i,e}^-$, as well as, $F_e^+ := \sum_{i \in I} F_{i,e}^+$ and $F_e^- := \sum_{i \in I} F_{i,e}^-$.

A *feasible* flow over time satisfies the following conditions (2), (3), (4), and (6). The *flow conservation constraints* are modeled for a commodity $i \in I$ and all nodes $v \neq t_i$ as

² Note that all results of this paper also hold for multigraphs.

$$\sum_{e \in \delta_v^+} f_{i,e}^+(\theta) - \sum_{e \in \delta_v^-} f_{i,e}^-(\theta) = \begin{cases} u_i(\theta), & \text{if } v = s_i \\ 0, & \text{if } v \neq s_i \end{cases} \quad \text{for all } \theta \in \mathbb{R}_{\geq 0}, \quad (2)$$

where $\delta_v^+ := \{vu \in E\}$ and $\delta_v^- := \{uv \in E\}$ are the sets of outgoing edges from v and incoming edges into v , respectively. For the sink node t_i of commodity i we require

$$\sum_{e \in \delta_{t_i}^+} f_{i,e}^+(\theta) - \sum_{e \in \delta_{t_i}^-} f_{i,e}^-(\theta) \leq 0 \quad \text{for all } \theta \in \mathbb{R}_{\geq 0}. \quad (3)$$

We assume that the queue operates at capacity which can be modeled by

$$f_e^-(\theta + \tau_e) = \begin{cases} v_e, & \text{if } q_e(\theta) > 0 \\ \min\{f_e^+(\theta), v_e\}, & \text{else} \end{cases} \quad \text{for all } e \in E, \theta \in \mathbb{R}_{\geq 0}. \quad (4)$$

Since $q_e'(\theta) = \sum_{i \in I} f_{i,e}^+(\theta) - \sum_{i \in I} f_{i,e}^-(\theta + \tau_e)$, this condition is equivalent to the following equation describing the queue length dynamics (cf. [5, Section 2.2]):

$$q_e'(\theta) = \begin{cases} f_e^+(\theta) - v_e, & \text{if } q_e(\theta) > 0 \\ \max\{0, f_e^+(\theta) - v_e\}, & \text{else} \end{cases} \quad \text{for all } e \in E, \theta \in \mathbb{R}_{\geq 0}. \quad (5)$$

Finally, we want the flow to follow a strict FIFO principle on the queues, which can be formalized by the following condition (see [20]):

$$f_{i,e}^-(\theta) = \begin{cases} f_e^-(\theta) \cdot \frac{f_{i,e}^+(\vartheta)}{f_e^+(\vartheta)} & \text{if } f_e^+(\vartheta) > 0, \\ 0 & \text{else,} \end{cases} \quad (6)$$

where $\vartheta := \min\{\vartheta \leq \theta | \vartheta + \tau_e + \frac{q_e(\vartheta)}{v_e} = \theta\}$ is the earliest point in time a particle can enter edge e in order to leave it at time θ . The quotient $\frac{q_e(\vartheta)}{v_e}$ is hereby the current waiting time to be spent in the queue of edge e . In other words, Constraint (6) ensures that the share of commodity i of the aggregated outflow rate at some point in time θ equals the share of commodity i of the aggregated inflow rate at the time the particles entered the edge.

Note however, that all results within this paper also hold if we relax the FIFO condition (6) to the following condition

$$F_{i,e}^-(\theta) \leq F_{i,e}^+(\theta - \tau_e) \quad \text{for all } i \in I, e \in E, \theta \in \mathbb{R}_{\geq 0}. \quad (7)$$

This condition allows for overtaking within a queue and only prevents flow from changing its commodity on an edge by requiring the total amount of flow of every commodity that has left an edge to not exceed the total amount of flow of this commodity that has reached the head of this edge up to that point in time.

We assume that, whenever an agent arrives at an intermediate node v at time θ , she is given the information about the current queue length $q_e(\theta)$ and transit time τ_e of all edges $e \in E$, and, based on this information, she computes a shortest v - t path and enters the first edge on this path (breaking potential ties arbitrarily). We define the *instantaneous travel time* of an edge e at time θ as $c_e(\theta) := \tau_e + \frac{q_e(\theta)}{v_e}$. We can now define commodity-specific node labels $\ell_{i,v}(\theta)$ corresponding to current shortest path distances from v to the sink t_i . For $i \in I, v \in V$ and $\theta \in \mathbb{R}_{\geq 0}$, define

$$\ell_{i,v}(\theta) := \begin{cases} 0, & \text{for } v = t_i \\ \min_{e=vw \in E} \{\ell_{i,w}(\theta) + c_e(\theta)\}, & \text{else.} \end{cases} \tag{8}$$

We say that edge $e = vw$ is *active* for $i \in I$ at time θ , if $\ell_{i,v}(\theta) = \ell_{i,w}(\theta) + c_e(\theta)$ and we denote the set of active edges for commodity i by $E_\theta^i \subseteq E$. We call a v - t_i path P an *active v - t_i path for commodity i at time θ* , if all edges of P are active for i at θ or, equivalently, $\sum_{e \in P} c_e(\theta) = \ell_{i,v}(\theta)$. For differentiation we call paths that are minimal with respect to the transit times τ *physical shortest paths*.

Now we are ready to formally define an instantaneous dynamic equilibrium for multi-commodity flows over time:

Definition 2.1 A feasible flow over time f is an *instantaneous dynamic equilibrium (IDE)*, if for all $i \in I, \theta \in \mathbb{R}_{\geq 0}$ and $e \in E$ it satisfies

$$f_{i,e}^+(\theta) > 0 \Rightarrow e \in E_\theta^i. \tag{9}$$

In other words, a feasible flow over time f is an IDE, if, whenever flow of commodity i enters an edge $e = vw$ at some point θ , this edge is contained in the set of active edges E_θ^i , i.e., e lies on a currently shortest path from v to t_i .

Note that, while the set of active edges E_θ^i changes over time, the set of nodes, from which t_i is reachable via E_θ^i (i.e. $\{v \in V | \ell_{i,v}(\theta) < \infty\}$) does not. In particular, this means that, whenever flow of commodity i enters an active edge $uv, v \neq t_i$, at least one of the edges in δ_v^+ will be active by the time the flow reaches v —although possibly different edges than those active when the flow left u .

3 Existence of IDE flows in single-sink networks

In this and the following chapter we only consider single-sink networks, i.e., networks where all commodities have one common sink node t . In this case all commodities have the same label function, which we will denote by ℓ_v . Since the origin of a particle in the network is not important for its route to the sink, we do not distinguish the commodities, and instead, only consider the aggregated flow functions f_e^+ and f_e^- . Furthermore, we restrict the network inflow functions u_i to be right-constant, where a function $u : [a, b) \rightarrow \mathbb{R}$ is *right-constant*, if for every $\theta \in [a, b)$ there exists an $\varepsilon > 0$ such that u is constant on $[\theta, \theta + \varepsilon)$. For this case, we will now describe an algorithm computing an IDE flow.

Let $f = (f^+, f^-)$ denote a feasible flow over time. We denote by

$$b_v^-(\theta) := \sum_{e \in \delta_v^-} f_e^-(\theta) + \sum_{i \in I: s_i=v} u_i(\theta) \tag{10}$$

the current inflow at node v at time θ . Moreover, let $\delta_v^+(\theta) := \delta_v^+ \cap E_\theta$ denote the set of outgoing edges of v that are active at time θ . The main idea of our algorithm works as follows. Starting from time $\theta = 0$ we compute inductively a sequence of intervals $[0, \theta_1), [\theta_1, \theta_2), \dots$ with $0 < \theta_i < \theta_{i+1}$ and corresponding *constant* edge inflows $(f_e^+(\theta))_e$ for $\theta \in [\theta_i, \theta_{i+1})$ that form together with the corresponding edge outflows $(f_e^-(\theta))_e$ an IDE. Suppose we are given an *IDE flow up to time* θ_k , that is, a tuple (f^+, f^-) of right-constant functions $f_e^+ : [0, \theta_k) \rightarrow \mathbb{R}_{\geq 0}$ and $f_e^- : [0, \theta_k + \tau_e) \rightarrow \mathbb{R}_{\geq 0}$ satisfying Constraints (2) to (4) and (9). Note that this is enough information to compute $F_e^+(\theta_k)$ and $F_e^-(\theta_k + \tau_e)$, and thus also $q_e(\theta_k)$, $c_e(\theta_k)$ and $\ell_v(\theta_k)$ for all $e \in E$ and $v \in V$. We now describe how to extend this feasible flow over time to the interval $[\theta_k, \theta_k + \varepsilon)$ for some $\varepsilon > 0$. The idea is that whenever there is positive inflow $b_v^-(\theta_k) > 0$ into some node $v \in V$, we assign this inflow to outgoing edges that are currently active. Since the node labels at the heads of these edges depend themselves on queue dynamics at other nodes along a currently shortest path towards t , we need to handle *time-varying* labels $\ell_w(\theta)$ when distributing the flow among the edges in $\delta_v^+(\theta)$. In the following, we describe how to define the flow-split in order to maintain the invariant that flow is only assigned to edges that are active for at least some interval even if adjacent labels vary linearly over time.

