Dyson Ferrari–Spohn diffusions and ordered walks under area tilts

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Abstract We consider families of non-colliding random walks above a hard wall, which are subject to a self-potential of tilted area type. We view such ensembles as effective models for the level lines of a class of $2+1$-dimensional discrete-height random surfaces in statistical mechanics. We prove that, under rather general assumptions on the step distribution and on the self-potential, such walks converge, under appropriate rescaling, to non-intersecting Ferrari–Spohn diffusions associated with limiting Sturm–Liouville operators. In particular, the limiting invariant measures are given by the squares of the corresponding Slater determinants.

Keywords Ordered random walks · Self-potentials · Invariance principles · Sturm–Liouville operators · Doob transforms · Ferrari–Spohn diffusions · Dyson diffusions · Slater determinants

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1 Introduction

Random walks under area tilts mimic phase separation lines in certain low-temperature two-dimensional lattice models of statistical mechanics, particularly in the regime of pre-wetting. A prototypical example is the two-dimensional Ising model in a large box with negative boundary conditions and a small positive magnetic field h. In such circumstances, the $\pm$-interface is pushed towards the boundary of the box and its fluctuations above flat segments of the boundary are expected to be of order $h^{-1/3}$. Rigorous justification of the latter claim is still an open problem (but see [21] for partial results in this direction). Instead, the papers [12,15] are devoted to a refined analysis of effective random walk models of such interfaces. In particular, the full scaling limits were identified in [15], for a large class of effective random walks, as Ferrari–Spohn diffusions [10].

In this paper, we consider ensembles of n non-colliding random walks which are subject to generalized area tilts. Precise definitions are given in Sect. 3. These ensembles are intended to model non-intersecting level lines for certain low-temperature $2+1$-dimensional interfaces (which themselves are intended to model two-dimensional random surfaces of lattice statistical mechanics). A prototypical example is the SOS model, see [3,4] and references therein, or even more so its version with bulk Bernoulli fields which was introduced in [13]. In either case, low-temperature level lines have the structure of Ising polymers whose effective random walk representation is discussed in [14] and is based on the general fluctuation theory of ballistic walks with self-interactions as developed in [16].

In Sect. 2, we introduce and briefly discuss the class of limiting objects, which we call Dyson Ferrari–Spohn diffusions. The latter can be alternatively described as Ferrari–Spohn diffusions conditioned to remain ordered, or as ergodic n-dimensional diffusions driven by the log-derivative of the Slater determinants of the corresponding Sturm–Liouville operators. The construction is well understood: We refer to [19, Section 2] for extensive details on determinantal random point fields in general and Fermi gas in particular, and to [7, Section 3] where such diffusions are discussed for specific kernels in the context of random matrix theory.

Properties of Dyson Ferrari–Spohn diffusions, in the form we need them, are formulated in Theorem 1. To keep our exposition self-contained and to stress the role played by the Karlin–McGregor formula, we sketch the proof.

Our effective model of ordered walks under area tilts is introduced in Sect. 3, while our main result, Theorem 2, is formulated in Sect. 3.4. In Sect. 3.6, we introduce the rescaling notation which is employed in all the subsequent arguments. The step-by-step structure of our arguments is explained in Sect. 3.7. The details of the proofs are given in Sects. 4–7 and in the “Appendix”. The organization of these sections is described in Sect. 3.8. Many of our technical estimates rely on strong approximation techniques and on a refinement of recent results on random walks in Weyl chambers and on cones [6,8].

2 Sturm–Liouville operators and Dyson Ferrari–Spohn diffusions

2.1 One particle

Consider the Sturm–Liouville operator

$$\mathsf{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - q(r), \tag{2.1}$$

where q is a non-negative symmetric C^2-potential satisfying

$$\lim_{|r|\to\infty} q(r) = \infty. \tag{2.2}$$

We can either think of L as being defined on $\mathbb{L}_2(\mathbb{R}_+)$ with zero boundary conditions at zero, or as being defined on $\mathbb{L}_2(\mathbb{R})$. It is a classical fact [5] that L has a complete orthonormal family of eigenfunctions

$$\mathsf{L}\varphi_i = -\mathsf{e}_i\varphi_i \quad 0 < \mathsf{e}_1 < \mathsf{e}_2 < \cdots \nearrow \infty. \tag{2.3}$$

The Krein–Rutman eigenfunction φ_1 is positive on $(0,\infty)$, respectively on $\mathbb{R}$.

In the case of the half-line, L has a closed self-adjoint extension from $\mathsf{C}_0(0,\infty)$, whereas in the case of $\mathbb{R}$ it has a closed self-adjoint extension from $\mathsf{C}_0(\mathbb{R})$. In both cases, the domain of the closure is given by

$$\mathcal{D}(\mathsf{L}) = \left\{ f = \sum_k f_k\varphi_k \ : \ \sum_k \mathsf{e}_k f_k^2 < \infty \right\}. \tag{2.4}$$

We proceed to discuss the half-line case only; the full-line case would be a literal repetition.

L is a generator of a contraction semigroup T_t on $\mathbb{L}_2(\mathbb{R}_+)$: For $f = \sum f_k\varphi_k$,

$$\mathsf{T}_t f(r) = \sum_k f_k \mathrm{e}^{-\mathsf{e}_k t}\varphi_k. \tag{2.5}$$

This semigroup has the following probabilistic representation: For $r > 0$, let $\hat{\mathbf{P}}^r$ be the (sub-probability) path measure of the Brownian motion started at r and killed upon hitting the origin. Then, for any $f \in \mathcal{D}(\mathsf{L})$ and any $t \geq 0$,

$$\mathsf{T}_t f(r) = \hat{\mathbf{E}}^r \left\{ \mathrm{e}^{-\int_0^t q(B(s))\,\mathrm{d}s} f(B(t)) \right\}. \tag{2.6}$$

Clearly, T_t is an integral operator with kernel h_t given by

$$h_t(r,s) = \sum_{k=1}^{\infty} \mathrm{e}^{-\mathsf{e}_k t}\varphi_k(r)\varphi_k(s). \tag{2.7}$$

2.2 n non-colliding particles

Let us fix $n \in \mathbb{N}$ and define

$$\mathbb{A}_n^+ = \{\underline{r} \in \mathbb{R}^n \ : \ 0 < r_1 < \cdots < r_n\}. \tag{2.8}$$

Let L_i be a copy of L acting on the i-th variable. Consider the closed self-adjoint extension of

$$\sum_{i=1}^n \mathsf{L}_i = \sum_{i=1}^n \left(\frac{1}{2} \frac{\partial^2}{\partial r_i^2} - q(r_i) \right) \tag{2.9}$$

from $\mathsf{C}_0(\mathbb{A}_n^+)$ and let T_t^+ be the corresponding contraction semigroup on $\mathbb{L}_2(\mathbb{A}_n^+)$.

In probabilistic terms, T_t^+ can be described as follows: For an n-tuple $\underline{r} \in \mathbb{A}_n^+$, set

$$\hat{\mathbf{E}}^{\underline{r}} = \hat{\mathbf{E}}^{r_1} \otimes \hat{\mathbf{E}}^{r_2} \otimes \cdots \otimes \hat{\mathbf{E}}^{r_n}.$$

Let $\underline{B}$ be the n-dimensional Brownian motion, and define

$$\tau = \min\{t \ : \ \underline{B}(t) \notin \mathbb{A}_n^+\}. \tag{2.10}$$

In other words, τ is the minimum between the first collision time and the first time the bottom trajectory exits from the positive semi-axis. Then,

$$\mathsf{T}_t^+ f(\underline{r}) = \hat{\mathbf{E}}_{\underline{r}} \big\{ \mathrm{e}^{-\sum_i \int_0^t q(B_i(s))\,\mathrm{d}s} f(\underline{B}(t)) \mathbb{1}_{\tau > t} \big\}. \tag{2.11}$$

T_t^+ is an integral operator on $\mathbb{L}_2(\mathbb{A}_n^+)$ and, by the Karlin–McGregor formula, its kernel κ_t is given by

$$\kappa_t(\underline{r}, \underline{s}) = \det\{h_t(r_i, s_j)\}. \tag{2.12}$$

2.3 Limiting behaviour

Let

$$\Delta(\underline{r}) = \det \begin{bmatrix} \varphi_1(r_1) & \varphi_2(r_1) & \cdots & \varphi_n(r_1) \\ \varphi_1(r_2) & \varphi_2(r_2) & \cdots & \varphi_n(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(r_n) & \varphi_2(r_n) & \cdots & \varphi_n(r_n) \end{bmatrix}. \tag{2.13}$$

Note that $\Delta \in \mathbb{L}_2(\mathbb{A}_n^+)$.

Theorem 1 *Set* $\mathsf{D}_n = \sum_{\ell=1}^n \mathsf{e}_\ell$. *Then,*

$$\lim_{t \to \infty} \mathrm{e}^{\mathsf{D}_n t} \kappa_t(\underline{r}, \underline{s}) = \Delta(\underline{r}) \Delta(\underline{s}). \tag{2.14}$$

Moreover, for any $f \in \mathbb{L}_2(\mathbb{A}_n^+)$ *and for any* $t > 0$*,*

$$\begin{aligned}
\lim_{T\to\infty} & \frac{\mathbb{E}_{\underline{r}}\left\{\mathrm{e}^{-\sum_i \int_0^T q(B_i(s))\,\mathrm{d}s} f(\underline{X}(t))\,\mathbb{1}_{T<\tau_+}\right\}}{\mathbb{E}_{\underline{r}}\left\{\mathrm{e}^{-\sum_i \int_0^T q(B_i(s))\,\mathrm{d}s}\,\mathbb{1}_{T<\tau_+}\right\}} \\
&= \frac{\mathrm{e}^{\mathsf{D}_n t}}{\Delta(\underline{r})}\mathbb{E}_{\underline{r}}\left\{\mathrm{e}^{-\sum_i \int_0^t q(B_i(s))\,\mathrm{d}s} f(\underline{B}(t))\Delta(\underline{B}(t))\,\mathbb{1}_{t<\tau_+}\right\} \\
&= \frac{\mathrm{e}^{\mathsf{D}_n t}}{\Delta(\underline{r})}\mathsf{T}_t^+(f\Delta)(\underline{r}) \overset{\Delta}{=} \mathsf{S}_t^+ f(\underline{r}). \qquad (2.15)
\end{aligned}$$

In its turn, S_t^+ *is a diffusion semigroup with transition kernel*

$$q_t(\underline{r}, \underline{s}) = \frac{\mathrm{e}^{\mathsf{D}_n t}}{\Delta(\underline{r})}\kappa_t(\underline{r}, \underline{s})\Delta(\underline{s}), \qquad (2.16)$$

which is self-adjoint on $\mathbb{L}_2(\mathbb{A}_n^+, \Delta^2)$*; the generator of the corresponding ergodic diffusion on* $\mathbb{A}_n^+$ *is given by*

$$\mathfrak{G}_n^+ = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial r_i^2} + \nabla\log(\Delta)(\underline{r})\cdot\nabla = \frac{1}{2\Delta^2(\underline{r})}\mathrm{div}\big(\Delta^2(\underline{r})\nabla\big). \qquad (2.17)$$

Proof Let us introduce the column vectors

$$b_\ell = \begin{bmatrix} \varphi_\ell(r_1) \\ \varphi_\ell(r_2) \\ \vdots \\ \varphi_\ell(r_n) \end{bmatrix} \qquad (\ell = 1, \dots, n)$$

and the volume form $F(c_1, \dots, c_n) = \det[c_1, \dots, c_n]$. Under our assumptions, (2.3), (2.7) and (2.12) imply, asymptotically as $t \to \infty$, that

$$\begin{aligned}
\mathrm{e}^{\mathsf{D}_n t}\kappa_t(\underline{r}, \underline{s})\big(1 + \mathrm{o}(1)\big) &= F\left(\sum_{\ell=1}^n \varphi_\ell(s_1)b_\ell, \sum_{\ell=1}^n \varphi_\ell(s_2)b_\ell, \dots, \sum_{\ell=1}^n \varphi_\ell(s_n)b_\ell\right) \\
&= F(b_1, \dots, b_n)\sum_\sigma (-1)^{\mathrm{sgn}(\sigma)}\prod_{\ell=1}^n \varphi_{\sigma_\ell}(s_\ell), \qquad (2.18)
\end{aligned}$$

and the first claim (2.14) follows. Above, σ runs over all permutations of $\{1, \dots, n\}$ and $\mathrm{sgn}(\sigma) = \pm 1$ denotes the signature of σ.

Modulo some technicalities, (2.15) follows from (2.14) and the Markov property.

Finally, $\mathfrak{G}_n$ in (2.17) is the generator of S_t^+, since the generator of T_t^+ is the closed self-adjoint extension of (2.9) from $\mathsf{C}_0(\mathbb{A}_n^+)$ and since, by direct computation,

$$\sum_{i=1}^{n}\left(\frac{1}{2}\frac{\partial^2}{\partial r_i^2} - q(r_i)\right)\Delta(\underline{r}) = -\mathsf{D}_n \Delta(\underline{r}). \tag{2.19}$$

□

2.4 Dyson diffusions for Sturm–Liouville operators

For every $n \in \mathbb{N}$, the diffusion

$$\mathrm{d}\underline{x}(t) = \mathrm{d}\underline{B}(t) + \nabla \log(\Delta)(\underline{x}(t))\,\mathrm{d}t, \tag{2.20}$$

with the generator $\mathfrak{G}_n$ described in Theorem 1, lives on $\mathbb{A}_n^+$ and is reversible with respect to $\Delta^2(\underline{r})\,\mathrm{d}\underline{r}$. In the sequel, we shall use $\mathbb{P}_n^+$ for its distribution on $\mathsf{C}\big((-\infty, +\infty), \mathbb{A}_n^+\big)$ and $\mathbb{P}_n^{+;T}$ for the restriction of this distribution to $\mathsf{C}\big([-T, T], \mathbb{A}_n^+\big)$. Without loss of generality, let us assume that $\Delta^2(\underline{r})$ is a probability density (on $\mathbb{A}_n^+$). Note that the latter has a determinantal structure:

$$\Delta^2(\underline{r}) = \det\{\mathsf{K}_n(r_i, r_j)\}, \tag{2.21}$$

where the kernel K_n is given by

$$\mathsf{K}_n(r, s) = \sum_{\ell=1}^{n} \varphi_\ell(r)\varphi_\ell(s). \tag{2.22}$$

In particular, the level density distribution is given by

$$\rho_n(r) = \frac{1}{n}K_n(r, r) = \frac{1}{n}\sum_{\ell=1}^{n} \varphi_\ell^2(r). \tag{2.23}$$

There are similar determinantal formulas for level spacing, gap probabilities, etc. We note that the unpublished work [2] contains results on the universality of scaling limits (as $n \to \infty$) in this general Sturm–Liouville context.

3 Ordered walks with area tilts

3.1 Underlying random walks and ordering of trajectories

The setup follows [15].

Let p_y be an irreducible random walk kernel on $\mathbb{Z}$. The probability of a finite trajectory $\mathbb{X} = (X_1, X_2, \ldots, X_k)$ is $\mathsf{p}(\mathbb{X}) = \prod_i p_{X_{i+1}-X_i}$. The product probability of n finite trajectories $\underline{\mathbb{X}} = (\mathbb{X}^1, \ldots, \mathbb{X}^n)$ is

$$\mathbf{P}(\underline{\mathbb{X}}) = \prod_{\ell=1}^{n} \mathsf{p}(\mathbb{X}^\ell). \tag{3.1}$$

Assumptions on p. Assume that

$$\sum_{z\in\mathbb{Z}} zp_z = 0 \quad \text{and } \mathsf{p} \text{ has finite exponential moments.} \tag{3.2}$$

In order to facilitate the notation, we shall assume in the sequel that the variance satisfies

$$\sigma^2 = \sum_{z\in\mathbb{Z}} z^2 p_z = 1. \tag{3.3}$$

This assumption simplifies formulas in our technical arguments. The main result remains valid, after an obvious change of the normalization, for all $\sigma \in (0, \infty)$; see Remark 2 after Theorem 2.

