Entanglement swapping in presence of dephasing

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1 Introduction The phenomenon of entanglement of quantum degrees of freedom occurring in open quantum systems has attracted considerable attention due to its prominent significance for both, fundamental physics and applications of quantum information processing [1]. The open system impacts the otherwise unitary time-evolution, giving rise to loss of coherence properties [2–5]. There have been proposed several procedures to sustain coherence and to minimize decoherence processes in such systems. E.g. the way how the system interacts with its outer environment decides about entenglement survival in the long time regime [6, 7]. In this work we consider one of the more challenging applications involving the concept of entanglement in quantum communication: the entanglement swapping [8]. This procedure, originally proposed in Ref. [9] has been developed both theoretically and experimentally in Refs. [10–13], allows to generate entanglement between qubits which neither come from the same source nor have they ever interacted before. In the following we focus on the (most) simple system composed of two pairs of non-interacting qubits which are coupled in a non-demolition manner to their own, independent, environments which are either of finite or infinite (thermodynamic) size. Put differently, the two qubits undergo pure dephasing only with those surroundings. We demonstrate that this deteriorating coherence mechanism in such a system does not render entanglement swapping ineffective even at long asymptotic measurements times, provided that the environments possess a superohmic spectrum. We also propose a method of controlling the swapped entanglement by a proper choice of the initial preparation of the surroundings.

There exist several quantitative measures of entanglement degree of bipartite systems [14]. Two common such measures are the concurrence [15] and the negativity [16], which can uniquely be related to each other [17]. In this work, we employ as a measure for physical entanglement the negativity; i.e., $N(\rho) = \max(0, -\sum_i \lambda_i)$ [16], where λ_i denote the negative eigenvalues of the partially transposed density matrix $\rho(t)$ of a pair of qubits [18]. For entangled mixed states, this negativity assumes positive values, whereas it identically vanishes for disentangled states.

2 Entanglement swapping Let us consider four qubits, denoted as $\{S_1, S_2, S_3, S_4\}$. Importantly, the four qubits do not interact with each other. At initial time of preparation $t = 0$, the two subsystems, formed by two pairs of qubits each, i.e., $A = \{S_1, S_2\}$ and $B = \{S_3, S_4\}$ are prepared in the separable state

$$\rho(0) = \rho_A(0) \otimes \rho_B(0), \tag{1}$$

wherein each pair A and B of the qubits is maximally entangled,

$$\rho_k(0) = \frac{1}{2} (|01\rangle + |10\rangle) (\langle 01| + \langle 10|)_k , \quad k = A, B. \tag{2}$$

The notation $|01\rangle$ means that for $k = A$, the qubit S_1 is in the state $|0\rangle$ and the qubit S_2 is in the state $|1\rangle$. *Mutatis mutandis* for $k = B$: the qubit S_3 is in the state $|0\rangle$ and the qubit S_4 is in the state $|1\rangle$. As there is no interaction acting between the pairs A and B, the separability remains preserved at any later time $t > 0$,

$$\rho(t) = \rho_A(t) \otimes \rho_B(t). \tag{3}$$

Let at time $t = \tau$ the entanglement swapping procedure being performed: it consists of projecting the system on the Bell state of the qubits $\{S_2, S_4\}$, i.e. measuring the system by means of the Bell state measurement [8]

$$\rho(\tau) \to \rho^{\text{swap}}(\tau) = \text{Tr}_{S_2,S_4} (|\Psi_{24}\rangle\langle\Psi_{24}|\rho(\tau)) , \tag{4}$$

where

$$|\Psi_{24}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)_{24} \tag{5}$$

is the symmetric Bell state of the qubits S_2 and S_4. As a result, one can create an entanglement between the qubits S_1 and S_3 which in fact have never interacted with each other before. Because the density matrix $\rho^{\text{swap}}(\tau)$ is a 4×4 matrix, we can utilize the well-known, established methods to analyze the degree of entanglement, e.g. one can evaluate the negativity $N(\rho^{\text{swap}})$. To do so, we have to specify the dynamics (the Hamiltonian) of the system of four qubits.

3 Hamiltonian and reduced dynamics The system of four qubits consists of two independent and identical subsystems A and B and which are characterized by the total Hamiltonian

$$H = H_{\text{A}} + H_{\text{B}}, \qquad H_{\text{A}} = H_1 + H_2, \qquad H_{\text{B}} = H_3 + H_4. \tag{6}$$

Because the subsystems A and B are identical, we describe the subsystem A only.