Assume that $b_v^-(\theta)$ is constant for $\theta \in [\theta_k, \theta_k + \varepsilon)$ for a node $v \in V$ and some $\varepsilon > 0$. Moreover, let $\delta_v^+(\theta_k) = \{vw_1, vw_2, \dots, vw_{p_k}\}$ for some $p_k \geq 1$ and $[p_k] := \{1, \dots, p_k\}$. Thus, we have

$$\ell_v(\theta_k) = c_{vw_i}(\theta_k) + \ell_{w_i}(\theta_k) \quad \text{for all } i \in [p_k]. \tag{11}$$

Suppose that the labels of nodes w_i change linearly after θ_k , that is, there are constants $a_{w_i} \in \mathbb{R}$ for $i \in [p_k]$ with

$$\ell_{w_i}(\theta) = \ell_{w_i}(\theta_k) + a_{w_i}(\theta - \theta_k) \quad \text{for all } \theta \in [\theta_k, \theta_k + \varepsilon).$$

Our goal is to find constant edge inflow rates $f_{vw_i}^+(\theta)$ during $[\theta_k, \theta_k + \varepsilon)$ satisfying the supply $b_v^-(\theta)$ and, for some $\varepsilon' > 0$, fulfilling the following invariant for all $i \in [p_k]$ and $\theta \in [\theta_k, \theta_k + \varepsilon')$:

$$f_{vw_i}^+(\theta) > 0 \quad \Rightarrow \quad \ell_v(\theta) = c_{vw_i}(\theta) + \ell_{w_i}(\theta), \tag{12}$$

$$f_{vw_i}^+(\theta) = 0 \quad \Rightarrow \quad \ell_v(\theta) \leq c_{vw_i}(\theta) + \ell_{w_i}(\theta). \tag{13}$$

Note that a constant inflow rate $f_{vw_i}^+$ implies by (5) that the queue length q_{vw_i} is piecewise linear. Hence, the instantaneous travel time c_{vw_i} is also piecewise linear on $[\theta_k, \theta_k + \varepsilon')$ for some $\varepsilon' > 0$, with derivative $c'_{vw_i}(\theta) = \frac{q'_{vw_i}(\theta)}{v_{vw_i}}$. Since all edges

vw_i are active at time θ_k , we have $\ell_v(\theta_k) = c_{vw_i}(\theta_k) + \ell_{w_i}(\theta_k)$ and, thus, a flow with constant inflow rates satisfies (12) and (13) for all $\theta \in [\theta_k, \theta_k + \varepsilon']$, if

$$f_{vw_i}^+(\theta_k) > 0 \Rightarrow \ell'_v(\theta_k) = c'_{vw_i}(\theta_k) + \ell'_{w_i}(\theta_k) \tag{14}$$

$$f_{vw_i}^+(\theta_k) = 0 \Rightarrow \ell'_v(\theta_k) \leq c'_{vw_i}(\theta_k) + \ell'_{w_i}(\theta_k). \tag{15}$$

This condition ensures that whenever an edge vw_i has positive inflow, the remaining distance towards t grows from θ_k onwards at the lowest speed.

For a given value $b_v^-(\theta_k)$ and given vector $(a_{w_i})_{i \in [p_k]}$ we consider the following optimization problem in variables x_{vw_i} for $i \in [p_k]$ in order to obtain inflow rates that satisfy the conditions (14) and (15).

$$\min \sum_{i=1}^{p_k} \int_0^{x_{vw_i}} \frac{g_{vw_i}(z)}{v_{vw_i}} + a_{w_i} dz \tag{OPT-}b_v^-(\theta_k)$$

$$\begin{aligned} \text{s.t. } & \sum_{i=1}^{p_k} x_{vw_i} = b_v^-(\theta_k) \\ & x_{vw_i} \geq 0 \text{ for all } i \in [p_k]. \end{aligned} \tag{16}$$

Each function g_{vw_i} maps an edge inflow rate z to the change of the queue size if a constant flow with rate z would enter the edge vw_i , i.e.,

$$g_{vw_i}(z) := \begin{cases} z - v_{vw_i} & \text{if } q_{vw_i}(\theta_k) > 0, \\ \max\{z - v_{vw_i}, 0\} & \text{if } q_{vw_i}(\theta_k) = 0. \end{cases} \tag{17}$$

Hence, $g_{vw_i}(f_{vw_i}^+(\theta_k))$ is the derivative of q_{vw_i} at θ_k (cf. Eq. (5)).

Lemma 3.1 *There exists an optimal solution $(x_{vw_i})_{i \in [p_k]}$ to $(OPT-}b_v^-(\theta_k))$ and for every optimal solution $f_{vw_i}^+(\theta_k) := x_{vw_i}$ satisfies (14) and (15) for all $i \in [p_k]$.*

Proof The objective function is continuous and the feasible region is non-empty and compact. Hence, by the theorem of Weierstraß at least one optimal solution exists. Moreover, the objective is differentiable over the feasible domain, thus, first order optimality conditions hold. Assigning a multiplier $\lambda \in \mathbb{R}$ to (16) and taking partial derivatives of the Lagrangian over the positive orthant, we obtain

$$x_{vw_i} > 0 \Rightarrow \frac{g_{vw_i}(x_{vw_i})}{v_{vw_i}} + a_{w_i} + \lambda = 0 \tag{18}$$

$$x_{vw_i} = 0 \Rightarrow \frac{g_{vw_i}(x_{vw_i})}{v_{vw_i}} + a_{w_i} + \lambda \geq 0. \tag{19}$$

These conditions imply (14) and (15) with $\ell'_v(\theta_k) := -\lambda$. □

Lemma 3.2 *Let $f = (f^+, f^-)$ be an IDE flow up to time $\theta_k \geq 0$ and suppose there are constant inflow rate functions $b_v^- : [\theta_k, \theta_k + \varepsilon) \rightarrow \mathbb{R}_{\geq 0}$ for some $\varepsilon > 0$ and all nodes $v \in V$ (in particular, this means $\varepsilon \leq \min\{\tau_e | e \in E\}$). Then, there exists some $\varepsilon' > 0$ such that f can be extended to an IDE flow up to time $\theta_k + \varepsilon'$ with all functions f_e^+ constant on the interval $[\theta_k, \theta_k + \varepsilon')$ and all functions f_e^- right-constant on the intervals $[\theta_k + \tau_e, \theta_k + \tau_e + \varepsilon')$.*

Proof First, it is possible to determine the queue lengths at time θ_k using Constraint (1) and from those the labels $\ell_v(\theta_k)$ can be obtained. Applying Lemma 3.1 on the nodes in order of increasing $\ell_v(\theta_k)$ values, we obtain the outflow rates and, therefore, the slope a_v of label ℓ_v for some interval right after θ_k . More precisely, we start with t (since $\ell_t(\theta_k) = 0$) for which we can define $f_e^+(\theta) = f_e^-(\theta + \tau_e) = 0$ for all outgoing edges $e \in \delta_t^+$ and all times $\theta \in [\theta_k, \theta_k + \varepsilon')$, where $\varepsilon' := \varepsilon$. Now we take some node v such that there exists an $\varepsilon' > 0$ and for all nodes w with strictly smaller label at time θ_k and all edges $e \in \delta_w^+$, we have already defined f_e^+ on some interval $[\theta_k, \theta_k + \varepsilon')$ and f_e^- on some interval $[\theta_k + \tau_e, \theta_k + \tau_e + \varepsilon')$ in such a way that on the interval $[\theta_k, \theta_k + \varepsilon')$

1. the labels $\ell_w(\theta)$ change linearly with slope a_w ,
2. no additional edges are added to the sets $\delta_w^+(\theta)$ of active edges leaving w ,
3. the functions f_e^+ and f_e^- for $e \in \delta_w^+$ are constant and right-constant, respectively, and
4. the functions f_e^+ and f_e^- for $e \in \delta_w^+$ satisfy Constraints (2), (4) and (9).

Let $\delta_v^+(\theta_k) := \{vw_1, vw_2, \dots, vw_{p_k}\}$ be the set of active edges at v at time θ_k . Then, at time θ_k , each w_i must have a strictly smaller label than v . Hence, they satisfy Properties (1)–(4). We can now apply Lemma 3.1 to determine the flows $f_{vw_i}^+(\theta_k)$. Additionally, we set $f_e^+(\theta_k) = 0$ for all non-active edges leaving v . Assuming that this flow remains constant on the whole interval $[\theta_k, \theta_k + \varepsilon')$, we can determine the first time $\hat{\theta} \geq \theta_k$, where an additional edge $vw \in \delta_v^+$ or $wv \in \delta_w^+$ with $w \in V$ a node with already defined label ℓ_w for the coming time interval becomes newly active. This can only happen after some positive amount of time has passed, i.e., for some $\hat{\theta} > \theta_k$, because

- at time θ_k the edge was non-active, and therefore $\ell_v(\theta_k) > c_{vw}(\theta_k) + \ell_w(\theta_k)$ or $\ell_w(\theta_k) > c_{wv}(\theta_k) + \ell_v(\theta_k)$, respectively,
- all labels change linearly (and thus continuously) and
- c_{vw} or c_{wv} is changing piecewise linearly, since the length of its queue does so as well (as both f_{vw}^+ and f_{wv}^- are piecewise constant).