Sets of trajectories. Let $u, v \in \mathbb{N}$. As in [15], $\mathcal{P}^{u,v}_{M,N,+}$ is used to denote the set of trajectories $\mathbb{X}$ starting at u at time M, ending at v at time N and staying positive during the time interval $\{M, \ldots, N\}$.

Let $\underline{u}, \underline{v} \in \mathbb{N}^n \cap \mathbb{A}^+_n$ and $M, N \in \mathbb{Z}$ with $M \le N$. Let $\mathcal{P}^{\underline{u},\underline{v}}_{M,N,+}$ be the family of n trajectories $\underline{\mathbb{X}}$ starting at $\underline{u}$ at time M, ending at $\underline{v}$ at time N and satisfying

$$0 < X^1_j < X^2_j < \cdots < X^n_j \quad \forall j \in \{M+1, \ldots, N-1\}. \tag{3.4}$$

For $N > 0$, we shall use the shorthand notations $\mathcal{P}^{\underline{u},\underline{v}}_{N,+} = \mathcal{P}^{\underline{u},\underline{v}}_{-N,N,+}$ and $\hat{\mathcal{P}}^{\underline{u},\underline{v}}_{N,+} = \mathcal{P}^{\underline{u},\underline{v}}_{1,N,+}$.

The model which we define below is a polymer measure over ordered trajectories from $\mathcal{P}^{\underline{u},\underline{v}}_{N,+}$.

3.2 The model

Let $\{V_\lambda\}_{\lambda>0}$ be a family of self-potentials $V_\lambda : \mathbb{N} \to \mathbb{R}_+$. For a finite trajectory $\mathbb{X} = (X_M, \ldots, X_N)$, let $\mathsf{p}(\mathbb{X}) = \prod_{i=M+1}^{N} \mathsf{p}(X_i - X_{i-1})$ be its probability for the underlying random walk, and let us introduce the tilted weights

$$\mathsf{w}_\lambda(\mathbb{X}) = \mathsf{e}^{-\sum_{i=M+1}^{N} V_\lambda(X_i)} \mathsf{p}(\mathbb{X}). \tag{3.5}$$

Given $u, v \in \mathbb{N}$ and $\lambda > 0$, define the partition functions and the probability distributions

$$Z^{u,v}_{N,+,\lambda} = \sum_{\mathbb{X} \in \mathcal{P}^{u,v}_{N,+}} \mathsf{w}_\lambda(\mathbb{X}) \text{ and } \mathbb{P}^{u,v}_{N,+,\lambda}(\mathbb{X}) = \frac{1}{Z^{uv}_{N,+,\lambda}} \mathsf{w}_\lambda(\mathbb{X}) \mathbb{1}_{\{\mathbb{X} \in \mathcal{P}^{u,v}_{N,+}\}}. \tag{3.6}$$

In the case of an n-tuple $\underline{\mathbb{X}} = (\mathbb{X}^1, \dots, \mathbb{X}^n)$ of trajectories, we consider the product weights $\mathsf{w}_\lambda(\underline{\mathbb{X}}) = \prod_{i=1}^n \mathsf{w}_\lambda(\mathbb{X}^i)$. If $\mathcal{S}$ is a finite or countable set of such tuples, then the corresponding restricted partition functions are denoted by

$$Z_\lambda[\mathcal{S}] = \sum_{\underline{\mathbb{X}} \in \mathcal{S}} \mathsf{w}_\lambda(\underline{\mathbb{X}}). \tag{3.7}$$

We shall use the shorthand notations $Z^{\underline{u},\underline{v}}_{N,+,\lambda} = Z_\lambda[\mathcal{P}^{\underline{u},\underline{v}}_{N,+}]$ and $\hat{Z}^{\underline{u},\underline{v}}_{N,+,\lambda} = Z_\lambda[\hat{\mathcal{P}}^{\underline{u},\underline{v}}_{N,+}]$. Finally, let us define the probability distribution $\mathbb{P}^{\underline{u},\underline{v}}_{N,+,\lambda}$ on $\mathcal{P}^{\underline{u},\underline{v}}_{N,+}$ by

$$\mathbb{P}^{\underline{u},\underline{v}}_{N,+,\lambda}(\underline{\mathbb{X}}) = \frac{1}{Z^{\underline{u},\underline{v}}_{N,+,\lambda}} \mathsf{w}_\lambda(\underline{\mathbb{X}}) \mathbb{1}_{\{\underline{\mathbb{X}} \in \mathcal{P}^{\underline{u},\underline{v}}_{N,+}\}}. \tag{3.8}$$

The term $\sum_{-N+1}^N V_\lambda(X_i)$ represents a generalized (non-linear) area below the trajectory $\mathbb{X}$. It reduces to (a multiple of) the usual area when $V_\lambda(x) = \lambda x$. As in [15], we make the following set of assumptions on V_λ:

3.3 Assumptions on V_λ and the scale H_λ

For any $\lambda > 0$, the function V_λ on $[0, \infty)$ is continuous, monotone increasing and satisfies

$$V_\lambda(0) = 0 \quad \text{and} \quad \lim_{x \to \infty} V_\lambda(x) = \infty. \tag{3.9}$$

In particular, the relation

$$H_\lambda^2 V_\lambda(H_\lambda) = 1 \tag{3.10}$$

determines unambiguously the quantity H_λ. Furthermore, we make the assumptions that $\lim_{\lambda \downarrow 0} H_\lambda = \infty$ and that there exists a function $q \in \mathsf{C}^2(\mathbb{R}^+)$ such that

$$\lim_{\lambda \downarrow 0} H_\lambda^2 V_\lambda(r H_\lambda) = q(r), \tag{3.11}$$

uniformly on compact subsets of $\mathbb{R}_+$. Note that H_λ, respectively H_λ^2, plays the role of the spatial, respectively temporal, scale in the invariance principle which is formulated below in Theorem 2.

Furthermore, we shall assume that there exist $\lambda_0 > 0$ and a (continuous non-decreasing) function $q_0 \geq 0$ with $\lim_{r \to \infty} q_0(r) = \infty$ such that, for all $\lambda \leq \lambda_0$,

$$H_\lambda^2 V_\lambda(r H_\lambda) \geq q_0(r) \text{ on } \mathbb{R}_+. \tag{3.12}$$

Finally, we assume that q_0 grows to ∞ sufficiently fast; namely, for any $\kappa > 0$,

$$\int_0^\infty \mathrm{e}^{-\kappa q_0(r)}\, \mathrm{d}r < \infty. \tag{3.13}$$

Presumably, our main results hold without assumption (3.13). However, since it is rather soft and since it implies the claim of the technically very convenient Lemma 2 below, we decided to keep it.

Remark 1 A natural class of examples of family of potentials satisfying assumptions (3.9)–(3.13) is given by $V_\lambda(x) = \lambda x^\alpha$ with $\alpha > 0$. For the latter, $H_\lambda = \lambda^{-1/(2+\alpha)}$ and $q(r) = q_0(r) = r^\alpha$. In this way, the case of linear area tilts $\alpha = 1$ corresponds to the familiar Airy rescaling $H_\lambda = \lambda^{-1/3}$.

3.4 The result

We set $h_\lambda = H_\lambda^{-1}$. The paths are rescaled as follows: For $t \in h_\lambda^2 \mathbb{Z}$, define

$$\underline{x}^\lambda(t) = h_\lambda \underline{X}_{H_\lambda^2 t} = \frac{1}{H_\lambda} \underline{X}_{H_\lambda^2 t}. \tag{3.14}$$

Then, extend $\underline{x}^\lambda$ to any $t \in \mathbb{R}$ by linear interpolation. In this way, given $T > 0$ and $\underline{u}, \underline{v}$, we can talk about the induced distribution $\mathbb{P}_{N,+,\lambda}^{\underline{u},\underline{v};T}$ on the space of continuous functions $\mathsf{C}\big([-T, T], \mathbb{A}_n^+\big)$.

Theorem 2 *Let λ_N be a sequence satisfying*

$$\lim_{N\to\infty} \lambda_N = 0 \quad \textit{and} \quad \lim_{N\to\infty} \frac{N}{H_{\lambda_N}^2} = \infty. \tag{3.15}$$

Fix any $C \in (0, \infty)$ and any $T > 0$. Then, the sequence of distributions $\mathbb{P}_{N,+,\lambda_N}^{\underline{u},\underline{v};T}$ converges weakly to the distribution $\mathbb{P}_n^{+;T}$ of the ergodic diffusion $\underline{x}(\cdot)$ in (2.20), uniformly in $v_n, u_n \le C H_{\lambda_N}$.

In the sequel, we shall denote $a_N := N/H_{\lambda_N}^2$; with this notation, the second statement in (3.15) becomes $\lim_{N\to\infty} a_N = \infty$.

Remark 2 If $\sigma \neq 1$ in (3.3), then the assertion of Theorem 2 may be adjusted as follows: Define H_λ and q as in (3.10) and (3.11), and set $q_\sigma(r) = q(\sigma r)$. Let λ_N be a sequence satisfying (3.15). Define $\underline{x}_\sigma$ to be the Dyson diffusion (2.20) for the Sturm–Liouville operator $L_\sigma = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - q_\sigma(r)$. Then $\underline{x}_\sigma$ is the weak limit of the rescaled process

$$\underline{x}_\sigma^{\lambda_N}(t) = \frac{1}{\sigma H_{\lambda_N}} \underline{X}_{H_{\lambda_N}^2 t}.$$

3.5 Non-strict constraints

In the sequel, we shall focus on the strict constraints expressed in (3.4). However, a rather straightforward modification of our arguments would imply that the conclusion still holds when the ordering in (3.4) is non-strict, that is, when we instead require that

$$0 \leq X_j^1 \leq X_j^2 \leq \cdots \leq X_j^n \quad \forall j \in \{M, \ldots, N\}. \tag{3.16}$$

Namely, let $\mathcal{P}^{\underline{u},\underline{v}}_{M,N,0}$ be the family of n trajectories $\underline{\mathbb{X}}$ starting at $\underline{u}$ at time M, ending at $\underline{v}$ at time N and satisfying (3.16). As in the case of strict ordering, we use abbreviation $\mathcal{P}^{\underline{u},\underline{v}}_{N,0} = \mathcal{P}^{\underline{u},\underline{v}}_{-N,N,0}$. Define [recall (3.1)]

$$Z^{\underline{u},\underline{v}}_{N,0,\lambda} = \sum_{\underline{\mathbb{X}} \in \mathcal{P}^{\underline{u},\underline{v}}_{N,0}} \mathrm{e}^{-\sum_{\ell=1}^{n} \sum_{i=-N}^{N} V_\lambda(X_i^\ell)} \mathbf{P}(\underline{\mathbb{X}})$$

and

$$\mathbb{P}^{\underline{u},\underline{v}}_{N,0,\lambda}(\underline{\mathbb{X}}) = \frac{\mathrm{e}^{-\sum_{\ell=1}^{n} \sum_{i=-N}^{N} V_\lambda(X_i^\ell)} \mathbf{P}(\underline{\mathbb{X}})}{Z^{\underline{u},\underline{v}}_{N,0,\lambda}} \mathbb{1}_{\{\underline{\mathbb{X}} \in \mathcal{P}^{\underline{u},\underline{v}}_{N,0}\}}.$$

Corollary 1 *Under the same assumptions, the conclusions of Theorem 2 hold for the family of measures* $\mathbb{P}^{\underline{u},\underline{v}}_{N,0,\lambda}$.

3.6 Rescaling and the corresponding notation

It will be convenient to adjust our notations to the running scales h_λ. Define:

$$\mathbb{N}_\lambda = h_\lambda \mathbb{N}, \quad \mathbb{A}^+_{n,\lambda} = \mathbb{A}^+_n \cap (\mathbb{N}_\lambda)^n \quad \text{and} \quad \mathbb{Z}_\lambda = h_\lambda^2 \mathbb{Z}. \tag{3.17}$$

In this way, $\underline{x}^\lambda(t)$ in (3.14) belongs to $\mathbb{A}^+_{n,\lambda}$ for every $t \in \mathbb{Z}_\lambda$.

For $a, b, t \in \mathbb{Z}_\lambda$ and $\underline{r}, \underline{s} \in \mathbb{A}^+_{n,\lambda}$, we shall write, with a slight abuse of notation,

$$\mathcal{P}^{\underline{r},\underline{s}}_{a,b,+,\lambda} \equiv \mathcal{P}^{H_\lambda \underline{r}, H_\lambda \underline{s}}_{H_\lambda^2 a, H_\lambda^2 b,+,\lambda} \text{ and similarly for } \mathcal{P}^{\underline{r},\underline{s}}_{t,+,\lambda} \text{ and } \hat{\mathcal{P}}^{\underline{r},\underline{s}}_{t,+,\lambda}. \tag{3.18}$$

The same conventions apply to partition functions (e.g., we shall write $\hat{Z}^{\underline{r},\underline{s}}_{t,+,\lambda} = Z_\lambda[\hat{\mathcal{P}}^{\underline{r},\underline{s}}_{t,+,\lambda}]$) and for probability distributions (e.g., we shall write $\mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda}$ for $a \in \mathbb{Z}_\lambda$ and $\underline{r}, \underline{s}, \in \mathbb{A}^+_{n,\lambda}$).

With the above notations, Theorem 2 can be restated as follows: Let

$$\lim_{N\to\infty} \lambda_N = 0 \quad \text{and} \quad \lim_{N\to\infty} a_N = \infty. \tag{3.19}$$

Then, the family of distributions $\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}$ converges weakly to the distribution $\mathbb{P}^{+;T}_n$ of the ergodic diffusion $\underline{x}(\cdot)$, uniformly in $r_n, s_n \le C$.

