Parameters of the fractional Fokker-Planck equation

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Introduction. – Heavy-tailed distributions, *i.e.*, probability distributions with power tails and infinite second moments, are an important tool for studying a number of physical, biological, economical and other systems whose behavior is determined by rare but large events [1–4]. In many cases the continuous-time dynamics of these systems can be effectively described by the (dimensionless) overdamped Langevin equation

$$\dot{x}(t) = f(x(t), t) + \xi(t), \tag{1}$$

where $x(t)$ $[x(0) = 0]$ is a state parameter of the system, $f(x,t)$ is a deterministic function, and $\xi(t)$ is a random noise defined by the infinitesimal increments $\Delta\eta(t) = \int_t^{t+\tau} \mathrm{d}t' \xi(t')$ $(\tau \to 0)$ that are assumed to be independent on non-overlapping intervals and distributed with a heavy-tailed distribution. Since the tails of these distributions cannot be neglected, the classical stochastic theory, which is based on the ordinary central-limit theorem, is not applicable to eq. (1). Specifically, if the increments are distributed according to a Lévy stable distribution [5], *i.e.*, $\xi(t)$ is a Lévy stable noise, then the probability density $P(x,t)$ that $x(t) = x$ satisfies the *fractional* Fokker-Planck equation [6–11] which can be written as

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} f(x,t) P(x,t) + \gamma \frac{\partial^\alpha}{\partial |x|^\alpha} P(x,t) + \gamma\beta \tan\frac{\pi\alpha}{2} \frac{\partial}{\partial x} \frac{\partial^{\alpha-1}}{\partial |x|^{\alpha-1}} P(x,t). \tag{2}$$

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Here, the Riesz derivative, $\partial^\alpha/\partial|x|^\alpha$, is defined as [12] $\partial^\alpha h(x)/\partial|x|^\alpha = -\mathcal{F}^{-1}\{|k|^\alpha h_k\}$, a pair $\mathcal{F}\{h(x)\} \equiv h_k = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ikx} h(x)$ and $\mathcal{F}^{-1}\{h_k\} \equiv h(x) = (1/2\pi) \times \int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx} h_k$ represents the Fourier transforms, and α, β and γ are the parameters of the stable distribution.

Because of the generalized central limit theorem [13], the Lévy stable distributions constitute an important but a particular class of heavy-tailed distributions. In this letter, we show that the fractional Fokker-Planck equation (2) is valid also for *all* noises $\xi(t)$ whose increments have heavy-tailed distributions. Explicit expressions for the parameters of eq. (2) are derived in terms of the asymptotic characteristics of these distributions.

Definitions and basic equations. – Our starting point is the generalized Fokker-Planck equation [14]

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} f(x,t) P(x,t) + \mathcal{F}^{-1}\{P_k(t) \ln S_k\}, \tag{3}$$

which corresponds to the Langevin equation (1) driven by an *arbitrary* noise. The term "arbitrary" means that the independent increments $\Delta\eta(j\tau) = \eta(j\tau + \tau) - \eta(j\tau) = \int_{j\tau}^{j\tau+\tau} \mathrm{d}t' \xi(t')$ $(\tau \to 0,\ j = 0,\ 1,\ \ldots)$ of the discrete-time noise generating process $\eta(n\tau) = \sum_{j=0}^{n-1} \Delta\eta(j\tau)$ $(n = 1, 2, \ldots)$ are distributed according to an arbitrary probability density function $p(\Delta\eta, \tau)$. In other words, $p(\Delta\eta, \tau)$ is the transition probability density of the process $\eta(n\tau)$. It is assumed that i) $p(\Delta\eta, \tau)$ is properly normalized, *i.e.*, $\int_{-\infty}^{\infty} \mathrm{d}(\Delta\eta)\, p(\Delta\eta, \tau) = 1$, ii) the first moment, if it exists, equals zero, *i.e.*, $\int_{-\infty}^{\infty} \mathrm{d}(\Delta\eta)\, p(\Delta\eta, \tau)\Delta\eta = 0$, and iii) $\lim_{\tau\to 0} p(\Delta\eta, \tau) = \delta(\Delta\eta)$, where $\delta(\cdot)$ is the Dirac δ