The subsystem A consists of a pair of qubits, i.e., $A = \{S_1, S_2\}$, and its Hamiltonian reads

$$H_1 = \omega_0 S_1^z + S_1^z \otimes \sum_{k=1}^{\infty} g_k (a_k^\dagger + a_k) + \sum_{k=1}^{\infty} \omega_k a_k^\dagger a_k \tag{7}$$

for the qubit S_1 and

$$H_2 = \frac{1}{2}|0\rangle\langle 0| \otimes K_+ + \frac{1}{2}|1\rangle\langle 1| \otimes K_- \tag{8}$$

for the qubit S_2.

The qubit S_1 is represented by the spin-1/2 operator S_1^z. It interacts with a heat bath R_1 modeled by an infinite quasi-free reservoir, composed of bosonic harmonic oscillators of angular frequencies ω_k, the operators a_k and $a_k^\dagger$ are Bose annihilation and creation operators. The strength of the interaction between the qubit S_1 and the k-th mode of the heat bath R_1 is described by the coupling constant g_k .

The qubit S_2 is coupled to its own environment R_2 represented by a finite (or infinite) quantum system and the interaction is described in terms of the operators $K_\pm$ which are elements of a Lie algebra $\mathcal{G}$ generating the symmetry group G [19],

$$K_\pm = \sum_{k=1}^{N} h_\pm^k(t) X_k \pm \varepsilon_0, \quad [X_r, X_j] = \sum_l C_{rj}^l X_l, \tag{9}$$

where $h_\pm^k(t)$ are scalar control functions and X_k are basis elements of the Lie algebra with the structural constants C_{rj}^l. For the qubit S_2, in its standard basis $\{|0\rangle, |1\rangle\}$, e.g. $S_2^z = (|0\rangle\langle 0| - |1\rangle\langle 1|)/2$. The energy levels of the qubit S_2 are $\varepsilon_0/2$ and $-\varepsilon_0/2$. The Hamiltonian for the subsystem $B = \{S_3, S_4\}$ (consisting of two qubits S_3 and S_4) has the same form as for the subsystem A, provided that the qubit S_3 is coupled to the heat bath R_3 and the qubit S_4 is coupled to the heat bath R_4.

The reduced dynamics of the qubits can be determined *exactly for arbitrary model parameters* [20–23] provided the initial state $\varrho(0)$ of the total system described by the Hamiltonian (6) can be factorized; namely,

$$\varrho(0) = \rho_1 \otimes \rho_A(0) \otimes \rho_2 \otimes \rho_3 \otimes \rho_B(0) \otimes \rho_4. \tag{10}$$

We next assume that the state ρ_1 of the heat bath R_1 is an equilibrium Gibbs state of temperature T_1 and the initial state of the environment R_2 is any state of the form $\rho_2 = |\Omega\rangle\langle\Omega|$. The state ρ_3 of the heat bath R_3 is an equilibrium Gibbs state of temperature T_3 and the initial state of the environment R_4 is a similar state as that for R_2. For a time $t > 0$, the state $\rho(t)$ of total system of four qubits assumes the form (cf. Eq. (5.19) in [20])

$$\begin{aligned} \rho(t) &= \Lambda(t)\rho(0) \\ &= \Lambda_1(t) \otimes \Lambda_2(t) \otimes \Lambda_3(t) \otimes \Lambda_4(t)\rho(0), \end{aligned} \tag{11}$$

where

$$\Lambda_n(t)\rho = C_n^{(1)}(t)\rho + 2C_n^{(2)}(t)[S_n^z, \rho] + 4C_n^{(3)}(t) S_n^z \rho S_n^z \tag{12}$$

for $n = 1,2,3,4$ and with ρ an arbitrary operator. The functions $C_n^{(i)}(t); i = 1,2,3$, read explicitly

$$C_n^{(1)}(t) = \frac{1}{2}[1 + F_n(t)\cos\phi_n(t)],$$
$$C_n^{(2)}(t) = \frac{1}{2}iF_n(t)\sin\phi_n(t),$$
$$C_n^{(3)}(t) = \frac{1}{2}[1 - F_n(t)\cos\phi_n(t)]. \quad (13)$$