Our proofs rely on the properties of the underlying rescaled random walks (without area tilts). The corresponding notation for the latter follows the above convention adopted for polymer measures: Given $\lambda > 0$ and $\underline{r} \in \mathbb{N}^n_\lambda$, we use $\hat{\mathbf{P}}^{\underline{r}}_\lambda$ for the law of the rescaled walk started at time zero at $\underline{r}$. The *restriction* of $\hat{\mathbf{P}}^{\underline{r}}_\lambda$ to the set of trajectories which stay in $\mathbb{A}^+_{n,\lambda}$ during the interval $[0, t]$ is denoted by $\hat{\mathbf{P}}^{\underline{r}}_{t,+,\lambda}$. When the end-point t is clear from the context, we will sometimes use the shorthand notation $\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}$. Finally, given $\underline{s} \in \mathbb{N}^n_\lambda$ and $t \in \mathbb{Z}_\lambda$, we use

$$\hat{\mathbf{P}}^{\underline{r},\underline{s}}_{t,\lambda} = \hat{\mathbf{P}}^{\underline{r}}_\lambda\big(\cdot \mid \underline{x}^\lambda(t) = \underline{s}\big) \quad \text{and} \quad \hat{\mathbf{P}}^{\underline{r},\underline{s}}_{t,+,\lambda} = \hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}\big(\cdot \mid \underline{x}^\lambda(t) = \underline{s}\big). \tag{3.20}$$

3.7 Structure of the argument

As $\lambda \downarrow 0$, the following notion of convergence is employed: Consider the spaces $\ell_2(\mathbb{N}_\lambda)$ and $\ell_2(\mathbb{A}^+_{n,\lambda})$ with scalar products

$$\langle f, g\rangle_{2,\lambda} = h_\lambda \sum_{r\in\mathbb{N}_\lambda} f(r)g(r) \text{ and, respectively, } \langle f, g\rangle_{2,\lambda} = h^n_\lambda \sum_{\underline{r}\in\mathbb{A}^+_{n,\lambda}} f(\underline{r})g(\underline{r}). \tag{3.21}$$

Let $\rho_\lambda : \mathbb{L}_2(\mathbb{R}_+) \to \ell_2(\mathbb{N}_\lambda)$ and $\rho_{\lambda,n} : \mathbb{L}_2(\mathbb{A}^+_n) \to \ell_2(\mathbb{A}^+_{n,\lambda})$ be linear contractions; for instance, to fix the ideas, set

$$\rho_\lambda u(r) = \frac{1}{h_\lambda}\int_{(r-h_\lambda)_+}^{r} u(s)\,\mathrm{d}s$$

and

$$\rho_{\lambda,n} u(\underline{r}) = \frac{1}{h^n_\lambda}\int_{(r_1-h_\lambda)_+}^{r_1}\cdots\int_{(r_n-h_\lambda)_+}^{r_n} u(\underline{s})\,\mathbb{1}_{\{\underline{s}\in\mathbb{A}^+_n\}}\,\mathrm{d}\underline{s}.$$

Above, $s_+ = s \vee 0$ for any $s \in \mathbb{R}$. Let us say that a sequence $u_\lambda \in \ell_2(\mathbb{N}_\lambda)$ converges to $u \in \mathbb{L}_2(\mathbb{R}_+)$, which we denote $u = \lim u_\lambda$, if

$$\lim_{\lambda\downarrow 0} \|u_\lambda - \rho_\lambda u\|_{2,\lambda} = 0. \tag{3.22}$$

The same definition applies for sequences $u_\lambda \in \ell_2(\mathbb{A}^+_{n,\lambda})$ and, accordingly, for the limiting u in $\mathbb{L}_2(\mathbb{A}^+_n)$. Note that, in both cases, if $u = \lim_{\lambda\downarrow 0} u_\lambda$ and $v = \lim_{\lambda\downarrow 0} v_\lambda$, then

$$\lim_{\lambda\downarrow 0}\langle u_\lambda, v_\lambda\rangle_{2,\lambda} = \int_0^\infty u(r)v(r)\,\mathrm{d}r,$$

respectively,

$$\lim_{\lambda\downarrow 0}\langle u_\lambda, v_\lambda\rangle_{2,\lambda} = \int_{\mathbb{A}_n^+} u(\underline{r})v(\underline{r})\,\mathrm{d}\underline{r}.$$

STEP 1. (*Convergence of one-dimensional and product semi-groups*) Recall that T_t is an integral operator whose kernel h_t is defined in (2.7). [15, Proposition 3] implies that if a sequence $f_\lambda \in \ell_2(\mathbb{N}_\lambda)$ converges to $f \in \mathbb{L}_2(\mathbb{R}_+)$, then, for any $t > 0$,

$$\lim_{\lambda\downarrow 0}\sum_{s\in\mathbb{N}_\lambda} \hat{Z}^{r,s}_{t,+,\lambda} f_\lambda(s) = \int_0^\infty h_t(r,s)f(s)\,\mathrm{d}s, \tag{3.23}$$

in the sense of (3.22) above. In particular, for any $f, g \in \mathsf{C}_0(\mathbb{R}_+)$,

$$\lim_{\lambda\downarrow 0} h_\lambda \sum_{r\in\mathbb{N}_\lambda}\sum_{s\in\mathbb{N}_\lambda} g(r)\hat{Z}^{r,s}_{t,+,\lambda} f(s) = \int_0^\infty\int_0^\infty g(r)h_t(r,s)f(s)\,\mathrm{d}r\mathrm{d}s. \tag{3.24}$$

We claim:

Proposition 1 *Assume that the sequence $f_\lambda \in \ell_2(\mathbb{A}^+_{n,\lambda})$ converges to $f \in \mathsf{C}_0(\mathbb{A}^+_n)$. Let σ be a permutation of $\{1,\ldots,n\}$. Then, for any $t > 0$,*

$$\lim_{\lambda\downarrow 0}\sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}}\prod_{i=1}^n \hat{Z}^{r_i,s_{\sigma_i}}_{t,+,\lambda} f_\lambda(\underline{s}) = \int_{\mathbb{A}_n^+}\prod_{i=1}^n h_t(r_i,s_{\sigma_i})f(\underline{s})\,\mathrm{d}\underline{s}, \tag{3.25}$$

in the sense of (3.22) above. In particular, let $f, g \in \mathsf{C}_0(\mathbb{A}^+_n)$. Then, for any $t > 0$,

$$\lim_{\lambda\downarrow 0} h^n_\lambda \sum_{\underline{r}\in\mathbb{A}^+_{n,\lambda}}\sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} g(\underline{r})\prod_{i=1}^n \hat{Z}^{r_i,s_{\sigma_i}}_{t,+,\lambda} f(\underline{s}) = \int_{\mathbb{A}_n^+}\int_{\mathbb{A}_n^+} g(\underline{r})\prod_{i=1}^n h_t(r_i,s_{\sigma_i})f(\underline{s})\,\mathrm{d}\underline{r}\mathrm{d}\underline{s}. \tag{3.26}$$

STEP 2. (*Karlin–McGregor formula and probabilistic estimates*) In order to apply a general Karlin–McGregor formula we need to introduce the set of non-colliding trajectories. More precisely, for $\underline{u}, \underline{v} \in \mathbb{N}^n$, let $\mathcal{A}^{\underline{u},\underline{v}}_{M,N,+}$ be the set of n trajectories $\underline{\mathbb{X}}$ starting at $\underline{u}$ at time M, ending at $\underline{v}$ at time N, staying positive during the time interval $\{M,\ldots,N\}$ and satisfying

$$X^\ell_j \neq X^k_j \quad \forall j \in \{M,\ldots,N\} \text{ and } \ell \neq k. \tag{3.27}$$

For each $N > 0$ we shall use for the shorthand notations $\mathcal{A}^{\underline{u},\underline{v}}_{N,+} = \mathcal{A}^{\underline{u},\underline{v}}_{-N,N,+}$ and $\hat{\mathcal{A}}^{\underline{u},\underline{v}}_{N,+} = \mathcal{A}^{\underline{u},\underline{v}}_{1,N,+}$. Furthermore, as in (3.18), we use notation $\mathcal{A}^{\underline{r},\underline{s}}_{t,+,\lambda}$ and $\hat{\mathcal{A}}^{\underline{r},\underline{s}}_{t,+,\lambda}$ for the rescaled trajectories.

Let $\underline{r}, \underline{s} \in \mathbb{A}^+_{n,\lambda}$. By an application of Karlin–McGregor formula (see [17, Section 5]),

$$\det\left\{\hat{Z}^{r_i,s_j}_{t,+,\lambda}\right\} = \sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} Z[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}]. \tag{3.28}$$

Above, $(\underline{s}_\sigma)_i \equiv s_{\sigma_i}$.

Recall our notation for rescaled norms: $\| f_\lambda\|^2_{2,\lambda} = h^n_\lambda \sum_{\underline{r}} f^2_\lambda(\underline{r})$. We claim:

Theorem 3 (a) *For any $t_0 > 0$ and for any non-trivial permutation $\sigma \neq \mathsf{Id}$,*

$$\lim_{\lambda\downarrow 0} \sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}] f_\lambda(\underline{s}) = 0, \tag{3.29}$$

in the sense of (3.22), uniformly in $t \geq t_0$ and in $\| f_\lambda\|_{2,\lambda} = 1$.

(b) *For any $t_0 > 0$,*

$$\lim_{\lambda\downarrow 0} \sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} \left(Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}}_{t,+,\lambda}] - Z_\lambda[\hat{\mathcal{P}}^{\underline{r},\underline{s}}_{t,+,\lambda}]\right) f_\lambda(\underline{s}) = 0, \tag{3.30}$$

as well, also uniformly in $t \geq t_0$ and $\| f_\lambda\|_{2,\lambda} = 1$.

Recall our notation $\hat{Z}^{\underline{r},\underline{s}}_{t,+,\lambda} = Z_\lambda[\hat{\mathcal{P}}^{\underline{r},\underline{s}}_{t,+,\lambda}]$ and $\kappa_t(\underline{r},\underline{s}) = \det\{h_t(r_i, s_j)\}$. Proposition 1 and Theorem 3 imply:

Theorem 4 *For any $t > 0$ and any sequence $f_\lambda \in \ell_2(\mathbb{A}^+_{n,\lambda})$ with $\lim_{\lambda\downarrow 0} f_\lambda = f$,*

$$\lim_{\lambda\downarrow 0} \sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} \hat{Z}^{\underline{r},\underline{s}}_{t,+,\lambda} f_\lambda(\underline{s}) = \int_{\mathbb{A}^+_n} \kappa_t(\underline{r},\underline{s}) f(\underline{s})\,\mathrm{d}\underline{s}. \tag{3.31}$$

In particular, for any $f, g \in \mathsf{C}_0(\mathbb{A}^+_n)$,

$$\lim_{\lambda\downarrow 0} h^n_\lambda \sum_{\underline{r}\in\mathbb{A}^+_{n,\lambda}} \sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} g(\underline{r}) Z^{\underline{r},\underline{s}}_{t,+,\lambda} f(\underline{s}) = \int_{\mathbb{A}^+_n}\int_{\mathbb{A}^+_n} g(\underline{r})\kappa_t(\underline{r},\underline{s}) f(\underline{s})\,\mathrm{d}\underline{r}\mathrm{d}\underline{s}. \tag{3.32}$$

STEP 3. (*Tightness*) We claim:

Proposition 2 *Fix any $T > 0$. Under the conditions of Theorem 2, the family $\{\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\}$ of probability distributions on $\mathsf{C}([-T, T], \mathbb{A}^+_n)$ is tight.*

STEP 4. (*Mixing*) Let d_{TV} denote the total variation distance. We claim:

Theorem 5 *For any $C < \infty$, there exist $c_1, c_2 > 0$ such that, for any $K > 0$,*

$$d_{\mathrm{TV}}\left(\mathbb{P}^{\underline{r},\underline{s};T}_{a,+,\lambda}, \mathbb{P}^{\underline{w},\underline{z};T}_{b,+,\lambda}\right) \leq c_1 \mathrm{e}^{-c_2 K}, \tag{3.33}$$

holds uniformly in λ small, $a, b \in \mathbb{Z}_\lambda$ with $a, b \geq (K+T)$ and uniformly in $\underline{r}, \underline{s}, \underline{w}, \underline{z} \in \mathbb{A}^+_{n,\lambda}$ with $r_n, s_n, w_n, z_n \leq C$.

STEP 5. (*Convergence of finite-dimensional distributions*) Fix $T > 0$. Let $\lambda_N \downarrow 0$ and let $a_N \in \mathbb{Z}_{\lambda_N}$ satisfy $\lim a_N = \infty$. Let $f, g \in \mathsf{C}_0(\mathbb{A}_n^+)$ be two non-negative and non-identically zero functions. For $M \in \mathbb{Z}_\lambda$, $M > T$, define the partition functions (rescaled as in (3.18))

$$Z_{M,+,\lambda}^{g,f} = h_\lambda^n \sum_{\underline{r}\in\mathbb{A}_n^+} \sum_{\underline{s}\in\mathbb{A}_n^+} g(\underline{r}) Z_{M,+,\lambda}^{\underline{r},\underline{s}} f(\underline{s}) > 0 \tag{3.34}$$

and let $\mathbb{P}_{M,+,\lambda}^{g,f;T}$ denote the corresponding induced probability distribution on $\mathsf{C}([-T,T],\mathbb{A}_n^+)$. By Theorem 5, under the conditions of Theorem 2,

$$\lim_{M\to\infty} \lim_{N\to\infty} d_{\mathrm{TV}}\big(\mathbb{P}_{a_N,+,\lambda_N}^{\underline{r},\underline{s};T}, \mathbb{P}_{M,+,\lambda_N}^{g,f;T}\big) = 0, \tag{3.35}$$

uniformly in $r_n, s_n \le C$.

Let now $-T \le t_1 < t_2 < \cdots t_m \le T$ and let $u_1, \ldots, u_m \in \mathsf{C}_0(\mathbb{A}_n^+)$. By Theorem 4,

$$\lim_{\lambda\downarrow 0} \mathbb{E}_{M,+,\lambda}^{g,f;T}\left(\prod_{i=1}^m u_i\big(\underline{x}^\lambda(t_i)\big)\right)$$
$$= \frac{\int g(\underline{r}) \int \kappa_{t_1+M}(\underline{r},\underline{r}^1) u_1(\underline{r}^1) \int \ldots u_m(\underline{r}^m) \int \kappa_{M-t_m}(\underline{r}^m,\underline{s}) f(\underline{s})\, \mathrm{d}\underline{s}\mathrm{d}\underline{r}^m \cdots \mathrm{d}\underline{r}}{\int\int g(\underline{r})\kappa_{2M}(\underline{r},\underline{s}) f(\underline{s})\, \mathrm{d}\underline{s}\mathrm{d}\underline{r}}. \tag{3.36}$$

Above, all integrals are over $\mathbb{A}_n^+$. In view of (2.14) and by the definition of the semigroup S_t^+ in (2.15), the formulas (3.35) and (3.36) imply:

Proposition 3 *Fix $T > 0$, $-T \le t_1 < t_2 < \cdots t_m \le T$ and let $u_1, \ldots, u_m$ be bounded continuous functions on $\mathbb{A}_n^+$. Let λ_N and a_N satisfy the assumptions of Theorem 2. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{a_N,+,\lambda_N}^{\underline{r},\underline{s}}\left(\prod_{i=1}^m u_i\big(\underline{x}^{\lambda_N}(t_i)\big)\right)$$
$$= \int \Delta^2(\underline{r}^1)u_1(\underline{r}^1)\int q_{t_2-t_1}(\underline{r}^1,\underline{r}^2)u_2(\underline{r}^2)\int\cdots\int q_{t_m-t_{m-1}}(\underline{r}^{m-1},\underline{r}^m)\,\mathrm{d}\underline{r}^m\cdots\mathrm{d}\underline{r}^1, \tag{3.37}$$

uniformly in $r_n, s_n \le C$. Above, q_t is the transition kernel of S_t^+, as defined in (2.16).

STEP 6. (*Conclusion of the Proof*) By Proposition 2, the sequence of measures $\big\{\mathbb{P}_{a_N,+,\lambda_N}^{\underline{r},\underline{s};T};\ r_n, s_n \le C\big\}$ on $\mathsf{C}([-T,T],\mathbb{A}_n^+)$ is tight for any $T > 0$ fixed. By Proposition 3, its finite-dimensional distributions converge to the finite-dimensional distributions of the Dyson diffusion $\underline{x}(\cdot)$ in (2.20). □

3.8 Organization of the technical part of the paper

Section 4 is devoted to the proof of Theorem 3. Propositions 1 and 2 will be proven in Sect. 5. The proof of Theorem 5 is by far the most technically loaded part of the paper, and it will be done in Sects. 6 and 7. This proof relies on the probabilistic estimates (I.1)–(I.3), which are formulated in Sect. 7.1 and are based on strong approximation techniques and on invariance principles for random walks in Weyl chambers. The derivation of (I.1)–(I.3) is relegated to the "Appendix".

4 Proof of Theorem 3

4.1 Preliminary estimates

Let us start with three preliminary estimates. The first one is just a rough local CLT estimate for the underlying random walk without area tilts: Recall that whenever we write quantities like $\hat{Z}^{r,s}_{t,+,\lambda}$, we are implicitly assuming that $t \in \mathbb{Z}_\lambda = h_\lambda^2\mathbb{Z}$ and that $r, s \in \mathbb{N}_\lambda = h_\lambda\mathbb{N}$.