function. The characteristic function $S_k = \langle e^{-ik\eta(1)} \rangle$ of $\eta(1) = \lim_{\tau\to 0} \sum_{j=0}^{[1/\tau]-1} \Delta\eta(j\tau)$ ($[1/\tau]$ denotes the integer part of $1/\tau$) is connected with the characteristic function $p_k(\tau) = \langle e^{-ik\Delta\eta(j\tau)} \rangle$ of $\Delta\eta(j\tau)$, *i.e.*, the Fourier transform of $p(\Delta\eta, \tau)$, via the relation [14]

$$\ln S_k = \lim_{\tau\to 0} \frac{1}{\tau}[p_k(\tau) - 1]. \tag{4}$$

The generalized central-limit theorem [13] implies that for a wide class of properly scaled probability densities $p(\Delta\eta, \tau)$ the characteristic function S_k corresponds to Lévy stable distributions. These distributions are described by four parameters [5]: an index of stability $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma \in (0, \infty)$, and a location parameter $\rho \in (-\infty, \infty)$, which, in accordance with the initial condition $P(x, 0) = \delta(x)$, is assumed to be zero. Therefore, excluding from consideration the singular case when $\alpha = 1$ and $\beta \neq 0$ simultaneously (in this case $|\ln S_k| = \infty$ if $k \neq 0$, see below, and as a consequence the system reaches the final state immediately), one obtains

$$S_k = \exp\left[-\gamma|k|^\alpha \left(1 + i\beta\, \mathrm{sgn}(k) \tan\frac{\pi\alpha}{2}\right)\right], \tag{5}$$

and eq. (3) reduces to eq. (2) [14].

In order to find the stable parameters in eq. (2), the transition probability density $p(\Delta\eta, \tau)$ must be specified. Next we consider a class of the functions $p(\Delta\eta, \tau)$ defined as

$$p(\Delta\eta, \tau) = \frac{1}{a(\tau)} p\left(\frac{\Delta\eta}{a(\tau)}\right). \tag{6}$$

Here, $a(\tau)$ is a positive scale function that vanishes at $\tau = 0$, and the rescaled transition probability density, $p(y)$, is an arbitrary probability density which satisfies the condition $\lim_{\epsilon\to 0} p(y/\epsilon)/\epsilon = \delta(y)$ and has zero first moment (if it exists). According to this definition, $p_k(\tau) = p_{ka(\tau)}$ and the normalization condition for $p(y)$, which is equivalent to $p_0 = 1$, yields $\ln S_0 = 0$. If $k \neq 0$, then $|\ln S_k| \in [0, \infty]$ and we can select three physically different situations depending on how quickly $a(\tau)$ tends to zero as $\tau \to 0$. First, if $p_{ka(\tau)} - 1 = o(\tau)$ (the scale function quickly vanishes), then $\ln S_k = 0$ and the noise is so weak that it does not effect the system at all. Second, if $p_{ka(\tau)} - 1$ tends to zero slower than τ (the scale function slowly vanishes), then $|\ln S_k| = \infty$, *i.e.*, the influence of the noise is so strong that the system relaxes instantaneously to the final state. Finally, the case we are primarily interested in corresponds to $p_{ka(\tau)} - 1 = O(\tau)$, *i.e.*, $0 < |\ln S_k| < \infty$ and the noise acts on the system in a non-trivial way.

Using eqs. (4), (6) and the representation $\ln S_k = R(k) + iI(k)$, we find the real,

$$R(k) = -\lim_{\tau\to 0} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}y\, p(y)[1 - \cos(ka(\tau)y)], \tag{7}$$

and imaginary,

$$I(k) = -\lim_{\tau\to 0} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}y\, p(y) \sin(ka(\tau)y), \tag{8}$$

parts of $\ln S_k$. In the next two sections we will evaluate $R(k)$ and $I(k)$ and express the stable parameters through a few main characteristics of $p(y)$ and $a(\tau)$. It should be noted in this context that the results are quite different for the cases with finite and infinite variance $\sigma^2 = \int_{-\infty}^{\infty} \mathrm{d}y\, p(y) y^2$ of the probability density $p(y)$. Mathematically, this difference arises from the fact that in the latter case the operations of taking the limit and integration in eqs. (7) and (8) do not commute.