If the difference $\hat{\theta} - \theta_k$ is smaller than the current ε' , we take it as our new ε' , otherwise we keep it as it is. In both cases, we extend f_e^+ onto the interval $[\theta_k, \theta_k + \varepsilon')$ for all $e \in \delta_v^+$ by setting $f_e^+(\theta) = f_e^+(\theta_k)$ for all $\theta \in [\theta_k, \theta_k + \varepsilon')$. This guarantees that the label of v changes linearly on this interval, no additional edges become active and the functions f_e^+ are constant. Also, f_e^+ satisfies Constraints (2) and (9) by definition. Finally, we define f_e^- as follows:

$$f_e^-(\theta + \tau_e) := \begin{cases} v_e, & \text{if } q_e(\theta_k) + (\theta - \theta_k)(f_e^+(\theta_k) - v_e) > 0, \\ f_e^+(\theta), & \text{else.} \end{cases}$$

Then, f_e^- is right-constant and together with f_e^+ satisfies Constraint (4). In summary, using this procedure we can extend f node by node to an IDE flow up to $\theta_k + \varepsilon'$ for some $\varepsilon' > 0$. □

Remark 3.3 The above lemma is the key for the following existence result of IDE flows. For the extension property, we used that travel times are strictly positive, because otherwise, for a given IDE flow up to time θ_k , the edge outflow rates are not well-defined for any proper interval after θ_k assuming zero travel times. Existence of IDE flows is not ruled out for $\tau_e \in \mathbb{R}_{\geq 0}$, but this would require the use of a more intricate way of describing the extensions or a completely different approach.

Theorem 3.4 *For any multi-source single-sink network with right-constant network inflow rate functions, there exists an IDE flow f with right-constant functions f_e^+ and f_e^- , $e \in E$.*

Proof Let \mathfrak{F}_0 be the set of tuples (f, θ) , with $\theta \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and f an IDE flow up to time θ with right-constant functions f_e^+ and f_e^- . Define $\hat{\theta}_0 := \sup\{\theta | \exists f \text{ s.t. } (f, \theta) \in \mathfrak{F}_0\}$. If $\hat{\theta}_0 = \infty$ we are done, so suppose $\hat{\theta}_0 < \infty$. There exists an IDE flow f_1 with $\theta_1 := \hat{\theta}_0/2$ such that $(f_1, \theta_1) \in \mathfrak{F}_0$. Now we define

$$\mathfrak{F}_1 := \{(f, \theta) \in \mathfrak{F}_0 | f|_{[0, \theta_1]} = f_1\}.$$

This set is not empty so we set $\hat{\theta}_1 := \sup\{\theta | \exists f \text{ s.t. } (f, \theta) \in \mathfrak{F}_1\}$. By Lemma 3.2 we know that $\hat{\theta}_1 > \theta_1$, and therefore $\hat{\theta}_1 \in (\theta_1, \hat{\theta}_0]$. Let $\theta_2 := (\hat{\theta}_1 - \theta_1)/2$. Going on we get a strict monotone increasing sequence $(\theta_i)_{i \in \mathbb{N}}$ and a non-increasing sequence $(\hat{\theta}_i)_{i \in \mathbb{N}}$ with $\theta_i < \hat{\theta}_i$ for all $i \in \mathbb{N}$ and $\hat{\theta}_i - \theta_i \leq \hat{\theta}_0/2^i \rightarrow 0$ for $i \rightarrow \infty$. Let θ^* be the limit of these two sequences. By taking pointwise limits of the sequence $(f_i)_{i \in \mathbb{N}}$ we can construct a flow f^* such that $(f^*, \theta^*) \in \mathfrak{F}_0$. By Lemma 3.2 we can extend f^* by some ε but this is a contradiction to the definition of $\hat{\theta}_i$ for all i with $\hat{\theta}_i \in [\theta^*, \theta^* + \varepsilon)$. Hence, $\hat{\theta}_0 = \infty$, which finishes the proof. □

Remark 3.5 While there always exists an IDE flow, this flow does not have to be unique. In fact, neither the flow f itself nor the label functions ℓ_v and the time of termination need to be unique. This is in contrast to dynamic equilibria, where at least the label functions are uniquely determined (see Cominetti et al. [5, Theorem 6]). An example for non-uniqueness is illustrated by the instance in Fig. 3.

Remark 3.6 The task to find suitable inflow rates x_e at a given time θ_k can also be formulated globally as follows: Find two vectors $(x_e)_{e \in E_{\theta_k}}, (a_v)_{v \in V}$ such that:

$$\begin{aligned} \sum_{vw \in E_{\theta_k}} x_e &= b_v^-(\theta_k) && \text{for all } v \in V \setminus \{t\}, \\ x_e &\geq 0 && \text{for all } e \in E_{\theta_k}, \\ a_t &= 0, \end{aligned}$$

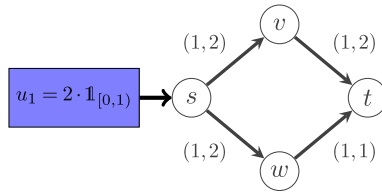


Fig. 3 An example of a graph with infinitely many distinct flows in IDE (all with different label functions and termination times). Two such flows are: all flow uses the above path or all flow uses the bottom path. Since the first edge on both paths has rate capacity 2, no queues will form on those edges. So both edges will lie on a shortest path as long as the flow has not arrived at node w

$$\begin{aligned}
 a_v &= \min_{e=vw \in E_{\theta_k}} \frac{g_e(x_e)}{v_e} + a_w && \text{for all } v \in V \setminus \{t\}, \\
 a_v &= \frac{g_e(x_e)}{v_e} + a_w && \text{for all } e = vw \in E_{\theta_k} \text{ with } x_e > 0,
 \end{aligned}$$

where

$$g_e(x_e) := \begin{cases} x_e - v_e & \text{if } q_e(\theta_k) > 0 \\ \max\{x_e - v_e, 0\} & \text{if } q_e(\theta_k) = 0. \end{cases}$$

This formulation is very similar to the *thin flow with resetting* formulation of dynamic equilibria (see [5,14]) and in the same way the existence of a solution can be shown via Kakutani’s fixed point theorem. We will extend this formulation to a multi-commodity version in order to show the existence of IDE flows in a multiple sink setting in Sect. 5.

We have proven now that an IDE flow always exists, but in order to actually compute an IDE flow we need to make Lemma 3.1 constructive, i.e. provide an algorithm to find an optimal solution to $(\text{OPT-}b_v^-(\theta_k))$ or, equivalently, a distribution of the inflow $b_v^-(\theta_k)$ satisfying Constraints (14) and (15). Since $(\text{OPT-}b_v^-(\theta_k))$ is a convex program there are various methods available—but thanks to the particular nice structure of $(\text{OPT-}b_v^-(\theta_k))$ even a simple water filling procedure suffices. We provide further details as well as an example for computing an IDE flow in the online appendix.

4 Termination of IDE flows in single-sink networks

In this section, we investigate the question, whether an IDE flow vanishes within finite time given finitely lasting network inflow rates. More precisely, given a time θ_0 , such that $\text{supp}(u_i) \subseteq [0, \theta_0]$ for every $i \in I$, we ask whether there exists a time $\hat{\theta} \geq \theta_0$, such that all injected flow actually reaches the sink within time $\hat{\theta}$. To answer this question, we first need to introduce some additional notation. For every edge $e \in E$, we define a function F_e^Δ denoting the total amount of flow currently on edge e (either waiting in its queue or traveling along the edge) for any time θ . As in [21], we call these the *edge load functions*:

$$F_e^\Delta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, \quad \theta \mapsto F_e^+(\theta) - F_e^-(\theta).$$

The function $F^\Delta(\theta) := \sum_{e \in E} F_e^\Delta(\theta)$ specifies the *total amount of flow in the network at time θ* . Furthermore, we define a function Z indicating the amount of flow that already reached the sink t by time θ :

$$Z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, \quad \theta \mapsto \sum_{e \in \delta_t^-} F_e^-(\theta) - \sum_{e \in \delta_t^+} F_e^+(\theta).$$

Note that for IDE flows the subtrahend is always 0 since edges leaving t are never active.

For every subset $W \subseteq V$ and any time θ a direct computation shows that we have

$$\sum_{e \in E(W)} F_e^\Delta(\theta) = \sum_{e \in \delta_W^-} F_e^-(\theta) + \sum_{i \in I: s_i \in W} U_i(\theta) - \sum_{e \in \delta_W^+} F_e^+(\theta) - \begin{cases} Z(\theta), & t \in W \\ 0, & \text{else,} \end{cases} \quad (20)$$

with $\delta_W^+ := \{vw \in E \mid w \in W, v \notin W\}$ and $\delta_W^- := \{vw \in E \mid v \notin W, w \in W\}$. In particular taking $W = V$ we get

$$F^\Delta(\theta) = \sum_{i \in I} U_i(\theta) - Z(\theta). \quad (21)$$

Since $Z'(\theta) = \sum_{e \in \delta_t^-} f_e^-(\theta) - \sum_{e \in \delta_t^+} f_e^+(\theta)$ is always non-negative by Constraint (3), it follows immediately that the total amount of flow in the network is non-increasing after time θ_0 . More generally, since all F_e^+ are non-decreasing, for all $W \subseteq V$ with $\delta_W^- = \emptyset$ we have

$$\sum_{e \in E(W)} F_e^\Delta(\theta_2) \leq \sum_{e \in E(W)} F_e^\Delta(\theta_1) \quad \text{for all } \theta_2 \geq \theta_1 \geq \theta_0. \quad (22)$$

In particular, for $\hat{\theta} \geq \theta_0$ with $F^\Delta(\hat{\theta}) = 0$, we have $F^\Delta(\hat{\theta}) = 0$ for all $\theta \geq \hat{\theta}$.