Lemma 1 *For any $t_0 > 0$, there exists a finite constant $c_1(t_0)$ such that*

$$\sup_{t\geq t_0}\sup_{r,s\in\mathbb{N}_\lambda} \hat{Z}^{r,s}_{t,+,\lambda} \leq c_1(t_0)h_\lambda, \tag{4.1}$$

for all λ sufficiently small.

Proof Indeed, since $V_\lambda \geq 0$, $Z^{r,s}_{t,+,\lambda} \leq \hat{\mathbf{P}}^r_\lambda\left(x^\lambda(t) = s\right)$. □

Next, following [15], let us introduce

$$\hat{Z}^{r,\emptyset}_{t,+,\lambda} = \sum_{s\in\mathbb{N}_\lambda} \hat{Z}^{r,s}_{t,+,\lambda}. \tag{4.2}$$

Lemma 2 *For any $t_0 > 0$, there exists a finite constant $c_2(t_0)$ such that*

$$\sup_{t\geq t_0} h_\lambda \sum_{r\in\mathbb{N}_\lambda} \hat{Z}^{r,\emptyset}_{t,+,\lambda} \leq c_2(t_0), \tag{4.3}$$

for all λ sufficiently small. Furthermore,

$$\lim_{K\to\infty}\sup_{t\geq t_0} h_\lambda \sum_{\substack{r\in\mathbb{N}_\lambda\\ r\geq K}} \hat{Z}^{r,\emptyset}_{t,+,\lambda} = 0, \tag{4.4}$$

uniformly in λ sufficiently small.

Proof Under Assumption (3.12), it is straightforward to check that there exists $\kappa = \kappa(t_0) > 0$ such that

$$\sup_{t\geq t_0} \hat{Z}^{r,\emptyset}_{t,+,\lambda} \leq \mathrm{e}^{-\kappa\min\{q_0(r/2), r^2\}}, \tag{4.5}$$

for all λ small and all $r \in \mathbb{N}_\lambda$. Both (4.3) and (4.4) follow now from (3.13).

Note that Lemma 2 is in general wrong without the additional Assumption (3.13). $\square$

The third estimate is again on the underlying random walk, or more precisely on two independent copies $\left(x^\lambda, y^\lambda\right)$ of this walk. Namely,

Lemma 3 *For any $\delta_0 \in \mathbb{R}_+$ and $K \in \mathbb{R}_+$ fixed,*

$$\lim_{\lambda\to 0} \max_{\delta\geq\delta_0} \max_{0<u<v\leq K} \hat{\mathbf{P}}_\lambda^u \otimes \hat{\mathbf{P}}_\lambda^v\left(x^\lambda(\delta) > y^\lambda(\delta)\,;\; x^\lambda(t) \neq y^\lambda(t)\, \forall\, t \in [0,\delta]\cap\mathbb{Z}_\lambda\right) = 0. \tag{4.6}$$

Proof The claim follows from [20, Theorem 1] and local limit asymptotics for random walks with exponential tails. $\square$

4.2 Proof of Theorem 3(a)

Pick $f_\lambda \in \ell_2(\mathbb{A}^+_{n,\lambda})$ with $\| f_\lambda\|^2_{2,\lambda} = 1$. Set

$$u_\lambda(\underline{r}) = \sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}] f_\lambda(\underline{s}). \tag{4.7}$$

In order to prove (3.29), we need to check that, whenever $\sigma \neq \mathsf{Id}$,

$$\lim_{\lambda\downarrow 0} h^n_\lambda \sum_{\underline{r}\in\mathbb{A}^+_{n,\lambda}} u_\lambda(\underline{r})^2 = 0, \tag{4.8}$$

uniformly in f with $\| f_\lambda\|^2_{2,\lambda} = 1$. By the Cauchy–Schwarz inequality and Lemma 1,

$$\begin{aligned}
u_\lambda(\underline{r})^2 &= \left(\sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}] f_\lambda(\underline{s})\right)^2 \\
&\leq \left(\sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}]\right)\left(\sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}] f^2_\lambda(\underline{s})\right) \\
&\leq c_1(t_0)^n \sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}].
\end{aligned} \tag{4.9}$$

If $\sigma \neq \mathsf{Id}$, then there exist $i < j$ such that $\sigma_i > \sigma_j$. In this case,

$$\sum_{\underline{s}\in\mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\sigma}_{t,+,\lambda}] \leq \chi^+_{t,\lambda}(r_i, r_j) \prod_{k\neq i,j} Z^{r_k,\emptyset}_{t,+,\lambda},$$

where the partition functions $Z^{r,\emptyset}_{t,+,\lambda}$ were defined in (4.2) and, for $\underline{r}, \underline{s} \in \mathbb{A}^{+}_{2,\lambda}$, we define $\underline{s}^* = (s_1, s_2)^* = (s_2, s_1)$ and

$$\chi^{+}_{t,\lambda}(\underline{r}) = \sum_{\underline{s} \in \mathbb{A}^{+}_{2,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}^*}_{t,+,\lambda}]. \tag{4.10}$$

In view of Lemma 2, (4.8) would follow once we check that

$$\lim_{\lambda \downarrow 0} h^2_\lambda \sum_{\underline{r} \in \mathbb{A}^{+}_{2,\lambda}} \chi^{+}_{t,\lambda}(\underline{r}) = 0. \tag{4.11}$$

Given $K > 0$, let us define

$$\epsilon_\lambda(K) = \sup_t h_\lambda \sum_{\substack{r \in \mathbb{N}_\lambda \\ r \geq K}} \hat{Z}^{r,\emptyset}_{t,+,\lambda}. \tag{4.12}$$

By (4.3) of Lemma 2,

$$h^2_\lambda \sum_{\underline{r} \in \mathbb{A}^{+}_{2,\lambda}} \chi^{+}_{t,\lambda}(\underline{r}) \leq \epsilon^2_\lambda(K) + 2\epsilon_\lambda(K) c_2(t_0) + h^2_\lambda \sum_{\substack{\underline{r} \in \mathbb{A}^{+}_{2,\lambda} \\ 0 < r_1 < r_2 \leq K}} \chi^{+}_{t,\lambda}(\underline{r}). \tag{4.13}$$

Next, by (4.4) of Lemma 2, the term $\epsilon_\lambda(K) \to 0$ as K tends to infinity, uniformly in λ small enough. Moreover, choosing $\delta = \frac{t}{K^2} \geq \frac{t_0}{K^2} \stackrel{\Delta}{=} \delta_0$, we infer from Lemma 3 that

$$\limsup_{\lambda \downarrow 0} \sup_{t \geq t_0} \max_{0 < r_1 < r_2 \leq K} \chi^{+}_{t,\lambda}(\underline{r}) = 0, \tag{4.14}$$

and hence the third term in (4.13) tends to zero (as λ tends to 0) for any K fixed. (4.11) follows. $\square$

4.3 Proof of Theorem 3 (b)

Fix $t_0 > 0$. We should check that

$$\lim_{\lambda \downarrow 0} \sup_{\|g_\lambda\|_{2,\lambda} = \|f_\lambda\|_{2,\lambda} = 1} h^n_\lambda \sum_{\underline{r},\underline{s} \in \mathbb{A}^{+}_{n,\lambda}} g_\lambda(\underline{r}) Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}}_{t,+,\lambda} \setminus \hat{\mathcal{P}}^{\underline{r},\underline{s}}_{t,+,\lambda}] f_\lambda(\underline{s}) = 0, \tag{4.15}$$

uniformly in $t > t_0$. By definition, any path $\underline{x}^\lambda \in \hat{\mathcal{A}}^{\underline{r},\underline{s}}_{t,+,\lambda} \setminus \hat{\mathcal{P}}^{\underline{r},\underline{s}}_{t,+,\lambda}$ [of the random walk in discrete $\mathbb{Z}_\lambda$-time, rescaled as in (3.14)] has to exit $\mathbb{A}^{+}_{n,\lambda}$ on its way from $\underline{r}$ to $\underline{s}$. Let τ_- and τ_+, $\tau_\pm \in \mathbb{Z}_\lambda$, be, respectively, the times of the first and the last visits to $\{\mathbb{A}^{+}_{n,\lambda}\}^c$. Again, by definition of $\hat{\mathcal{A}}^{\underline{r},\underline{s}}_{t,+,\lambda}$, the points $\underline{x}^\lambda(\tau_\pm)$ belong to

$$[\mathbb{A}^{+}_{n,\lambda}]_{\sigma_\pm} = \{\underline{\mathsf{w}}_{\sigma_\pm} : \underline{\mathsf{w}} \in \mathbb{A}^{+}_{n,\lambda}\},$$

for some permutation $\sigma_\pm \neq \mathrm{Id}$, which of course depends on the particular realization of $\underline{x}^\lambda$. Since either

$$\text{(i) } \tau_- \le t/2 \quad \text{or} \quad \text{(ii) } t - \tau_+ \le t/2,$$

(4.15) follows by the same arguments as employed for the proof of (3.29) [although, in case (ii), the latter should be applied to the reversed walk].

Indeed, let us fix a permutation $\sigma \neq \mathrm{Id}$. Consider the following modification of (4.7): Set $\eta = \sigma^{-1}$ and

$$u_\lambda(\underline{r}) = \max_{u \in [t/2,t]} \sum_{\underline{s} \in \mathbb{A}^+_{n,\lambda}} Z_\lambda[\hat{\mathcal{A}}^{\underline{r},\underline{s}_\eta}_{u,+,\lambda}] f_\lambda(\underline{s}). \tag{4.16}$$

For $\underline{r}, \underline{s} \in \mathbb{A}^+_{n,\lambda}$ define,

$$\rho_\lambda(\underline{r}, \underline{s}) = Z_\lambda\big[\underline{x}^\lambda(0) = \underline{r},\ \underline{x}^\lambda(\tau_-) = \underline{s}_\sigma\big].$$

Above, $\{\underline{x}^\lambda(0) = \underline{r},\ \underline{x}^\lambda(\tau_-) = \underline{s}_\sigma\}$ is the set of trajectories started at time zero in $\underline{r}$ and arriving, at their first exit from $\mathbb{A}^+_{n,\lambda}$, to the point $\underline{s}_\sigma \in [\mathbb{A}^+_{n,\lambda}]_\sigma$. Clearly, for any $\lambda \ge 0$,

$$\sum_{\underline{s}} \rho_\lambda(\underline{r}, \underline{s}) \le 1. \tag{4.17}$$

Furthermore, under Assumption (3.13), there exists a constant c_3 such that

$$\sum_{\underline{r}} \rho_\lambda(\underline{r}, \underline{s}) \le c_3, \tag{4.18}$$

for all λ small enough.

We can now bound from above the contribution of (i) with $x^\lambda(\tau_-) \in [\mathbb{A}^+_{n,\lambda}]_\sigma$ to the sum in (4.15): applying the Cauchy–Schwarz inequality and the bounds (4.17) and (4.18),

$$h^n_\lambda \sum_{\underline{r},\underline{s}} g_\lambda(\underline{r}) \rho_\lambda(\underline{r}, \underline{s}) u_\lambda(\underline{s}) \le \sqrt{c_3} \|g_\lambda\|_{2,\lambda} \|u_\lambda\|_{2,\lambda}$$

and one can then proceed as in the Proof of Theorem 3(a) to show that $\lim_{\lambda\downarrow 0} \|u_\lambda\|_{2,\lambda} = 0$, uniformly in f such that $\|f_\lambda\|_{2,\lambda} \le 1$. □

5 Proof of Propositions 1 and 2

5.1 Proof of Proposition 1

Let us consider $f \in \mathsf{C}_0(\mathbb{R}^n_+)$ and $f_\lambda \in \ell_{2,\lambda}(\mathbb{N}^n_\lambda)$. The convergence $\lim_{\lambda\downarrow 0} f_\lambda = f$ is still defined via (3.22) with

$$\rho_{\lambda,n} u(\underline{r}) = \frac{1}{h^n_\lambda} \int_{(r_1 - h_\lambda)_+}^{r_1} \cdots \int_{(r_n - h_\lambda)_+}^{r_n} u(\underline{s})\, \mathrm{d}\underline{s}. \tag{5.1}$$

If $u_\lambda \in \ell_2(\mathbb{N}^n_\lambda)$ converges to $u \in \mathbb{L}_2(\mathbb{R}^n_+)$, then, evidently, $\tilde u_\lambda \stackrel{\Delta}{=} u_\lambda \mathbb{I}_{\mathbb{A}^+_{n,\lambda}} \in \ell_2(\mathbb{A}^+_{n,\lambda})$ converges to $\tilde u \stackrel{\Delta}{=} u \mathbb{I}_{\mathbb{A}^+_n} \in \mathbb{L}_2(\mathbb{A}^+_n)$. Hence, (3.25) will follow if we check that

$$\lim_{\lambda\downarrow 0} \sum_{\underline{s}\in\mathbb{N}^n_\lambda} \prod_{i=1}^n \hat Z^{r_i,s_i}_{t,+,\lambda} f_\lambda(\underline{s}) = \int_{\mathbb{R}^n_+} \prod_{i=1}^n h_t(r_i,s_i) f(\underline{s}) \mathrm{d}\underline{s}, \tag{5.2}$$

whenever $f \in \mathsf{C}_0(\mathbb{R}^n_+)$ and $\lim_{\lambda\downarrow 0} f_\lambda = f$.

Next, we may assume without loss of generality that $f_\lambda = \rho_{\lambda,n} f$. Hence, there exists $R > 0$ such that both f and f_λ vanish for $r \notin [0,R]^n \stackrel{\Delta}{=} \Omega_R$. In other words, we can restrict our attention to $f_\lambda \in \ell_{2,\lambda}(\mathbb{N}^n_\lambda \cap \Omega_R)$ and $f \in \mathsf{C}_0(\Omega_R)$.

The rest is a monotone class argument based on (3.23): Let $\mathcal{H}_R$ be the family of bounded measurable functions on Ω_R such that (5.2) holds. By (3.23), the family $\mathcal{H}_R$ contains all the products $\prod_{i=1}^n f_i(r_i)$ of bounded and measurable functions $f_1,\ldots,f_n$ on $[0,R]$. In particular, $\mathbb{I}_{\Omega_R} \in \mathcal{H}_R$. Next, by linearity, $f, g \in \mathcal{H}_R$ clearly implies that $af + bg \in \mathcal{H}_R$ for any $a, b \in \mathbb{R}$. Finally, if

$$0 \le f^{(1)} \le f^{(2)} \le \cdots$$

is a non-decreasing family of functions from $\mathcal{H}_R$ and if $f = \lim f^{(k)}$ exists and is bounded, then $\lim_{k\to\infty} \|f - f^{(k)}\|_2 = 0$. Since $\rho_{\lambda,n}$ are contractions, $\|f_\lambda - f^{(k)}_\lambda\|_{2,\lambda} \le \|f - f^{(k)}\|_2$ for all $\lambda > 0$. On the one hand, in view of Lemma 1 and (2), an application of the Cauchy–Schwarz inequality yields

$$\begin{aligned} h^n_\lambda \sum_{\underline{r}} \Big(\sum_{\underline{s}} \prod_{i=1}^n \hat Z^{r_i,s_i}_{t,+,\lambda} u_\lambda(\underline{s}) \Big)^2 &\le \Big(h^n_\lambda \sum_{\underline{r},\underline{s}} \prod_{i=1}^n \hat Z^{r_i,s_i}_{t,+,\lambda} \Big) \Big(c_1(t_0)^n h^n_\lambda \sum_{\underline{s}} u_\lambda(\underline{s})^2 \Big) \\ &\le \big(c_1(t_0) c_2(t_0)\big)^n \|u_\lambda\|^2_{2,\lambda}, \end{aligned} \tag{5.3}$$

uniformly in $t \ge t_0$. In particular,

$$\Big\| \sum_{\underline{s}} \prod_{i=1}^n \hat Z^{r_i,s_i}_{t,+,\lambda} \big(f_\lambda(\underline{s}) - f^{(k)}_\lambda(\underline{s})\big) \Big\|_{2,\lambda} \le \big(c_1(t_0) c_2(t_0)\big)^n \|f - f^{(k)}\|_2,$$

uniformly in $t \ge t_0$. On the other hand,

$$\Big\| \int_{\mathbb{R}^n_+} \prod_{i=1}^n h_t(r_i,s_i) \big(f(\underline{s}) - f^{(k)}(\underline{s})\big) \,\mathrm{d}\underline{s} \Big\|_2 \le \|f - f^{(k)}\|_2.$$

(5.2) follows, for instance, by a diagonal procedure.