For the qubits S_1 and S_3, we then obtain the result [20]

$$\phi_1(t) = \phi_3(t) = \omega_0 t, \quad F_n(t) = \exp[-f_n(t)], \quad (14)$$

where the damping function

$$f_n(t) = \int_0^\infty d\omega \frac{J_n(\omega)}{\omega^2} \coth(\hbar\omega\beta_n/2)[1 - \cos\omega t] \quad (15)$$

for $n = 1,3$. The parameters $\beta_n = 1/k_B T_n$ and k_B is the Boltzmann constant. The spectral functions $J_n(\omega)$ are assumed to take the form [23,24]

$$J_n(\omega) = \lambda_n\, \omega^{1+\mu_n} \exp(-\omega/\omega_n^c), \quad \mu > -1, \quad (16)$$

where the cut-off frequency ω_n^c determines the largest energy scale of the heat bath R_n. The parameter λ_n is the coupling constant of the qubit S_n and R_n. The spectral exponent μ_n characterizes low frequency properties of the heat baths R_n. According to the classification proposed in Ref. [24], the heat bath is called sub-Ohmic for $\mu \in (-1,0)$, Ohmic for $\mu = 0$ and super-Ohmic for $\mu \in (0,\infty)$. This classification shall be reflected in the dynamical properties of entanglement.

For the qubits S_2 and S_4, we find with $\rho_2 = |\Omega_2\rangle\langle\Omega_2|$ and $\rho_4 = |\Omega_4\rangle\langle\Omega_4|$ the result for phase [25]

$$\phi_n(t) = \varepsilon_0 t + \arg\left[\langle\Omega_n|T_n^\dagger(g_-^{(n)}(t))T_n(g_+^{(n)}(t))|\Omega_n\rangle\right] \quad (17)$$

and the amplitude

$$F_n(t) = |\langle\Omega_n|T_n^\dagger(g_-^{(n)}(t))T_n(g_+^{(n)}(t))|\Omega_n\rangle| \quad (18)$$

for $n = 2,4$, where $T^{(n)}(\cdot)$ is a representation of the group $G^{(n)}$ acting in the space of the controlling system R_n and functions $g_\pm^{(n)}(t) \in G^{(n)}$ depend on the specific form given in eq. (9). A relevant example will be given below.

4 Entanglement swapping in environments producing pure dephasing The swapped entanglement, quantified by the negativity, for qubits with reduced dynamics (11)-(17) reads

$$N = N(\rho^{\text{swap}})(\tau) = \frac{1}{2}F_1(\tau)F_2(\tau)F_3(\tau)F_4(\tau). \quad (19)$$

In the absence of coupling to the environments, when $F_n = 1$ for $n = 1,2,3,4$, there occurs maximal entanglement swapping with $N = 1/2$. The effect of the coupling to the environments producing pure dephasing is two–fold. The infinite baths cause dissipation of information. This effect can either result in a full deterioration of the entanglement swapping, or only partially, i.e. such that the swapping is still effective (i.e. the qubits 1 and 3 become entangled) at arbitrary time instants of measurement.

First let us consider the case when the subsystems A and B are coupled to the infinite baths R_1 and R_3 only (i.e. $F_2(\tau) = F_4(\tau) = 1$). For convenience, the baths have exactly the same characteristics, i.e. $\lambda_1 = \lambda_3 = \lambda, \mu_1 = \mu_3 = \mu, \omega_1^c = \omega_3^c = \omega_c$. We assume first that the baths operate at vanishing temperature,i.e., $T_1 = T_3 = 0$. In order to make the work self-contained we quote here the formulas derived in [20]. For the Ohmic bath ($\mu = 0$), one obtains

$$F_1(\tau) = F_3(\tau) = (1 + \omega_c^2\tau^2)^{-\lambda/2}. \quad (20)$$

For the sub-Ohmic and super-Ohmic baths one finds instead

$$F_1(\tau) = F_3(\tau) = \exp\{-\lambda\Gamma(\mu)\omega_c^\mu \times[1 - (1 + \omega_c^2\tau^2)^{-\mu/2}\cos(\mu\arctan(\omega_c\tau))]\}, \quad (21)$$

where $\Gamma(z)$ is the Euler gamma function. One can deduce that for both Ohmic ($\mu = 0$) and sub-Ohmic ($\mu \in (-1,0)$) reservoirs, $F_1(\infty) = F_3(\infty) = 0$. In consequence, the long-time negativity is zero, $N = 0$, and there occurs no entanglement swapping in the asymptotic long-time limit. For the super-Ohmic bath ($\mu > 0$), however,

$$F_1(\infty) = F_3(\infty) = \exp(-\lambda\Gamma(\mu)\omega_c^\mu) \neq 0. \quad (22)$$

It follows that in this case the long-time negativity remains positive, $N > 0$ and the information does not deteriorate completely, i.e., one can obtain a non-vanishing swapped entanglement also at $\tau \to \infty$, as depicted in Fig. 1.