Definition 4.1 We say a feasible flow f *terminates*, if there exists a time $\hat{\theta} \geq \theta_0$ with $F^\Delta(\hat{\theta}) = 0$, i.e., the network is empty at time $\hat{\theta}$ (and remains empty for all later times).

Before we turn to the main termination result, we need several technical lemmas. Since all of their statements are quite intuitive, we will skip their proofs here and refer the interested reader to the online appendix for the full details. The first lemma shows that all flow on an edge eventually leaves the edge (ignoring the identities of the flow particles).

Lemma 4.2 *Let f be a feasible flow over time, $\theta_1 \in \mathbb{R}_{\geq 0}$, $e \in E$ and any $\lambda \in [0, F_e^\Delta(\theta_1)]$. Then there exists a time $\theta_2 \geq \theta_1$ such that a flow volume of at least λ leaves e during the interval $[\theta_1, \theta_2]$, i.e., $F_e^-(\theta_2) - F_e^-(\theta_1) \geq \lambda$.*

Using this, one can show that for acyclic networks every feasible flow over time terminates. This result will serve as a building block for the more general result that in a single-sink network all IDE flows terminate.

Lemma 4.3 *Let G be an acyclic graph and assume finitely lasting bounded inflow functions. Then, every feasible flow over time terminates.*

As the next step one shows that if the sum of all edge loads between a node v and the sink t are small enough (and, thus, in particular all queues on edges between v and t are small), then an IDE flow can not be diverted away from the physically shortest paths towards t . Since those physically shortest paths form a time independent acyclic subgraph, we are then able to apply Lemma 4.3 to the flow inside this subgraph. In order to formalize this in the next lemma, we need the minimal non-zero difference between two path lengths $\tau_\Delta := \min\{\tau(P) - \tau(P') > 0 \mid u \in V, P, P' \text{ two } u\text{-}t \text{ paths}\}$ and the minimal rate capacity $\nu_{\min} := \min\{\nu_e \mid e \in E\}$.

Lemma 4.4 *If, for some node $v \in V$ and some time $\theta \in \mathbb{R}_{\geq 0}$, every physical shortest v - t path (i.e., w.r.t. τ) has total flow volume of less than $\tau_\Delta \cdot \nu_{\min}$, then all active v - t paths at time θ are also physical shortest v - t paths, i.e., if $\sum_{e \in P} F_e^\Delta(\theta) < \tau_\Delta \cdot \nu_{\min}$ for all physical shortest v - t paths P , then the following holds:*

$$P' \text{ is an active } v\text{-}t \text{ path at time } \theta \implies P' \text{ is a physical shortest } v\text{-}t \text{ path.}$$

Corollary 4.5 *Let f be an IDE flow with $F^\Delta(\hat{\theta}) < \tau_\Delta \cdot \nu_{\min}$ for some $\hat{\theta} \geq \theta_0$. Then, f terminates.*

With the help of these results we can now prove the main result of this section.

Theorem 4.6 *For multi-source single-sink networks, any IDE flow f with finitely lasting bounded network inflow rates u_i terminates.*

Proof Let $W \subseteq V$ be a subset of nodes with the following properties:

1. For every $w \in W$, all physical shortest w - t paths only use edges in $E(W)$.
2. There is a θ_W such that for all $\theta \geq \theta_W$ and $e \in E(W)$, we have $F_e^\Delta(\theta) < \frac{\tau_\Delta \cdot \nu_{\min}}{|E|}$.

We show that for every such $W \neq V$, there exists a node $v \in V \setminus W$ such that $W \cup \{v\}$ also has the two properties. Since $W = \{t\}$ satisfies the two properties this shows that $W = V$ exhibits those as well and, in particular, there exists some time θ_V with $F_e^\Delta(\theta_V) < \frac{\tau_\Delta \cdot \nu_{\min}}{|E|}$ for all edges $e \in E$, and therefore $\sum_{e \in E} F_e^\Delta(\theta_V) < \tau_\Delta \cdot \nu_{\min}$. Hence, by Corollary 4.5, f terminates.

Let $W \subsetneq V$ be a set of nodes fulfilling both properties and $v \in V \setminus W$ be the node with the shortest distance to t with respect to τ of all nodes in $V \setminus W$. Then, all physically shortest v - t paths only use edges from $E(W \cup \{v\})$, so the first property holds for $W \cup \{v\}$. Since the second property holds for W we know that for all $\theta \geq \theta_W$, we have

$$\sum_{e \in E(W)} q_e(\theta) \leq \sum_{e \in E(W)} F_e^\Delta(\theta) < \frac{|E(W)| \cdot \tau_\Delta \cdot \nu_{\min}}{|E|} \leq \tau_\Delta \cdot \nu_{\min}.$$

Lemma 4.4 implies that for every node $w \in W$, all active edges leaving w have to be in $E(W)$, i.e., $\delta_w^+ \cap E_\theta \subseteq E(W)$. Since f is an IDE flow this implies $f_e^+(\theta) = 0$

for all $e \in \delta_W^+$, and thus $F_e^+(\theta) = F_e^+(\theta_W)$ for all those edges. We now assume by contradiction that the second property does not hold for $W \cup \{v\}$, so there is an edge $e \in \delta_v^+ \cap \delta_W^-$ and a sequence of times $\theta_1 < \theta_2 < \dots$ with $F_e^\Delta(\theta_k) \geq \frac{\tau_\Delta \cdot v_{\min}}{|E|}$ for all $k \in \mathbb{N}$ and $\theta_k \rightarrow \infty$ for $k \rightarrow \infty$. From Lemma 4.2, we get times $\theta'_k \geq \theta_k$ with $F_e^-(\theta'_k) - F_e^-(\theta_k) \geq F_e^\Delta(\theta_k) \geq \frac{\tau_\Delta \cdot v_{\min}}{|E|}$. By possibly taking a subsequence, we can assume $\theta'_{k-1} \leq \theta_k$ for all k , and thus,

$$\begin{aligned} F_e^-(\theta'_k) &\geq F_e^\Delta(\theta_k) + F_e^-(\theta_k) \geq F_e^\Delta(\theta_k) + F_e^-(\theta'_{k-1}) \geq \dots \\ &\geq \sum_{j=1}^k F_e^\Delta(\theta_j) \geq k \cdot \frac{\tau_\Delta \cdot v_{\min}}{|E|}. \end{aligned}$$

Hence $F_e^-(\theta_k)$ tends to infinity as k grows larger. On the other hand (20) states that

$$F_e^-(\theta_k) = \sum_{e' \in E(W)} F_{e'}^\Delta(\theta_k) + \sum_{e' \in \delta_W^+} F_{e'}^+(\theta_k) + Z(\theta_k) - \sum_{i \in I: s_i \in W} U_i(\theta_k) - \sum_{e' \in \delta_W^- \setminus \{e\}} F_{e'}^-(\theta_k)$$

which in turn is bounded from above as all positive summands are bounded as well:

- $\sum_{e' \in E(W)} F_{e'}^\Delta(\theta_k) \leq \frac{|E(W)|}{|E|}$ since $\theta_k \geq \hat{\theta}$ and W has the second property.
- $\sum_{e' \in \delta_W^+} F_{e'}^+(\theta_k) = \sum_{e' \in \delta_W^+} F_{e'}^+(\hat{\theta})$ as shown above.
- $Z(\theta_k) \stackrel{(21)}{\leq} \sum_{i \in I} U_i(\theta_k) = \sum_{i \in I} U_i(\theta_0) < \infty$.

This is a contradiction. So the second property must also hold for $W \cup \{v\}$, which concludes the proof. □

5 Existence of IDE flows in multi-sink networks

We now turn to the general model with multiple sinks. In this section, we will present a proof for the existence of IDE flows in such networks. It is similar to the proof for single-sink networks in Sect. 3 using the concept of thin flows, and thus, it is constructive in the same sense. We also develop a second proof using an infinite dimensional variational inequality. While not constructive, it allows for much more general network inflow rate functions u_i (and interestingly, avoids the need for a limit argument to extend the flow for all times altogether). The formal result appears at the end of this section but details for this second approach are given in the online appendix.

5.1 Right-constant network inflow rates

First, we want to show that IDE flows exist in networks with multiple sinks, still under the assumption that the network inflow rates u_i are right-constant. As before, we show that for a given IDE flow up to some point in time θ_k it is possible to extend it by some $\varepsilon > 0$. In contrast to the case of a single sink, we need to determine all inflow rates $f_{i,e}^+$ and the slopes $a_{i,k}$ of the current shortest path distances at the same time.