5.2 Proof of Proposition 2

Note that our proof of Theorem 5 below, and hence our proof of Proposition 3, does not rely on the tightness property which we are trying to establish here. By Proposition 3 the one-dimensional projections of $\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}$, that is, the distributions of $\underline{x}^{\lambda_N}(t)$ for each fixed $|t| \le T$, converge.

Then, according to [1, Theorem 8.3], the family $\{\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\}$ is tight if for all positive γ and β there exist $\delta \in (0, 1)$ and N_0 such that, uniformly in $t \in [-T, T]$,

$$\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\left(\sup_{s\in[t,t+\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(t)| \ge \gamma\right) \le \delta\beta, \quad N \ge N_0. \tag{5.4}$$

Since a_N tends to infinity, it suffices to prove (5.4) for $t = 0$.

Recall that T is fixed. We may assume that $a_N \gg T$. Now, the exponential mixing bound (3.33) implies that the following holds uniformly in $M \ge 2T$:

$$\begin{aligned}
&\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \ge \gamma\right) \\
&\quad \le e^{-c_1 M} + \mathbb{P}^{g,f;T}_{M,+,\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \ge \gamma\right).
\end{aligned} \tag{5.5}$$

Since the potentials V_{λ_N} in the definition of tilted measures are non-negative, the latter probability is controlled in terms of the underlying random walk:

$$\begin{aligned}
&\mathbb{P}^{g,f;T}_{M,+,\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \ge \gamma\right) \\
&\quad \le \frac{1}{Z^{g,f}_{M,+,\lambda_N}} \mathbf{P}_{\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \ge \gamma\right).
\end{aligned}$$

It follows from Theorem 4 and the definition of the kernel κ_t that there exists $c_2 = c_2(g, f)$ such that

$$Z^{g,f}_{M,+,\lambda_N} \ge e^{-c_2 M}.$$

From these estimates and (5.5), we conclude that

$$\begin{aligned}
&\mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \ge \gamma\right) \\
&\quad \le e^{-c_1 M} + e^{c_2 M}\mathbf{P}_{\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \ge \gamma\right).
\end{aligned}$$

An application of the standard functional CLT [recall our assumption (3.3) on unit variance] yields the inequality

$$\limsup_{N\to\infty} \mathbf{P}_{\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \geq \gamma\right) \leq e^{-\gamma^2/4\delta}.$$

Consequently,

$$\limsup_{N\to\infty} \mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \geq \gamma\right) \leq e^{-c_1 M} + e^{c_2 M - \gamma^2/4\delta}.$$

Choosing $M = \frac{\gamma^2}{8c_2\delta}$ (and assuming that the parameters are tuned in such a way that $M \geq 2T$), we finally obtain

$$\limsup_{N\to\infty} \mathbb{P}^{\underline{r},\underline{s};T}_{a_N,+,\lambda_N}\left(\sup_{s\in[0,\delta]} |\underline{x}^{\lambda_N}(s) - \underline{x}^{\lambda_N}(0)| \geq \gamma\right) \leq 2e^{-c_3\gamma^2/8\delta},$$

where $c_3 = \min\{\frac{c_1}{c_2}, 1\}$. Thus, (5.4) is proved.

6 Proof of Theorem 5

Throughout this section, we shall assume that $H_\lambda^2 \in \mathbb{N}$; this implies that $\mathbb{Z} \subset \mathbb{Z}_\lambda$. In particular, the values of rescaled walks $\underline{x}^\lambda(\ell)$ in discrete $\mathbb{Z}_\lambda$-time are well defined for any $\ell \in \mathbb{Z}$.

6.1 Regular set $\mathbb{A}_n^{+,\mathsf{r}}$, regular intervals and good blocks

Fix $\eta < \infty$ large enough and $\epsilon > 0$ small enough. The regular subset $\mathbb{A}_n^{+,\mathsf{r}} \subset \mathbb{A}_n^+$ is defined as (under the convention that $x_0 \equiv 0$):

$$\mathbb{A}_n^{+,\mathsf{r}} = \left\{\underline{x} \in \mathbb{A}_n^+ \ : \ x_n \leq \eta \quad \text{and} \quad \min_{i\leq n}(x_i - x_{i-1}) \geq \epsilon\right\}. \tag{6.1}$$

The notion of regular interval is defined relative to a given continuous $\mathbb{A}_n^+$-valued function $\underline{x}(\cdot)$. Namely, an interval $[\ell, \ell+1]$ is said to be *regular* if

$$\underline{x}(\ell), \underline{x}(\ell+1) \in \mathbb{A}_n^{+,\mathsf{r}} \quad \text{and} \quad \max_{t\in[\ell,\ell+1]} x_n(t) \leq 2\eta. \tag{6.2}$$

Consider now the intervals $D_\ell = [2\ell, 2(\ell+1)]$, which we shall call *blocks*. A block is a union of two successive unit length intervals,

$$D_\ell = [2\ell, 2\ell+1] \cup [2\ell+1, 2(\ell+1)] \stackrel{\Delta}{=} D_\ell^- \cup D_\ell^+.$$

We shall say that D_ℓ is *good* if both D_ℓ^+ and D_ℓ^- are regular. If the notion of *goodness* is defined with respect to random trajectories, namely $\underline{x}^\lambda(\cdot)$, then we shall also use D_ℓ for the corresponding event.

Lemma 4 *Define* $\mathbb{A}_{n,\lambda}^{+,\mathsf{r}} = \mathbb{A}_n^{+,\mathsf{r}} \cap \mathbb{A}_{n,\lambda}^+$. *There exist two constants* c_1, c_2 *such that*

$$c_1 h_\lambda^n \le \mathbb{P}_{0,2,+,\lambda}^{\underline{r},\underline{z}}\big(\underline{x}^\lambda(1) = \underline{s} \bigm| D_0\big) \le c_2 h_\lambda^n, \tag{6.3}$$

uniformly in λ *small and in* $\underline{r}, \underline{s}, \underline{z} \in \mathbb{A}_{n,\lambda}^{+,\mathsf{r}}$.

We prove Lemma 4 in Sect. 7.2.

6.2 Good blocks for a couple of trajectories

Consider now a couple of independent trajectories $\big(\underline{x}^\lambda(\cdot), \underline{y}^\lambda(\cdot)\big)$ [rescaled as in (3.14)], distributed according to

$$\mathbb{P}_{a,+,\lambda}^{\underline{r},\underline{s}} \otimes \mathbb{P}_{b,+,\lambda}^{\underline{u},\underline{w}}.$$

Set $3M = \min\{a, b\}$. For $D_\ell \subset [-2M, 2M]$, let us define

$$\mathfrak{D}_\ell = \{D_\ell \text{ is good for both } \underline{x}^\lambda \text{ and } \underline{y}^\lambda\} \quad \text{and} \quad \mathcal{M}_0 = \sum_{-M \le \ell \le M-1} \mathbb{I}_{\mathfrak{D}_\ell}. \tag{6.4}$$

Lemma 5 *There exist* $\nu > 0$ *and* $\kappa > 0$ *such that*

$$\mathbb{P}_{a,+,\lambda}^{\underline{r},\underline{s}} \otimes \mathbb{P}_{b,+,\lambda}^{\underline{u},\underline{w}}(\mathcal{M}_0 \le \nu M) \le \mathrm{e}^{-\kappa M}, \tag{6.5}$$

uniformly in λ *small,* M *large and* $r_n, s_n, u_n, v_n \le C$.

The proof of Lemma 5 is relegated to Sect. 7.3.

6.3 A coupling argument

Fix λ small, a negative integer $a < -2T$ and $\underline{r} \in \mathbb{A}_{n,\lambda}^{+,\mathsf{r}}$. For $K \in \mathbb{N}$ and $\underline{u}, \underline{v} \in \mathbb{A}_{n,\lambda}^{+,\mathsf{r}}$, define

$$\mathbb{Q}_K^{\underline{u},\underline{v}}(\cdot) = \mathbb{P}_{a,2K+1,+,\lambda}^{\underline{r},\underline{u}} \otimes \mathbb{P}_{a,2K+1,+,\lambda}^{\underline{r},\underline{v}}\big(\cdot \bigm| \mathfrak{D}_K^-\big), \tag{6.6}$$

where, similarly to (6.4), we define

$$\mathfrak{D}_\ell^\pm = \{D_\ell^\pm \text{ is regular for both } \underline{x}^\lambda \text{ and } \underline{y}^\lambda\}.$$

In this way, $\mathfrak{D}_\ell = \mathfrak{D}_\ell^- \cap \mathfrak{D}_\ell^+$. As before, the number $\mathcal{M}_0$ of good blocks D_ℓ for $\ell \in \{1, \ldots, K-1\}$ is defined by

$$\mathcal{M}_0 = \sum_{\ell=1}^{K-1} \mathbb{I}_{\mathfrak{D}_\ell}. \tag{6.7}$$

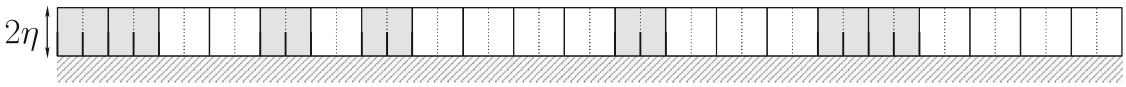

Fig. 1 Decomposition of the line into blocks. The *shaded blocks* represent jointly good blocks. Note that, in this case, the couple $(\underline{x}^\lambda(\cdot), \underline{y}^\lambda(\cdot))$ must be such that all trajectories stay inside the *shaded area* above jointly good blocks and cross the bold line segments in such a way that their n paths $x_1^\lambda, \dots, x_n^\lambda$, resp. $y_1^\lambda, \dots, y_n^\lambda$, are ϵ-separated. Consequently, $\underline{x}^\lambda(\cdot)$ and $\underline{y}^\lambda(\cdot))$ can be coupled with positive probability, independently over each such jointly good block

Let $\mathcal{F}_T = \mathcal{F}_T^\lambda$ be the σ-algebra generated by rescaled trajectories (3.14) on $[-2T, 0]$. For the σ-algebra generated by a couple of such trajectories $\big(\underline{x}^\lambda(\cdot), \underline{y}^\lambda(\cdot)\big)$, we use $\mathcal{F}_T \times \mathcal{F}_T$. Given $A \in \mathcal{F}_T$, $A \times \Omega$ stands for the event that $\underline{x}^\lambda(\cdot) \in A$ without further restrictions on $\underline{y}^\lambda(\cdot)$; $\Omega \times A$ is defined similarly. Define

$$\psi(m) = \sup_{K>m} \sup_{\underline{u},\underline{v}\in\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}} \sup_{A\in\mathcal{F}_T} \left\{\mathbb{Q}_K^{\underline{u},\underline{v}}\big(A \times \Omega; \mathcal{M}_0 \geq m\big) - \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\Omega \times A; \mathcal{M}_0 \geq m\big)\right\}. \tag{6.8}$$

Lemma 6 *There exists $\delta > 0$, which does not depend on λ, a and $\underline{r}$, such that*

$$\psi(m) \leq (1-\delta)^m. \tag{6.9}$$

Proof The idea of the proof is sketched in Fig. 1.

Let K, $\underline{u}$ and $\underline{v}$ be as above. Define

$$\tau = \max\{\ell < K \ : \ \mathfrak{D}_\ell \text{ occurs}\}.$$

For any $\mathcal{A} \in \mathcal{F}_T \times \mathcal{F}_T$, consider the decomposition

$$\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\mathcal{A}; \mathcal{M}_0 \geq m\big) = \sum_{\ell=1}^{K-1} \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\mathcal{A}; \mathcal{M}_0 \geq m; \tau = \ell\big). \tag{6.10}$$

In its turn, let us decompose each summand in (6.10) as

$$\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\mathcal{A}; \mathcal{M}_0 \geq m; \tau = \ell\big) = \sum_{\underline{x},\underline{y}\in\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}} \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\mathcal{A}; \mathcal{M}_0 \geq m; \tau = \ell; \{\underline{x}, \underline{y}\}_\ell\big), \tag{6.11}$$

where we used the shorthand notation

$$\{\underline{x}, \underline{y}\}_\ell = \big\{\underline{x}^\lambda(2\ell+1) = \underline{x}\, ;\ \underline{y}^\lambda(2\ell+1) = \underline{y}\big\}.$$

Now the Markov property implies that

$$\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\mathcal{A}; \mathcal{M}_0 \geq m \;\big|\; \tau = \ell; \{\underline{x}, \underline{y}\}_\ell\big) = \mathbb{Q}_\ell^{\underline{x},\underline{y}}\big(\mathcal{A}; \mathcal{M}_0 \geq m-1\big). \tag{6.12}$$

Therefore,

$$\mathbb{Q}_K^{\underline{u},\underline{v}}(\mathcal{A};\mathcal{M}_0 \geq m) = \sum_{\ell=1}^{K-1} \sum_{\underline{x},\underline{y}\in\mathbb{A}_{n,\lambda}^{+,\mathrm{r}}} \mathbb{Q}_\ell^{\underline{x},\underline{y}}(\mathcal{A};\mathcal{M}_0 \geq m-1) \times \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{x},\underline{y}\}_\ell \bigm| \tau=\ell\big)\mathbb{Q}_K^{\underline{u},\underline{v}}(\tau=\ell). \tag{6.13}$$

This means that, for any $A \in \mathcal{F}_T$, and for any $m, K, \underline{u}, \underline{v}$ in question, the following holds:

$$\begin{aligned} &\mathbb{Q}_K^{\underline{u},\underline{v}}(A\times\Omega;\mathcal{M}_0 \geq m) - \mathbb{Q}_K^{\underline{u},\underline{v}}(\Omega\times A;\mathcal{M}_0 \geq m) \\ &\quad = \sum_{\ell,\underline{x},\underline{y}} \big\{\mathbb{Q}_\ell^{\underline{x},\underline{y}}(A\times\Omega;\mathcal{M}_0 \geq m-1) - \mathbb{Q}_\ell^{\underline{x},\underline{y}}(\Omega\times A;\mathcal{M}_0 \geq m-1)\big\} \\ &\qquad \times \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{x},\underline{y}\}_\ell \bigm| \tau=\ell\big)\mathbb{Q}_K^{\underline{u},\underline{v}}(\tau=\ell). \end{aligned} \tag{6.14}$$

Since, evidently,

$$\mathbb{Q}_\ell^{\underline{x},\underline{y}}(A\times\Omega;\mathcal{M}_0 \geq m-1) = \mathbb{Q}_\ell^{\underline{y},\underline{x}}(\Omega\times A;\mathcal{M}_0 \geq m-1), \tag{6.15}$$

all the terms with $\underline{x} = \underline{y}$ in (6.14) vanish. On the other hand, each unordered pair $\underline{x} \neq \underline{y}$ is encountered exactly twice. Hence, again in view of (6.15), the contribution of each *unordered pair* $\underline{x} \neq \underline{y}$ to the right-hand side of (6.14) is equal to

$$\begin{aligned} &\big\{\mathbb{Q}_\ell^{\underline{x},\underline{y}}(A\times\Omega;\mathcal{M}_0 \geq m-1) - \mathbb{Q}_\ell^{\underline{x},\underline{y}}(\Omega\times A;\mathcal{M}_0 \geq m-1)\big\} \\ &\quad \times \big\{\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{x},\underline{y}\}_\ell \bigm| \tau=\ell\big) - \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{y},\underline{x}\}_\ell \bigm| \tau=\ell\big)\big\}. \end{aligned} \tag{6.16}$$