When temperatures of the bosonic baths are nono-zero, $T_1 > 0, T_3 > 0$, then the entanglement swapping can also be performed effectively still at asymptotic long times; but the environments need to be super-Ohmic with $\mu > 1$ [25].

For the system considered so far, the negativity (and the swapped entanglement) is a monotonic decreasing function of measurement time t. The results are depicted with in Fig. 2. The saturation of the swapped entanglement at asymptotic large time is, for superohmic baths, readily observed. Any additional bath influences cause a decrease of the swapped entanglement which, remarkably, remains still finite provided the bath is superohmic. If one wants to manipulate the entanglement swapping in a desired way (e. g. via modulating or maintaining the entanglement in a desired interval), a control method of how to achieve this has to be devised. Below, we propose one of the possible scenarios how to decrease and increase in time t the entanglement swapping by controlling the dynamics by an external finite quantum system; i.e. by a proper choice of the controlling Hamiltonians H_2 and H_4. As an example, we consider a control scheme with a single bosonic mode. This situation may typically occur for circuit-quantum electrodynamics (circuit-QED); see in Refs. [13,26]. Put differently, let in Eq. (9) for both S_2 and S_4, the operators $K_\pm$

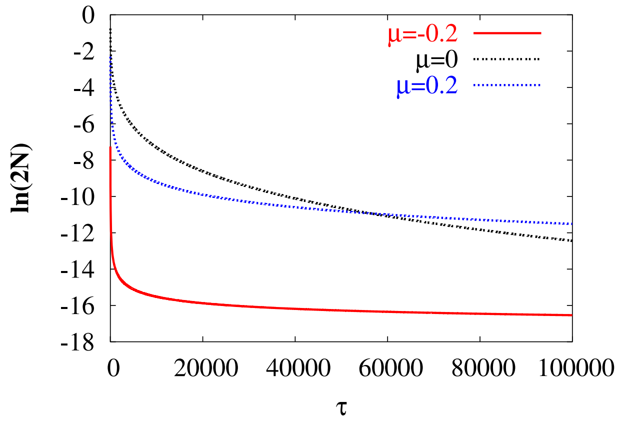

Figure 1 Entanglement swapped at the dimensionless time $\tau = \omega_0 t$ in the system interacting with one bosonic baths ($T = 0$) with different μ, fixed $\lambda = 1$ and $\omega_c/\omega_0 = 10^3$.

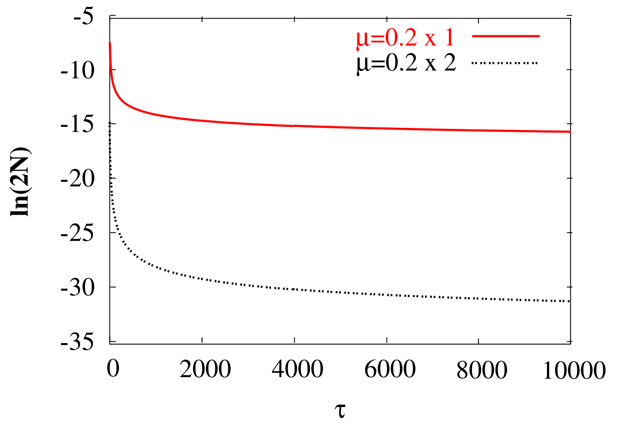

Figure 2 Entanglement swapping performed at the dimensionless time $\tau = \omega_0 t$ for the system interacting with one (solid line, red online) or two (dotted line or black online) super-Ohmic bosonic baths ($T = 0$) with $\mu = 0.2$, $\lambda = 1$ and $\omega_c/\omega_0 = 10^3$.