Density functions with finite variance. – If the variance σ^2 is finite, then, taking first the limit and then integrating, from eq. (7) we obtain

$$R(k) = -\frac{k^2\sigma^2}{2} \lim_{\tau\to 0} \frac{a^2(\tau)}{\tau}. \tag{9}$$

Since, by assumption, the first moment of $p(y)$ equals zero, for calculating $I(k)$ we temporarily assume that the third moment, $m_3 = \int_{-\infty}^{\infty} \mathrm{d}y\, p(y) y^3$, exists. This yields

$$I(k) = -\frac{k^3 m_3}{6} \lim_{\tau\to 0} \frac{a^3(\tau)}{\tau}. \tag{10}$$

As is seen from eqs. (9) and (10), the condition $0 < |\ln S_k| < \infty$ holds only if $a^2(\tau) \sim q\tau$ $(0 < q < \infty)$. In this case $I(k) = 0$, $\mathcal{F}^{-1}\{P_k(t) \ln S_k\} = (\sigma^2 q/2)\partial^2 P(x, t)/\partial x^2$, and the generalized Fokker-Planck equation (3) reduces to the ordinary Fokker-Planck equation [15]

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} f(x, t) P(x, t) + \gamma \frac{\partial^2}{\partial x^2} P(x, t), \tag{11}$$

which has the form of eq. (2) with

$$\alpha = 2, \quad \gamma = \frac{\sigma^2 q}{2}. \tag{12}$$

If the third moment m_3 does not exist, then $I(k)$ can be evaluated by the method described in the next section, yielding the same result: $I(k) = 0$. Thus, if the increments of the noise generating process have *finite* variance, the Langevin equation (1) is always associated with the *ordinary* Fokker-Planck equation (11). In this case the noise $\xi(t)$ is white, *i.e.*, it has a constant power spectral density at all frequencies, and γ is the white noise intensity. In particular, for Gaussian white noise of intensity D characterized by the Gaussian probability density $p(y) = (4\pi D)^{-1/2} \exp[-y^2/(4D)]$ and the scale function $a(\tau) = \tau^{1/2}$, we obtain $\sigma^2 = 2D$, $q = 1$, and so $\gamma = D$.

Density functions with infinite variance. – For noises with $\sigma^2 = \infty$ the power spectral density does not exist. To find $R(k)$ and $I(k)$ in this case, we consider a class of probability densities $p(y)$ whose asymptotic behavior is characterized by heavy tails, *i.e.*,

$$p(y) \sim \frac{h_\pm}{|y|^{1+\Lambda_\pm}} \quad (y \to \pm\infty), \tag{13}$$

with $\Lambda = \min\{\Lambda_+, \Lambda_-\} \in (0,2]$ and $h_\pm > 0$. Let us first calculate $R(k)$ at $\Lambda \in (0,2)$. In this case, representing eq. (7) in the form

$$R(k) = -\lim_{\tau\to 0} \frac{1}{\tau a(\tau)|k|} \int_0^\infty \mathrm{d}z \left[p\left(\frac{z}{a(\tau)|k|}\right) + p\left(\frac{-z}{a(\tau)|k|}\right)\right](1-\cos z), \tag{14}$$

replacing $p(y)$ by the asymptotic formula (13), and using the integral relation

$$\int_0^\infty \mathrm{d}z \frac{1-\cos z}{z^{1+\Lambda}} = \frac{\pi}{2\Gamma(1+\Lambda)\sin(\pi\Lambda/2)} \tag{15}$$

($\Lambda \in (0,2)$, $\Gamma(\cdot)$ is the gamma function), we find

$$R(k) = -|k|^\Lambda \frac{\pi(h_+\delta_{\Lambda\Lambda_+} + h_-\delta_{\Lambda\Lambda_-})}{2\Gamma(1+\Lambda)\sin(\pi\Lambda/2)} \lim_{\tau\to 0} \frac{a^\Lambda(\tau)}{\tau}, \tag{16}$$

where $\delta_{\Lambda\Lambda_\pm}$ is the Kronecker symbol.