On the other hand, by Lemma 4,

$$c_1 \leq \frac{\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{x},\underline{y}\}_\ell \bigm| \tau=\ell\big)}{\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{y},\underline{x}\}_\ell \bigm| \tau=\ell\big)} \leq c_2, \tag{6.17}$$

uniformly in all the situations in question. Set $\delta = \frac{c_1}{c_2}$. Then, (6.17) implies that the expression in (6.16) is bounded above by

$$\begin{aligned} &\psi(m-1)(1-\delta)\max\big\{\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{x},\underline{y}\}_\ell \bigm| \tau=\ell\big), \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{y},\underline{x}\}_\ell \bigm| \tau=\ell\big)\big\} \\ &\quad \leq \psi(m-1)(1-\delta)\big\{\mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{x},\underline{y}\}_\ell \bigm| \tau=\ell\big) + \mathbb{Q}_K^{\underline{u},\underline{v}}\big(\{\underline{y},\underline{x}\}_\ell \bigm| \tau=\ell\big)\big\}. \end{aligned} \tag{6.18}$$

Since $\mathbb{Q}_K^{\underline{u},\underline{v}}$ is a probability measure, substituting (6.18) into (6.14) yields the conclusion (6.9) of the lemma. $\square$

6.4 Conclusion of the proof

We are in a position to conclude the proof of Theorem 5. Let $a, b \geq (T + K)$ and $\underline{r}, \underline{s}, \underline{w}, \underline{z} \in \mathbb{A}^+_{n,\lambda}$ with $r_n, s_n, w_n, z_n \leq C$. Let A be an event generated by the rescaled trajectories of (3.14) on $[-T, T]$. Then,

$$\mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda}(A) - \mathbb{P}^{\underline{w},\underline{z}}_{b,+,\lambda}(A) = \mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda} \otimes \mathbb{P}^{\underline{w},\underline{z}}_{b,+,\lambda}\left(\mathbb{I}_{A\times\Omega} - \mathbb{I}_{\Omega\times A}\right).$$

Consider a pair of trajectories $\left(x^\lambda(\cdot), y^\lambda(\cdot)\right)$, sampled from $\mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda} \otimes \mathbb{P}^{\underline{w},\underline{z}}_{b,+,\lambda}$. Let $\mathcal{M}_\pm$ be the number of jointly good blocks D_ℓ with $T \leq 2\ell \leq T + K - 2$, respectively $-T - K \leq 2\ell \leq -T - 2$.

By Lemma 5, there exist $\nu', \kappa' > 0$ such that, up to the $2\mathrm{e}^{-\kappa' K}$ correction, we may restrict our attention to the event

$$E_K = \{\mathcal{M}_+ \geq \nu' K\} \cap \{\mathcal{M}_- \geq \nu' K\}.$$

On the other hand, by Lemma 6,

$$\left|\mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda} \otimes \mathbb{P}^{\underline{w},\underline{z}}_{b,+,\lambda}\left(\mathbb{I}_{E_K}\left(\mathbb{I}_{A\times\Omega} - \mathbb{I}_{\Omega\times A}\right)\right)\right| \leq (1-\delta)^{\nu' K}.$$

Our target exponential mixing bound (3.33) follows.

7 Proof of Lemmas 4 and 5

7.1 Probabilistic estimates

Our proofs of Lemmas 4 and 5 rely on strong approximation techniques and on refined information on random walks in Weyl chambers. There are three inputs, (I.1)–(I.3), which are stated below, but proved in the "Appendix". In the sequel, we fix η sufficiently large; in particular, $\eta > C$, where C is the constant which appears in Theorem 2. Furthermore, we fix $\epsilon > 0$ sufficiently small.

First of all, we claim that, for any $0 < a < b < \infty$, there exists $\nu = \nu(a, b) > 0$ such that

$$\hat{\mathbf{P}}^{\underline{r}}_{t,+,\lambda}\left(\underline{x}^\lambda(t) = \underline{z}, \max_{s\in[0,t]} x^\lambda_n(s) \leq 2\eta\right) \geq \nu h^n_\lambda, \tag{I.1}$$

uniformly in $t \in [a, b]$, $\underline{r}, \underline{z} \in \mathbb{A}^{+,\mathrm{r}}_{n,\lambda}$ and λ small.

Next, let

$$\tau = \inf\{t \geq 0 : \underline{x}^\lambda \notin \mathbb{A}^+_{n,\lambda}\} \tag{7.1}$$

be the first exit time of the path from $\mathbb{A}^+_{n,\lambda}$. We then claim that, for any $0 < a < b < \infty$, there exists $\rho = \rho(a, b)$ such that the following two lower bounds hold uniformly in $t \in [a, b]$, $\underline{u} \in \mathbb{A}^+_{n,\lambda}$ with $u_n \leq \eta$ and in λ sufficiently small:

$$\hat{\mathbf{P}}^{\underline{u}}_{t,\lambda}\Big(\max_{s\in[0,t]} x_n^\lambda(s) \le 2\eta \;\Big|\; \tau > t\Big) \ge \rho, \tag{I.2}$$

$$\hat{\mathbf{P}}^{\underline{u}}_{t,+,\lambda}\Big(\underline{x}^\lambda(t) \in \mathbb{A}^{+,r}_{n,\lambda} \;\Big|\; \max_{s\in[0,t]} x_n^\lambda(s) \le 2\eta\,,\; \tau > t\Big) \ge \rho. \tag{I.3}$$

7.2 Proof of Lemma 4

Since, by definition, the area-tilt of every path in D_0 is uniformly bounded, it suffices to prove the lemma for random walks without area tilts. That is, we need to show that

$$c_1 h_\lambda^n \le \hat{\mathbf{P}}^{\underline{r},\underline{s}}_{2,+,\lambda}(\underline{x}^\lambda(1) = \underline{z} \,|\, D_0) \le c_2 h_\lambda^n, \tag{7.2}$$

uniformly in $\underline{r}, \underline{s}, \underline{z} \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$.

We first note that upper bounds for $\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(D_0, \underline{x}^\lambda(2) = \underline{s})$ and $\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^\lambda(1) = \underline{z}, D_0, \underline{x}^\lambda(2) = \underline{s})$ follow from the classical inequalities for concentration functions. Indeed, by [9, Theorem 6.2], there exists a constant c_3 such that

$$\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^\lambda(t) = \underline{y}) \le \hat{\mathbf{P}}^{\underline{r}}_{\lambda}(\underline{x}^\lambda(t) = \underline{y}) \le \frac{c_3}{t^{n/2}} h_\lambda^n, \tag{7.3}$$

uniformly in $\underline{r}$, $\underline{y}$ and $t \ge 0$.

Consequently,

$$\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(D_0, \underline{x}^\lambda(2) = \underline{s}) \le \hat{\mathbf{P}}^{\underline{r}}_{\lambda}(\underline{x}^\lambda(2) = \underline{s}) \le \frac{c_3}{2^{n/2}} h_\lambda^n \tag{7.4}$$

and

$$\begin{aligned} \hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^\lambda(1) = \underline{z}, D_0, \underline{x}^\lambda(2) = \underline{s}) &\le \hat{\mathbf{P}}^{\underline{r}}_{\lambda}(\underline{x}^\lambda(1) = \underline{z}, \underline{x}^\lambda(2) = \underline{s}) \\ &= \hat{\mathbf{P}}^{\underline{r}}_{\lambda}(\underline{x}^\lambda(1) = \underline{z})\hat{\mathbf{P}}^{\underline{z}}_{\lambda}(\underline{x}^\lambda(1) = \underline{s}) \\ &\le c_3^2 h_\lambda^{2n}, \end{aligned} \tag{7.5}$$

uniformly in $\underline{r}, \underline{s}, \underline{z} \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$.

The corresponding matching lower bounds follow from (I.1). Indeed,

$$\begin{aligned} &\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^\lambda(1) = \underline{z}, D_0, \underline{x}^\lambda(2) = \underline{s}) \\ &\quad = \hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^\lambda(1) = \underline{z}, \max_{t\le 1} x_n^\lambda(t) \le 2\eta)\hat{\mathbf{P}}^{\underline{z}}_{+,\lambda}(\underline{x}^\lambda(1) = \underline{s}, \max_{t\le 1} x_n^\lambda(t) \le 2\eta) \\ &\quad \overset{(I.1)}{\ge} c_4 h_\lambda^{2n}, \end{aligned} \tag{7.6}$$

for any $\underline{r}, \underline{z}, \underline{s} \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$. Since the cardinality

$$\left|\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}\right| \ge c_5(\epsilon)\eta^n h_\lambda^{-n}, \tag{7.7}$$

we infer, by summing over $\underline{z}$ in (7.6), that

$$\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^{\lambda}(2) = \underline{s},\, D_0) \geq c_6 h^n_{\lambda}. \tag{7.8}$$

It remains to note that the lower bound in (7.2) follows from (7.4) and (7.6), and that the upper bound in (7.2) follows from (7.5) and (7.8).

7.3 Proof of Lemma 5

The proof of Lemma 5 proceeds in two steps.

Consider the 5-blocks

$$D^{(5)}_{\ell} \stackrel{\Delta}{=} D_{5\ell-2} \cup D_{5\ell-1} \cup D_{5\ell} \cup D_{5\ell+1} \cup D_{5\ell+2},$$

where $\ell \in \{-\lfloor M/5\rfloor, \ldots, \lfloor M/5\rfloor\} \subset \mathbb{Z}$.

Let us say that a 5-block $D^{(5)}_{\ell}$ is pre-good (relative to a trajectory $\underline{x}^{\lambda}(\cdot)$) if both

$$\min_{t\in D_{5\ell-2}} x^{\lambda}_n(t), \quad \min_{t\in D_{5\ell+2}} x^{\lambda}_n(t) \leq \eta. \tag{7.9}$$

Given a couple of trajectories $\underline{x}^{\lambda}$ and $\underline{y}^{\lambda}$, let $\tilde{\mathfrak{D}}^{(5)}_{\ell}$ denote the event that $D^{(5)}_{\ell}$ is pre-good for both $\underline{x}^{\lambda}$ and $\underline{y}^{\lambda}$.

Step 1. Note that the definitions are set up in such a way that $D_{5\ell}$ is the middle section of $D^{(5)}_{\ell}$. We claim that there exists $\rho_1 = \rho_1(\eta, \epsilon) > 0$ such that

$$\mathbb{P}^{\underline{r},\underline{s}}_{-4,6,+,\lambda}\big(D_0 \text{ is good} \bigm| D^{(5)}_0 \text{ is pre-good}\big) \geq \rho_1, \tag{7.10}$$

uniformly in $\underline{r}, \underline{s} \in \mathbb{A}^+_{n,\lambda}$ and λ sufficiently small.

As a result, for any $\ell \in \{-\lfloor M/5\rfloor, \ldots, \lfloor M/5\rfloor\} \subset \mathbb{Z}$,

$$\mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda} \otimes \mathbb{P}^{\underline{u},\underline{w}}_{b,+,\lambda}\big(\mathfrak{D}_{5\ell} \bigm| \tilde{\mathfrak{D}}^{(5)}_{\ell}\big) \geq \rho_1^2, \tag{7.11}$$

uniformly in $\underline{r}, \underline{s}, \underline{u}, \underline{v} \in \mathbb{A}^+_{n,\lambda}$ and λ small enough.

By the Markov property, this means that any jointly pre-good 5-block gives rise to a good block in its middle section with probability at least ρ_1^2, regardless of the behavior of trajectories outside this particular pre-good 5-block.

Step 2. In this second step, we control the density of jointly pre-good 5-blocks $D^{(5)}_{\ell}$ which lie inside $[-2M, 2M]$. Define

$$\mathcal{M}^{(5)}_0 = \sum_{j=-\lfloor M/5\rfloor}^{\lfloor M/5\rfloor} \mathbb{I}_{\mathfrak{D}^{(5)}_{\ell}}.$$

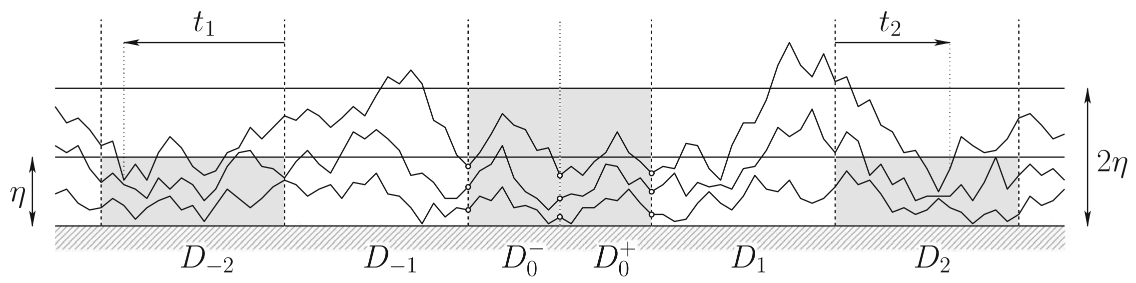

Fig. 2 A picture of the 5-block $D_0^{(5)}$. In the picture, this 5-block is pre-good, since the top-most path (here $n = 3$) visits the *shaded areas* in the blocks D_{-2} and D_2; the corresponding random variables t_1 and t_2 are also represented. The event D_0 also occurs: the top-most path stays inside the *shaded area* above the block $D_0 = D_0^- \cup D_0^+$ and the $n = 3$ paths stay ϵ apart from each other and the *bottom* wall at the boundary of D_0^- and D_0^+ (the corresponding positions of the 3 paths there are marked with *dots*)

We claim that there exist $\nu^{(5)} > 0$ and $\kappa^{(5)} > 0$ such that

$$\mathbb{P}^{\underline{r},\underline{s}}_{a,+,\lambda} \otimes \mathbb{P}^{\underline{u},\underline{w}}_{b,+,\lambda}\left(\mathcal{M}_0^{(5)} \le \nu^{(5)} M\right) \le \mathrm{e}^{-\kappa^{(5)} M}. \tag{7.12}$$

uniformly in λ small, M large, $a, b \ge 3M$ and $r_n, s_n, u_n, v_n \le C$.

Evidently, (7.11) and (7.12) imply the target bound (6.5).

7.4 Proof of (7.10)

We are going to show that

$$\mathbb{P}^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}(D_0) \ge \rho_1, \tag{7.13}$$

uniformly in $\underline{u}, \underline{v}$ with $u_n, v_n \le \eta$, $T_1, T_2 \in [0, 2]$ and λ sufficiently small. The target (7.10) is an immediate consequence by the Gibbs property and conditioning on the left-most $t_- \in [-4, -2]$ and the right-most $t_+ \in [4, 6]$ such that $x_n^\lambda(t_-), x_n^\lambda(t_+) \le \eta$; see Fig. 2.

The proof boils down to deriving an appropriate upper bound on the partition function $Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}$ and an appropriate matching lower bound on the constrained partition function $Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}[D_0]$.

In the sequel, $\underline{\tilde{x}}^\lambda$ stands for the *reversed* random walk with transition probabilities $\tilde{p}_z = p_{-z}$. Let $\tilde{\tau}$ be the first exit time of $\underline{\tilde{x}}^\lambda$ from $\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$. We assume that the constants ν and ρ in the probabilistic estimates (I.1)–(I.3) are chosen in such a way that the corresponding bounds hold for the reflected process as well.