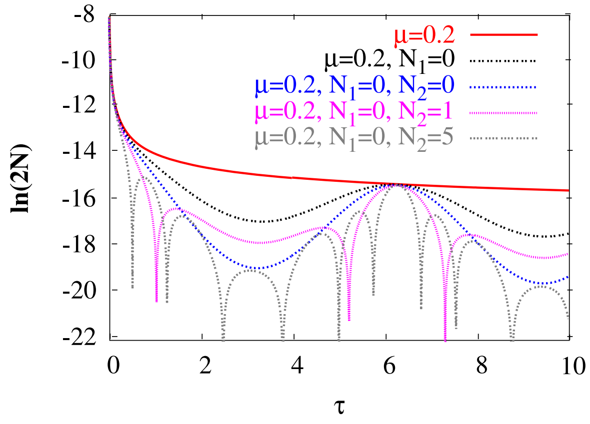

Figure 3 Entanglement swapping at $\tau = \omega_0 t$ in the system interacting with the single bosonic bath ($T = 0$) with $\mu = 0.2$ and none (solid line, red online), one (the case with N_1 only) and two (the case with N_1 and N_2) finite controlling systems with $\varepsilon_0 = \omega_0$ prepared in a number eigenstate either $|N_1\rangle$ (one controlling system) or $|N_1, N_2\rangle$ (two controlling finite systems), respectively. $\lambda = 1$ and $\omega_c/\omega_0 = 10^3$.

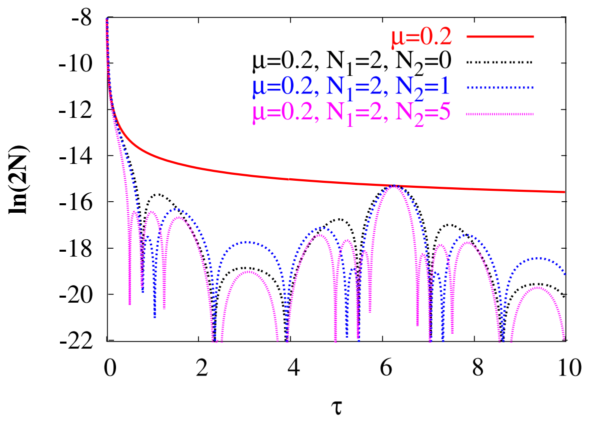

Figure 4 Entanglement swapped at $\tau = \omega_0 t$ in the system interacting with the single bosonic bath ($T = 0$) with $\mu = 0.2$ and two finite controlling systems with $\varepsilon_0 = \omega_0$ prepared in a number eigenstate $|N_1, N_2\rangle$. $\lambda = 1$ and $\omega_c/\omega_0 = 10^3$.

be of the form

$$K_\pm = a^\dagger a \pm \gamma_\pm (a + a^\dagger) \pm 1. \tag{23}$$

In the following we limit ourselves to an isotropic coupling $\gamma_\pm = \gamma$. The corresponding functions $F_2(t)$ and $F_4(t)$ then read with

$$F_2(t) = F_4(t) = |\langle \Omega | D(\alpha(t)) | \Omega \rangle|, \tag{24}$$

where

$$\alpha(t) = \gamma[1 - \exp(it)] \tag{25}$$

and $D(z)$ is the displacement operator [19,27]

$$D(z) = \exp(z a^\dagger - z^* a). \tag{26}$$

This operator generates the set of standard coherent states. The function $F_2(t) = F_4(t)$ is the Weyl function studied intensively in the context of interference phenomena [28] and for mesoscopic devices controlled by non-classical external fields [29]. The effect of the control when applied in a particularly simple case with $|\Omega\rangle$ being a number (Fock)-eigenstate, is shown in Figs. 3 and 4 for one and two controlling systems. The non-monotonic entanglement of the qubits swapped at the time τ results from the non-Markovian properties of the evolving qubits. The only *non-local* operation performed on the system is the Bell state mesurment. The time evolution of uncoupled qubits is clearly local and hence, in agreement with the common wisdom, cannot increase bipartite entanglement, which is never larger than the initial one. As the time homogeneity of the non-Markovian system is broken the local operations related to the time evolution transform the reduced density matrices on the time interval $t = 0 \rightarrow t = \tau$. The resulting swapped entanglement can be effectively designed: one can predict the qubit–qubit correlations by a proper choice of the initial state of the controlling boson. Such a method can be of great importance when the measurment time τ cannot easily and precisely be adjused. There is an obvious adventage of using several controlling bosons instead of a single harmonic oscillator coupled to a single qubit: the rich structure of control scenarios can be mantained with the help of only lowest excitations with small N.

5 Conclusions In summary, we studied the problem of origination of qubit–qubit entanglement via entanglement swapping when the qubits are interacting with a surrounding environment producing dephasing onto the qubit dynamics. It has been shown that under certain conditions this procedure still remains effective even for asymptotically long times t: in particular, the decoherence of entanglement information is asymptotically not entirely destroyed for superohmic environments at non-vanishing, finite temperatures. The effective control by a suitable initial preparation of a controlling finite quantum system (e.g. via a coupling to a single mode bosonic oscillator) can be performed, leading to oscillatory-like behavior of the entanglement obtained by a swapping measurement.

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