At $\Lambda = 2$ the integral in eq. (15) does not exist. Therefore, for calculating $R(k)$ in this case, we present the integral in eq. (14) as a sum of two integrals over the intervals $(0, a(\tau)|k|\xi)$ and $(a(\tau)|k|\xi, \infty)$ with $\xi = O(1)$. This yields $R(k) = R_1(k,\xi) + R_2(k,\xi)$, where

$$R_1(k,\xi) = -\frac{k^2}{2} \int_0^\xi \mathrm{d}y\, [p(y) + p(-y)]y^2 \lim_{\tau\to 0} \frac{a^2(\tau)}{\tau} \tag{17}$$

and

$$R_2(k,\xi) = -k^2(h_+\delta_{2\Lambda_+} + h_-\delta_{2\Lambda_-}) \lim_{\tau\to 0} \frac{a^2(\tau)}{\tau} \times \int_{a(\tau)|k|\xi}^\infty \mathrm{d}z \frac{1-\cos z}{z^3}. \tag{18}$$

Since $\int_{a(\tau)|k|\xi}^\infty \mathrm{d}z\, (1-\cos z) z^{-3} \sim (1/2)\ln[1/a(\tau)] \to \infty$ as $\tau \to 0$, the first term, $R_1(k,\xi)$, can be neglected in comparison with the second, $R_2(k,\xi)$, yielding

$$R(k) = -k^2 \frac{h_+\delta_{2\Lambda_+} + h_-\delta_{2\Lambda_-}}{2} \lim_{\tau\to 0} \frac{a^2(\tau)}{\tau} \ln\frac{1}{a(\tau)}. \tag{19}$$

In order to find explicit expressions for the imaginary part of $\ln S_k$, we first rewrite eq. (8) in the form

$$I(k) = -\lim_{\tau\to 0} \frac{1}{\tau a(\tau) k} \int_0^\infty \mathrm{d}z \left[p\left(\frac{z}{a(\tau)|k|}\right) - p\left(\frac{-z}{a(\tau)|k|}\right)\right] \sin z. \tag{20}$$

Then, assuming that $\Lambda \in (0,1)$, we substitute eq. (13) into eq. (20). Finally, taking into account that

$$\int_0^\infty \mathrm{d}z \frac{\sin z}{z^{1+\Lambda}} = \frac{\pi}{2\Gamma(1+\Lambda)\cos(\pi\Lambda/2)} \tag{21}$$

if $\Lambda \in (0,1)$, eq. (20) can be reduced to

$$I(k) = -k|k|^{\Lambda-1} \frac{\pi(h_+\delta_{\Lambda\Lambda_+} - h_-\delta_{\Lambda\Lambda_-})}{2\Gamma(1+\Lambda)\cos(\pi\Lambda/2)} \lim_{\tau\to 0} \frac{a^\Lambda(\tau)}{\tau}. \tag{22}$$

If $\Lambda \in (1,2]$ then the integral in eq. (21) diverges at the lower limit of integration, and the described approach becomes inapplicable. To generalize it to $\Lambda \in (1,2]$, we use the condition $\int_{-\infty}^\infty \mathrm{d}y\, p(y) y = 0$, which permits us to replace $\sin z$ by $\sin z - z$ in eq. (20). As a consequence, we arrive to the integral $\int_0^\infty \mathrm{d}z\, (\sin z - z) z^{-1-\Lambda}$ that can be calculated by the same formula (21), *i.e.*,

$$\int_0^\infty \mathrm{d}z \frac{\sin z - z}{z^{1+\Lambda}} = \frac{\pi}{2\Gamma(1+\Lambda)\cos(\pi\Lambda/2)}. \tag{23}$$

Thus, the representation (22) remains valid for $\Lambda \in (1,2]$ as well.