STEP 1. (*An upper bound on* $Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}$) Since we are dealing with non-negative potentials,

$$\begin{aligned}
&Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda} \\
&\le \hat{\mathbf{P}}^{\underline{u}}_{2+T_1,\lambda}(\tau > 2 + T_1) \max_{\underline{r},\underline{s} \in \mathbb{A}^+_{n,\lambda}} \hat{\mathbf{P}}^{\underline{r}}_{2,\lambda}(\underline{x}^\lambda(2) = \underline{s})\, \hat{\mathbf{P}}^{\underline{v}}_{2+T_2,\lambda}(\tilde{\tau} > 2 + T_2) \\
&\le c_1 h_\lambda^n \hat{\mathbf{P}}^{\underline{u}}_{2+T_1,\lambda}(\tau > 2 + T_1)\, \hat{\mathbf{P}}^{\underline{v}}_{2+T_2,\lambda}(\tilde{\tau} > 2 + T_2).
\end{aligned} \tag{7.14}$$

The second inequality follows from the concentration bound (7.3).

STEP 2. (*A lower bound on* $Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}[D_0]$) By our assumption (3.12),

$$
\begin{aligned}
& Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}(D_0) \\
& \quad \geq \mathrm{e}^{-10nq_0(2\eta)} \mathbf{P}^{\underline{u}}_{-2-T_1,4+T_2,+,\lambda}\big(D_0, \max_{t\in[-2-T_1,4+T_2]} x^{\lambda}_n(t) \leq 2\eta, \underline{x}^{\lambda}(4+T_2) = \underline{v}\big).
\end{aligned}
\tag{7.15}
$$

Above $\hat{\mathbf{P}}^{\underline{v}}_{s,t,+,\lambda}$ is the provisional notation for the restriction of the law of the rescaled walk started at time s at $\underline{v}$ to the set of trajectories which stay inside $\mathbb{A}^{+}_{n,\lambda}$ during the time interval $[s,t]$.

The probability on the right-hand side of (7.15) is bounded below by the following product of three factors:

$$
\begin{aligned}
& \hat{\mathbf{P}}^{\underline{u}}_{2+T_1,\lambda}\big(\tau > 2+T_1, \max_{t\leq 2+T_1} x^{\lambda}_n(t) \leq 2\eta, \underline{x}^{\lambda}(2+T_1) \in \mathbb{A}^{+,\mathrm{r}}_{n,\lambda}\big) \\
& \quad \times \min_{\underline{r},\underline{s}\in\mathbb{A}^{+,\mathrm{r}}_{n,\lambda}} \hat{\mathbf{P}}^{\underline{r}}_{2,+,\lambda}(D_0, \underline{x}^{\lambda}(2) = \underline{s}) \\
& \quad \times \hat{\mathbf{P}}^{\underline{v}}_{2+T_2,\lambda}\big(\tilde{\tau} > 2+T_2, \max_{t\leq 2+T_2} \tilde{x}^{\lambda}_n(t) \leq 2\eta, \underline{\tilde{x}}^{\lambda}(2+T_2) \in \mathbb{A}^{+,\mathrm{r}}_{n,\lambda}\big).
\end{aligned}
\tag{7.16}
$$

On the one hand, in view of (7.8), the middle factor is bounded below by $c_2 h^n_{\lambda}$. On the other hand, the probabilistic bounds (I.2), (I.3) imply that the left-most factor in (7.16) is bounded below by $\rho^2 \hat{\mathbf{P}}^{\underline{u}}_{2+T_1,\lambda}(\tau > 2+T_1)$. Similarly, the right-most factor in (7.16) is bounded below by $\rho^2 \hat{\mathbf{P}}^{\underline{v}}_{2+T_2,\lambda}(\tilde{\tau} > 2+T_2)$. Hence,

$$
Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}[D_0] \geq c_3 \rho^4 h^n_{\lambda} \, \hat{\mathbf{P}}^{\underline{u}}_{2+T_1,\lambda}(\tau > 2+T_1) \, \hat{\mathbf{P}}^{\underline{v}}_{2+T_2,\lambda}(\tilde{\tau} > 2+T_2). \tag{7.17}
$$

Since

$$
\mathbb{P}^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}(D_0) = \frac{Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}[D_0]}{Z^{\underline{u},\underline{v}}_{-2-T_1,4+T_2,+,\lambda}},
$$

(7.13) directly follows from (7.14) and (7.17). □

7.5 Proof of (7.12)

We start by deriving a lower bound on partition functions, as this will allow us to exclude sets of pathological trajectories.

Lemma 7 *There exist constants $c_1 = c_1(n)$ and $c_2 = c_2(n,\eta)$ and a sufficiently large value $T_0 = T_0(\eta)$ such that, for all $T \geq T_0$,*

$$
Z^{\underline{w},\underline{z}}_{T,+,\lambda} \geq c_2 \, e^{-c_1 T} \, \hat{\mathbf{P}}^{\underline{w}}_{2T,+,\lambda}(\underline{x}^{\lambda}(2T) = \underline{z}), \tag{7.18}
$$

uniformly in λ small and in $z_n, w_n \le \eta$.

Proof The point is that constant c_1 does not depend on η, only on the dimension n. The dependence of c_1 on n is expressed in terms of the dependence of ϵ [in the definition of the regular set $\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$, see (6.1)] on n. We shall work with a fixed small value of $\epsilon > 0$ which satisfies

$$n\epsilon < 1. \tag{7.19}$$

In the sequel, we consider $T > 2$. Let $\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}(\alpha) = \mathbb{A}^{+,\mathsf{r}}_{n,\lambda} \cap \{\underline{x} : x_n \le \alpha\}$. Consider the events

$$\begin{aligned}\mathcal{E}_- &= \Big\{ \max_{t\in[0,1]} x_n^\lambda(t) \le 2\eta\,,\, \underline{x}(1) \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}(1)\Big\},\\ \mathcal{E}_+ &= \Big\{ \max_{t\in[2T-1,2T]} x_n^\lambda(t) \le 2\eta\,,\, \underline{x}(2T-1) \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}(1)\Big\}\end{aligned} \tag{7.20}$$

and

$$\mathcal{E}_T = \Big\{ \max_{t\in[1,2T-1]} x_n^\lambda(t) < 2\Big\}.$$

On the one hand, by (3.12),

$$Z^{\underline{w},\underline{z}}_{T,+,\lambda} \ge \mathrm{e}^{-2nq_0(2\eta)-2nq_0(2)T}\, \hat{\mathbf{P}}^{\underline{w}}_{2T,+,\lambda}\big(\mathcal{E}_-, \mathcal{E}_T, \mathcal{E}_+, \underline{x}^\lambda(2T) = \underline{z}\big). \tag{7.21}$$

On the other hand,

$$\hat{\mathbf{P}}^{\underline{w}}_{2T,+,\lambda}\left(\underline{x}^\lambda(2T) = \underline{z}\right) \le c_3 \hat{\mathbf{P}}^{\underline{w}}_{1,\lambda}\left(\tau > 1\right) \hat{\mathbf{P}}^{\underline{z}}_{1,\lambda}\left(\tilde{\tau} > 1\right) \frac{h_\lambda^n}{T^{n/2}}. \tag{7.22}$$

Above, we relied on the concentration bound (7.3).

In order to compare the probabilities appearing in (7.21) and (7.22), note that an application of (I.1)–(I.3) (and the observation that, as in (7.7), the cardinality $\left|\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}(1)\right| \ge c_4(\epsilon) h_\lambda^{-n}$) yields

$$\begin{aligned}\hat{\mathbf{P}}^{\underline{w}}_{2T,+,\lambda}(\mathcal{E}_-, \mathcal{E}_T, \mathcal{E}_+, \underline{x}^\lambda(2T) = \underline{z}) &\ge c_5 \hat{\mathbf{P}}^{\underline{w}}_{1,\lambda}(\tau > 1) \hat{\mathbf{P}}^{\underline{z}}_{1,\lambda}(\tilde{\tau} > 1)\\ &\times \min_{\underline{u},\underline{v}\in\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}(1)} \hat{\mathbf{P}}^{\underline{u}}_{2(T-1),+,\lambda}\Big(\max_{t\in[0,2(T-1)]} x_n^\lambda(t) < 2\,,\, \underline{x}^\lambda(2(T-1)) = \underline{v}\Big).\end{aligned} \tag{7.23}$$

However,

$$\min_{\underline{u},\underline{v}\in\mathbb{A}^{+,\mathsf{r}}_{n,\lambda}(1)} \hat{\mathbf{P}}^{\underline{u}}_{2(T-1),+,\lambda}\left(\max_{t\in[0,2(T-1)]} x_n^\lambda(t) < 2,\ \underline{x}^\lambda(2(T-1)) = \underline{v}\right) \ge \mathrm{e}^{-c_6(\epsilon)T} h_\lambda^n. \tag{7.24}$$

Indeed, consider n walks x_ℓ^λ, $\ell = 1, \ldots, n$, going from u_ℓ to v_ℓ inside space-time tubes of width $\epsilon/4$ centered around the space-time segments $[(u_\ell, 0), (v_\ell, 2(T-1)]$. By construction, these walks stay in $\mathbb{A}^{+}_{n,\lambda} \cap \{\underline{x} : x_n < 2\}$. By a coarse splitting into

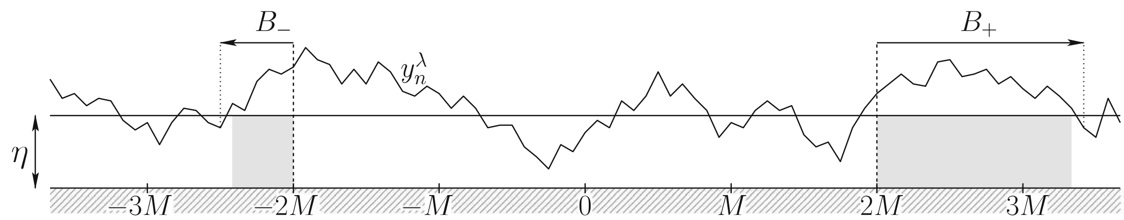

Fig. 3 The definition of the random variables B_- and B_+. Note that the *shaded areas* are necessarily below the path y_n^λ and thus contribute to the area-tilt

time-blocks of lengths of order ϵ^2, we bound from below the probability of staying within such tubes by $\mathrm{e}^{-c_6(\epsilon)T}$. Applying the local CLT for the last step, we bound from below the probability of ending up in $\underline{z}$ by a multiple of h_λ^n. (7.24) follows.

The bound (7.18) is a direct consequence of (7.22) and (7.21), (7.23) and (7.24). □

Let us resume the proof of (7.12). Without loss of generality, we shall assume that $a = 3M$ and $b \geq a$. In the sequel, the trajectory $\underline{x}^\lambda$ is sampled from $\mathbb{P}_{a,+,\lambda}^{\underline{r},\underline{s}}$ and $\underline{y}^\lambda$ is sampled from $\mathbb{P}_{b,+,\lambda}^{\underline{u},\underline{v}}$. Recall that $r_n, s_n, u_n, v_n \leq C \leq \eta$.

In principle, b can be much larger than M. Let us verify that one can restrict our attention to the case where b is of the same order as M. Define the random variables $B_\pm \geq 0$ via (see Fig. 3)

$$-2M - B_- = \max\{t \leq -2M \,:\, y_n^\lambda(t) \leq \eta\} \tag{7.25}$$

and, accordingly, $2M + B_+ = \min\{t \geq 2M \,:\, y_n^\lambda(t) \leq \eta\}$.

By the Gibbs property,

$$\begin{aligned}&\mathbb{P}_{b,+,\lambda}^{\underline{u},\underline{v}}(B_\pm = b_\pm)\\ &\quad\leq \max_{w_n,z_n\leq\eta} \mathbb{P}_{-2M-b_-,2M+b_+,+,\lambda}^{\underline{w},\underline{z}}\Big(\min_{t\in(-2M-b_-,-2M]} x_n^\lambda(t) \wedge \min_{t\in[2M,2M+b_+)} x_n^\lambda(t) > \eta\Big).\end{aligned}$$

Therefore, in view of (3.12),

$$\mathbb{P}_{b,+,\lambda}^{\underline{u},\underline{v}}(B_\pm = b_\pm) \leq \mathrm{e}^{-(b_-+b_+)q_0(\eta)} \max_{w_n,z_n\leq\eta} \frac{\hat{\mathbf{P}}_{T,+,\lambda}^{\underline{w}}(\underline{x}^\lambda(T) = \underline{z})}{\hat{Z}_{T,+,\lambda}^{\underline{w},\underline{z}}}, \tag{7.26}$$

where we have set $T = 4M + b_- + b_+$. Using the lower bound (7.18) on $\hat{Z}_{T,+,\lambda}^{\underline{w},\underline{z}}$, we conclude that

$$\mathbb{P}_{b,+,\lambda}^{\underline{u},\underline{v}}(B_\pm = b_\pm) \leq c_7(\epsilon)\mathrm{e}^{c_8(\epsilon)M-(b_-+b_+)q_0(\eta)}. \tag{7.27}$$

Therefore, if we choose η so large that

$$q_0(\eta) > 2c_8(\epsilon), \tag{7.28}$$

then we may ignore the case $b_\pm \geq M$.

Consequently, (7.12) will follow once we check that

$$\mathbb{P}^{\underline{r},\underline{s}}_{3M,+,\lambda} \otimes \mathbb{P}^{\underline{u},\underline{w}}_{-b_1,b_2,+,\lambda}(\mathcal{M}_0^{(5)} \le \nu^{(5)} M) \le \mathrm{e}^{-\kappa^{(5)} M}, \tag{7.29}$$

uniformly in λ small, M large, $b_1, b_2 \in [2M, 3M]$ and $r_n, s_n, u_n, v_n \le \eta$.

If we choose $\nu^{(5)}$ to be sufficiently small, for instance smaller than 0.2, then, by (3.12),

$$\begin{aligned}
&\mathbb{P}^{\underline{r},\underline{s}}_{3M,+,\lambda} \otimes \mathbb{P}^{\underline{u},\underline{w}}_{b,+,\lambda}(\mathcal{M}_0^{(5)} \le \nu^{(5)} M) \\
&\quad \le \mathrm{e}^{-\frac{q_0(\eta)}{10} M} \frac{\hat{\mathbf{P}}^{\underline{r}}_{6M,+,\lambda}(\underline{x}^{\lambda}(3M) = \underline{s}) \hat{\mathbf{P}}^{\underline{u}}_{b_1+b_2,+,\lambda}(\underline{y}^{\lambda}(b) = \underline{w})}{Z^{\underline{r},\underline{s}}_{3M,+,\lambda} Z^{\underline{u},\underline{w}}_{-b_1,b_2,+,\lambda}}.
\end{aligned} \tag{7.30}$$

Taking η and M large enough and applying (7.18), we arrive to (7.29). □

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Appendix 1: Strong approximation techniques

In order to prove (I.1), we are going to apply strong approximation techniques from [6]. By rescaling, it is sufficient to consider the case $t = 1$.

In the sequel, $\hat{\mathbf{P}}^{\underline{r}}_{+}$ denotes the restriction of the law of the n-dimensional Brownian motion $\underline{B}$ started at $\underline{r}$ to the set $\mathbb{A}_n^+$.