First of all, we denote the current inflow at a node v for commodity i by

$$b_{i,v}^-(\theta_k) := \sum_{e \in \delta_v^-} f_{i,e}^-(\theta_k) + \mathbb{1}_{v=s_i} \cdot u_i(\theta_k),$$

where $\mathbb{1}_{v=s_i} = 1$ if $v = s_i$ and $\mathbb{1}_{v=s_i} = 0$ otherwise. With this we can extend the idea of IDE thin flows (inspired by the *thin flows with resetting* for dynamic equilibria; see [5,14]) introduced in Remark 3.6 to the multi-commodity setting.

Definition 5.1 (*IDE thin flows*) For a given IDE flow up to time θ_k we call a pair of vectors $(x, a) \in \mathbb{R}_{\geq 0}^{I \times E} \times \mathbb{R}^{I \times V}$ an *IDE thin flow* if it satisfies:

$$\sum_{e \in \delta_v^+} x_{i,e} = b_{i,v}^-(\theta_k) \quad \text{f.a. } i \in I \text{ and } v \in V \setminus \{t_i\}, \quad \text{(TF1)}$$

$$x_{i,e} = 0 \quad \text{f.a. } i \in I \text{ and } e \in E \setminus E_{\theta_k}^i, \quad \text{(TF2)}$$

$$a_{i,t_i} = 0 \quad \text{f.a. } i \in I, \quad \text{(TF3)}$$

$$a_{i,v} = \min_{\substack{e \in E_{\theta_k}^i \\ e=vw}} \frac{g_e(\sum_{j \in I} x_{j,e})}{v_e} + a_{i,w} \quad \text{f.a. } v \in V \setminus \{t_i\}, \quad \text{(TF4)}$$

$$a_{i,v} = \frac{g_e(\sum_{j \in I} x_{j,e})}{v_e} + a_{i,w} \quad \text{f.a. } e = vw \in E_{\theta_k}^i \text{ with } x_{i,e} > 0, \quad \text{(TF5)}$$

where

$$g_e(x_e) := \begin{cases} x_e - v_e & \text{if } q_e(\theta_k) > 0, \\ \max\{x_e - v_e, 0\} & \text{if } q_e(\theta_k) = 0. \end{cases}$$

As a first step, we show that such an IDE thin flow always exists for every reasonable network (i.e., every node v with $b_{i,v}^-(\theta_k) > 0$ can reach t_i within $E_{\theta_k}^i$) and all current inflow vectors $b^-(\theta_k)$. We do this by utilizing the following fixed point theorem by Kakutani [13]:

Theorem 5.2 (Kakutani’s Fixed Point Theorem) *Let K be a compact, convex and non-empty subset of \mathbb{R}^n , $n \in \mathbb{N}$, and $\Gamma: K \rightarrow 2^K$, such that for every $x \in K$ the image $\Gamma(x)$ is non-empty and convex and the set $\{(x, y) | x \in K, y \in \Gamma(x)\}$ is closed. Then there exists a fixed point x^* of Γ , i.e., $x^* \in \Gamma(x^*)$.*

With this theorem we can prove the following lemma:

Lemma 5.3 *For all possible queues $q(\theta_k) \in \mathbb{R}_{\geq 0}^E$, acyclic edge sets $E_{\theta_k}^i \subseteq E$ and all current inflow rates $b^-(\theta_k) \in \mathbb{R}_{\geq 0}^{I \times V}$, such that every node v with $b_{i,v}^-(\theta_k) > 0$ can reach t_i within $E_{\theta_k}^i$, there exists an IDE thin flow (x, a) .*

Proof For every vector $x \in \mathbb{R}_{\geq 0}^{I \times E}$ satisfying (TF1) and (TF2) there exist uniquely defined node labels $a \in \mathbb{R}_{\geq 0}^{I \times V}$ that fulfil (TF3) and (TF4). Existence follows since $E_{\theta_k}^i$ is acyclic and the uniqueness follows from the fact that for every v there is a v - t_i -path within $E_{\theta_k}^i$. This mapping $x \mapsto a$ is continuous. So the only difficult part is to satisfy (TF5).

Let K be the set of vectors x satisfying (TF1) and (TF2), i.e.,

$$K := \left\{ x \in \mathbb{R}_{\geq 0}^{I \times E} \mid \begin{array}{l} \sum_{e \in \delta_v^+} x_{i,e} = b_{i,v}^-(\theta_k) \text{ for all } i \in I \text{ and } v \in V \setminus \{t_i\} \\ \text{and } x_{i,e} = 0 \quad \text{for all } i \in I \text{ and } e \in E \setminus E_{\theta_k}^i \end{array} \right\}.$$

Clearly, K is compact, convex and non-empty.

We define a set-valued function $\Gamma : K \rightarrow 2^K$ as follows:

$$\Gamma(x) = \left\{ y \in K : y_{i,e} = 0 \text{ for all } e \in E_{\theta_k}^i \text{ with } a_{i,v} < \frac{g_e \left(\sum_{j \in I} x_{j,e} \right)}{v_e} + a_{i,w} \right\}$$

where a are the labels corresponding to x . Then $\Gamma(x)$ is non-empty and convex. For non-emptiness note that every node v with $b_{i,v}^- > 0$, except t_i , has at least one outgoing edge with $a_{i,v} = g_e \left(\sum_{j \in I} x_{j,e} \right) / v_e + a_{i,w}$, so y can send everything into this edge. Convexity is clear as well since x determines which edges can be used and which not and these are fixed within $\Gamma(x)$.

Finally, we show that $\{(x, y) \mid x \in K, y \in \Gamma(x)\}$ is a closed set. Therefore, let $(x^n, y^n)_{n \in \mathbb{N}}$ be a sequence in this set that converges in $\mathbb{R}^{I \times E} \times \mathbb{R}^{I \times E}$. Since K is compact, both sequences separately converge to some points x and y in K . Let $(a^n)_{n \in \mathbb{N}}$ be the sequence of associated node labels of x^n and a the node label of x . Since $x \mapsto a$ is continuous we have $a = \lim_{n \rightarrow \infty} a^n$. We need to show that $y \in \Gamma(x)$. Suppose for contradiction that there is a commodity $i \in I$ and an $e = vw \in E_{\theta_k}^i$ with $y_{i,e} > 0$ and $a_{i,v} < g_e \left(\sum_{j \in I} x_{j,e} \right) / v_e + a_{i,w}$. But since g_e is continuous, there has to be an $n_0 \in \mathbb{N}$ such that $y_{i,e}^n > 0$ and $a_{i,v}^n < g_e \left(\sum_{j \in I} x_{j,e}^n \right) / v_e + a_{i,w}^n$ for all $n \geq n_0$. This is a contradiction to $y^n \in \Gamma(x^n)$.

Hence, by Kakutani’s fixed point theorem (Theorem 5.2) there exists an $x^* \in K$ with $x^* \in \Gamma(x^*)$, which forms together with the associated node label a^* an IDE thin flow. □

Consider an IDE flow f up to time θ_k where the inflow rates $f_{i,e}^+$ are right-constant. Due to the continuity of q_e and $a_{i,v}$ we can determine the active arcs $E_{\theta_k}^i$, as well as, the current node inflows $b_{i,v}^-(\theta_k)$ since the feasibility conditions (4) and (6) determine unique outflow rates $f_{i,e}^-(\theta_k)$ for given inflow rates $f_{i,e}^+$ from the past.

In order to extend f , we consider an IDE thin flow (x, a) and extend the inflow rates and current shortest path distances for all $i \in I, e \in E$ and $v \in V$ by

$$f_{i,e}^+(\theta_k + \xi) := x_{i,e} \quad \text{and} \quad \ell_{i,v}(\theta_k + \xi) := \ell_{i,v}(\theta_k) + \xi \cdot a_{i,v} \quad \text{for all } \xi \in [0, \alpha).$$

We call this extended flow over time an α -extension.

To ensure that we end up with an IDE up to time $\theta + \alpha$, the following requirements on the size α of the *extension phase* $[\theta_k, \theta_k + \alpha)$ must be satisfied:

First of all, the queues can never be negative, and therefore, the phase ends as soon as a queue depletes:

$$q_e(\theta_k) + \alpha \cdot \left(\sum_{j \in I} x_{j,e} - v_e \right) \geq 0 \quad \text{for all } e \in E \text{ with } q_e(\theta_k) > 0. \quad (23)$$

Furthermore, the phase ends as soon as an inactive edge gets active. Since queues can build up on inactive edges as well (due to flow of other commodities), we need to take into account the changing rate of a queue, as well. Hence, for all $i \in I$ and $e = vw \in E \setminus E_{\theta_k}^i$ we have:

$$\ell_{i,v}(\theta_k) + \alpha \cdot a_{i,v} \leq \tau_e + \frac{q_e(\theta_k)}{v_e} + \alpha \cdot \frac{g_e \left(\sum_{j \in I} x_{j,e} \right)}{v_e} + \ell_{i,w}(\theta_k) + \alpha \cdot a_{i,w}. \quad (24)$$

Finally, the current node inflow should stay constant during a phase:

$$b_{i,v}^-(\theta_k + \xi) = b_{i,v}^-(\theta_k) \quad \text{for all } i \in I \text{ and } v \in V \setminus \{t_i\} \text{ and all } \xi \in [0, \alpha). \quad (25)$$

We call $\alpha > 0$ *feasible* if it satisfies (23), (24) and (25).