At $\Lambda = 1$ both approaches developed for $\Lambda \in (0,1)$ and $\Lambda \in (1,2]$ are not applicable (the integrals in eqs. (21) and (23) are divergent). Therefore, to calculate $I(k)$ at $\Lambda = 1$, we use the method applied to find $R(k)$ at $\Lambda = 2$. According to that we write $I(k) = I_1(k,\xi) + I_2(k,\xi)$, where

$$I_1(k,\xi) = -k \int_0^\xi \mathrm{d}y\, [p(y) - p(-y)]y \lim_{\tau\to 0} \frac{a(\tau)}{\tau} \tag{24}$$

and

$$I_2(k,\xi) = -k(h_+\delta_{1\Lambda_+} - h_-\delta_{1\Lambda_-}) \lim_{\tau\to 0} \frac{a(\tau)}{\tau} \int_{a(\tau)|k|\xi}^\infty \mathrm{d}z \frac{\sin z}{z^2}. \tag{25}$$

Then, using the asymptotic formula $\int_{a(\tau)|k|\xi}^\infty \mathrm{d}z\, z^{-2} \sin z \sim \ln[1/a(\tau)]$ that occurs as $\tau \to 0$, one obtains

$$I(k) = -k(h_+\delta_{1\Lambda_+} - h_-\delta_{1\Lambda_-}) \lim_{\tau\to 0} \frac{a(\tau)}{\tau} \ln\frac{1}{a(\tau)}. \tag{26}$$

Next, on the basis of the above-derived results we can express the parameters of the fractional Fokker-Planck equation (2) through the asymptotic characteristics of the probability density $p(y)$ (at $y \to \pm\infty$) and the scale function $a(\tau)$ (at $\tau \to 0$). If $\Lambda \in (0,1)$ or $\Lambda \in (1,2)$ then, using eqs. (16) and (22) and the definition of the Riesz derivative according to which

$$\mathcal{F}^{-1}\{|k|^\Lambda P_k(t)\} = -\frac{\partial^\Lambda}{\partial |x|^\Lambda} P(x,t), \qquad \mathcal{F}^{-1}\{ik|k|^{\Lambda-1} P_k(t)\} = -\frac{\partial}{\partial x}\frac{\partial^{\Lambda-1}}{\partial |x|^{\Lambda-1}} P(x,t), \tag{27}$$

we obtain that the generalized Fokker-Planck equation (3) reduces to the fractional one (2) with

$$\alpha = \Lambda, \qquad \beta = \frac{h_+\delta_{\Lambda\Lambda_+} - h_-\delta_{\Lambda\Lambda_-}}{h_+\delta_{\Lambda\Lambda_+} + h_-\delta_{\Lambda\Lambda_-}}, \qquad \gamma = \frac{\pi(h_+\delta_{\Lambda\Lambda_+} + h_-\delta_{\Lambda\Lambda_-})}{2\Gamma(1+\Lambda)\sin(\pi\Lambda/2)}\, q, \tag{28}$$

and $q = \lim_{\tau\to 0} a^{\Lambda}(\tau)/\tau$. The condition $0 < |\ln S_k| < \infty$ assumes that $0 < q < \infty$, and so the scale parameter must be proportional to $\tau^{1/\Lambda}$, *i.e.*, $a(\tau) \propto \tau^{1/\Lambda}$. We note also that $\beta = (h_+ - h_-)/(h_+ + h_-)$ if $\Lambda_+ = \Lambda_- = \Lambda$, $\beta = 1$ if $\Lambda_- > \Lambda_+ = \Lambda$, and $\beta = -1$ if $\Lambda_+ > \Lambda_- = \Lambda$.