Define

$$O_\epsilon(\underline{z}) = \left\{ \underline{y} \,:\, |y_i - z_i| \le \frac{\epsilon}{3} \text{ for all } i \right\}.$$

It follows easily from [6, Lemma 17] that

$$\begin{aligned}
&\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^{\lambda}(1-\gamma) \in O_\epsilon(\underline{z}),\ \max_{t \le 1-\gamma} x_n^{\lambda}(t) \le 2\eta) \\
&\quad = \hat{\mathbf{P}}^{\underline{r}}_{+}(\underline{B}(1-\gamma) \in O_\epsilon(\underline{z}),\ \max_{t \le 1-\gamma} B_n(t) \le 2\eta) + o(1),
\end{aligned}$$

uniformly in $\underline{r}, \underline{z} \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$. This implies that there exists a constant $c(\epsilon, \eta, \gamma) > 0$ such that

$$\hat{\mathbf{P}}^{\underline{r}}_{+,\lambda}(\underline{x}^{\lambda}(1-\gamma) \in O_\epsilon(\underline{z}),\ \max_{t \le 1-\gamma} x_n^{\lambda}(t) \le 2\eta) \ge c(\epsilon, \eta, \gamma), \tag{7.31}$$

for all $\underline{r}, \underline{z} \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda}$. Since $O_\epsilon(\underline{z})$ is separated from the boundary of $\mathbb{A}_n^+$, we may choose γ so small that the probability that the random walk x^{λ} started at $\underline{y} \in O_\epsilon(\underline{z})$ has, at time γ, the value $\underline{z}$ and leaves $\mathbb{A}^+_{n,\lambda}$ before time γ is quite small. This heuristic is made precise in [6, Lemma 29]. In our notation, we can state that result as follows: There exist $a > 0$ and $c_1 < \infty$ such that

$$\begin{aligned}&\hat{\mathbf{P}}^{\underline{y}}_{+,\lambda}(\underline{x}^{\lambda}(\gamma)=\underline{z}, \max_{t\le\gamma} x_n^{\lambda}(t)\le 2\eta)\\&\quad\ge \hat{\mathbf{P}}^{\underline{y}}_{\lambda}(\underline{x}^{\lambda}(\gamma)=\underline{z}, \max_{t\le\gamma} x_n^{\lambda}(t)\le 2\eta) - c_1\gamma^{-n/2}e^{-a\epsilon^2/\gamma}h_{\lambda}^n.\end{aligned}$$

By a similar argument, one can show that

$$\begin{aligned}&\hat{\mathbf{P}}^{\underline{y}}_{\lambda}(\underline{x}^{\lambda}(\gamma)=\underline{z}, \max_{t\le\gamma} x_n^{\lambda}(t)\le 2\eta)\\&\quad\ge \hat{\mathbf{P}}^{\underline{y}}_{\lambda}(\underline{x}^{\lambda}(\gamma)=\underline{z}) - c_2\gamma^{-n/2}e^{-a\eta^2/\gamma}h_{\lambda}^n.\end{aligned}$$

Finally, by the standard local limit theorem,

$$\hat{\mathbf{P}}^{\underline{y}}_{\lambda}(\underline{x}^{\lambda}(\gamma)=\underline{z}) \ge c_3\gamma^{-n/2}h_{\lambda}^n.$$

As a result, we have the bound

$$\hat{\mathbf{P}}^{\underline{y}}_{+,\lambda}(\underline{x}^{\lambda}(\gamma)=\underline{z}, \max_{t\le\gamma} x_n^{\lambda}(t)\le 2\eta) \ge c_4 h_{\lambda}^n,$$

uniformly in $\underline{y}, \underline{z} \in \mathbb{A}^{+,\mathrm{r}}_{n,\lambda}$. Combining this bound with (7.31), we infer that

$$\hat{\mathbf{P}}^{\underline{r}}_{1,+,\lambda}(\underline{x}^{\lambda}(1)=\underline{z}, \max_{t\le 1} x_n^{\lambda}(t)\le 2\eta) \ge c_5 h_{\lambda}^n, \tag{7.32}$$

uniformly in $\underline{r}, \underline{z} \in \mathbb{A}^{+,\mathrm{r}}_{n,\lambda}$. □

Appendix 2: Invariance principles for random walks in Weyl chambers

Conditional limit theorems and conditional invariance principles for random walks in different cones have been studied in [6] and [8]. All the results in these papers are proved in the case when the non-rescaled walk starts at a fixed point. In this paragraph, we give certain improvements of these results to the case when the starting point of the non-rescaled walk may grow (but we shall consider walks in Weyl chambers only).

More precisely, we shall consider the following subsets of the Euclidean space:

- chamber of type A: $\{x : x_1 < x_2 < \cdots < x_n\}$;
- chamber of type C: $\{x : 0 < x_1 < x_2 < \cdots < x_n\}$;
- chamber of type D: $\{x : |x_1| < x_2 < \cdots < x_n\}$.

Let u_W denote the unique (up to a constant multiplier) positive harmonic function on W:

- if W is the chamber of type A, then $u_W(x) = \prod_{i<j}(x_j - x_i)$;
- if W is the chamber of type C, then $u_W(x) = \prod_k x_k \prod_{i<j}(x_j^2 - x_i^2)$;
- if W is the chamber of type D, then $u_W(x) = \prod_{i<j}(x_j^2 - x_i^2)$.

Proposition 4 *Let W be a Weyl chamber of type A, C or D. Let τ be the first exit time from W, that is,*

$$\tau = \inf\{t > 0 : \underline{x}^{\lambda}(t) \notin W\}.$$

Then, as $\underline{r} = \underline{r}_{\lambda} \to 0$,

$$\hat{\mathbf{P}}_{\lambda}^{\underline{r}}(\underline{x}^{\lambda}(1) \in \cdot \mid \tau > 1) \to \mu \quad \textit{weakly},$$

where μ is the probability measure on W with density proportional to $u_W(x)e^{-|x|^2/2}$.

Furthermore, under $\hat{\mathbf{P}}_{\lambda}^{\underline{r}}$, $\underline{x}^{\lambda}$ converges weakly on $C[0, 1]$ to the Brownian meander in W started at zero.

By "Brownian meander in W", we mean a Brownian motion conditioned on staying in W up to time one. If the starting point lies inside W, then one has a condition of positive probability. However, if the starting point lies on the boundary of W, then the probability of the condition is zero and it is not at all clear how one can construct such a process. Garbit [11] has constructed Brownian meanders started at zero for a quite large class of cones. This class includes Weyl chambers.

Proof The main difference with [6, Theorem 3] is that we find the limit for conditional distributions without determining the asymptotic behavior of $\hat{\mathbf{P}}_{\lambda}^{\underline{r}}(\tau > 1)$. (Recall once again that [6, Theorem 3] is proven under the assumption $\underline{r} = h_{\lambda}a$ for some fixed $a \in W$.)

Fix some $\epsilon \in (0, 1/2)$ and define the stopping time

$$\nu_{\lambda,\epsilon} = \inf\{t > 0 : \underline{x}^{\lambda}(t) \in W_{\lambda,\epsilon}\},$$

where

$$W_{\lambda,\epsilon} = \{x \in W : \operatorname{dist}(x, \partial W) \geq H_{\lambda}^{-2\epsilon}\}.$$

According to [6, Lemma 14],

$$\hat{\mathbf{P}}_{\lambda}^{\underline{r}}(\tau > H_{\lambda}^{-2\epsilon}, \nu_{\lambda,\epsilon} > H_{\lambda}^{-2\epsilon}) \leq e^{-c_1 H_{\lambda}^{2\epsilon}}, \tag{7.33}$$

uniformly in $\underline{r}$. Since we consider lattice random walks, there exists $\underline{r}_0$ such that

$$\hat{\mathbf{P}}_{\lambda}^{\underline{r}}(\tau > 1) \geq \hat{\mathbf{P}}_{\lambda}^{\underline{r}_0}(\tau > 1).$$

(If W is of type A or C, then we may take $r_0 = h_{\lambda}(1, 2, \ldots, n)$, while if W is of type D, then we may take $r_0 = h_{\lambda}(0, 1, \ldots, n-1)$.) According to [6, Theorem 1],

$$\hat{\mathbf{P}}_{\lambda}^{\underline{r}_0}(\tau > 1) \sim C_1 h_{\lambda}^{p},$$

where p is a positive constant depending on the type of W only. Consequently,

$$\hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > 1) \geq C_2 h_\lambda^p, \tag{7.34}$$

uniformly in $\underline{r}$. Combining (7.33) and (7.34), we infer that

$$\frac{\hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > H_\lambda^{-2\epsilon}, \nu_{\lambda,\epsilon} > H_\lambda^{-2\epsilon})}{\hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > 1)} \to 0, \quad \lambda \downarrow 0, \tag{7.35}$$

uniformly in $\underline{r}$. Furthermore, it follows from the exponential Doob inequality that

$$\hat{\mathbf{P}}_\lambda^{\underline{r}}\Big(\max_{t \leq H_\lambda^{-2\epsilon}} |\underline{x}^\lambda(t) - \underline{x}^\lambda(0)| > \theta_\lambda\Big) \leq e^{-c_2\theta_\lambda^2 H_\lambda^\epsilon},$$

where $\theta_\lambda \to 0$ sufficiently slowly. This implies that, whenever $|\underline{r}| \leq \theta_\lambda$,

$$\frac{\hat{\mathbf{P}}_\lambda^{\underline{r}}\big(\max_{t \leq H_\lambda^{-2\epsilon}} |\underline{x}^\lambda(t)| > 2\theta_\lambda\big)}{\hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > 1)} \to 0. \tag{7.36}$$

It follows now from (7.35) and (7.36) that, uniformly in $\underline{r}$,

$$\hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > 1) = (1 + o(1))\, \hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > 1, \nu_{\lambda,\epsilon} \leq H_\lambda^{-2\epsilon}, \max_{t \leq \nu_{\lambda,\epsilon}} |\underline{x}(t)| \leq 2\theta_\lambda) \tag{7.37}$$

and

$$\begin{aligned}&\hat{\mathbf{P}}_\lambda^{\underline{r}}(\underline{x}^\lambda(1) \in A, \tau > 1)\\ &\quad = (1 + o(1))\, \hat{\mathbf{P}}_\lambda^{\underline{r}}(\underline{x}^\lambda(1) \in A, \tau > 1, \nu_{\lambda,\epsilon} \leq H_\lambda^{-2\epsilon}, \max_{t \leq \nu_{\lambda,\epsilon}} |\underline{x}(t)| \leq 2\theta_\lambda)\end{aligned} \tag{7.38}$$

for any compact $A \subset W$.

Using the Markov property at time $\nu_{\lambda,\epsilon}$ and applying [6, Lemma 20], we obtain from (7.37) and (7.38)

$$\hat{\mathbf{P}}_\lambda^{\underline{r}}(\tau > 1) = (c_3 + o(1))\, h_\lambda^p\, \hat{\mathbf{E}}_\lambda^{\underline{r}}\big[u_W(\underline{x}^\lambda(\nu_{\lambda,\epsilon}));\ \nu_{\lambda,\epsilon} \leq H_\lambda^{-2\epsilon}, \max_{t \leq \nu_{\lambda,\epsilon}} |\underline{x}(t)| \leq 2\theta_\lambda\big]$$

and

$$\begin{aligned}\hat{\mathbf{P}}_\lambda^{\underline{r}}(\underline{x}^\lambda(1) \in A, \tau > 1) &= (c_4 + o(1))\, h_\lambda^p \int_A u_W(z) e^{-|z|^2/2} \mathrm{d}z\\ &\times \hat{\mathbf{E}}_\lambda^{\underline{r}}\big[u_W(\underline{x}^\lambda(\nu_{\lambda,\epsilon}));\ \nu_{\lambda,\epsilon} \leq H_\lambda^{-2\epsilon}, \max_{t \leq \nu_{\lambda,\epsilon}} |\underline{x}(t)| \leq 2\theta_\lambda\big].\end{aligned}$$

Thus, the proof of the first statement is completed.

To prove the functional convergence, it suffices to repeat the proof of [8, Theorem 1] using (7.35) and (7.36) instead of the corresponding estimates therein. □

Corollary 2 *Let W be the chamber of type C. If $\underline{r} = \underline{r}_\lambda \to \underline{r}^* \in \partial W$, then the sequence $\hat{\mathbf{P}}^{\underline{r}}_\lambda(\underline{x}^\lambda(1) \in A \,|\, \tau > 1)$ converges weakly. The densities of limiting laws on W has are uniformly bounded. Moreover, $\underline{x}^\lambda$ converges weakly on $C[0,1]$ towards the Brownian meander in W started at $\underline{r}^*$.*

Proof We just split the original set of random walks into a finite number of subsets in such a way that the differences of coordinates of the starting points in every block converge to zero and the differences of coordinates from different blocks stay bounded away from zero. Then, the probability that different blocks do not intersect is bounded away from zero and, consequently, the conditioning on $\{\tau > 1\}$ is equivalent to conditioning every block on staying in the corresponding chamber. (If $r_1^* > 0$, then every block is a random block in a chamber of type A, while if $r_1^* = 0$, then the lowest block is a random walk in a chamber of type C and all other blocks are random walks in chambers of type A.) □

Proof (Proof of (I.2)) Assume that there exists a sequence $\underline{r}(j)$ such that

$$\hat{\mathbf{P}}^{\underline{r}(j)}_{1,\lambda}\big(\max_{t\le 1} x_n^\lambda(t) \le 2\eta \;\big|\; \tau > 1\big) \to 0.$$

Since we are looking at starting points $\underline{r}$ with $r_n \le \eta$, there exists a convergent subsequence $\underline{r}(j_k)$. Let $\underline{r}^*$ denote the limiting point. It follows immediately from the usual functional CLT that the case $\underline{r}^* \in W$ is impossible. But, if $\underline{r}^* \in \partial W$, then we may use Corollary 2 to conclude that the Brownian meander in W started at $\underline{r}^*$ leaves the set $\{\underline{x} \in W : x_n \le 2\eta\}$ with probability one. However, this would contradict [18, Theorem 3.2]. Thus,

$$\inf_{\underline{r}:r_n\le\eta} \hat{\mathbf{P}}^{\underline{r}}_{1,\lambda}\big(\max_{t\le 1} x_n^\lambda(t) \le 2\eta \;\big|\; \tau > 1\big) > 0, \tag{7.39}$$

which implies (I.2). □

Proof (Proof of (I.3)) Fix some $\epsilon > 0$ and define

$$W_{\le\epsilon} = \{\underline{x} \in W : |x_{i+1} - x_i| \le \epsilon \text{ for some } i \ge 0\}.$$

Assume that there exists a sequence $\underline{r}(j)$ such that

$$\hat{\mathbf{P}}^{\underline{r}(j)}_{1,\lambda}\big(\underline{x}^\lambda(1) \in W_{\le\epsilon}\,;\, \max_{t\le 1} x_n^\lambda(t) \le 2\eta \;\big|\; \tau > 1\big) \ge \epsilon^{1/2}.$$

We may again assume that $\underline{r}(j)$ converges to $\underline{r}^*$ and this limiting point can not lie in W. But, if $\underline{r}^*$ is on the boundary of W, then the conditions of Corollary 2 are satisfied and the contradiction follows now from the boundedness of the density of the limiting law and the fact that $vol(W_{\le\epsilon} \cap \{\underline{x} : x_n \le 2\eta\}) \le C_3\eta^{n-1}\epsilon$.

As a consequence we have that, for all ϵ small enough,

$$\inf_{\underline{r}:r_n\le\eta} \hat{\mathbf{P}}^{\underline{r}}_{1,+,\lambda}\big(\underline{x}^\lambda(1) \in \mathbb{A}^{+,\mathsf{r}}_{n,\lambda},\, \max_{t\le 1} x_n^\lambda(t) \le 2\eta \;\big|\; \tau > 1\big) \ge 1 - \epsilon^{1/2}. \tag{7.40}$$

Combining (7.39) and (7.40), we conclude that (I.3) holds for $t = 1$ and all ϵ sufficiently small. Using Brownian scaling, we conclude that (I.3) is valid for all $t > 0$. $\square$

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