It is easy to see that such a feasible $\alpha > 0$ always exists since $\ell_{i,v}(\theta_k) < \tau_e + \frac{q_e(\theta_k)}{v_e} + \ell_{i,w}(\theta_k)$ for all $i \in I$ and $e = vw \in E \setminus E_{\theta_k}^i$. Furthermore, the functions $b_{i,v}^-$ are right-constant, since $f_{i,e}^+$ as well as u_i are right-constant. Since $\tau_e > 0$ for all $e \in E$ we have that $b_{i,v}^-$ is well-defined and constant on some small interval $[\theta_k, \theta_k + \varepsilon)$.

Lemma 5.4 *Given an IDE flow up to time θ_k , an IDE thin flow (x, a) at time θ_k and a feasible $\alpha > 0$. Then the α -extension is an IDE flow up to time $\theta_{k+1} := \theta_k + \alpha$ and the extended ℓ -functions denote the current shortest path distances.*

Proof First note, that the feasibility conditions are satisfied, since the outflow rates $f_{i,e}^-$ are exactly defined that way. Furthermore, flow conservation holds since

$$\begin{aligned} \sum_{e \in \delta_v^+} f_{i,e}^+(\theta_k + \xi) &= \sum_{e \in \delta_v^+} x_{i,e} = b_{i,v}^-(\theta_k) = b_{i,v}^-(\theta_k + \xi) \\ &= \sum_{e \in \delta_v^-} f_{i,e}^-(\theta_k + \xi) + \mathbb{1}_{v=s_i} \cdot u_i(\theta_k + \xi) \end{aligned}$$

for all $v \in V \setminus \{t_i\}$, $i \in I$ and all $\xi \in [0, \alpha)$.

Next we show that the ℓ labels satisfy Eq. (8). Given a point in time $\theta_k + \xi$ with $\xi \in [0, \alpha)$ we have by (5) applied on the total inflow rate $f_e^+(\theta_k + \xi)$ that

$$q'_e(\theta_k + \xi) = \left\{ \begin{array}{ll} f_e^+(\theta_k + \xi) - v_e & \text{if } q_e(\theta_k + \xi) > 0, \\ \max\{f_e^+(\theta_k + \xi) - v_e, 0\} & \text{else,} \end{array} \right\} = g_e \left(\sum_{j \in I} x_{j,e} \right).$$

Note that we have $q_e(\theta_k + \xi) = q_e(\theta_k) + \xi \cdot g_e \left(\sum_{j \in I} x_{j,e} \right)$ since $q'_e(\theta_k + \xi)$ is constant for $\xi \in [0, \alpha)$.

For non-active arcs $e = vw \notin E^i_{\theta_k}$ we have by (24) that

$$\begin{aligned} \ell_{i,v}(\theta_k + \xi) &= \ell_{i,v}(\theta_k) + \xi \cdot a_{i,v} \\ &\leq \tau_e + \frac{q_e(\theta_k)}{v_e} + \xi \cdot \frac{g_e \left(\sum_{j \in I} x_{j,e} \right)}{v_e} + \ell_{i,w}(\theta_k) + \xi \cdot a_{i,w} \\ &= \tau_e + \frac{q_e(\theta_k + \xi)}{v_e} + \ell_{i,w}(\theta_k + \xi). \end{aligned}$$

For active arcs $e = vw \in E^i_{\theta_k}$ we have by (TF4) that

$$\begin{aligned} \ell_{i,v}(\theta_k + \xi) &= \ell_{i,v}(\theta_k) + \xi \cdot a_{i,v} \\ &\leq \tau_e + \frac{q_e(\theta_k)}{v_e} + \ell_{i,w}(\theta_k) + \xi \cdot \left(\frac{g_e \left(\sum_{j \in I} x_{j,e} \right)}{v_e} + a_{i,w} \right) \\ &= \tau_e + \frac{q_e(\theta_k + \xi)}{v_e} + \ell_{i,w}(\theta_k + \xi). \end{aligned}$$

Since there has to be one active arc that satisfies (TF4) with equality, the same arc satisfies the inequality above with equality, which shows that (8) holds. In other words, the extended ℓ labels denote the current shortest path distances in the α -extension.

Finally, we show that the α -extension satisfies the IDE condition (9): For all $\xi \in [0, \alpha)$ and all arcs $e = vw \in E$ we have that $f_{i,e}^+(\theta_k + \xi) > 0$ implies that $x_{i,e} > 0$, and therefore,

$$a_{i,v}(\theta_k + \xi) = a_{i,v} \stackrel{\text{(TF5)}}{=} \frac{g_e \left(\sum_{j \in I} x_{j,e} \right)}{v_e} + a_{i,w} = \frac{q'_e(\theta_k + \xi)}{v_e} + a_{i,w}(\theta_k)$$

for all $\xi \in [0, \alpha)$. Hence,

$$\ell_{i,v}(\theta_k + \xi) = \tau_e + \frac{q_e(\theta_k + \xi)}{v_e} + \ell_{i,w}(\theta_k + \xi)$$

which shows $e \in E^i_{\theta_k + \xi}$. Hence, the α -extension is indeed an IDE flow up to time $\theta_{k+1} = \theta_k + \alpha$. □

Theorem 5.5 *Consider a multi-source multi-sink network with a finite set of commodities I and right-constant network inflow functions u_i . Then, there exists an IDE flow f with right-constant inflow rate functions $f_{i,e}^+$.*

The proof is exactly the same as the proof for Theorem 3.4 but this time we extend the IDE flow by Lemma 5.4.

Since Kakutani’s fixed point theorem doesn’t give much insight into how to construct such multi-commodity thin flows, we want to give a brief idea how to do this with a mixed integer program. For this we introduce two different types of Boolean decision variables. For every $i \in I$ and $e \in E_{\theta_k}^i$ we have $y_{i,e} \in \{0, 1\}$ and for every $e \in E$ with $q_e(\theta_k) = 0$ we have $z_e \in \{0, 1\}$.

$$\begin{aligned}
 y_{i,e} = 1 &\Leftrightarrow x_{i,e} = 0, && \text{thus, (TF5) does not apply,} \\
 z_e = 1 &\Leftrightarrow \sum_{j \in I} x_{j,e} - v_e \leq 0, && \text{thus, } g_e \left(\sum_{j \in I} x_{j,e} \right) = 0.
 \end{aligned}$$

When these decision variables are guessed correctly, the task to find an IDE thin flow is a simple linear program. It remains open if the complete thin flow can be computed efficiently.

5.2 Arbitrary network inflow rates

Now we turn to the second approach showing that IDE flows also exist for bounded network inflow functions in the L^2 -function space.

Theorem 5.6 *Consider a multi-source multi-sink network with a finite set of commodities I and bounded network inflow functions u_i with $u_i|_{[a,b]} \in L^2([a,b])$ for all $a < b$ and $i \in I$. Then there exists an IDE flow f with bounded inflow rate functions $f_{i,e}^+$.*

The proof idea is to reformulate this problem as an infinite-dimensional variational inequality and then use the existence theorem by Brézis [3, Theorem 24]. A solution to the variational inequality corresponds one-to-one to a multi-sink IDE flow. For all the details including a brief introduction into the Hilbert-space L^2 and weak-strong-continuous functions we refer to the online appendix.

6 Termination of IDE flows in multi-sink networks

We show that there are instances in which all IDE flows do not terminate. We first observe that while the proofs of Lemma 4.3 and Corollary 4.5 can easily be adapted to the multi-sink case (so it is still true that all flows in an acyclic network and all IDE flows with total volume less than $\tau_{\Delta} v_{\min}$ eventually terminate), this is not true for the proof of Theorem 4.6.

Theorem 6.1 *There is a multi-source multi-sink network with two sinks and all edge transit times and rate capacities equal to 1, where any IDE flow does not terminate.*

To construct such an instance we make use of several gadgets. The first one, gadget A , will serve as the main building block and is depicted in Fig. 5. It consists of two cycles with one common edge $v_1 v_2$ and one player i with sink node t (outside the gadget and reachable from the nodes v_2, v_5 and v_7 via some paths P_2, P_5 and P_7 , respectively) and a constant network inflow rate of 2 on the interval $[0, 1)$ at node v_1 . Our goal will be to embed this gadget into a larger instance in such a way, that for any

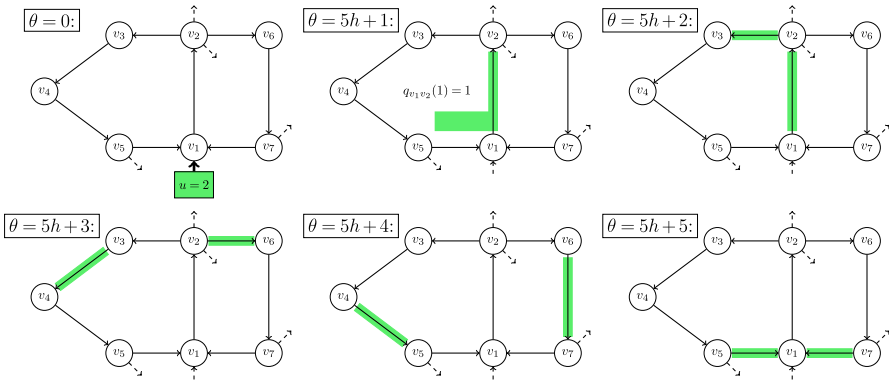


Fig. 4 The desired flow pattern in gadget A at times $\theta = 0, 1, 2, 3, 4, 5, \dots$