According to eqs. (16) and (26), at $\Lambda = 1$ and $h_+\delta_{1\Lambda_+} \neq h_-\delta_{1\Lambda_-}$ the condition $0 < |\ln S_k| < \infty$ holds only if $a(\tau) \propto \tau/\ln(1/\tau)$. In this case $R(k) = 0$ and eq. (3) takes the form

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x}[f(x,t) + f_0]P(x,t), \tag{29}$$

where

$$f_0 = q(h_+\delta_{1\Lambda_+} - h_-\delta_{1\Lambda_-}) \tag{30}$$

and $q = \lim_{\tau\to 0} a(\tau)\ln[1/a(\tau)]/\tau$. Equation (29) presents a very unexpected and truly remarkable result. It shows that, in contrast to common belief, a given type of noise *does not* generate a random dynamics of the system, but rather acts as a *constant* force f_0. This force arises from the difference in the asymptotic behavior of the probability density $p(y)$ at $y \to \infty$ and $y \to -\infty$. If $f_0 = 0$, *i.e.*, $h_+ = h_- = h$ and $\Lambda_+ = \Lambda_- = 1$, then eq. (29) becomes invalid, and the next terms of the asymptotic expansion of $p(y)$ as $y \to \pm\infty$ must be taken into account. For the special case of symmetric noise, when $p(y) = p(-y)$, eqs. (8) and (16) yield $I(k) = 0$ and $R(k) = -|k|\pi hq$, respectively, and thus eq. (3) can also be written in the form of eq. (2) with

$$\alpha = 1, \quad \beta \tan\frac{\pi\alpha}{2} = 0, \quad \gamma = \pi hq, \tag{31}$$

and $q = \lim_{\tau\to 0} a(\tau)/\tau$, *i.e.*, $a(\tau) \sim q\tau$. We note in this context that if the same scale function, $a(\tau) \sim q\tau$, is chosen for asymmetric noise with $h_+ \neq h_-$ then $|I(k)| = \infty$ and, consequently, the system reaches the final state immediately. This result clarifies the nature of instabilities in the numerical simulation of $x(t)$ which occur in the present case (see ref. [16] and references therein).

Finally, at $\Lambda = 2$ a non-trivial action of the noise $\xi(t)$ takes place only if the scale function satisfies the condition $a^2(\tau) \sim 2q\tau/\ln(1/\tau)$ as $\tau \to 0$. In this case $\lim_{\tau\to 0} a^2(\tau)\ln[1/a(\tau)]/\tau = q$, $\lim_{\tau\to 0} a^2(\tau)/\tau = 0$, and, as it follows from eqs. (19) and (22), $R(k) = -k^2(h_+\delta_{2\Lambda_+} + h_-\delta_{2\Lambda_-})q/2$ and $I(k) = 0$. Therefore, at $\Lambda = 2$ eq. (3) takes the form of the ordinary Fokker-Planck equation (11), *i.e.*, eq. (2) with

$$\alpha = 2, \quad \gamma = \frac{h_+\delta_{2\Lambda_+} + h_-\delta_{2\Lambda_-}}{2}\, q. \tag{32}$$

It should be emphasized that although eq. (11) is the same for $\sigma^2 \neq \infty$ and $\Lambda = 2$, these two cases are quite different because at $\Lambda = 2$ the variance σ^2 of $p(y)$ is infinite. This difference results in different dependence of the scale parameter γ on $p(y)$. Namely, while in the former case it is proportional to σ^2 (see eq. (12)), *i.e.*, an *integral* characteristic of $p(y)$, in the latter case the scale parameter is determined by the *tails* of $p(y)$ (see eq. (32)).

Thus, we have shown that i) each noise whose increments have a heavy-tailed distribution acts on the system the same (in the sense of the probability density $P(x,t)$) as a *certain* Lévy stable noise, and ii) the action of each Lévy stable noise can be reproduced by *different* noises characterized by different distributions of the noise increments. We have determined the parameters of the fractional Fokker-Planck equation (2) that corresponds to the overdamped Langevin equation (1). If the transition probability density of the noise-generating process is heavy tailed, then these parameters are expressed through the characteristics of the tails. Otherwise, the fractional Fokker-Planck equation reduces to the ordinary one. These theoretical results seems to be especially important for the simplification and validation of numerical simulations of the Langevin systems driven by noises with heavy-tailed distributions of the increments.

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