IDE flow, the flow associated with player i will exhibit the following flow pattern for all $h \in \mathbb{N}$ (see Fig. 4):

1. **On the interval $[5h, 5h + 1)$:** All flow generated at v_1 (for $h = 0$) or arriving at v_1 (for $h > 0$) enters the edge to v_2 at a rate of 2, half of it directly starting to travel along the edge, half of it building up a queue of length 1 at time $5h + 1$.
2. **On the interval $[5h + 1, 5h + 2)$:** The flow arriving at node v_2 enters the edge to v_3 because v_2, v_3, v_4, v_5, P_5 is currently the shortest path to t . The length of the queue of edge $v_1 v_2$ decreases until it reaches 0 at time $5h + 2$.
3. **On the interval $[5h + 2, 5h + 3)$:** The flow arriving at node v_2 enters the edge to v_6 because v_2, v_6, v_7, P_7 is currently the shortest path to t .
4. **On the interval $[5h + 4, 5h + 5)$:** The flows arriving at nodes v_5 and v_7 enter the respective edges towards node v_1 because v_5, v_1, v_2, P_2 as well as v_7, v_1, v_2, P_2 are currently the shortest paths to get to t .
5. **On the interval $[5h + 5, 5h + 6)$:** There is a total inflow of 2 at node v_1 , which enters the edge to v_2 . Thus, the pattern repeats.

The effect of this behavior is, that other particles outside the gadget, who want to travel through this gadget along the central vertical path, will estimate an additional waiting time as indicated by the diagram displayed inside gadget A in Fig. 5 (next to the vertical red path). Now, in order to actually guarantee the described behavior, we need to embed gadget A into a larger instance in such a way, that for any IDE flow the following assumptions hold:

1. The only edges leaving A are the start edges of the four dashed paths indicated in Fig. 5.
2. The three (blue) paths P_2, P_5 and P_7 are of the same length L (w.r.t. τ_ℓ).
3. For all $h \in \mathbb{N}$
 - the unique shortest v_2 - t path for all $\theta \in [5h + 1, 5h + 2)$ is v_2, v_3, v_4, v_5, P_5 ,
 - the unique shortest v_2 - t path for all $\theta \in (5h + 2, 5h + 3)$ is v_2, v_6, v_7, P_7 ,
 - the unique shortest v_5 - t path for all $\theta \in [5h + 4, 5h + 5)$ is v_5, v_1, v_2, P_2 and
 - the unique shortest v_7 - t path for all $\theta \in [5h + 4, 5h + 5)$ is v_7, v_1, v_2, P_2 .

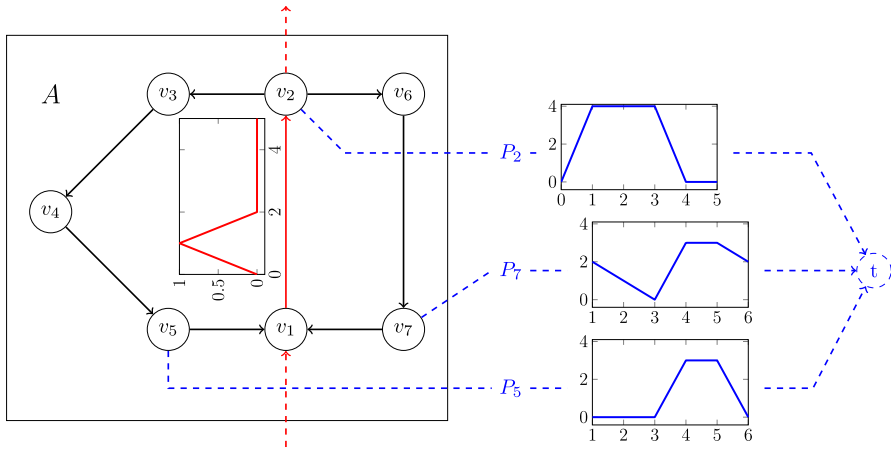


Fig. 5 Gadget A (the dashed paths and nodes are not part of the gadget). The (red) diagram inside the box A indicates the waiting time on edge $v_1 v_2$ [and therefore on the (red) vertical path through the gadget], provided that the flow originating inside this gadget follows the flow pattern indicated in Fig. 4. The (blue) diagrams on the right indicate the desired waiting times on the paths P_2 , P_5 and P_7 , respectively, which in turn ensure that the flow inside the gadget does indeed follow the desired flow pattern (color figure online)

Note that at time $\theta = 5h + 2$ we do not require that there is only one unique v_2-t path. This is due to the fact that waiting times always change continuously and therefore when the shortest v_2-t path changes from one path to another, there needs to be a time where both paths are equally long. Thus, there cannot always be a unique shortest path. However, this does not influence the overall flow pattern, since those discrete points in time form a set of measure zero, and thus, only allow for flow of volume zero to escape the overall flow pattern.

In order to satisfy the assumptions (1)–(3), we will now construct three types of gadgets B_2 , B_5 and B_7 for the three paths P_2 , P_5 and P_7 , each of equal length and on which any IDE flow induces waiting times as shown by the respective diagrams on the right side in Fig. 5.

To build these gadgets we need time shifted versions of gadget A , which we denote by A^{+k} . Such a gadget is constructed the same way as gadget A above, with the only difference that the support of the network inflow rate function u_i is shifted to the interval $[k \bmod 5, k \bmod 5 + 1)$. Gadget B_2 now consists of the concatenation of four gadgets of type A^{+0} , four gadgets of type A^{+1} and four gadgets of type A^{+2} in series along their vertical paths through them with three edges between each two gadgets (see Fig. 6).

Similarly, gadget B_5 consists of three copies of A^{+3} -type gadgets, three copies of A^{+4} -type gadgets and additional $6 \cdot 4$ edges to ensure that the vertical path has the same length as the one of gadget B_2 . Finally, gadget B_7 consists of three copies of A^{+3} -type gadgets, three copies of A^{+4} -type gadgets, two copies of A^{+5} -type gadgets, one copy of A^{+6} -type gadgets and additional $3 \cdot 4$ edges to ensure that the vertical path has the same length as the one of gadget B_2 .

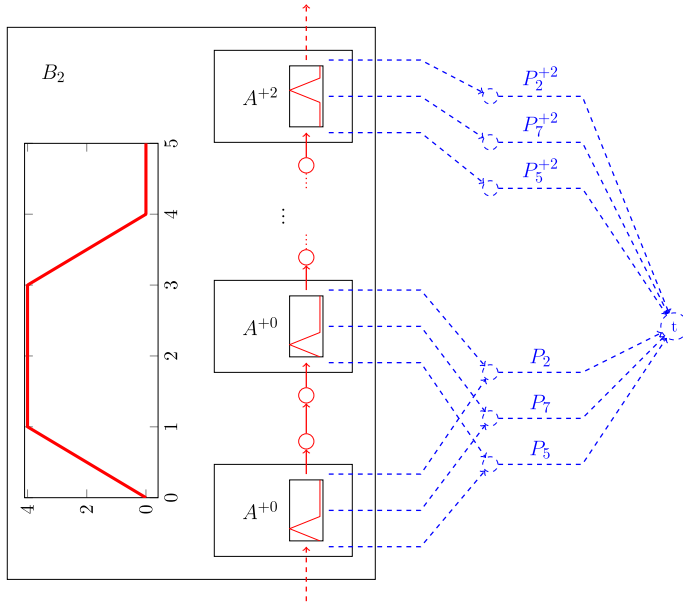


Fig. 6 Gadget B_2 consisting of four copies of each of the types A^{+0}, A^{+1}, A^{+2} . The diagram inside the box of gadget B_2 indicates the waiting time on the vertical path through gadget B_2 , provided that within all of the used gadgets A , the flow follows the flow pattern from Fig. 4. The dashed parts are not part of the gadgets and only sketch how this gadget needs to be embedded in a larger instance

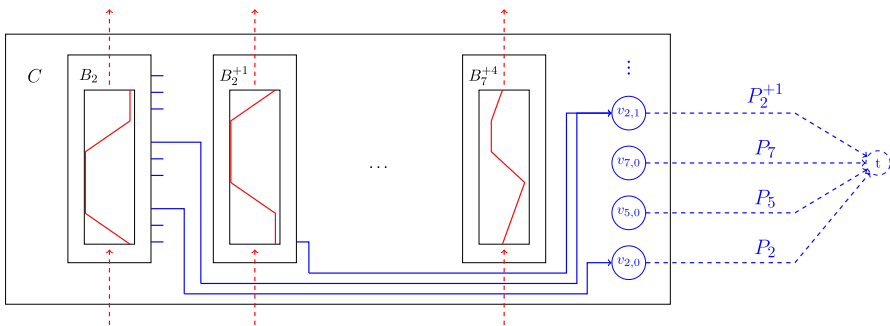


Fig. 7 Gadget C

We again use the notation B_j^{+k} to refer to a time shifted version of gadget B_j —i.e. with all used gadgets A shifted by additional k time steps. Next, we build a gadget C by just taking one copy of each B_j^{+k} for all $j \in \{2, 5, 7\}$ and $k = 0, 1, 2, 3, 4$ (see Fig. 7).

Finally, taking two copies of this gadget, C and C' , and two additional nodes, t and t' , where t will be the sink node for all players in C and t' the sink node for all players in C' , we can build our entire graph as indicated by Fig. 8. We connect the top edges of the gadgets B_j^{+k} in gadget C' with the sink t and use those gadgets' respective vertical paths as the P_j^{+k} paths for gadget C and vice versa.

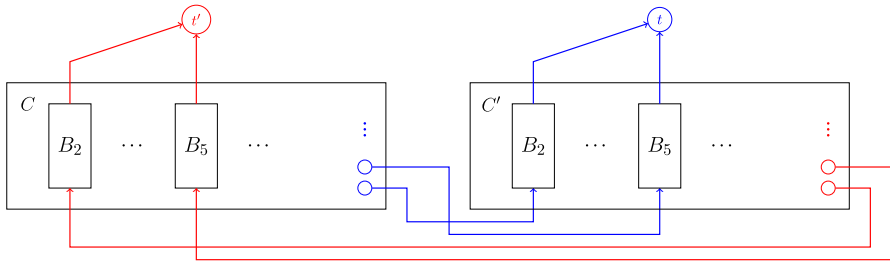


Fig. 8 The whole graph

The correctness of our construction (i.e. that any IDE flow on this instance does not terminate) can be proven by a careful but mostly straightforward analysis using the following important observation. A formal proof of this observation as well as of the main theorem can be found in the online appendix.

Observation 6.2 If a flow in some A^{+k} -type gadget (with $k \in \{0, 1, 2, 3, 4\}$) follows the desired flow pattern for all unit time intervals between k and some $\theta \in \mathbb{N}_0, \theta \geq k$, the induced waiting time on edge $v_1 v_2$ of this gadget (and therefore on the vertical path through this gadget) will follow the waiting time function indicated by the diagram in Fig. 5 (shifted by k) for the next unit time interval $[\theta, \theta + 1)$, independent of the evolution of the flow in this interval.

Although the network constructed for the above proof contains several distinct source nodes, this is in fact not necessary in order to get a network with non-terminating IDEs. I.e. Theorem 6.1 can be strengthened as follows.

Theorem 6.3 *There exists a single-source multi-sink network with two sinks, where any IDE flow does not terminate.*

This can be accomplished by adding a single super-source to the network constructed above and connecting it in an appropriate way to the original source nodes. The detailed construction can be found in the online appendix.

7 Summary and open problems

We introduced in this paper the concept of IDE flows and investigated two key questions: existence and termination of IDE flows. Regarding the former, we gave in Sect. 3 an extension-algorithm leading to the existence of IDE flows for single-sink instances. While the extension-algorithm is constructive, it is not clear if finitely many calls of the algorithm suffice to compute an IDE—at the moment existence relies on a limit argument. Especially for restricted graph classes (series-parallel graphs or acyclic graphs) we expect finiteness. For multi-source multi-sink instances, we gave in Sect. 5 a general existence theorem—also based on an extension property. For piece-wise constant network inflow rate functions this extension can be achieved by solving a set of equations (called IDE thin flows), which we can obtain algorithmically by solving a

linear mixed-integer program. The extension property for general network inflow rates, however, relies on a solution to a variational inequality rendering it non-constructive.

Regarding termination of IDE flows, we showed in Theorem 4.6 that for single-sink networks, all IDE flows terminate. In a forthcoming paper [10], we give quantitative upper bounds of $\mathcal{O}(U\tau(G))$ for the time such an IDE flow needs to terminate, where $U := \sum_{i \in I} \int_0^\infty u_i(\theta) d\theta$ is the total network inflow and $\tau(G) := \sum_{e \in E} \tau_e$ the sum of all edge transit times. On the other hand, we can give lower bounds of order $\Theta(U \log \tau(G))$, see [10] for details. In Sect. 6, we gave an example for a multi-sink network, where no IDE flow terminates. By Theorem 6.3, we know that only a single-source and two sinks are needed for this effect to appear (and by Theorem 4.6 we also know that this is the minimal number of sinks necessary). However, the underlying graph is quite complex and it would be interesting to see, whether there are certain graph classes beside acyclic ones, where termination is guaranteed even in the multi-sink case (e.g., planar graphs, series parallel graphs, ...).

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References

1. Bhaskar, U., Fleischer, L., Anshelevich, E.: A Stackelberg strategy for routing flow over time. *Games Econ. Behav.* **92**, 232–247 (2015)
2. Boyce, D.E., Ran, B., LeBlanc, L.J.: Solving an instantaneous dynamic user-optimal route choice model. *Transp. Sci.* **29**(2), 128–142 (1995)
3. Brézis, H.: Équations et inéquations non linéaires dans les espaces vectoriels en dualité. *Ann. Inst. Fourier* **18**(1), 115–175 (1968)
4. Cominetti, R., Correa, J.R., Larré, O.: Existence and uniqueness of equilibria for flows over time. In: *International Colloquium on Automata, Languages, and Programming*, pp. 552–563. Springer, Berlin (2011)
5. Cominetti, R., Correa, J.R., Larré, O.: Dynamic equilibria in fluid queueing networks. *Oper. Res.* **63**(1), 21–34 (2015)
6. Cominetti, R., Correa, J.R., Olver, N.: Long term behavior of dynamic equilibria in fluid queueing networks. In: *Integer Programming and Combinatorial Optimization—19th International Conference, IPCO 2017, Waterloo, ON, Canada, June 26–28, 2017, Proceedings*, pp. 161–172 (2017)
7. Ford, L.R., Fulkerson, D.R.: *Flows in Networks*. Princeton University Press, Princeton (1962)
8. Friesz, T.L., Bernstein, D., Smith, T.E., Tobin, R.L., Wie, B.-W.: A variational inequality formulation of the dynamic network user equilibrium problem. *Oper. Res.* **41**(1), 179–191 (1993)

9. Friesz, T.L., Luque, J., Tobin, R.L., Wie, B.-W.: Dynamic network traffic assignment considered as a continuous time optimal control problem. *Oper. Res.* **37**(6), 893–901 (1989)
10. Graf, L., Harks, T.: Termination bounds for instantaneous dynamic equilibrium flows. Working paper (2019)
11. Hamdouch, Y., Marcotte, P., Nguyen, S.: A strategic model for dynamic traffic assignment. *Netw. Spat. Econ.* **4**(3), 291–315 (2004)
12. Ismaili, A.: Routing games over time with FIFO policy. In: Devanur, N.R., Lu, P. (eds.) *Web and Internet Economics*, pp. 266–280. Springer, Berlin (2017). [arXiv:1709.09484](https://arxiv.org/abs/1709.09484)
13. Kakutani, S.: A generalization of brouwer's fixed point theorem. *Duke Math. J.* **8**(3), 457–459 (1941)
14. Koch, R., Skutella, M.: Nash equilibria and the price of anarchy for flows over time. *Theory Comput. Syst.* **49**(1), 71–97 (2011)
15. Marcotte, P., Nguyen, S., Schoeb, A.: A strategic flow model of traffic assignment in static capacitated networks. *Oper. Res.* **52**(2), 191–212 (2004)
16. Meunier, F., Wagner, N.: Equilibrium results for dynamic congestion games. *Transp. Sci.* **44**(4), 524–536 (2010)
17. Peeta, S., Ziliaskopoulos, A.: Foundations of dynamic traffic assignment: the past, the present and the future. *Netw. Spat. Econ.* **1**(3), 233–265 (2001)
18. Ran, B., Boyce, D.E.: *Modeling Dynamic Transportation Networks: An Intelligent Transportation System Oriented Approach*. Springer, Berlin (1996)
19. Ran, B., Boyce, D.E., LeBlanc, L.J.: A new class of instantaneous dynamic user-optimal traffic assignment models. *Oper. Res.* **41**(1), 192–202 (1993)
20. Sering, L., Skutella, M.: Multi-source multi-sink nash flows over time. In: Borndörfer, R., Storandt, S. (eds.) *18th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2018)*, Volume 65 of OASiCS—OpenAccess Series in Informatics, pp 12:1–12:20. Dagstuhl, Wadern (2018). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik
21. Sering, L., Vargas-Koch, L.: Nash flows over time with spillback. In: *Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms*. ACM (2019)
22. Skutella, M.: An introduction to network flows over time. In: *Research Trends in Combinatorial Optimization, Bonn Workshop on Combinatorial Optimization, November 3–7, 2008, Bonn, Germany*, pp. 451–482 (2008)
23. Unnikrishnan, A., Waller, S.: User equilibrium with recourse. *Netw. Spat. Econ.* **9**(4), 575–593 (2009)
24. Vickrey, W.S.: Congestion theory and transport investment. *Am. Econ. Rev.* **59**(2), 251–60 (1969)
25. Zhu, D., Marcotte, P.: On the existence of solutions to the dynamic user equilibrium problem. *Transp. Sci.* **34**(4), 402–414 (2000)

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